EXPONENTIAL SEPARATION
AND PRINCIPAL FLOQUET BUNDLES
FOR LINEAR PARABOLIC EQUATIONS ON GENERAL
BOUNDED DOMAINS: NONDIVERGENCE CASE

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ABSTRACT. We consider the Dirichlet problem for linear nonautonomous second order parabolic equations of nondivergence type on general bounded domains with bounded measurable coefficients. Under such minimal regularity assumptions, we establish the existence of a principal Floquet bundle exponentially separated from a complementary invariant bundle. As a special case of our main theorem, assuming the coefficients are time-periodic, we obtain a new result on the existence of a principal eigenvalue of an associated (time-periodic) parabolic eigenvalue problem. We also show the existence of a uniform spectral gap between the principal eigenvalue and the rest of the spectrum for a class of time-periodic uniformly parabolic operators. Finally, we prove the uniqueness of positive entire solutions in the class of solutions whose supremum norms do not grow superexponentially as time goes to negative infinity.

1. INTRODUCTION

In this paper we consider linear nonautonomous nondivergence structure parabolic equations

$$
\begin{aligned}
u_t - Lu = 0 &\quad \text{in} \quad \Omega \times J, \\
u = 0 &\quad \text{on} \quad \partial\Omega \times J.
\end{aligned}
\tag{1.1}
$$

Here $\Omega \subset \mathbb{R}^N$ is an arbitrary bounded domain, J is an open interval in $\mathbb{R}$, and L is a time-dependent second-order elliptic operator of the nondivergence form

$$
Lu = Mu + c_0 u = a_{ij}(x,t)D_{ij}u + b_i(x,t)D_i u + c_0(x,t)u
\tag{1.2}
$$

(we use the summation convention and the notation $D_i = \partial/\partial x_i$). We assume that the coefficients of L satisfy the following conditions.

(L1) $a_{ij} \in C(\Omega \times \mathbb{R})$, b_i, c_0 are measurable and for some positive constants α_0, β_0 we have

$$|a_{ij}(x,t)|, \ |b_i(x,t)|, \ |c_0(x,t)| < \beta_0 \quad (i,j = 1,\ldots,N, \ x \in \Omega, \ t \in \mathbb{R}),$$
$$a_{ij}(x,t)\xi_i\xi_j \geq \alpha_0|\xi|^2 \qquad (\xi \in \mathbb{R}^N, \ x \in \Omega, \ t \in \mathbb{R}).$$

In case L is a *time-independent* (elliptic) or *time-periodic* operator, the existence of an associated principal eigenvalue and eigenfunction is well known and has been thoroughly studied (see, for example, [1, 3, 4, 7, 11]).

If L is *time-dependent* with general time-dependence, one can still define objects analogous to principal eigenvalues and principal eigenfunctions, the relevant concepts being the *principal spectrum*, the *principal Floquet bundle* and the associated *exponential separation*. Investigation of the principal Floquet bundles and exponential separation has been carried out in [13, 14, 15, 16, 17, 22, 24, 25, 26, 27, 31, 33]. Typically, one proves the existence of two time-dependent spaces $X^1(t)$, $X^2(t)$, $t \in \mathbb{R}$, which are subspaces of a suitable Banach space X from which one takes the initial conditions; $X^1(t)$ is the (one-dimensional) span of a positive function, $X^2(t) \setminus \{0\}$ does not contain any nonnegative function and the subspaces are complementary to one another:

$$(1.3) \qquad\qquad\qquad X = X^1(t) \oplus X^2(t) \quad (t \in \mathbb{R}).$$

Moreover, $X^1(t)$ is uniquely determined by Ω and L and the bundles $X^i(t)$, $t \in \mathbb{R}$, $i = 1, 2$ are invariant for (1.1): if $u_1(t)$, $u_2(t)$ are solutions with $u_i(s) \in X^i(s)$, then $u_i(t) \in X^i(t)$ for all $t > s$. Finally, if $u_i(s) \in X^i(s)$ $(i = 1, 2)$, assuming u_1 is nontrivial, we have the estimate (which we call the exponential separation)

$$(1.4) \qquad\qquad \frac{||u_2(\cdot,t)||_X}{||u_1(\cdot,t)||_X} \leq Ce^{-\gamma(t-s)}\frac{||u_2(\cdot,s)||_X}{||u_1(\cdot,s)||_X} \quad (t \geq s),$$

where $C, \gamma > 0$ are some positive constants. The one dimensional bundle $X^1(t)$, $t \in \mathbb{R}$, also known as the principal Floquet bundle, is an analogue of the principal eigenfunction and its uniqueness parallels the simplicity of the principal eigenvalue in the time-independent (or time-periodic) case. Moreover, exponential separation is a generalization of the fact that the principal eigenvalue of a time-independent elliptic (or time-periodic parabolic) operator is smaller than the real part of any other eigenvalue. We refer the reader to [5,6, 12–18, 22–27, 31–36] for the motivation and applications of the study of invariant bundles and exponential separation in nonautonomous parabolic equations with general time-dependence.

The motivation for our present work comes from the previous works on exponential separation cited above. In all of the references, where nondivergence equations are considered, some degree of smoothness of the domain is assumed to derive an estimate as in (1.4). The least restrictive of these references is [17], where the domain is assumed to be only Lipschitz and the regularity assumptions on the coefficients are the same as in (L1). In particular, at least for Lipschitz domains, the smoothness of the coefficients is irrelevant in establishing (1.3) and (1.4). It seems therefore natural to ask whether exponential separation can be derived without any smoothness assumptions on the domain. We have recently answered this question in the affirmative in [13] for divergence form equations. For nondivergence form equations on general domains such results are only available if L in (1.2) has time-independent coefficients [3], a much simpler case than the one with general time-dependence considered here.

It is our goal, inspired by the development in [3] and [13], to establish the existence of a principal Floquet bundle exponentially separated from a complementary invariant bundle for equations (1.1), whose coefficients satisfy (L1), on *arbitrary* bounded domains. Such an undertaking should also reveal what is the true underlying principle behind the results on exponential separation.

We find it instructive to first discuss some of the results proved in [17], where Ω is assumed to be Lipschitz. This will also enable us to show what difficulties we have to face in the general case. We denote by $u(\cdot, t; s, u_0)$ $(t \geq s)$ the solution of (1.1) with the initial condition $u(\cdot, s) = u_0$, where $u_0 \in C_0(\bar{\Omega})$, the space of continuous functions vanishing on $\partial \Omega$. The following result is a special case of [17, Theorem 2.2].

Theorem 1.1. *Let Ω be a Lipschitz bounded domain. Let $u_0 \in C_0(\bar{\Omega})$ be such that the solution $u(\cdot, t, s, u_0)$ vanishes somewhere in Ω for all $t > s$. Moreover, let v be a positive solution of* (1.1) *on $\Omega \times (s - 1, \infty)$. Then there are constants $C, \gamma > 0$ such that*

$$(1.5) \qquad \frac{||u(\cdot, t; s, u_0)||_{L^\infty(\Omega)}}{||v(\cdot, t)||_{L^\infty(\Omega)}} \leq C e^{-\gamma (t-s)} \frac{||u_0||_{L^\infty(\Omega)}}{||v(\cdot, s)||_{L^\infty(\Omega)}} \quad (t \geq s).$$

Moreover, C and γ depend only on N, α_0 and β_0 in (L1) *and on the Lipschitz characteristics of Ω.*

Roughly speaking, Theorem 1.1 states that on Lipschitz domains sign-changing solutions are exponentially dominated by positive solutions. As a consequence of this theorem the authors of [17] have given the following characterization of the space $X^2(t)$ in the decomposition (1.3):

$$(1.6) \qquad X^2(t) := \{u_0 \in C_0(\bar{\Omega}) : u(\cdot, \tilde{t}, t, u_0) \text{ has a zero in } \Omega \text{ for all } \tilde{t} > t\}.$$

The proofs of both Theorem 1.1 and (1.6) crucially depend on a new elliptic-type Harnack inequality, which was also established in [17] (see [17, Theorem 2.1]); we state this inequality in Theorem 3.7 below. By means of examples it has been shown in [13] that there are bounded domains Ω such that, even if $L = \triangle$ (the Laplace operator), the following happens:

(i) Theorem 3.7 (with "$G = \Omega$") fails,
(ii) the inequality (1.5) fails for some u and v as in Theorem 1.1,
(iii) (1.6) cannot be used as a characterization of the space $X^2(t)$ for which (1.3) and (1.4) hold.

Thus, according to (ii), on arbitrary bounded domains one cannot expect that sign-changing solutions are exponentially dominated by positive solutions. In fact, the examples in [13] show that in general there are solutions which change sign for all times but become "asymptotically" positive, a phenomenon which does not occur on Lipschitz domains (see Proposition 6.11 and Remark 6.12). Therefore, as can be expected, the solutions of (1.1) on general bounded domains exhibit a more complicated behavior than those on Lipschitz domains, and the methods of [17] are not applicable here. All of the above mentioned difficulties have been overcome in [13] for divergence form equations. Among the basic tools we used there were the weak Green's function and the existence of a well defined adjoint equation. For the nondivergence form equations, however, we do not have these tools at our disposal and thus are faced with additional difficulties. Finally, unlike in the case of divergence form equations, where the basic solution space is "naturally"

given, it is not at all obvious how one should define solutions of (1.1) on arbitrary bounded domains. The difficulties mentioned in this paragraph make the results to be presented particularly interesting and nontrivial.

In our main result, Theorem 2.2 below, we establish the existence of principal Floquet bundles and exponential separation under the sole assumption of boundedness of the spatial domain (assuming the coefficients in (1.1) satisfy (L1)). In particular, we give a new characterization of the space $X^2(t)$ in the decomposition (1.3). Instead of (1.6) (which we know may fail in general), we characterize it as the space of those initial conditions u_0 such that for all $T > t$ the corresponding solution $u(\cdot, \tilde{t}; t, u_0)$ vanishes somewhere in $\bar{D}_2$ for some $\tilde{t} \geq T$, where D_2 is a fixed (sufficiently large) subdomain of Ω (and D_2 is independent of u_0). Another interesting point is that in Theorem 2.2 we establish a time-dependent decomposition of a time-dependent space of initial conditions; it is different from previous results of this kind where the space of initial conditions is always fixed (see (1.3)). That this space is time-dependent is a natural consequence of our choice of the definition of a solution of (1.1) (see Definition 4.7); this definition mimics in a "time-dependent" manner the definition introduced in [3].

We present two applications of Theorem 2.2. Proposition 2.5 states the uniqueness of positive entire (defined for all $t \in \mathbb{R}$) solutions of (1.1). Next, as a special case of Theorem 2.2, assuming the coefficients of L are time-periodic, we get a new result on the existence of a principal eigenvalue for time-periodic parabolic operators on nonsmooth domains (with nonsmooth coefficients). This application is treated in Corollary 2.6. This corollary also gives additional properties of the principal eigenvalue and the associated eigenfunction. In particular, we prove the existence of a uniform spectral gap between the principal eigenvalue and the rest of the spectrum for a class of time-periodic parabolic operators.

Similarly as in [3], an important feature of this paper is that in all our estimates the constants depend only on N, α_0, β_0 in (L1) and on Ω but not on the smoothness of Ω (or the coefficients). From this standpoint, many of our results are new even if the domain is assumed smooth. Among the basic tools we use here are the maximum principle, the interior Harnack inequality, along with considerations of the behavior of solutions of (1.1) on "small" subdomains of Ω. In this respect, we have greatly benefited from recent results established in [30] (see Lemma 3.5 and Theorem 3.6). Let us also point out that, despite the difficulties mentioned above, we do rely on the results established in [17]. The general approach taken in this paper can be described as follows. We take a smooth subdomain G of Ω, which is close to Ω (in a suitable sense), and obtain appropriate estimates of solutions of (1.1) with "$\Omega = G$" independently of the smoothness of G. Such estimates are obtained, in part, due to being able to control the behavior of solutions on small subdomains near the boundary ∂G. Once such estimates are established, we let G converge to Ω in a suitable sense to get these estimates for the solutions of (1.1). Let us note that the above described strategy has also been used in [3]. Here, however, the matters are significantly more complicated by the time-dependent nature of the problem.

In this paper we only consider the Dirichlet problem. Related results for the oblique derivative problem can be found in [14, 24, 31, 33]. Let us mention here that the techniques developed in this paper can also be applied to related problems on unbounded domains. Such results will appear elsewhere.

The paper is organized as follows. Section 2 contains the statements of our main results. Section 3 contains some useful reference material, which will be used in later sections. In Section 4 we construct a function w_Ω, which is used to define in what sense the solutions of (1.1) attain the boundary values. In Section 5 we prove Theorem 2.1 and establish some other properties of positive entire (defined for all $t \in \mathbb{R}$) solutions of (1.1). Sections 6 and 7 serve as a preparation for the proof of Theorem 2.2. More precisely, Section 6 is devoted to proving several estimates of positive solutions of (1.1); some of them might be of independent interest (see Theorem 6.1, Lemma 6.4, Proposition 6.11). Section 7 then contains some estimates of sign-changing solutions. Section 8 contains proofs of Theorem 2.2 and Corollary 2.4. Finally, Sections 9 and 10 contain proofs of Proposition 2.5 and Corollary 2.6, respectively.

2. STATEMENT OF THE MAIN RESULTS

In this section and throughout the paper, whenever we refer to a solution of (1.1) we mean a solution as defined in Definition 4.7. We call a solution of (1.1) an *entire* solution if it is a solution of (1.1) on $\Omega \times \mathbb{R}$. The first result is the following theorem, which establishes the existence of a "well behaved" positive entire solution of (1.1).

Theorem 2.1. *There exists a positive entire solution φ_Ω of* (1.1) *such that the following holds. If D is a subdomain of Ω with $\bar{D} \subset \Omega$, then*

$$(2.1) \qquad \frac{\varphi_\Omega(x,t)}{||\varphi_\Omega(\cdot,t)||_{L^\infty(\Omega)}} \geq C(D) \qquad ((x,t) \in D \times \mathbb{R}).$$

The constant $C(D) > 0$ depends only on D, Ω and N, α_0 and β_0 in (L1).

Our main result is the following theorem on exponential separation and principal Floquet bundles. Below we denote by $u(\cdot, t; s, u_0)$ the solution of (1.1) with the initial condition $u(\cdot, s) = u_0$ (see Definition 4.7). The space $X(s)$ appearing in Theorem 2.2 is simply the space of all (appropriate) initial conditions $u(\cdot, s) = u_0$ such that the associated initial value problem has a solution $u(\cdot, t; s, u_0)$ defined for all $t \geq s$ (for the precise definition, see (4.24)).

Theorem 2.2. *The positive entire solution φ_Ω from Theorem* 2.1 *is uniquely determined by the condition* (2.1) *and by the assumption $||\varphi_\Omega(\cdot, 0)||_{L^\infty(\Omega)} = 1$. There exists a subdomain D_2 of Ω with $\bar{D}_2 \subset \Omega$ such that the following statements hold true.*

(i) *Let $X(s)$ be as in* (4.24) *and set*

$$X^1(s) := \mathrm{span}\{\varphi_\Omega(\cdot, s)\},$$
$$X^2(s) := \{u_0 \in X(s) : \textit{for all } T > s \textit{ the function } u(\cdot, t; s, u_0)$$
$$\textit{has a zero in } \bar{D}_2 \textit{ for some } t \geq T\}.$$

These sets are closed subspaces of $X(s)$. They are invariant under (1.1) *in the following sense: if $i \in \{1, 2\}$, $u_0 \in X^i(s)$, then $u(\cdot, t; s, u_0) \in X^i(t)$ $(t \geq s)$. Moreover, $X^1(s)$, $X^2(s)$ are complementary subspaces of $X(s)$:*

$$(2.2) \qquad\qquad X(s) = X^1(s) \oplus X^2(s) \quad (s \in \mathbb{R}).$$

(ii) *There are constants $C, \gamma > 0$ depending only on Ω and N, α_0 and β_0 in (L1) such that for any $u_0 \in X^2(s)$ one has*

$$(2.3) \qquad \frac{||u(\cdot, t; s, u_0)||_{L^\infty(\Omega)}}{||\varphi_\Omega(\cdot, t)||_{L^\infty(\Omega)}} \leq C\, e^{-\gamma\,(t-s)} \frac{||u_0||_{L^\infty(\Omega)}}{||\varphi_\Omega(\cdot, s)||_{L^\infty(\Omega)}} \quad (t \geq s).$$

Remark 2.3. Let us point out that it follows from the proof of Theorem 2.2 that the domain D_2, whose existence is claimed, can in fact be any (smooth) domain D_2 with $\bar{D}_2 \subset \Omega$ such that the (Lebesgue) measure $|\Omega \setminus \bar{D}_2| < \delta$, where $\delta > 0$ depends only on Ω and N, α_0 and β_0 in (L1).

We refer to the collection of the one-dimensional spaces $X^1(s)$, $s \in \mathbb{R}$, as the *principal Floquet bundle* of (1.1) and to $X^2(s)$, $s \in \mathbb{R}$, as its *complementary Floquet bundle*. Property (ii) as stated is an *exponential separation* between these two bundles. As discussed in the introduction, the existence of the Floquet bundles with exponential separation extends in a natural way the properties of the principal eigenvalue of time-independent (or time-periodic) parabolic problems. The positive entire solution φ_Ω serves as an analogue of the principal eigenfunction.

The following corollary gives an equivalent characterization of the space $X^2(s)$ in Theorem 2.2.

Corollary 2.4. *Let $X^2(s)$ $(s \in \mathbb{R})$ and D_2 be as in Theorem 2.2. Then for any subdomain D of Ω with $D_2 \subseteq D \subset \bar{D} \subset \Omega$ the following holds:*

$$X^2(s) = \{u_0 \in X(s) : \text{for all } T > s \text{ the function } u(\cdot, t; s, u_0)$$
$$(2.4) \qquad\qquad \text{has a zero in } \bar{D} \text{ for some } t \geq T\}.$$

As a consequence of the exponential separation estimate (2.3) we will derive the following uniqueness result.

Proposition 2.5. *Let φ_1, φ_2 be positive entire solutions of (1.1) such that for some $c > 0$ and $i = 1, 2$,*

$$(2.5) \qquad\qquad \varphi_i(x, t) \leq c\, e^{c\,|t|} \qquad ((x, t) \in \Omega \times \mathbb{R}).$$

Then there exists a constant $q > 0$ such that $\varphi_1 \equiv q\,\varphi_2$.

The uniqueness of positive entire solutions has also been proved in [13, 15, 17, 25, 29, 31]. We believe that the assumption (2.5) is not merely of a technical nature. We conjecture that (if $N \geq 2$) there are positive entire solutions, which grow superexponentially as time goes to negative infinity. We point out that the uniqueness results on smooth (at least Lipschitz) domains [15, 17, 25, 29, 31] exclude this possibility.

We now show another nontrivial application of Theorem 2.2, establishing the existence of a principal eigenvalue for a time-periodic parabolic eigenvalue problem. Suppose the coefficients of L are T-periodic in t for some $T > 0$. Consider the eigenvalue problem

$$
\begin{aligned}
u_t - Lu &= \lambda u && \text{in} \quad \Omega \times \mathbb{R}, \\
(2.6) \qquad u &\overset{w_\Omega}{=} 0 && \text{on} \quad \partial\Omega \times \mathbb{R}, \\
u(\cdot, t) &= u(\cdot, t + T) && \text{in} \quad \Omega \times \mathbb{R},
\end{aligned}
$$

where the boundary condition $u \overset{w_\Omega}{=} 0$ on $\partial\Omega \times \mathbb{R}$ is defined in Section 4 (it reduces to $u = 0$ on $\partial\Omega \times \mathbb{R}$ if Ω is sufficiently regular). The principal eigenvalue λ_1 of

this problem is the eigenvalue which is real and has a positive eigenfunction. The following corollary states the existence of such an eigenvalue for (2.6).

Corollary 2.6. *Assume the coefficients of L are T-periodic in t for some $T > 0$. Then there exists a unique eigenvalue λ_1 of (2.6) with the following properties.*

(i) *It is real and it has a unique (up to constant multiples) eigenfunction φ_1 which is positive in $\Omega \times \mathbb{R}$ and it satisfies the estimate (2.1).*

(ii) *$|\lambda_1|$ is bounded by a constant depending only on Ω and N, α_0 and β_0 in (L1).*

(iii) *Let $\lambda \neq \lambda_1$ be any eigenvalue of (2.6) with a corresponding eigenfunction φ_λ. Then*

$$\text{(2.7)} \qquad \qquad \mathrm{Re}(\lambda) - \lambda_1 \geq \gamma > 0,$$

where $\gamma = \gamma(\Omega, N, \alpha_0, \beta_0)$ is as in (2.3). Moreover, if φ_λ is real, then $\varphi(\cdot, \cdot)$ has a zero in $\bar{D}_2 \times [t, t + T]$ for all $t \in \mathbb{R}$, where D_2 is as in Theorem 2.2.

In the time-independent case, the existence of the principal eigenvalue for non-divergence form elliptic operators on general bounded domains has been proved in [3]. Even if we restrict ourselves to the autonomous case, Corollary 2.6 contains [3, Theorem 2.1] and provides an improvement of [3, Theorem 2.3]. In the time-periodic case, our result is new in this generality. Indeed, for nondivergence form equations with periodic time-dependence the existence of the principal eigenvalue has been previously known only in the case of smooth coefficients and domain (see [11]). For more general results of this kind for divergence form equations, we refer the reader to [7, 9].

3. Preliminary results

Let $\Omega \subset \mathbb{R}^N$ be a fixed bounded domain and L be an operator as in (1.2) with coefficients satisfying (L1). We will use the letter G to denote a smooth (at least Lipschitz) subdomain of Ω. For such a domain and an interval $J \subseteq \mathbb{R}$ consider a general linear problem

$$\text{(3.1)} \qquad \qquad u_t - Lu = 0 \quad \text{in} \quad G \times J,$$

$$\text{(3.2)} \qquad \qquad u = 0 \quad \text{on} \quad \partial G \times J.$$

Let us specify what notions of (super)solutions we use in this section. First, whenever we refer to (super)solutions of (3.1), we mean a function $u \in W^{2,1,N+1}_{\mathrm{loc}}(G \times J)$ such that (3.1) ((3.1) with "=" replaced by "$\geq$") is satisfied almost everywhere. Second, a (super)solution of (3.1), (3.2) is in addition continuous on $\bar{G} \times \bar{J}$ and satisfies (3.2) ((3.2) with "=" replaced by "$\geq$"). Below we often use standard maximum and comparison principles for (super)solutions of (3.1) as found in [21], for example.

The following results will be used frequently throughout the paper. To simplify the exposition we will make the following convention.

Convention. The statement "the constant C depends on the structure of L" means that C is determined by the quantities N, α_0, β_0 appearing in the condition (L1). Next, throughout the paper we often omit (some) arguments of the functions under consideration; for example, we use $u(t)$ instead of $u(\cdot, t)$ if there is no danger of confusion. Finally, whenever a constant $C > 0$ may be assumed greater (or smaller) than 1 we assume it is such without further notice. For example, the

constant C in Theorem 3.1 below may be assumed greater than 1 and the constant $C_2(D, d)$ in Corollary 5.3 may be assumed less than 1.

First, we state the interior Harnack inequality ([19]).

Theorem 3.1. *Let v be a nonnegative solution of $u_t - Lu = 0$ on $\Omega \times (0, \infty)$. Suppose $0 < \delta < T$ and let D be a subdomain of Ω such that $\mathrm{dist}(\bar{D}, \partial\Omega) = d > 0$. Let $t, s \in \mathbb{R}$ be such that $s \geq \delta^2$, where $T \geq t - s \geq \delta^2$. Then there is a positive constant C depending only on δ, T, D, d and the structure of L such that one has*

$$(3.3) \qquad \sup_{x \in D} v(x, s) \leq C \inf_{x \in D} v(x, t).$$

Consider now the following initial value problem:

$$(3.4) \qquad \begin{aligned} u_t - Lu &= 0 && \text{in} \quad G \times (0, \infty), \\ u &= 0 && \text{on} \quad \partial G \times (0, \infty), \\ u &= u_0 && \text{in} \quad G \times \{0\}, \end{aligned}$$

where G is a smooth subdomain of Ω. The following result is standard if G is smooth. For Lipschitz G it has been proved in [17].

Proposition 3.2. *Let G be a Lipschitz bounded domain. Assume $u_0 \in C_0(\bar{G})$. Then the initial value problem (3.4) has a unique solution u contained in $W^{2,1,p}_{\mathrm{loc}}(G \times (0, \infty)) \cap C(\bar{G} \times [0, \infty))$ for all $p > 1$. It satisfies the following estimate:*

$$(3.5) \qquad \|u(t)\|_{L^\infty(G)} \leq e^{mt} \|u_0\|_{L^\infty(G)} \quad (t \in [0, \infty)),$$

where $m = \sup_{G \times (0, \infty)}(c_0) \leq \beta_0$ (β_0 is as in (L1)).

Remark 3.3. Obviously, there is nothing special about the interval $(0, \infty)$ in the above stated results. These results hold in the same way on any interval (s, ∞) with $s \in \mathbb{R}$.

In the proof of existence of a principal Floquet bundle we will use in an essential way the following result of Varadhan (as attributed to him in [2]).

Lemma 3.4. *Given any $a_0 > 0$, $b_0 \geq 1$, there exists $\delta > 0$ determined only by a_0, b_0, N, and $\mathrm{diam}(\Omega)$ such that for any closed set $K \subset \Omega$ with $|K| \leq \delta$ there exists a smooth function g on Ω such that $1 \leq g \leq 2$ and for any symmetric positive definite matrix (a_{ij}) with*

$$(3.6) \qquad \det(a_{ij}) \geq a_0^N$$

one has

$$(3.7) \qquad a_{ij} D_{ij} g + b_0(|\nabla g| + g) < 0 \quad (x \in K).$$

For any $a \in \mathbb{R}$ we use the following notation throughout the paper:

$$a^+ = \max\{a, 0\}, \quad a^- = \max\{-a, 0\}.$$

The next result (see [30, Lemma 3.1]) will be used frequently throughout the paper.

Lemma 3.5. *Given any $k > 0$ there is a constant $\delta > 0$ determined only by the structure of L, $\mathrm{diam}(\Omega)$, and k such that for any open set $U \subset \Omega$ with $|U| < \delta$ the following holds. If $v \in C(\bar{U} \times [\tau, T])$ is a supersolution of a problem (3.1), (3.2) with "$G = U$" and $J = (\tau, T)$, then*

$$(3.8) \qquad \|v^-(\cdot, t)\|_{L^\infty(U)} \leq 2 e^{-k(t-\tau)} \|v^-(\cdot, \tau)\|_{L^\infty(U)} \quad (t \in [\tau, T]).$$

Before we state the next result we need to introduce some notation. For a domain $D \subset \Omega$, we define the inner radius of D to be

$$(3.9) \qquad \mathrm{inrad}(D) := \sup\{\rho > 0 : B(x_0, \rho) \subset D \text{ for some } x_0 \in D\},$$

where $B(x_0, \rho)$ is the ball in $\mathbb{R}^N$ of radius ρ centered at x_0. If D_0 and D are bounded sets in $\mathbb{R}^N$, the notation $D_0 \subset\subset D$ means $\bar{D}_0 \subset D$. The following result will be crucial in our considerations; it has been proved in [30] (see Theorem 3.7 in [30] and the remarks following it). We state it in a simplified form tailored to our needs.

Theorem 3.6. *Fix $\rho \in (0, \mathrm{diam}(\Omega)/2)$. Then there exists δ determined only by ρ, $\mathrm{diam}(\Omega)$ and the structure of L such that the following holds. Given any $d > 0$, $\theta > 0$, there exists a positive constant $\mu = \mu(N, \mathrm{diam}(\Omega), \alpha_0, \beta_0, d, \theta, \rho)$ with the following properties. If $D \subset G$ are subdomains of Ω satisfying*

$$(3.10) \qquad\qquad \mathrm{inrad}(D) > \rho, \ |G \setminus \bar{D}| < \delta,$$

$$(3.11) \qquad\qquad \mathrm{dist}(\bar{D}, \partial G) > d,$$

if $v \in C(\bar{G} \times [\tau, \infty))$ is a solution of a problem (3.1), (3.2) (with $J = (\tau, \infty)$), and if

$$(3.12) \qquad\qquad v(x, t) > 0 \quad ((x, t) \in \bar{D} \times [\tau, \tau + 8\theta)),$$

$$(3.13) \qquad ||v^-(\cdot, \tau)||_{L^\infty(G \setminus \bar{D})} \leq \mu \, ||v||_{L^\infty(D \times (\tau + \theta, \tau + 2\theta))},$$

then the following statements hold true:

- (s1) *$v(x, t) > 0$ $((x, t) \in \bar{D} \times [\tau, \infty))$;*
- (s2) *for any domain $H \subset\subset G$ with $H \cap D \neq \emptyset$ there is T_H such that $v(\cdot, t) > 0$ on H for all $t \geq T_H$.*

Finally, we state some results from [17]. First, a Harnack type inequality for quotients of positive solutions of (3.1), (3.2) (see [17, Theorem 2.1]).

Theorem 3.7. *Let G be a Lipschitz bounded domain. Let $\delta_0 > 0$ and let u_1, u_2 ($u_2 \not\equiv 0$) be two nonnegative solutions of (3.1), (3.2) on $G \times (0, \infty)$. Then for all $t \geq \delta_0$ the following estimate holds:*

$$\sup_{x \in G} \frac{u_1(x, t)}{u_2(x, t)} \leq C_G \inf_{x \in G} \frac{u_1(x, t)}{u_2(x, t)},$$

with a constant $C_G > 1$ depending only on δ_0, the structure of L and on the Lipschitz characteristics of G.

Recall that u is an *entire* solution of (3.1), (3.2), if u is a solution of (3.1), (3.2) on $G \times \mathbb{R}$. We then have the following theorem (see [17]).

Theorem 3.8. *Let G be a Lipschitz bounded domain. There exists a unique (up to constant multiples) positive entire solution φ_G of (3.1), (3.2). Moreover, for every domain $D \subset\subset G$ there exists $C_G(D) > 0$ such that*

$$(3.14) \qquad \frac{\varphi_G(x, t)}{||\varphi_G(\cdot, t)||_{L^\infty(G)}} \geq C_G(D) \qquad ((x, t) \in D \times \mathbb{R}).$$

The constant $C_G(D)$ depends only on D, the structure of L and on the Lipschitz characteristics of G.

4. CONSTRUCTION OF w_Ω

Before we proceed to prove the existence of a positive entire solution of (1.1) we need to define first what we mean by a solution of (1.1); most importantly, we need to explain in what sense the solutions of (1.1) satisfy the boundary condition on $\partial\Omega$ if the boundary $\partial\Omega$ is "rough". We cannot require, in general, that the zero boundary values be attained (continuously) at all points of the boundary.

Let us recall here the approach used in [3]. In that paper, assuming the coefficients of L in (1.2) (and hence of M) are time-independent, the authors proved the existence of a bounded positive solution w_0 of

$$(4.1) \qquad\qquad -Mu = 1 \quad \text{in} \quad \Omega,$$

$$(4.2) \qquad\qquad u = 0 \quad \text{on} \quad \partial\Omega,$$

which attains the boundary values in some sense on $\partial\Omega$. Then, for any $f \in L^N(\Omega)$, a solution of $Lu = f$ in Ω, $u = 0$ on $\partial\Omega$ is defined to be a function $u \in W^{2,N}_{loc}(\Omega)$, for which $Lu = f$ almost everywhere in Ω and the boundary condition is attained in the following sense:

$$u \stackrel{w_0}{=} 0,$$

which means that $u(x_j) \to 0$ for all sequences $x_j \to \partial\Omega$ such that $w_0(x_j) \to 0$. It has also been proved in [3] that w_0 attains the zero boundary value continuously at each point $y \in \partial\Omega$ which admits a strong barrier; the definition of a strong barrier at a boundary point is as follows.

Definition 4.1. A point $y \in \partial\Omega$ is said to admit a strong barrier if for some ball $B(y, r)$ there is in $B(y, r) \cap \Omega = V$ a positive function $h \in W^{2,N}_{loc}(V)$ satisfying $Mh \leq -1$, which can be extended continuously to the point y with $h(y) = 0$.

Remark 4.2. K. Miller [28] has shown that any point $y \in \partial\Omega$ at which $\partial\Omega$ satisfies an exterior cone condition admits a strong barrier $h \in C^2(V)$ such that $Mh \leq -1$ on V for *all* operators M whose coefficients satisfy (L1); here V is some fixed relative neighborhood of y in Ω.

Here we will adopt a similar "time-dependent" approach. Our goal is to prove the existence of a positive function $w_\Omega \in \bigcap_{p>1} W^{2,1,p}_{loc}(\Omega \times \mathbb{R}) \cap C(\Omega \times \mathbb{R}) \cap L^\infty(\Omega \times \mathbb{R})$ which satisfies (with M as in (1.2))

$$(4.3) \qquad\qquad (w_\Omega)_t - Mw_\Omega = 1 \quad \text{in} \quad \Omega \times \mathbb{R},$$

$$(4.4) \qquad\qquad w_\Omega = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R},$$

and the boundary values are attained in a similar sense as for the function w_0 above. The function w_Ω will then be used in Definition 4.7 of solutions of (1.1).

To prove the existence of w_Ω we use a limiting procedure. Let $G \subset\subset \Omega$ be a smooth domain and let $T > 0$. Consider the following problem:

$$(4.5) \qquad u_t - Mu = 1 \quad \text{in} \quad G \times (-T, \infty),$$

$$(4.6) \qquad u = 0 \quad \text{on} \quad (\partial G \times (-T, \infty)) \cup (G \times \{-T\}).$$

By [21, Theorem 7.17] this problem has a unique solution $w_{G,T} \in \bigcap_{p>1} W^{2,1,p}(G \times (-T, T)) \cap C(\bar{G} \times [-T, T])$, which is in addition positive by the maximum principle. Let us prove a uniform bound on $w_{G,T}$ independent of $G \subset\subset \Omega$

and $T > 0$. Without loss of generality we may assume $\inf_{x=(x_1,\dots,x_N)\in\Omega} x_1 = 0$. Define the function

$$(4.7) \qquad u_{G,T} = e^{((1+\beta_0)/\alpha_0)\operatorname{diam}(\Omega)} - e^{((1+\beta_0)/\alpha_0)x_1} - w_{G,T}.$$

A simple calculation shows that $(u_{G,T})_t - Mu_{G,T} \geq 0$ on $G\times(-T,\infty)$, $u_{G,T} \geq 0$ on $(\partial G\times(-T,\infty))\cup(G\times\{-T\})$. The maximum principle then implies that $u_{G,T} \geq 0$ on $G\times(-T,\infty)$. We have thus proved

$$(4.8) \qquad 0 < w_{G,T}(x,t) \leq e^{((1+\beta_0)/\alpha_0)\operatorname{diam}(\Omega)} \quad ((x,t)\in G\times(-T,\infty)),$$

whenever $G \subset\subset \Omega$ is smooth and $T > 0$. The inequality (4.8) and standard parabolic regularity estimates imply that if $T_i \to \infty$, then, at least for a subsequence, $w_{G,T_i} \to w_G$ uniformly on compact subsets of $\bar{G}\times\mathbb{R}$, $w_G \in \bigcap_{p>1} W_{loc}^{2,1,p}(G\times(-T,T))\cap C(\bar{G}\times[-T,T])$ for all $T > 0$, and w_G satisfies

$$(4.9) \qquad (w_G)_t - Mw_G = 1 \quad \text{in} \quad G\times\mathbb{R},$$

$$(4.10) \qquad w_G = 0 \quad \text{on} \quad \partial G\times\mathbb{R},$$

and

$$(4.11) \qquad 0 < w_G(x,t) \leq e^{((1+\beta_0)/\alpha_0)\operatorname{diam}(\Omega)} \quad ((x,t)\in G\times\mathbb{R}).$$

Let us now prove the uniqueness of w_G in case G is smooth.

Proposition 4.3. *If $G \subset\subset \Omega$ is smooth, then there exists a unique w_G satisfying (4.9), (4.10), (4.11). More generally, for any bounded domain G, if w_1 and w_2 are bounded solutions of (4.9), (4.10) such that $w_i \in C(\bar{G}\times\mathbb{R})$, $i=1,2$, then $w_1 \equiv w_2$.*

Proof. Since we have already proved the existence of w_G it remains to prove the uniqueness statement. Let w_1, w_2 be as in the statement of the proposition. For $a > 1$ set $v_a = a\,w_1 - w_2$. The boundedness of w_i, $i=1,2$, implies that there exists $\epsilon_a > 0$ such that $(v_a)_t - Mv_a = a - 1 \geq -\epsilon_a\,v_a$ on $G\times\mathbb{R}$; by continuity of w_1, w_2, we also have $v_a = 0$ on $\partial G\times\mathbb{R}$. By the maximum principle we get

$$(4.12) \qquad ||v_a^-(\cdot,t)||_{L^\infty(G)} \leq e^{-\epsilon_a(t-\tau)}\,||v_a^-(\cdot,\tau)||_{L^\infty(G)} \quad (t\geq\tau).$$

Sending $\tau \to -\infty$ and using the boundedness of v_a we obtain $v_a^- \equiv 0$; hence $a\,w_1 \geq w_2$. Since this is true for any $a > 1$ we get $w_1 \geq w_2$. By symmetry of the argument also $w_2 \geq w_1$ and therefore $w_1 \equiv w_2$. Finally, if G is smooth, then any bounded solution w of (4.9), (4.10) belongs to $C(\bar{G}\times\mathbb{R})$ and hence has to be identically equal to w_G, completing the proof. $\qquad\square$

Next, we give a monotonicity result for w_G.

Proposition 4.4. *Let $G_1 \subseteq G_2$ be two smooth bounded domains and let w_{G_1}, w_{G_2} be the corresponding solutions given by Proposition 4.3. Then $w_{G_1} \leq w_{G_2}$ on $G_1\times\mathbb{R}$.*

Proof. Fix $a > 1$. Then the function $\tilde{v}_a = a\,w_{G_2} - w_{G_1}$ is bounded and hence, for some $\epsilon_a > 0$, we have $(\tilde{v}_a)_t - M\tilde{v}_a = a - 1 \geq -\epsilon_a v_a$ on $G_1\times\mathbb{R}$ and also $\tilde{v}_a = a\,w_{G_2} \geq 0$ on $\partial G_1\times\mathbb{R}$. By the maximum principle,

$$||\tilde{v}_a^-(\cdot,t)||_{L^\infty(G_1)} \leq e^{-\epsilon_a(t-\tau)}\,||\tilde{v}_a^-(\cdot,\tau)||_{L^\infty(G_1)} \quad (t\geq\tau).$$

The boundedness of $\tilde{v}_a$ then implies $\tilde{v}_a^- \equiv 0$ on $G_1\times\mathbb{R}$; therefore $a\,w_{G_2} \geq w_{G_1}$. Since this is true for any $a > 1$, we get $w_{G_2} \geq w_{G_1}$. $\qquad\square$

We are ready to construct the solution w_Ω. Let $\Omega_j \subset\subset \Omega_{j+1} \subset\subset \Omega$ be a sequence of smooth subdomains such that $\bigcup_{j=1}^\infty \Omega_j = \Omega$, and let w_{Ω_j} be the corresponding solutions given by Proposition 4.3. The uniform bound and monotonicity of w_{Ω_j} with respect to j and standard parabolic regularity imply that there exists a function $w_\Omega = \lim_{j\to\infty} w_{\Omega_j}$, $w_\Omega \in \bigcap_{p>1} W_{loc}^{2,1,p}(\Omega \times \mathbb{R}) \cap C(\Omega \times \mathbb{R})$, and (4.9) and (4.11) are satisfied with G replaced by Ω. Moreover, w_Ω vanishes continuously at each point $y \in \partial\Omega$ where $\partial\Omega$ satisfies an exterior cone condition. To prove this fact one can use Remark 4.2 and arguments very similar to those in [3] (see the discussion preceding [3, Remark 3.1]), and we omit the details.

Remark 4.5. The above defined function w_Ω is independent of the approximating sequence Ω_j. Indeed, this follows immediately from Proposition 4.4.

The next proposition states that the boundary behavior of w_Ω depends only on a neighborhood of $\partial\Omega$ (cf. [3, Lemma 3.1]).

Proposition 4.6. *Let G_1, G_2 be smooth domains such that $G_1 \subset\subset G_2 \subset\subset \Omega$. Then there exists a constant $C = C(G_1, G_2, \Omega) > 0$, depending only on G_1, G_2, Ω and on the structure of L, such that*

$$(4.13) \qquad w_{\Omega \setminus \bar{G}_1}(x,t) \leq w_\Omega(x,t) \leq C\, w_{\Omega \setminus \bar{G}_1}(x,t) \quad ((x,t) \in (\Omega \setminus \bar{G}_2) \times \mathbb{R}),$$

where $w_{\Omega \setminus \bar{G}_1}$ is the function as constructed above but corresponding to $\Omega \setminus \bar{G}_1$.

Proof. We note first that the existence of w_Ω can be established for any bounded open set Ω by repeating the above arguments for each connected component of Ω. Therefore the function $w_{\Omega \setminus \bar{G}_1}$ is well defined. Similarly, for Proposition 4.4 to hold we only need that G_1, G_2 in that proposition be bounded open sets with $G_1 \subseteq G_2$. This implies the first inequality in (4.13).

We now prove the remaining inequality. Let Ω_j, $j \geq 1$, be a strictly increasing sequence of smooth subdomains of Ω such that their union is equal to Ω. We may assume that all of these domains contain G_2 and $\mathrm{dist}(\bar{G}_2, \partial\Omega_j) \geq \mathrm{dist}(\bar{G}_2, \partial\Omega)/2$ for all $j \geq 1$. We have $w_{\Omega_j \setminus \bar{G}_1}(x,t) \geq \tilde{C} > 0$ for all $x \in \partial G_2$, $t \in \mathbb{R}$, $j \geq 1$ and a constant $\tilde{C}$ depending only on G_1, G_2, Ω and the structure of L. This fact follows from a result of Krylov and Safonov (see [20, Section 4.1, Lemma 1 and Theorem 2], for example). Therefore, using the boundary condition and boundedness of w_{Ω_j}, we have $v_j = a\, w_{\Omega_j \setminus \bar{G}_1} - w_{\Omega_j} \geq 0$ on $\partial(\Omega_j \setminus \bar{G}_2) \times \mathbb{R}$ for all $j \geq 1$ and some sufficiently large $a > 1$ depending only on G_1, G_2, Ω and the structure of L. The same arguments as in the proof of Proposition 4.3 (see (4.12)) show that $v_j \geq 0$ on $(\Omega_j \setminus \bar{G}_2) \times \mathbb{R}$. Letting $j \to \infty$, we obtain the right hand side inequality in (4.13). $\qquad\square$

We will now define what we mean by a solution of (1.1) (or an associated initial value problem) for a general bounded domain Ω. First, we say that

$$(4.14) \qquad\qquad\qquad u \overset{w_\Omega}{=} 0 \quad \text{on } \partial\Omega \times [t_0, t_1],$$

if $u(x_j, t_j) \to 0$ for all sequences $x_j \to \partial\Omega$ and $t_j \in [t_0, t_1]$ such that $w_\Omega(x_j, t_j) \to 0$. Let $Y = C(\Omega) \cap L^\infty(\Omega)$ be endowed with the supremum norm. Then Y is a closed subspace of $L^\infty(\Omega)$. Define the set

$$(4.15) \qquad Y(s) := \{u_0 \in C(\Omega) \cap L^\infty(\Omega) : u_0 \overset{w_\Omega}{=} 0 \text{ on } \partial\Omega \times [s]\} \quad (s \in \mathbb{R}).$$

Endowing $Y(s)$ with the supremum norm it is easy to prove that it is a closed subspace of Y.

Definition 4.7. Let $s < T$, $s, T \in \mathbb{R}$, and let $u_0 \in Y(s)$. A function u is a solution of the initial value problem

$$(4.16) \qquad\qquad \begin{aligned} u_t - Lu &= 0 \quad \text{in} \quad \Omega \times (s, T), \\ u &= 0 \quad \text{on} \quad \partial\Omega \times (s, T), \\ u &= u_0 \quad \text{in} \quad \Omega \times \{s\}, \end{aligned}$$

if $u \in W_{loc}^{2,1,N+1}(\Omega \times (s, T)) \cap C(\Omega \times [s, T)) \cap L^\infty(\Omega \times (s, T))$, u satisfies $u_t - Lu = 0$ almost everywhere in $\Omega \times (s, T)$, and $u \overset{w_\Omega}{=} 0$ on $\partial\Omega \times [s, \tau]$ for all $\tau \in [s, T)$. Further, a function u is a solution of (1.1) if u is a solution of an initial value problem (4.16) for all $s < T$ with $s, T \in J$.

Remark 4.8. Proposition 4.6 shows that the boundary condition (4.14) and thus also Definition 4.7 depends only on a neighborhood of the boundary $\partial\Omega$. If Ω is sufficiently regular, say, Lipschitz or, more generally, satisfying an exterior cone condition at each $y \in \partial\Omega$, then $Y(s) = C_0(\bar\Omega)$ for all $s \in \mathbb{R}$ and Definition 4.7 reduces to the usual definition of a (strong) solution of (1.1).

The following example illustrates that the boundary condition (4.14) used in our definition of solutions of (1.1) can in general be time-dependent.

Example. Let us demonstrate that, unlike the case of smooth domains, the attainment of $w_\Omega(\cdot, t)$ of the boundary values on $\partial\Omega$ may depend substantially on $t \in \mathbb{R}$. For any $N \geq 2$ let $\Omega = B_1 - \{0\}$, a punctured unit ball in $\mathbb{R}^N$ centered at the origin. Define the coefficients of the operator L to be

$$(4.17) \qquad a_{ij}(x, t) = \delta_{ij} + g(t)\frac{x_i\,x_j}{|x|^2}, \; b_i \equiv 0, \; c_0 \equiv 0 \quad ((x, t) \in \Omega \times \mathbb{R}),$$

where $g(t) \equiv N - 1$ if $t \leq 1 - N$, $g(t) = -t$ for $t \in [1 - N, 0]$, and $g(t) \equiv 0$ if $t > 0$. Then L satisfies (L1) with $\alpha_0 = 1$ and $\beta_0 = N$. Let w_0 be the function satisfying (4.1), (4.2) with M equal to the operator defined in (4.17) with $t \leq 1 - N$, which we denote for now by M_0. It is not difficult to see that then $w_\Omega(\cdot, t) \equiv w_0$ for all $t \leq 1 - N$. Indeed, this can be proved using the fact that w_Ω and w_0 are obtained as the (monotone) limits of suitable solutions on smooth subdomains Ω_j converging to Ω. Next, a simple computation shows that $\tilde{u}(x) = 1 - |x|^{1/N}$ is a positive smooth bounded function on Ω, vanishing on ∂B_1, and satisfying $M_0\tilde{u} = 0$ in Ω. The existence of $\tilde{u}$, the definition of the refined maximum principle (see [3]) and [3, Lemma 3.2] imply that there exists a sequence $x_j \to 0$ such that $w_0(x_j) = w_\Omega(x_j, t) \to 0$ $(t \leq 1 - N)$. On the other hand, for $t > 0$, w_Ω is a positive bounded solution of $u_t - \triangle u = 1$ on a punctured ball and thus (see [8]) can be uniquely extended to a solution of the same equation on B_1. Therefore, for all $t > 0$, we have $w(x, t) \geq \alpha(t) > 0$ on a neighborhood of the origin, which shows that, in general, the boundary behavior of $w_\Omega(\cdot, t)$ may change substantially in t (see also Remark 4.14). In particular, for $s > 0$ the solutions of the initial value problem (4.16) exist for all $u_0 \in C_0(\bar B_1) = Y(s)$, whereas for $s \leq 1 - N$ we have $Y(s) \subsetneq C_0(\bar B_1)$.

Let us conclude this section with a discussion of properties of solutions of (1.1) as defined in Definition 4.7. We first show that they still satisfy a modified version of the comparison principle. In the following proposition we say

$$(4.18) \qquad\qquad \liminf u \overset{w_\Omega}{\geq} 0 \qquad \text{on } \partial\Omega \times [s, T)$$

if for any $\tau \in [s, T]$ we have $\liminf u(x_j, t_j) \geq 0$ for all sequences $x_j \to \partial\Omega$ and $t_j \in [s, \tau]$ such that $w_\Omega(x_j, t_j) \to 0$.

Proposition 4.9. *Let $u_0 \geq 0$ in Ω, and suppose that $u \in W_{loc}^{2,1,N+1}(\Omega \times (s, T)) \cap C(\Omega \times [s, T))$, $u(\cdot, s) = u_0$, u is bounded below on $\Omega \times (s, \tau)$ for each $\tau < T$, and*

$$(4.19) \qquad\qquad u_t - Lu \geq 0 \qquad in \ \Omega \times (s, T),$$

$$(4.20) \qquad\qquad \liminf u \overset{w_\Omega}{\geq} 0 \qquad on \ \partial\Omega \times [s, T).$$

Then $u \geq 0$ on $\Omega \times (s, T)$.

Remark 4.10. By [3, Remark 1.1] the requirement that u be bounded below cannot be omitted.

Proof of Proposition 4.9. Without loss of generality we will assume in this proof that $c_0 \leq 0$. Fix $\tau \in [s, T)$ and denote $A = \sup_{\Omega \times (s, \tau)} u^-$. A simple contradiction argument using (4.20) shows that for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$(4.21) \qquad\qquad u \geq -\epsilon - \frac{A}{\delta(\epsilon)} \, w_\Omega \qquad on \ \Omega \times [s, \tau].$$

Now fix $x_0 \in \Omega$ and let $x_0 \in \Omega_j \subset\subset \Omega_{j+1} \subset\subset \Omega$ be a sequence of smooth subdomains such that $\bigcup_{j=1}^\infty \Omega_j = \Omega$. Let w_{Ω_j} be as given by Proposition 4.3. Define

$$(4.22) \qquad\qquad v = u + \epsilon + \frac{A}{\delta(\epsilon)} \, (w_\Omega - w_{\Omega_j}) \qquad on \ \Omega_j \times [s, \tau].$$

Using (4.21) and the fact that $w_{\Omega_j} = 0$ on $\Omega_j \times \mathbb{R}$, we get $v \geq 0$ on $\partial\Omega_j \times [s, \tau]$. Moreover, by the assumption $u_0 \geq 0$ and by the monotonicity of w_Ω with respect to domain inclusion (Proposition 4.4), we also have $v \geq 0$ on $\Omega_j \times \{s\}$. A straightforward computation, employing $c_0 \leq 0$ and Proposition 4.4, shows that $v_t - Lv \geq 0$ on $\Omega_j \times (s, \tau)$. By the usual comparison principle, we then have $v \geq 0$ on $\Omega_j \times [s, \tau]$. This fact leads, upon letting $j \to \infty$, to the conclusion that $u(x_0, t) \geq -\epsilon$ for all $t \in [s, \tau]$ and all $\epsilon > 0$. Therefore $u \geq 0$ on $\Omega \times (s, \tau)$ and since $\tau < T$ was arbitrary the proof is complete. $\qquad\square$

As an immediate consequence we obtain the following corollary.

Corollary 4.11. *Let u be a solution of* (4.16). *Then*

$$(4.23) \qquad\qquad \|u(t)\|_{L^\infty(\Omega)} \leq e^{m(t-s)} \|u_0\|_{L^\infty(\Omega)} \quad (t \in [s, T]),$$

where $m = \sup_{\Omega \times (s, T)}(c_0) \leq \beta_0$ (β_0 is as in (L1)). In particular, the solutions of (4.16) *are unique (if they exist).*

Next, with $Y(s)$ as defined in (4.15), set

$$(4.24) \qquad X(s) := \{u_0 \in Y(s) : (4.16) \text{ has a solution for all } T > s\} \quad (s \in \mathbb{R}).$$

Corollary 4.11 together with standard parabolic regularity estimates implies that if $X(s)$ is endowed with the supremum norm, then it becomes a closed subspace of $Y(s)$.

Remark 4.12. We note, similarly as in Remark 4.8, that if Ω is sufficiently regular, then $X(s) = Y(s) = C_0(\bar\Omega)$, $s \in \mathbb{R}$. However, in the example appearing after Remark 4.8 one has $X(s) = Y(s) = C_0(\bar{B}_1)$ for $s > 0$ and $X(s) \subseteq Y(s) \subsetneq C_0(\bar{B}_1)$ whenever $s \leq 1 - N$, and thus the spaces $X(s), Y(s)$ in general do depend nontrivially on t.

Let $\Omega_j \subset\subset \Omega_{j+1} \subset\subset \Omega$ be a sequence of smooth subdomains such that $\bigcup_{j=1}^\infty \Omega_j = \Omega$. For any integer $j \geq 2$ let ξ_j be a smooth function compactly supported in Ω_j such that $\xi_j \equiv 1$ on Ω_{j-1} and $0 \leq \xi_j \leq 1$. The following result will be useful in the sequel.

Proposition 4.13. *Let ξ_j, Ω_j be as above. Let $u_0 \in X(s)$ and let u be the corresponding solution of (4.16). Let u_j be the solution of (4.16) with Ω replaced by Ω_j and with u_0 replaced by $\xi_j u_0$. Then (a subsequence of) the sequence u_j converges uniformly to u on compact subsets of $\Omega \times [s, \infty)$.*

Proof. Without loss of generality suppose $c_0 \leq 0$ in this proof. Note that by Proposition 3.2 and standard parabolic regularity (a subsequence of) the sequence u_j converges to some $\tilde{u}$ locally uniformly on $\Omega \times (s, \infty)$ and $\tilde{u} \in \bigcap_{p>1} W_{loc}^{2,1,p}$ $(\Omega \times (s, \infty)) \cap C(\Omega \times (s, \infty))$, $\tilde{u}_t - L\tilde{u} = 0$ in $\Omega \times (s, \infty)$, and $\tilde{u}$ is bounded on $\Omega \times (s, T)$ for all $T > s$. The definition of ξ_j implies that if $m \geq n$, then $u_m(\cdot, s) - u_n(\cdot, s) \equiv 0$ on Ω_{n-1}. Employing the same barrier arguments as, for example, in the proof of [17, Proposition 5.4], one can prove that in fact (passing to a subsequence) the u_j converge uniformly to $\tilde{u}$ on compact subsets of $\Omega \times [s, \infty)$. Thus $\tilde{u} \in C(\Omega \times [s, \infty))$ and $\tilde{u}(x, s) = \lim_{j\to\infty} u_j(x, s) = \lim_{j\to\infty} \xi_j(x) u_0(x) = u_0(x)$ for all $x \in \Omega$. To conclude that $\tilde{u} = u$ we still need to show that $\tilde{u} \overset{w_\Omega}{=} 0$ on $\partial\Omega \times [s, T]$ for all $T > s$.

We denote $A = ||u_0||_{L^\infty(\Omega)}$. The definition of ξ_j and a simple contradiction argument show that for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for each j we have

$$|u_j(x, s)| \leq |u_0(x)| \leq \epsilon + \frac{A}{\delta(\epsilon)}\, w_\Omega(x, s) \qquad (x \in \Omega_j).$$

Set

$$v(x, t) = \epsilon + \frac{A}{\delta(\epsilon)}\, w_\Omega(x, t) - u_j(x, t) \qquad ((x, t) \in \Omega_j \times [s, \infty)).$$

Then an easy computation shows that $v_t - Lv \geq 0$ on $\Omega_j \times (s, \infty)$ and v is nonnegative on the parabolic boundary of that set. The usual comparison principle then gives that $v \geq 0$ on $\Omega_j \times [s, \infty)$. The same conclusion can be reached if the minus sign in the definition of v is replaced by the plus sign. As a consequence we obtain that for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$(4.25) \qquad |\tilde{u}(x, t)| = \lim_{j\to\infty} |u_j(x, t)| \leq \epsilon + \frac{A}{\delta(\epsilon)}\, w_\Omega(x, t) \qquad ((x, t) \in \Omega \times [s, \infty)).$$

Obviously, (4.25) implies $\tilde{u} \overset{w_\Omega}{=} 0$ on $\partial\Omega \times [s, T]$ for all $T > s$. Thus, by Corollary 4.11, we must have $\tilde{u} \equiv u$. $\square$

Remark 4.14. We remark that if u_0 is smooth and compactly supported in Ω we may take $\xi_j \equiv 1$ (on Ω_j) in the above proof, showing that $u_0 \in X(s)$, $s \in \mathbb{R}$. However, smooth functions compactly supported in Ω may not be dense in $X(s)$; this situation occurs (for any $s > 0$) in the example appearing after Remark 4.8 (see also Remark 4.12).

5. Existence of a positive entire solution

In this section we prove the existence of a "well behaved" positive entire solution φ_Ω of (1.1). We will obtain φ_Ω as the limit of a (sub)sequence of positive entire solutions φ_{Ω_j} given by Theorem 3.8 with "$G = \Omega_j$", where Ω_j is a sequence of smooth subdomains of Ω converging (in the Hausdorff metric) to Ω. To be able

to pass to the limit we need an estimate as in (3.14) with C_G independent of the smoothness of G. We establish such an estimate in Lemma 5.2 below. Before we state it let us introduce some notation.

In the remainder of this paper we fix

$$(5.1) \qquad x_\Omega \in \Omega \text{ with } \operatorname{dist}(x_\Omega, \partial\Omega) = \operatorname{inrad}(\Omega), \quad c_\Omega \geq 1,$$

where $\operatorname{inrad}(\Omega)$ is defined in (3.9) and c_Ω is the constant in Theorem 3.1 with "$D = B(x_\Omega, \operatorname{inrad}(\Omega)/4)$", "$\Omega = B(x_\Omega, \operatorname{inrad}(\Omega)/2)$", "$\delta = 1$", "$T = 2$". Note that c_Ω depends only on $\operatorname{inrad}(\Omega)$ and the structure of L, and we can assume without loss of generality that $c_\Omega \geq 1$. With α_0, β_0 as in (L1), set $a_0 = \alpha_0$ and $b_0 = 3(\sqrt{N}\beta_0 + \log c_\Omega)$ in Lemma 3.4 and let $\delta = \delta_0 > 0$ be as in that lemma. The constant δ_0 depends only on $\operatorname{inrad}(\Omega)$, $\operatorname{diam}(\Omega)$, and the structure of L. Now choose a smooth domain D_0 such that

$$(5.2) \qquad B(x_\Omega, \operatorname{inrad}(\Omega)/2) \subseteq D_0 \subset\subset \Omega \quad \text{and} \quad |\Omega \setminus D_0| < \delta_0.$$

From now on we will work with this D_0.

Remark 5.1. In addition to D_0 chosen above, in the course of this paper we fix (smooth) domains D_i, $i = 1, 2$ such that $D_i \subset\subset D_{i+1} \subset\subset \Omega$, $i = 0, 1, 2$. As we shall see, their choice is only determined by requiring that $|\Omega \setminus D_i| < \delta$, where $\delta > 0$ is a number depending only on Ω and the structure of L. Thus any dependence expressed in terms of D_i (or quantities related to it, such as $d_i = \operatorname{dist}(\bar{D}_i, \partial\Omega)$) can be stated as a dependence on Ω and the structure of L only. We use this observation without further notice.

Lemma 5.2. *Let D_0 be as in (5.2). Let D, G be domains such that G is smooth and $D_0 \subseteq D \subset\subset G \subseteq \Omega$. Then, if $\operatorname{dist}(\bar{D}, \partial G) \geq d > 0$, we have*

$$(5.3) \qquad \frac{\varphi_G(x,t)}{\|\varphi_G(\cdot,t)\|_{L^\infty(G)}} \geq C_1(D,d) \qquad ((x,t) \in D \times \mathbb{R}),$$

where φ_G is as in Theorem 3.8. The constant $C_1(D,d) > 0$ depends only on $\operatorname{inrad}(\Omega)$, D, d, and the structure of L.

Proof. Let D, G, φ_G be as in the statement of the lemma and let D_0 be as in (5.2). Fix $t_0 \in \mathbb{R}$. By the Harnack inequality (3.3) we have for any $t \leq t_0 - 2$,

$$\sup_{x \in D} \varphi_G(x,t) \leq C \varphi_G(x_\Omega, t+1) \leq \cdots \leq C c_\Omega^{[t_0-t-2]} \varphi_G(x_\Omega, t + [t_0 - t - 2] + 1)$$

$$(5.4) \qquad \leq C^2 c_\Omega^{[t_0-t-2]} \inf_{x \in D} \varphi_G(x, t_0) \leq C^2 e^{(\log c_\Omega)(t_0-t)} \inf_{x \in D} \varphi_G(x, t_0),$$

where x_Ω, c_Ω are as in (5.1), $C > 0$ depends only on D, d, and the structure of L, and $[t]$ denotes the integer part of a real number t.

Our goal is to show now that, roughly speaking, the inequality (5.4) is preserved if the supremum on the left hand side is taken over all of G. For that purpose let g be as in Lemma 3.4 with a_0, b_0, $\delta = \delta_0$ as chosen in the paragraph preceding Remark 5.1 and with $K = \bar{G} \setminus D_0$ (note that by smoothness of G one has $|K| = |G \setminus D_0| \leq |\Omega \setminus D_0| < \delta_0$). Denote $c_2 = C^2 \inf_{x \in D} \varphi_G(x, t_0)$. Define the following function:

$$(5.5) \qquad v_G(x,t) = \frac{(c_2 - e^{-(\log c_\Omega)(t_0-t)} \varphi_G(x,t))}{g(x)} + \frac{c_2}{2} \quad (x \in \bar{G} \setminus D, \ t \leq t_0 - 2).$$

We will show that this function is nonnegative on $(\bar{G} \setminus D) \times (-\infty, t_0 - 2]$. First, we have

$$L(v_G \, g) = g \, (a_{ij} D_{ij} v_G + \hat{b}_i D_i v_G + \hat{c}_0 v_G),$$

where

$$\hat{b}_i = (a_{ij} + a_{ji}) D_j g / g + b_i,$$
$$\hat{c}_0 = (a_{ij} D_{ij} g + b_i D_i g) / g + c_0 = Lg/g,$$

and, by Lemma 3.4 and our choice of a_0, b_0 above,

$$\hat{c}_0 = Lg/g \leq (\sqrt{N} \beta_0 - b_0)(|\nabla g| + g)/g \leq \sqrt{N} \beta_0 - b_0 \leq -2\beta_0 - 3 \log c_\Omega.$$

Next,

$$(v_G)_t \, g = -(\log c_\Omega) \, e^{-(\log c_\Omega)(t_0 - t)} \varphi_G - e^{-(\log c_\Omega)(t_0 - t)} (\varphi_G)_t$$
$$= (\log c_\Omega)(v_G g - \frac{c_2}{2} g - c_2) + L(-e^{-(\log c_\Omega)(t_0 - t)} \varphi_G)$$
$$= (\log c_\Omega)(v_G g - \frac{c_2}{2} g - c_2) + L(v_G g - \frac{c_2}{2} g - c_2).$$

Using the above relations and the fact that $1 \leq g \leq 2$ we get

$$(v_G)_t = a_{ij} D_{ij} v_G + \hat{b}_i D_i v_G + (\hat{c}_0 + \log c_\Omega) v_G$$
$$+ c_2(-\frac{1}{2} \log c_\Omega - \frac{1}{g} \log c_\Omega - \frac{1}{2} \hat{c}_0 - \frac{c_0}{g})$$
$$\geq a_{ij} D_{ij} v_G + \hat{b}_i D_i v_G + (\hat{c}_0 + \log c_\Omega) v_G,$$

where $\hat{c}_0 + \log c_\Omega \leq -2\beta_0 - 2 \log c_\Omega \leq -2\beta_0$ and the inequality holds on $(G \setminus \bar{D}) \times (-\infty, t_0 - 2)$. Note that (5.4) and the definition of c_2 imply

$$(5.6) \qquad\qquad v_G \geq 0 \quad \text{on} \quad \bar{D} \times (-\infty, t_0 - 2].$$

Hence, since $\varphi_G = 0$ on $\partial G \times \mathbb{R}$, $v_G \geq 0$ on $\partial(G \setminus \bar{D}) \times (-\infty, t_0 - 2]$. The maximum principle then implies

$$(5.7) \qquad ||v_G^-(\cdot, t)||_{L^\infty(G \setminus \bar{D})} \leq e^{-2\beta_0(t - \tau)} ||v_G^-(\cdot, \tau)||_{L^\infty(G \setminus \bar{D})} \quad (t_0 - 2 \geq t \geq \tau).$$

By the definition of v_G and Theorem 3.8 one has $(\tau \leq t_0 - 2)$

$$||v_G^-(\cdot, \tau)||_{L^\infty(G \setminus \bar{D})} \leq e^{-(\log c_\Omega)(t_0 - \tau)} ||\varphi_G(\cdot, \tau)||_{L^\infty(G \setminus \bar{D})}$$
$$\leq \frac{e^{-(\log c_\Omega)(t_0 - \tau)}}{C_G(D)} \sup_{x \in D} \varphi_G(x, \tau),$$

where $C_G(D)$ is as in (3.14). Applying (5.4) with $t = \tau$ to the last inequality above and using the definition of c_2, we obtain for any $\tau \leq t_0 - 2$,

$$(5.8) \qquad\qquad ||v_G^-(\cdot, \tau)||_{L^\infty(G \setminus \bar{D})} \leq \frac{c_2}{C_G(D)}.$$

Sending τ to $-\infty$ in (5.7) and taking (5.8) into account, we get that $v_G^- \equiv 0$ on $(G \setminus \bar{D}) \times (-\infty, t_0 - 2]$ and hence (using also (5.6)) $v_G \geq 0$ on $G \times (-\infty, t_0 - 2]$. Elementary inequalities then give $||\varphi_G(\cdot, t_0 - 2)||_{L^\infty(G)} \leq 2c_2 \, e^{2 \log c_\Omega}$. Using finally the maximum principle and the definition of c_2 we derive

$$||\varphi_G(\cdot, t_0)||_{L^\infty(G)} \leq e^{2\beta_0} ||\varphi_G(\cdot, t_0 - 2)||_{L^\infty(G)} \leq 2c_2 \, e^{2(\beta_0 + \log c_\Omega)}$$
$$= 2 \, C^2 \, e^{2(\beta_0 + \log c_\Omega)} \inf_{x \in D} \varphi_G(x, t_0),$$

which implies (5.3) with $C_1 = (1/2) C^{-2} e^{-2(\beta_0 + \log c_\Omega)}$. In view of the dependences of c_Ω and C, the constant C_1 depends only on $\operatorname{inrad}(\Omega)$, D, d, and the structure of L. The proof is complete. $\qquad \square$

As a simple consequence we obtain the following corollary.

Corollary 5.3. *Let D_0, D, G, d, φ_G be as in Lemma 5.2. Then*

$$(5.9) \qquad e^{\beta_0} \geq \frac{||\varphi_G(t+\tau)||_{L^\infty(G)}}{||\varphi_G(t)||_{L^\infty(G)}} \geq C_2(D,d) \qquad (t \in \mathbb{R},\ \tau \in [0,1]),$$

and the constant $C_2(D,d) > 0$ depends only on $\operatorname{inrad}(\Omega)$, D, d, and the structure of L.

Proof. The inequality on the left follows from the maximum principle (Proposition 3.2). By the Harnack inequality and Lemma 5.2 we have for any $x \in D$,

$$C \varphi_G(x, t+\tau) \geq \varphi_G(x, t+\tau-1) \geq C_1(D,d) ||\varphi_G(\cdot, t+\tau-1)||_{L^\infty(G)},$$

where $C > 0$ depends only on D, d, and the structure of L. Again using the maximum principle, we get $||\varphi_G(\cdot, t)||_{L^\infty(G)} \leq e^{\beta_0} ||\varphi_G(\cdot, t+\tau-1)||_{L^\infty(G)}$ for $\tau \in [0,1]$, which in combination with the above inequalities implies the right hand side inequality in (5.9) with $C_2(D,d) = C^{-1} C_1(D,d) e^{-\beta_0}$. $\qquad \square$

We next state a lemma, which will enable us to prove that φ_Ω to be defined below is a solution of (1.1) on $\Omega \times \mathbb{R}$.

Lemma 5.4. *Let D_0, D, G, d, φ_G be as in Lemma 5.2 and w_G as in Proposition 4.3. Then*

$$(5.10) \qquad \frac{\varphi_G(x,t)}{||\varphi_G(\cdot,t)||_{L^\infty(G)}} \leq C_3(D,d)\, w_G(x,t) \qquad ((x,t) \in G \times \mathbb{R}),$$

where $C_3 = 2(\beta_0 - \log C_2)/C_2$ and C_2 is as in Corollary 5.3.

Proof. Fix $t_0 \in \mathbb{R}$. Note that Corollary 5.3 implies that

$$(5.11) \qquad \frac{||\varphi_G(t)||_{L^\infty(G)}}{||\varphi_G(t_0)||_{L^\infty(G)}} \leq (1/C_2)\, e^{\log(1/C_2)(t_0 - t)} \qquad (t \leq t_0),$$

and we may assume without loss of generality that $C_2 < 1$. We set

$$v(x,t) = w_G(x,t) - \frac{C_2\, e^{\log(1/C_2)(t-t_0)}}{2(\beta_0 - \log C_2)} \frac{\varphi_G(x,t)}{||\varphi_G(t_0)||_{L^\infty(G)}} \quad (x \in G,\ t \leq t_0).$$

Then by the bound on w_G (see (4.11)) and (5.11), the function v is bounded and $v = 0$ on $\partial G \times (-\infty, t_0]$. Moreover, an easy computation shows that for some positive ϵ we have

$$(5.12) \qquad v_t - Mv \geq \frac{1}{2} \geq -\epsilon\, v \quad \text{on} \quad G \times (-\infty, t_0).$$

By the maximum principle we get

$$||v^-(\cdot, t_0)||_{L^\infty(G)} \leq e^{-\epsilon(t_0 - \tau)} ||v^-(\cdot, \tau)||_{L^\infty(G)} \quad (t_0 \geq \tau).$$

Sending $\tau \to -\infty$ and using the boundedness of v we obtain $v^-(\cdot, t_0) \equiv 0$ on G. Using this fact, we obtain (5.10) as claimed. $\qquad \square$

We are ready to state the following existence theorem, which in particular yields Theorem 2.1 (cf. [3, Theorem 2.1]).

Theorem 5.5. *There exists a positive entire solution φ_Ω of (1.1) such that the following holds. If D is a subdomain of Ω with $\bar{D} \subset \Omega$, then*

$$(5.13) \qquad C_4(D) \leq \frac{\varphi_\Omega(x,t)}{||\varphi_\Omega(\cdot,t)||_{L^\infty(\Omega)}} \leq C_5 w_\Omega(x,t) \qquad ((x,t) \in D \times \mathbb{R}).$$

The constant $C_4(D) > 0$ depends only on D, Ω and the structure of L. The constant C_5 depends only on Ω and the structure of L. Moreover, φ_Ω satisfies the estimate (5.9) (with "$G = \Omega$") with a constant $C_2 > 0$ depending only on Ω and the structure of L.

Proof. Let $\Omega_j \subset\subset \Omega_{j+1} \subset\subset \Omega$ be a sequence of smooth subdomains such that $\bigcup_{j=1}^\infty \Omega_j = \Omega$, and for each j let φ_{Ω_j} be the positive entire solution given by Theorem 3.8 and satisfying $||\varphi_{\Omega_j}(\cdot,0)||_{L^\infty(\Omega_j)} = 1$. The above results along with standard parabolic regularity guarantee that, at least for a subsequence, the functions φ_{Ω_j} converge locally uniformly on $\Omega \times \mathbb{R}$ to a positive entire solution φ_Ω of (1.1), which in addition satisfies the bounds (5.13), (5.9). The proof is complete. $\qquad \square$

6. Stabilizing properties of positive solutions

For D_0 as in (5.2) denote

$$(6.1) \qquad\qquad\qquad d_0 = \text{dist}(\bar{D}_0, \partial\Omega) > 0.$$

We start with the following theorem.

Theorem 6.1. *Let D_0 be as in (5.2). Then there exists a domain D_1 with $D_0 \subset\subset D_1 \subset\subset \Omega$ and such that the following holds. Let D, G be domains with G smooth and such that $D_1 \subseteq D \subset\subset G \subseteq \Omega$ and $\text{dist}(\bar{D}, \partial G) \geq d > 0$. Let v be a positive solution of (3.1), (3.2) on $G \times (-1, \infty)$. Then there exists a constant $C_6(D,d) > 0$ and $t_0(v) \in [0, 2 + \log_2(||v(0)||_{L^\infty(G)}/(\inf_{x \in D} v(x,0)))]$ such that for any $t \geq t_0(v)$ we have*

$$(6.2) \qquad\qquad \frac{v(x,t)}{||v(\cdot,t)||_{L^\infty(G)}} \geq C_6(D,d) \qquad (x \in D).$$

The constant $C_6(D,d)$ depends only on $\text{inrad}(\Omega)$, D_0, d_0, D, d and the structure of L.

Remark 6.2. For a similar result for divergence form parabolic equations we refer the reader to [13, Lemma 5.1]. A stronger version of the above estimate is contained in [17, Lemma 3.9] under the assumption that Ω is Lipschitz.

We prove Theorem 6.1 via two lemmas, which will also be useful in the proof of Theorem 2.2. Before we state them let us specify the domain D_1 in Theorem 6.1. Denote

$$(6.3) \qquad\qquad\qquad \tilde{C}_2 = C_2(D_0, d_0/8),$$

where d_0 is as in (6.1) and $C_2(D_0, d_0/8)$ is as in Corollary 5.3. Next, with c_Ω as in (5.1), let $k > 0$ be so large that $2e^{-k} < (\tilde{C}_2/16c_\Omega)$ and let $\delta > 0$ be the corresponding number given by Lemma 3.5. We now choose a smooth domain D_1 with $D_0 \subset\subset D_1 \subset\subset \Omega$ such that $\text{dist}(\bar{D}_0, \partial D_1) \geq d_0/8$ and $|\Omega \setminus D_1| < \delta$. For future use we also note that, with x_Ω as in (5.1), $B(x_\Omega, \text{inrad}(\Omega)/2) \subset D_1$, which follows from (5.2) and $D_0 \subset D_1$. Thus, setting $\rho = \text{inrad}(\Omega)/4$ in Theorem 3.6 and making D_1 larger if necessary, we may assume $|\Omega \setminus D_1| < \delta$, where $\delta =$

$\delta(N, \operatorname{diam}(\Omega), \alpha_0, \beta_0, \operatorname{inrad}(\Omega)/4)$ is as in Theorem 3.6. This is the domain D_1 that will serve our purpose.

For x_Ω, c_Ω as above and v, D, d as in Theorem 6.1 define the function

$$(6.4) \qquad \eta(t) = v(x_\Omega, t)/||v(\cdot, t)||_{L^\infty(G)} \qquad (t > -1),$$

and set

$$(6.5) \qquad \tilde{C}(D, d) = \frac{C_1(D, d)\,(\tilde{C}_2)^2}{16\,c_\Omega\,e^{2\beta_0}},$$

where $C_1(D, d)$ is as in Lemma 5.2 and $\tilde{C}_2$ is as in (6.3). Our first lemma in this section states that if a positive solution of (3.1), (3.2) is concentrating near the boundary of G over a unit of time, then its norm drops significantly during that time interval.

Lemma 6.3. *Let v, d, D, G be as in Theorem 6.1, let $\tilde{C}(D, d)$, $\tilde{C}_2$, c_Ω, and η be as in (6.5) and (6.4), respectively. Further, for $k > 0$ let δ be as in Lemma 3.5 and suppose $|G \setminus \bar{D}| < \delta$. Suppose for some $t_0 \geq 0$ and $\alpha \in (0, 1]$ we have*

$$(6.6) \qquad \frac{||v(t)||_{L^\infty(D)}}{||v(t)||_{L^\infty(G)}} \leq \alpha\,\tilde{C}(D, d) \qquad (t \in [t_0, t_0 + 1]).$$

Then

$$(6.7) \qquad \frac{||v(t_0 + 1)||_{L^\infty(G)}}{||v(t_0)||_{L^\infty(G)}} \leq \min\left\{ \frac{\alpha\,\tilde{C}_2}{16\,c_\Omega} + 2\,e^{-k}, \frac{||\varphi_G(t_0 + 1)||_{L^\infty(G)}}{2||\varphi_G(t_0)||_{L^\infty(G)}} \right\},$$

and

$$(6.8) \qquad \eta(t_0 + 1) > 2\,\eta(t_0).$$

Proof. We start with the proof of the first inequality in (6.7). Define the function

$$u(x, t) = \left(\frac{\alpha\,\tilde{C}_2}{16\,c_\Omega\,e^{\beta_0}} \frac{||v(t_0)||_{L^\infty(G)}}{||\varphi_G(t_0)||_{L^\infty(G)}} \right) \varphi_G(x, t) - v(x, t) \quad ((x, t) \in G \times [t_0, \infty)).$$

We are going to prove now that $u(x, t) \geq 0$ for all $x \in \partial D$, $t \in [t_0, t_0 + 1]$. First, by the assumption (6.6) and by the maximum principle, we have $||v(t)||_{L^\infty(D)} \leq \tilde{C}(D, d)||v(t)||_{L^\infty(G)} \leq \tilde{C}(D, d)e^{\beta_0}\,||v(t_0)||_{L^\infty(G)}$ for $t \in [t_0, t_0 + 1]$. Next, using Lemma 5.2 we get $\varphi_G(x, t) \geq C_1(D, d)||\varphi_G(t)||_{L^\infty(G)}$ for all $x \in D$ and by Corollary 5.3 (with $D = D_0$ and $d = d_0/8$) we obtain $||\varphi_G(t)||_{L^\infty(G)} \geq \tilde{C}_2||\varphi_G(t_0)||_{L^\infty(G)}$ for all $t \in [t_0, t_0 + 1]$ and $\tilde{C}_2$ as in (6.3). Combining these inequalities, one easily concludes that $u(x, t) \geq 0$ for all $x \in \partial D$, $t \in [t_0, t_0 + 1]$. Thus u is a supersolution of (3.1), (3.2) with G replaced by $G \setminus \bar{D}$ and since $|G \setminus \bar{D}| < \delta$ we can use Lemma 3.5 with the corresponding k. Therefore

$$||u^-(t_0 + 1)||_{L^\infty(G \setminus \bar{D})} \leq 2\,e^{-k}\,||u^-(t_0)||_{L^\infty(G \setminus \bar{D})} \leq 2\,e^{-k}\,||v(t_0)||_{L^\infty(G)}.$$

Now observe that an elementary estimate and the maximum principle give

$$||u^-(t_0 + 1)||_{L^\infty(G \setminus \bar{D})} \geq ||v(t_0 + 1)||_{L^\infty(G \setminus \bar{D})}$$
$$- \left(\frac{\alpha\,\tilde{C}_2}{16\,c_\Omega\,e^{\beta_0}} \frac{||v(t_0)||_{L^\infty(G)}}{||\varphi_G(t_0)||_{L^\infty(G)}} \right) ||\varphi_G(t_0 + 1)||_{L^\infty(G)}$$
$$\geq ||v(t_0 + 1)||_{L^\infty(G \setminus \bar{D})} - \frac{\alpha\,\tilde{C}_2}{16\,c_\Omega}\,||v(t_0)||_{L^\infty(G)}.$$

Since $\alpha \leq 1$ and we may assume $\tilde{C}(D,d) < 1$, (6.6) with $t = t_0 + 1$ implies $||v(t_0 + 1)||_{L^\infty(G\setminus\bar{D})} = ||v(t_0 + 1)||_{L^\infty(G)}$. Using this fact and the above estimates we arrive at

$$(6.9) \qquad ||v(t_0 + 1)||_{L^\infty(G)} \leq \left(\frac{\alpha \tilde{C}_2}{16\, c_\Omega} + 2\, e^{-k} \right) ||v(t_0)||_{L^\infty(G)},$$

which gives the first inequality in (6.7).

To prove the remaining inequality in (6.7), note that now we may assume $2\, e^{-k} < \tilde{C}_2/(16\, c_\Omega)$. Indeed, this follows from our choice of D_1 in the paragraph following Remark 6.2 and from the fact that $|G \setminus \bar{D}| \leq |\Omega \setminus \bar{D}_1|$. Thus, by Corollary 5.3 (with $D = D_0$ and $d = d_0/8$), we also get

$$(6.10) \qquad \frac{\alpha \tilde{C}_2}{16\, c_\Omega} + 2\, e^{-k} \leq \frac{\tilde{C}_2}{8\, c_\Omega} \leq \frac{1}{8} \frac{||\varphi_G(t_0 + 1)||_{L^\infty(G)}}{||\varphi_G(t_0)||_{L^\infty(G)}},$$

which together with (6.9) implies the second inequality in (6.7).

To prove (6.8), we first use the Harnack inequality to obtain

$$v(x_\Omega, t_0) \leq c_\Omega\, v(x_\Omega, t_0 + 1).$$

This inequality in combination with (6.9) and the left inequality in (6.10) yields $\eta(t_0 + 1) \geq (8/\tilde{C}_2)\, \eta(t_0)$. Since we may assume that $\tilde{C}_2 < 1$ (see (6.3) and Corollary 5.3), (6.8) follows. The proof is thus complete. $\qquad\square$

Lemma 6.4. *Let v, d, D, G be as in Theorem* 6.1 *and suppose that for some $\kappa > 0$, $t_0 \geq 0$ we have*

$$(6.11) \qquad \frac{||v(t_0)||_{L^\infty(D)}}{||v(t_0)||_{L^\infty(G)}} \geq \kappa.$$

Then there exists a constant $C_7(D,d) > 0$ such that

$$(6.12) \qquad \frac{v(x,t)}{||v(t)||_{L^\infty(G)}} \geq C_7(D,d)\, \kappa \qquad (x \in D,\, t \geq t_0 + 1).$$

The constant $C_7(D,d)$ depends only on $\mathrm{inrad}(\Omega)$, D_0, d_0, D, d, *and the structure of L.*

Proof. Fix $\kappa \in (0,1]$ as in the statement of Lemma 6.4. By the Harnack inequality and the maximum principle we have the following implication for any $t \geq 0$ and any $\tilde{\kappa} \in (0,1]$:

$$(6.13) \qquad \frac{||v(t)||_{L^\infty(D)}}{||v(t)||_{L^\infty(G)}} \geq \tilde{\kappa} \;\Rightarrow\; \eta(t + \tau) \geq C\,\tilde{\kappa} \quad (\tau \in [1,3]),$$

where $C > 0$ depends only on D, d, and the structure of L and η is as in (6.4). By Lemma 6.3 we also have for any $t \geq 0$:

$$(6.14) \qquad \frac{||v(\tau)||_{L^\infty(D)}}{||v(\tau)||_{L^\infty(G)}} \leq \tilde{C}(D,d) \text{ for } \tau \in [t, t+1] \;\Rightarrow\; \eta(t+1) > 2\,\eta(t).$$

For C, $\tilde{C}(D,d)$ as above we define the open set

$$A = \{t > t_0 + 1 : \eta(t) < C\,\tilde{C}(D,d)\,\kappa\}.$$

Suppose the interval (a,b) is a connected component of A. The assumption (6.11) inserted into the implication (6.13) with $t = t_0$ and $\tilde{\kappa} = \kappa$ gives that $\eta(t_0 + 1) \geq C\kappa > C\,\tilde{C}(D,d)\,\kappa$ (because $\tilde{C}(D,d) < 1$). Therefore $a > t_0 + 1$ and by continuity

of η we have $\eta(a) = C\,\tilde{C}(D,d)\,\kappa$. Now, the definition of A, the contrapositive of (6.13) with $\tilde{\kappa} = \tilde{C}(D,d)\kappa$ and (6.14) imply that if $l \leq b-a-1$ is a positive integer, then

$$C\,\tilde{C}(D,d)\,\kappa > \eta(a+l) > 2^l\,\eta(a) = 2^l\,C\,\tilde{C}(D,d)\,\kappa,$$

which is impossible. Therefore necessarily $b - a < 2$. This shows that for each $t \geq t_0 + 1$ there exists $r \in [t, t+2]$ such that $\eta(r) \geq C\,\tilde{C}(D,d)\,\kappa$. Then by the Harnack inequality and the maximum principle, $\inf_{x \in D} v(x,s)/\|v(s)\|_{L^\infty(G)} \geq C_7\,\kappa$ for $s \in [t+3, t+6]$ and some $C_7 > 0$ depending only on $\operatorname{inrad}(\Omega)$, D_0, d_0, D, d, and the structure of L. Hence (6.12) holds for any $t \geq t_0 + 4$, and for $t \in [t_0+1, t_0+4]$, it follows from (6.11), the Harnack inequality and the maximum principle. $\square$

Proof of Theorem 6.1. First of all, we have $\eta(0) \geq \inf_{x \in D} v(x,0)/\|v(0)\|_{L^\infty(G)}$. Lemma 6.3 with $\alpha = 1$ and the fact that $\eta \leq 1$ then guarantee that there exists a smallest

$$(6.15) \qquad \tau_0 \in [0, 1 + \log_2(\|v(0)\|_{L^\infty(G)}/(\inf_{x \in D} v(x,0)))]$$

for which

$$(6.16) \qquad \frac{\|v(\tau_0)\|_{L^\infty(D)}}{\|v(\tau_0)\|_{L^\infty(G)}} \geq \tilde{C}(D,d),$$

where $\tilde{C}(D,d)$ is as in (6.5). The inequality (6.2) then follows from Lemma 6.4 with $\kappa = \tilde{C}(D,d)$, and we can take $t_0(v) = \tau_0 + 1$. $\square$

We will need the following corollary in the sequel.

Corollary 6.5. *Let* v, d, D, G *be as in Theorem* 6.1 *and suppose that for some* $\kappa > 0$, $t_0 \geq 0$ *we have*

$$\frac{\|v(t_0)\|_{L^\infty(D)}}{\|v(t_0)\|_{L^\infty(G)}} \geq \kappa.$$

Then

$$(6.17) \qquad \frac{\|v(t)\|_{L^\infty(G)}}{\|v(s)\|_{L^\infty(G)}} \geq \kappa\,C_8(D,d)\,e^{-(\log c_\Omega)(t-s)} \qquad (t \geq s \geq t_0 + 1),$$

where c_Ω *is as in* (5.1) *and the constant* $C_8(D,d) > 0$ *depends only on* $\operatorname{inrad}(\Omega)$, D_0, d_0, D, d, *and the structure of* L.

Proof. By the Harnack inequality and by our assumption on v we obtain

$$C_H\,v(x, t_0 + \tau) \geq \|v(t_0)\|_{L^\infty(D)} \geq \kappa\,\|v(t_0)\|_{L^\infty(G)} \quad (x \in D, \tau \in [1,3]),$$

where $C_H > 0$ depends only on D, d, and the structure of L. The maximum principle gives $\|v(t_0 + \tau)\|_{L^\infty(G)} \leq e^{3\beta_0}\,\|v(t_0)\|_{L^\infty(G)}$ for all $\tau \in [1,3]$. The latter two sets of inequalities imply the following estimate:

$$\frac{\|v(t_0 + \tau_2)\|_{L^\infty(G)}}{\|v(t_0 + \tau_1)\|_{L^\infty(G)}} \geq \frac{\kappa}{C_H e^{3\beta_0}} \quad (1 \leq \tau_1 \leq \tau_2 \leq 3).$$

To finish the proof observe that if $t \geq s \geq t_0 + 3$, then by the Harnack inequality, (6.12) and the maximum principle,

$$
\begin{aligned}
||v(t)||_{L^\infty(G)} \geq v(x_\Omega, t) &\geq \frac{1}{c_\Omega} \, v(x_\Omega, t-1) \\
&\geq \cdots \geq \frac{1}{c_\Omega^{[t-s]+1}} \, v(x_\Omega, t-[t-s]-1) \\
&\geq \frac{C_7(D, d)\, \kappa}{c_\Omega^{[t-s]+1}} \, ||v(t-[t-s]-1)||_{L^\infty(G)} \\
&\geq \frac{C_7(D, d)\, \kappa}{e^{\beta_0}\, c_\Omega^{t-s+1}} \, ||v(s)||_{L^\infty(G)},
\end{aligned}
$$

where x_Ω and c_Ω are as in (5.1) and $[t]$ denotes the integer part of a real number t. The above inequalities imply (6.17). $\qquad\square$

The following lemma compares the growth of a positive solution of (3.1), (3.2) with the growth of any other solution.

Lemma 6.6. *Let* v, d, D, G *be as in Theorem* 6.1 *and for* $t_0 \geq 0$ *let* u *be a solution of* (3.1), (3.2) *on* $G \times (t_0, \infty)$ *with* $u(\cdot, t_0) \in C_0(\bar{G})$. *There exists a constant* $C_9(D, d) \in (0, 1)$ *depending only on* $\mathrm{inrad}(\Omega)$, $\mathrm{diam}(\Omega)$, D_0, d_0, D, d, *and the structure of* L *such that the following holds. Suppose that for some* $\kappa > 0$ *we have*

$$
||v(t_0)||_{L^\infty(D)} \geq \kappa \, ||v(t_0)||_{L^\infty(G)} \geq C_9^{-1}(D, d)||u(t_0)||_{L^\infty(G)}.
$$

Then

$$
(6.18) \qquad\qquad v(x, t) > |u(x, t)| \qquad (x \in \bar{D},\ t \geq t_0 + 1)
$$

and

$$
(6.19) \qquad\qquad ||v(t)||_{L^\infty(G)} \geq (\kappa/3)\, ||u(t)||_{L^\infty(G)} \qquad (t \geq t_0 + 1).
$$

Proof. For u, v, t_0 as in the lemma define the function

$$
(6.20) \qquad \tilde{u}(x, t) = v(x, t) - u(x, t) \quad ((x, t) \in G \times [t_0 + 1, \infty)).
$$

To prove (6.18) we are going to apply Theorem 3.6 to $\tilde{u}$ with $\rho = \mathrm{inrad}(\Omega)/4$, $\theta = 1/8$, $d = \mathrm{dist}(\bar{D}, \partial G)/2$, and $\tau = t_0 + 1$. Note that our choice of D_1 above (see the paragraph immediately following Remark 6.2) and the fact that $D_1 \subseteq D$ imply that the assumptions (3.10) and (3.11) in Theorem 3.6 are satisfied. We need to verify the remaining assumptions in this theorem.

By positivity of v and the maximum principle we have

$$
(6.21) \quad ||\tilde{u}^-(t_0 + r)||_{L^\infty(G)} \leq ||u(t_0 + r)||_{L^\infty(G)} \leq e^{2\beta_0}\, ||u(t_0)||_{L^\infty(G)} \quad (r \in [1, 2]).
$$

By the Harnack inequality and the assumption on u, v we have

$$
\begin{aligned}
||v(t_0 + r)||_{L^\infty(G)} \geq v(x, t_0 + r) &\geq C_H\, ||v(t_0)||_{L^\infty(D)} \\
(6.22) \qquad\qquad &\geq C_H\, C_9^{-1}(D, d)||u(t_0)||_{L^\infty(G)} \quad (x \in D,\ r \in [1, 2]),
\end{aligned}
$$

where $C_H > 0$ depends only on D, d and the structure of L. Hence for all $x \in \bar{D}$ one has

$$
\begin{aligned}
||\tilde{u}||_{L^\infty(D \times (t_0 + \frac{9}{8}, t_0 + \frac{10}{8}))} \geq \tilde{u}(x, t_0 + \frac{9}{8}) &\geq v(x, t_0 + \frac{9}{8}) - e^{2\beta_0} ||u(t_0)||_{L^\infty(G)} \\
&\geq (C_H \, C_9^{-1}(D, d) - e^{2\beta_0}) ||u(t_0)||_{L^\infty(G)} \\
&= \mu^{-1} \, e^{2\beta_0} ||u(t_0)||_{L^\infty(G)},
\end{aligned}
$$
(6.23)

where μ is as in Theorem 3.6 and $C_9(D, d) = \mu \, (\mu + 1)^{-1} C_H e^{-2\beta_0}$ depends only on $\mathrm{inrad}(\Omega)$, $\mathrm{diam}(\Omega)$, D, d, and the structure of L. The inequality (6.21) with $r = 1$ and (6.23) show that (3.13) in Theorem 3.6 is satisfied. Finally, we note that if we replace $t_0 + \frac{9}{8}$ in (6.23) by any $t \in [t_0 + 1, t_0 + 2]$, then all of these inequalities, except for the first one, remain valid. This gives that $\tilde{u}$ is positive on $\bar{D} \times [t_0 + 1, t_0 + 2]$ and thus also the assumption (3.12) in Theorem 3.6 is satisfied. Therefore, by $(s1)$ in Theorem 3.6, $v(x, t) > u(x, t)$ for all $(x, t) \in \bar{D} \times [t_0 + 1, \infty)$. The same arguments as above apply if we define $\tilde{u} = v + u$, showing that also $v(x, t) > -u(x, t)$ for all $(x, t) \in \bar{D} \times [t_0 + 1, \infty)$. The proof of (6.18) is finished.

Using (6.17) with $s = t_0 + 1$ and (6.22) with $r = 1$ we obtain

(6.24)
$$
||v(t)||_{L^\infty(G)} \geq \frac{\kappa \, C_8(D, d) \, C_H}{C_9(D, d)} e^{-(\log c_\Omega)(t - t_0 - 1)} ||u(t_0)||_{L^\infty(G)} \quad (t \geq t_0 + 1).
$$

Since we have already proved (6.18) we may apply Lemma 3.5 to $\tilde{u} = v \pm u$ with $U = G \setminus \bar{D}$ and k such that $k > \log c_\Omega + \log 32$. The inequality for k follows from the fact that $D_1 \subseteq D$ and our choice of D_1 (see the paragraph following Remark 6.2). As a consequence we get for any $t \geq t_0 + 1$,

$$
\begin{aligned}
||(v(t) \pm u(t))^-||_{L^\infty(G \setminus \bar{D})} &\leq 2e^{-k(t - t_0 - 1)} ||(v(t_0 + 1) \pm u(t_0 + 1))^-||_{L^\infty(G \setminus \bar{D})} \\
&\leq 2e^{-k(t - t_0 - 1)} ||u(t_0 + 1)||_{L^\infty(G)} \\
&\leq 2e^{\beta_0} e^{-k(t - t_0 - 1)} ||u(t_0)||_{L^\infty(G)}.
\end{aligned}
$$
(6.25)

Using $k > \log c_\Omega + \log 32$, (6.24) and (6.25), making $C_9(D, d)$ smaller if necessary, we obtain

$$
||(v(t) \pm u(t))^-||_{L^\infty(G \setminus \bar{D})} \leq (1/\kappa) ||v(t)||_{L^\infty(G)} \qquad (t \geq t_0 + 1).
$$

Clearly,

$$
\pm u(x, t) - v(x, t) \leq ||(v(t) \mp u(t))^-||_{L^\infty(G \setminus \bar{D})} \quad (x \in G \setminus \bar{D}, \, t \geq t_0 + 1).
$$

The last two inequalities combined imply

$$
||u(t)||_{L^\infty(G \setminus \bar{D})} \leq (1 + 1/\kappa) ||v(t)||_{L^\infty(G)} \quad (t \geq t_0 + 1).
$$

Therefore, using also (6.18), we derive

$$
\begin{aligned}
||u(t)||_{L^\infty(G)} &\leq ||u(t)||_{L^\infty(D)} + ||u(t)||_{L^\infty(G \setminus \bar{D})} \\
&\leq ||v(t)||_{L^\infty(G)} + (1 + 1/\kappa) ||v(t)||_{L^\infty(G)} \\
&= (2 + 1/\kappa) ||v(t)||_{L^\infty(G)} \qquad (t \geq t_0 + 1),
\end{aligned}
$$

which implies (6.19) since $\kappa \in (0, 1]$. $\qquad\square$

The next lemma compares the growth of any positive solution v of (3.1), (3.2) with the growth of φ_G.

Lemma 6.7. *Let v, d, D, G, D_1 be as in Theorem 6.1 and assume $D = D_1$. Let τ_0 be as in (6.15) with $D = D_1$. Then there exist constants $C_{10}(D_1, d)$, $C_{11}(D_1, d) > 1$ such that the following inequalities hold:*

(6.26)
$$\frac{||v(t)||_{L^\infty(G)}}{||\varphi_G(t)||_{L^\infty(G)}} \leq C_{10}(D_1, d) \frac{||v(0)||_{L^\infty(G)}}{||\varphi_G(0)||_{L^\infty(G)}} \qquad (t \geq 0),$$

(6.27) $\quad (C_{11})^{-1} \leq \dfrac{||v(t)||_{L^\infty(G)}}{||\varphi_G(t)||_{L^\infty(G)}} \dfrac{||\varphi_G(s)||_{L^\infty(G)}}{||v(s)||_{L^\infty(G)}} \leq C_{11} \qquad (t \geq s \geq \tau_0 + 1).$

These constants depend only on $\mathrm{inrad}(\Omega)$, $\mathrm{diam}(\Omega)$, D_0, d_0, D_1, d, *and the structure of* L.

Remark 6.8. Heuristically, as soon as $v(x, \tau_0)$ becomes comparable with its norm $||v(\tau_0)||_{L^\infty(G)}$ for some $x \in D_1$ and $\tau_0 \geq 0$, then the norms $||v(t)||_{L^\infty(G)}$ and $||\varphi_G(t)||_{L^\infty(G)}$ will grow at the same rate for all $t \geq \tau_0 + 1$.

Proof of Lemma 6.7. Let $\tau_0 \geq 0$ be as defined in (6.15) with $D = D_1$. Let $t \in [0, \tau_0 + 1]$. Then by the maximum principle and Corollary 5.3 we have

$$\frac{||v(t)||_{L^\infty(G)}}{||\varphi_G(t)||_{L^\infty(G)}} \leq \frac{e^{2\beta_0}}{\tilde{C}_2^2} \frac{||v([t-1])||_{L^\infty(G)}}{||\varphi_G([t-1])||_{L^\infty(G)}},$$

where $\tilde{C}_2$ is as in (6.3), $[t-1]$ denotes the integer part of $t-1$ and we make a temporary convention that $[t-1] = 0$ if $t \in [0, 1]$. Note that, by the definition of τ_0 and since $[t-1] \leq [\tau_0]$, the inequality (6.6) (with $D = D_1$ and $\alpha = 1$) is satisfied for all $t \in [0, [\tau_0]]$. We may therefore use Lemma 6.3 (with $D = D_1$) to obtain (still respecting our temporary convention)

$$\frac{||v([t-1])||_{L^\infty(G)}}{||\varphi_G([t-1])||_{L^\infty(G)}} \leq \frac{1}{2^{[t-1]}} \frac{||v(0)||_{L^\infty(G)}}{||\varphi_G(0)||_{L^\infty(G)}}.$$

Clearly, the above inequalities show that (6.26) holds for all $t \in [0, \tau_0 + 1]$.

Let us prove (6.26) in the remaining case, i.e., for $t \geq \tau_0 + 1$. The choice of τ_0 above implies that (6.16) holds with $D = D_1$, i.e., we have

(6.28)
$$\frac{||v(\tau_0)||_{L^\infty(D_1)}}{||v(\tau_0)||_{L^\infty(G)}} \geq \tilde{C}(D_1, d).$$

By Lemma 5.2 also

(6.29)
$$\frac{||\varphi_G(\tau_0)||_{L^\infty(D_1)}}{||\varphi_G(\tau_0)||_{L^\infty(G)}} \geq C_1(D_1, d).$$

We are going to apply Lemma 6.6 with $\kappa = \min\{\tilde{C}(D_1, d), C_1(D_1, d)\}$, $t_0 = \tau_0$, and with

(6.30) $\qquad v(x, t) = \dfrac{\varphi_G(x, t)}{\kappa \, C_9(D_1, d) \, ||\varphi_G(\tau_0)||_{L^\infty(G)}}, \;\; u(x, t) = \dfrac{v(x, t)}{||v(\tau_0)||_{L^\infty(G)}}.$

To avoid confusion, we remark that the v in the definition of u in (6.30) is as in the statement of Lemma 6.7 and has nothing to do with the v appearing on the left in

(6.30). Applying Lemma 6.6, we obtain

$$(6.31) \qquad \frac{||\varphi_G(t)||_{L^\infty(G)}}{\kappa\, C_9(D_1, d)\, ||\varphi_G(\tau_0)||_{L^\infty(G)}} \geq \frac{\kappa\, ||v(t)||_{L^\infty(G)}}{3||v(\tau_0)||_{L^\infty(G)}} \quad (t \geq \tau_0 + 1),$$

which finishes the proof of (6.26).

Finally, note that we may interchange the roles of φ_G and v in (6.30). We then get (6.31) with the roles of φ_G and v interchanged. These two versions of (6.31) yield (6.27). $\qquad\square$

For future use we state a corollary of Lemma 6.7.

Corollary 6.9. *Let d, D, G, D_1 be as in Theorem 6.1 and assume $D = D_1$. Let u be any solution of (3.4) with $u_0 \in C_0(\bar{G})$. Then*

$$(6.32) \qquad \frac{||u(t)||_{L^\infty(G)}}{||\varphi_G(t)||_{L^\infty(G)}} \leq \tilde{C}_{10}(D_1, d)\, \frac{||u_0||_{L^\infty(G)}}{||\varphi_G(0)||_{L^\infty(G)}} \qquad (t \geq 0),$$

where $\tilde{C}_{10}(D_1, d) = e^{\beta_0}\, C_{10}(D_1, d)/C_2(D_1, d)$ and C_2 and C_{10} are as in Corollary 5.3 (with $D = D_1$) and Lemma 6.7, respectively.

Proof. By uniqueness of the solutions of (3.4) we can write

$$(6.33) \qquad u(x, t) = u_1(x, t; 0, u_0^+) - u_2(x, t; 0, u_0^-)$$

for all $x \in G$ and $t \geq 0$. Here, $u_1(x, t; 0, u_0^+)$ is the (nonnegative) solution of (3.4) with $u_0 = u_0^+$ (u_2 is defined analogously). By the maximum principle $||u(t)||_{L^\infty(G)} \leq e^{\beta_0}\, ||u_0||_{L^\infty(G)}$ for all $t \in [0, 1]$ and by Corollary 5.3 we have $||\varphi_G(t)||_{L^\infty(G)} \geq C_2(D_1, d)\, ||\varphi_G(0)||_{L^\infty(G)}$. Hence the inequality (6.32) holds for $t \in [0, 1]$ and with $\tilde{C}_{10}$ replaced by $e^{\beta_0}/C_2(D_1, d)$. If $t > 1$ we have by Lemma 6.7 and by what we have just said above,

$$\begin{aligned} \frac{||u(t)||_{L^\infty(G)}}{||\varphi_G(t)||_{L^\infty(G)}} &\leq \frac{\max\{||u_1(t)||_{L^\infty(G)}, ||u_2(t)||_{L^\infty(G)}\}}{||\varphi_G(t)||_{L^\infty(G)}} \\ &\leq C_{10}(D_1, d)\, \frac{\max\{||u_1(1)||_{L^\infty(G)}, ||u_2(1)||_{L^\infty(G)}\}}{||\varphi_G(1)||_{L^\infty(G)}} \\ &\leq \frac{e^{\beta_0}\, C_{10}(D_1, d)}{C_2(D_1, d)}\, \frac{\max\{||u_0^+||_{L^\infty(G)}, ||u_0^-||_{L^\infty(G)}\}}{||\varphi_G(0)||_{L^\infty(G)}} \\ &= \frac{e^{\beta_0}\, C_{10}(D_1, d)}{C_2(D_1, d)}\, \frac{||u_0||_{L^\infty(G)}}{||\varphi_G(0)||_{L^\infty(G)}}. \end{aligned}$$

$\qquad\square$

Proposition 4.13 gives the following corollary.

Corollary 6.10. *Theorem 6.1, Lemmas 6.3–6.7, and Corollaries 6.5 and 6.9 remain valid if G is replaced by Ω, i.e., if v is a positive solution of (1.1) on $\Omega \times (-1, \infty)$ and if u is a solution of (1.1) or an associated initial value problem (4.16). All of these results hold in the corresponding forms on arbitrary intervals (s, ∞), $s \in \mathbb{R}$.*

We conclude this section with a result which shows "eventual" positivity of all solutions of (3.1), (3.2), which stay positive on a sufficiently large subdomain of a smooth domain G.

Proposition 6.11. *Let D_1 and G be as in Theorem* 6.1 *and let u be a solution of* (3.1), (3.2) *on $G \times (s, \infty)$ such that $u(x,t) > 0$ for all $x \in D_1$ and $t \geq t_0 > s$. Then there exists $t_G(u) > t_0$ such that $u(x,t) > 0$ for all $x \in G$ and $t \geq t_G(u)$.*

Remark 6.12. The proof of Proposition 6.11 shows that it is sufficient to assume that G is a Lipschitz (bounded) domain. We point out, however, that if G is not smooth enough, then Proposition 6.11 need not hold. In [13] we have constructed an example of a bounded domain Ω satisfying a (uniform) exterior cone condition at each point of $\partial\Omega$ and such that if $L = \triangle$ in (1.1), then there exists a solution u of (1.1) on $\Omega \times (0, \infty)$ with the following properties. For all $t > 0$ we have $u(x(t), t) = 0$ for some $x(t) \in \Omega$, and for any subdomain $D \subset\subset \Omega$ there exists $t_D > 0$ such that $u(x,t) > 0$ for all $x \in D$ and $t \geq t_D$.

Proof of Proposition 6.11. By Lemma 3.5 (with "$U = G \setminus \bar{D}_1$") we have

$$\|u^-(t)\|_{L^\infty(G \setminus \bar{D}_1)} \leq 2\, e^{-k(t-t_0-1)} \|u^-(t_0+1)\|_{L^\infty(G \setminus \bar{D}_1)} \quad (t \geq t_0 + 1),$$

where k is chosen in the paragraph following Remark 6.2. On the other hand, with x_Ω and c_Ω as in (5.1), we have by the Harnack inequality,

$$(6.34) \qquad u(x_\Omega, t) \geq e^{-(\log c_\Omega)\,(t-t_0-1)}\, u(x_\Omega, t_0+1) \qquad (t \geq t_0 + 2).$$

Since $k > \log c_\Omega$, the above inequalities imply that there exists $\tilde{t} \geq t_0 + 2$ such that

$$(6.35) \qquad u(x_\Omega, \tilde{t}+1) > e^{\beta_0}\, C_G\, \|u^-(\tilde{t})\|_{L^\infty(G \setminus \bar{D}_1)},$$

where C_G is as in Theorem 3.7 (with $\delta_0 = 1$ in that theorem). By uniqueness of the solutions of (3.4) we have

$$(6.36) \qquad u(x,t) = u_1(x, t; \tilde{t}, u^+(\tilde{t})) - u_2(x, t; \tilde{t}, u^-(\tilde{t}))$$

for all $x \in G$ and $t \geq \tilde{t}$. Clearly

$$(6.37) \qquad u^+(x,t) \leq u_1(x, t; \tilde{t}, u^+(\tilde{t})) \quad (x \in G,\ t \geq \tilde{t}),$$

$$(6.38) \qquad u^-(x,t) \leq u_2(x, t; \tilde{t}, u^-(\tilde{t})) \quad (x \in G,\ t \geq \tilde{t}).$$

If $u^-(\cdot, \tilde{t}) \equiv 0$ on Ω, then the conclusion follows. We may therefore assume $u^-(\cdot, \tilde{t}) \not\equiv 0$ and proceed as follows. Theorem 3.7 applied to u_1 and u_2 gives

$$C_G \inf_{x \in G} \frac{u_1(x, \tilde{t}+1)}{u_2(x, \tilde{t}+1)} \geq \sup_{x \in G} \frac{u_1(x, \tilde{t}+1)}{u_2(x, \tilde{t}+1)}.$$

Since $x_\Omega \in D_1$, the positivity assumption on u in D_1 implies $u^+(x_\Omega, \tilde{t}+1) = u(x_\Omega, \tilde{t}+1)$. Then, by (6.37), the supremum on the right may be estimated from below by

$$\frac{u(x_\Omega, \tilde{t}+1)}{\|u_2(\tilde{t}+1)\|_{L^\infty(G)}} \geq \frac{u(x_\Omega, \tilde{t}+1)}{e^{\beta_0}\, \|u^-(\tilde{t})\|_{L^\infty(G)}} = \frac{u(x_\Omega, \tilde{t}+1)}{e^{\beta_0}\, \|u^-(\tilde{t})\|_{L^\infty(G \setminus \bar{D}_1)}} > C_G,$$

where we have also used the maximum principle, positivity of $u(\cdot, \tilde{t})$ in D_1, and (6.35). The above inequalities imply $u_1(x, \tilde{t}+1) > u_2(x, \tilde{t}+1)$ for all $x \in G$, which in view of (6.36) means that $u(x, \tilde{t}+1) > 0$ for all $x \in G$. By the maximum principle and the Harnack inequality we then get $u(x,t) > 0$ for all $x \in G$ and $t \geq t_G(u)$, where $t_G(u) = \tilde{t} + 1$. We conclude by noting that for the above proof to work it suffices to assume that G is a Lipschitz domain. This follows from Proposition 3.2 and Theorem 3.7. $\qquad\square$

7. Some estimates of sign-changing solutions

We feel that it may be helpful for the reader if we give some heuristic arguments justifying the lemmas to follow. To establish the estimate (2.3) we use the following strategy. For a smooth domain G (sufficiently close to Ω) and a given time-interval we analyze the relative decrease of u, a sign-changing solution of (3.4), with respect to φ_G during that time-interval. More precisely, if $||u(t+5)||_{L^\infty(G)}/||u(t)||_{L^\infty(G)} \ll ||\varphi_G(t+5)||_{L^\infty(G)}/||\varphi_G(t)||_{L^\infty(G)}$, then we simply move on from t to $t+5$. If, on the other hand, $||u(t+\tau)||_{L^\infty(G)}/||u(t)||_{L^\infty(G)} \approx ||\varphi_G(t+\tau)||_{L^\infty(G)}/||\varphi_G(t)||_{L^\infty(G)}$ for $\tau \in [0,5]$ (up to a certain multiple), that is, if u does not drop "too much" compared to φ_G, then Lemma 7.1 below guarantees that the solutions $u(\cdot, \tilde{t}; t, u^+(t))$, $u(\cdot, \tilde{t}; t, u^-(t))$ are comparable to each other away from the boundary of G for all $\tilde{t} \geq t + T_0$ and some $T_0 \geq 1$ independent of u_0 and G. Lemma 7.2 then guarantees that there is enough "cancellation" between the positive and negative parts of u to result in dropping of the norm of u with respect to φ_G.

Denote

$$(7.1) \qquad d_1 = \mathrm{dist}(\bar{D}_1, \partial\Omega) > 0,$$

where D_1 is as in Section 6. Let

$$(7.2) \qquad \tilde{C}_{10} = \tilde{C}_{10}(D_1, d_1/8),$$

where $\tilde{C}_{10}(D_1, d_1/8)$ is as in Corollary 6.9. Let $k > 0$ be so large that (with $\tilde{C}_2$ as in (6.3))

$$(7.3) \qquad 2\, e^{-k} < (\tilde{C}_2)^5/(4e^{4\beta_0}(\tilde{C}_{10})^2)$$

and let $\delta > 0$ be the corresponding number given by Lemma 3.5. Let us fix a smooth domain D_2 with

$$(7.4) \qquad D_1 \subset\subset D_2 \subset\subset \Omega,$$

and such that $|\Omega \setminus D_2| < \delta$ and $\mathrm{dist}(\bar{D}_1, \partial D_2) \geq d_1/8$. In the following lemma we use that for any solution u of (3.4) we may write

$$(7.5) \qquad u(x,t) = u_1(x,t;0,u_0^+) - u_2(x,t;0,u_0^-) \quad ((x,t) \in G \times [0,\infty)).$$

Lemma 7.1. *Let G be a smooth domain such that $D_2 \subset\subset G \subseteq \Omega$ and $\mathrm{dist}(\bar{D}_2, \partial G) \geq d > 0$. Let u be a solution of (3.4) with $u_0 \in C_0(\bar{G})$ and such that $u(\cdot, t)$ vanishes somewhere in G for all $t > 0$. Suppose that*

$$(7.6) \qquad \frac{||u(\tau)||_{L^\infty(G)}}{||u_0||_{L^\infty(G)}} \geq \frac{(\tilde{C}_2)^5}{2\,(\tilde{C}_{10})^2} \qquad (\tau \in [0,5]),$$

where $\tilde{C}_2$ and $\tilde{C}_{10}$ are as in (6.3) and (7.2), respectively. Then, with u_1 and u_2 as in (7.5), there exist constants $C_{12} \in (0,1)$ and $T_0 > 1$ with the following properties. We have

$$(7.7) \qquad \frac{u_i(x,t)}{||u_j(t)||_{L^\infty(G)}} \geq C_{12} \qquad (x \in D_2,\, t \geq T_0,\, i,j = 1,2).$$

The constants C_{12} and T_0 depend only on $\mathrm{inrad}(\Omega)$, $\mathrm{diam}(\Omega)$, D_0, d_0, D_1, d_1, D_2, d, and the structure of L.

Proof. Since $||u(\tau)||_{L^\infty(G)} \leq \max\{||u_1(\tau)||_{L^\infty(G)}, ||u_2(\tau)||_{L^\infty(G)}\}$ for all $\tau \geq 0$, (7.6) with $\tau = 5$ implies that we may assume, say,

$$(7.8) \qquad ||u_1(5)||_{L^\infty(G)} \geq \frac{(\tilde{C}_2)^5}{2\,(\tilde{C}_{10})^2}\,||u_0||_{L^\infty(G)}.$$

We claim that for some $t \in [1, 2]$ we must have

$$(7.9) \qquad \frac{||u_1(t)||_{L^\infty(D_2)}}{||u_1(t)||_{L^\infty(G)}} > \frac{C_1(D_2, d)\,(\tilde{C}_2)^6}{16\,c_\Omega\,e^{6\,\beta_0}\,(\tilde{C}_{10})^2},$$

where $C_1(D_2, d)$ is as in Lemma 5.2, $\tilde{C}_2$, $\tilde{C}_{10}$ are as above and c_Ω is as in (5.1). Indeed, suppose (7.9) is not true. This contradiction assumption allows us to use Lemma 6.3 with $D = D_2$, $\alpha = (\tilde{C}_2)^4/e^{4\beta_0}(\tilde{C}_{10})^2$ and k as in (7.3). Using that lemma and (7.3) we get

$$||u_1(5)||_{L^\infty(G)} \leq e^{3\beta_0}\,||u_1(2)||_{L^\infty(G)} \leq e^{3\beta_0}\left(\frac{\alpha\,\tilde{C}_2}{16\,c_\Omega} + 2\,e^{-k}\right)||u_1(1)||_{L^\infty(G)}$$

$$< e^{3\beta_0}\left(\frac{(\tilde{C}_2)^5}{16\,c_\Omega\,e^{4\beta_0}(\tilde{C}_{10})^2} + \frac{(\tilde{C}_2)^5}{4\,e^{4\beta_0}\,(\tilde{C}_{10})^2}\right)||u_1(1)||_{L^\infty(G)}$$

$$\leq \frac{(\tilde{C}_2)^5}{2\,e^{\beta_0}\,(\tilde{C}_{10})^2}\,||u_1(1)||_{L^\infty(G)} \leq \frac{(\tilde{C}_2)^5}{2\,(\tilde{C}_{10})^2}\,||u_0^+||_{L^\infty(G)}$$

$$\leq \frac{(\tilde{C}_2)^5}{2\,(\tilde{C}_{10})^2}\,||u_0||_{L^\infty(G)},$$

contradicting (7.8). Thus (7.9) holds and we note that the constant on the right hand side of that inequality depends only on $\mathrm{inrad}(\Omega)$, $\mathrm{diam}(\Omega)$, D_0, d_0, D_1, d_1, D_2, d and the structure of L. Lemma 6.4 with $D = D_2$ and κ equal to the right hand side in (7.9) then gives

$$(7.10) \qquad \frac{u_1(x, t)}{||u_1(t)||_{L^\infty(G)}} \geq C \qquad (x \in D_2, t \geq 3),$$

and $C > 0$ depends only on $\mathrm{inrad}(\Omega)$, $\mathrm{diam}(\Omega)$, D_0, d_0, D_1, d_1, D_2, d and the structure of L.

Our next goal is to prove that a similar estimate holds for u_2 and that the norms $||u_1(t)||_{L^\infty(G)}$ and $||u_2(t)||_{L^\infty(G)}$ are comparable (in a suitable sense) for all $t \geq T_0$ and some $T_0 > 1$ with the required dependence. With C as in (7.10), suppose that the assumptions of Lemma 6.6 are satisfied for $D = D_2$, $v = u_1$, $u = u_2$, $\kappa = C$ and some $t_0 \geq 3$. We would then have $u(x, t) = u_1(x, t) - u_2(x, t) > 0$ for all $x \in \bar{D}_2$ and $t \geq t_0 + 1$. Then Proposition 6.11 would imply $u(x, \tilde{t}) > 0$ for all $x \in G$ and $\tilde{t} \geq t_G(u)$, contradicting the assumption that $u(\cdot, t)$ vanishes somewhere in G for all $t > 0$. Thus, necessarily

$$(7.11) \qquad ||u_2(t)||_{L^\infty(G)} > C_9(D_2, d)\,C\,||u_1(t)||_{L^\infty(G)} \quad (t \geq 3),$$

where $C_9(D_2, d)$ is as in Lemma 6.6.

To motivate the estimates below we note that if we knew that (7.10) holds with u_1 replaced by u_2 and all $t \geq s_0$ for an appropriate s_0, then we would get (7.11) for all $t \geq s_0$ with the roles of u_1 and u_2 interchanged, which in combination with (7.10) (valid for both u_1 and u_2) would imply the conclusion of the lemma. On the other hand, if $u_2(\cdot, t)$ were very small in D_2 compared to $||u_2(t)||_{L^\infty(G)}$ on a long

time-interval, then by Lemma 6.3 the norm $||u_2(t)||_{L^\infty(G)}$ would be decaying very fast, eventually leading to a contradiction with (7.11).

We claim

$$(7.12) \qquad (C_{11})^{-1} \leq \frac{||u_1(s)||_{L^\infty(G)}}{||\varphi_G(s)||_{L^\infty(G)}} \frac{||\varphi_G(t)||_{L^\infty(G)}}{||u_1(t)||_{L^\infty(G)}} \leq C_{11} \qquad (t \geq s \geq 4),$$

where $C_{11} > 1$ depends only on $\mathrm{inrad}(\Omega)$, $\mathrm{diam}(\Omega)$, D_0, d_0, D_1, d_1, D_2, d and the structure of L. Indeed, making use of (7.10), this can be proved in exactly the same way as we proved (6.27) (see also Remark 6.8). We are now going to use Lemma 6.3 with $D = D_2$. Suppose that for some positive integer l the inequality (6.6) with $\alpha = 1$ (and with $D = D_2$) is satisfied for u_2 and all $t \in [5, 5 + l]$. Using Lemma 6.3 l times followed by an application of (7.12) and (7.8), we compute

$$\begin{aligned}
||u_2(5+l)||_{L^\infty(G)} &\leq \frac{||u_2(5)||_{L^\infty(G)}}{2^l} \frac{||\varphi_G(5+l)||_{L^\infty(G)}}{||\varphi_G(5)||_{L^\infty(G)}} \\
&\leq \frac{e^{5\beta_0} ||u_0||_{L^\infty(G)}}{2^l} \frac{C_{11} ||u_1(5+l)||_{L^\infty(G)}}{||u_1(5)||_{L^\infty(G)}} \\
&\leq \frac{C_{11} e^{5\beta_0} 2 (\tilde{C}_{10})^2 ||u_1(5)||_{L^\infty(G)}}{2^l (\tilde{C}_2)^5} \frac{||u_1(5+l)||_{L^\infty(G)}}{||u_1(5)||_{L^\infty(G)}} \\
&= \frac{C_{11} e^{5\beta_0} 2 (\tilde{C}_{10})^2}{2^l (\tilde{C}_2)^5} ||u_1(5+l)||_{L^\infty(G)}.
\end{aligned}$$

Comparing this estimate with (7.11), we get an upper bound on l, call it l_0, depending only on $\mathrm{inrad}(\Omega)$, $\mathrm{diam}(\Omega)$, D_0, d_0, D_1, d_1, D_2, d and the structure of L. Therefore, for some $t \in [5, l_0 + 6]$ the inequality (6.6) (with $\alpha = 1$ and $D = D_2$) is violated. Then by Lemma 6.4 we get

$$(7.13) \qquad \frac{u_2(x, t)}{||u_2(t)||_{L^\infty(G)}} \geq C \qquad (x \in D_2, t \geq l_0 + 7),$$

and $C > 0$ depends only on $\mathrm{inrad}(\Omega)$, $\mathrm{diam}(\Omega)$, D_0, d_0, D_1, d_1, D_2, d and the structure of L.

Having established (7.13), the same proof as above shows that (7.11) holds with the roles of u_1 and u_2 interchanged and with $t \geq l_0 + 9$. Set $T_0 = l_0 + 9$. The inequalities (7.10), (7.13), and (7.11) (for both u_1 and u_2 and $t \geq T_0$) imply that (7.7) holds. The proof is complete. $\qquad \square$

The following lemma will be crucial in our considerations.

Lemma 7.2. *Let u, G, d, $\tilde{C}_2$, $\tilde{C}_{10}$ be as in Lemma 7.1 and suppose that (7.6) holds. There exists a constant $T_1 > 0$, depending only on $\mathrm{inrad}(\Omega)$, $\mathrm{diam}(\Omega)$, D_0, d_0, D_1, d_1, D_2, d and the structure of L, with the following property. For any $t \geq 2T_1$ we have*

$$(7.14) \qquad ||u(t; T_1, u^+(T_1))||_{L^\infty(G)} \leq \sigma_0 ||u(t; 0, u^+(0))||_{L^\infty(G)},$$

$$(7.15) \qquad ||u(t; T_1, u^-(T_1))||_{L^\infty(G)} \leq \sigma_0 ||u(t; 0, u^-(0))||_{L^\infty(G)},$$

where $\sigma_0 = 1 - C_{12}/4$ and C_{12} is as in Lemma 7.1.

Proof. Set

$$(7.16) \qquad \omega_1 = 1 - C_{12}/2.$$

With T_0 as in Lemma 7.1 set $\widetilde{T}_0 = T_0 + 1$ and consider the function

$$(7.17) \qquad \tilde{u}(x,t) = \omega_1 u(x,t;0,u^+(0)) - u(x,t) \quad (x \in G,\ t \geq \widetilde{T}_0).$$

By uniqueness of the solutions of (3.4) we have

$$(7.18) \qquad u(x,t) = u(x,t;0,u^+(0)) - u(x,t;0,u^-(0)) \quad (x \in G,\ t \geq 0).$$

This fact and (7.7) with $i = 2$, $j = 1$ imply

$$\tilde{u}(x,t) = u(x,t;0,u^-(0)) - (C_{12}/2)\, u(x,t;0,u^+(0))$$

$$\geq C_{12}\, ||u(t;0,u^+(0))||_{L^\infty(G)} - (C_{12}/2)\, u(x,t;0,u^+(0))$$

$$(7.19) \qquad \geq (C_{12}/2)\, ||u(t;0,u^+(0))||_{L^\infty(G)} > 0 \quad (x \in D_2,\ t \geq \widetilde{T}_0).$$

We can therefore use Lemma 3.5 with "$U = G \setminus \bar{D}_2$". We get

$$(7.20) \qquad ||\tilde{u}^-(t)||_{L^\infty(G \setminus \bar{D}_2)} \leq 2\, e^{-k(t-\widetilde{T}_0)}\, ||\tilde{u}^-(\widetilde{T}_0)||_{L^\infty(G \setminus \bar{D}_2)} \quad (t \geq \widetilde{T}_0),$$

where k is as in (7.3) and, with c_Ω as in (5.1), we may assume

$$(7.21) \qquad\qquad\qquad e^{-k} < 1/(2c_\Omega).$$

This follows from the fact that $D_1 \subset D_2$ and from the discussion contained in the paragraph following Remark 6.2. Obviously, by the definition of $\tilde{u}$ and (7.18) we have

$$(7.22) \qquad ||\tilde{u}^-(\widetilde{T}_0)||_{L^\infty(G \setminus \bar{D}_2)} \leq ||u^+(\widetilde{T}_0)||_{L^\infty(G)} \leq ||u(\widetilde{T}_0;0,u^+(0))||_{L^\infty(G)}.$$

Next, Lemma 7.1 (with $i = j = 1$) and Corollary 6.5 (with $s = \widetilde{T}_0$) imply that for any $t \geq \widetilde{T}_0$ one has

$$(7.23) \qquad ||u(\widetilde{T}_0;0,u^+(0))||_{L^\infty(G)} \leq C\, e^{(\log c_\Omega)(t-\widetilde{T}_0)}\, ||u(t;0,u^+(0))||_{L^\infty(G)},$$

where $C > 0$ depends only on $\mathrm{inrad}(\Omega)$, $\mathrm{diam}(\Omega)$, D_0, d_0, D_1, d_1, D_2, d and the structure of L. On the other hand, the definition of $\tilde{u}$ and (7.19) give that for all $x \in G$ and $t \geq \widetilde{T}_0$ one also has

$$(7.24) \qquad ||\tilde{u}^-(t)||_{L^\infty(G \setminus \bar{D}_2)} = ||\tilde{u}^-(t)||_{L^\infty(G)} \geq u^+(x,t) - \omega_1 u(x,t;0,u^+(0)),$$

which clearly implies the following inequality ($t \geq \widetilde{T}_0$):

$$(7.25) \qquad ||u^+(t)||_{L^\infty(G)} \leq ||\tilde{u}^-(t)||_{L^\infty(G \setminus \bar{D}_2)} + \omega_1\, ||u(t;0,u^+(0))||_{L^\infty(G)}.$$

By (7.20), (7.22), (7.23) and the first equality in (7.24) we have ($t \geq \widetilde{T}_0$)

$$(7.26) \qquad ||\tilde{u}^-(t)||_{L^\infty(G)} \leq 2\, C\, e^{(-k+\log c_\Omega)(t-\widetilde{T}_0)}\, ||u(t;0,u^+(0))||_{L^\infty(G)}.$$

Using (7.25) and (7.26), we obtain ($t \geq \widetilde{T}_0$)

$$(7.27) \qquad ||u^+(t)||_{L^\infty(G)} \leq (\omega_1 + 2\, C\, e^{(-k+\log c_\Omega)(t-\widetilde{T}_0)})\, ||u(t;0,u^+(0))||_{L^\infty(G)}.$$

We now fix $t_1 > \widetilde{T}_0 + 1$ (to be determined below) and define the function v to be the solution of (3.1), (3.2) on $G \times [t_1, \infty)$ with the nonnegative initial condition given for all $x \in G$ by

(7.28)

$$v(x,t_1) = \omega_1 u(x,t_1;0,u^+(0)) - u^+(x,t_1) + (\omega_1 u(x,t_1;0,u^+(0)) - u^+(x,t_1))^-.$$

By the uniqueness of the solutions of the initial value problems we can write

$$(7.29) \qquad v(x,t) = v_1(x,t) + v_2(x,t) \quad (x \in G,\ t \geq t_1),$$

where $v_1(x,t) = \omega_1\, u(x,t;0,u^+(0)) - u(x,t;t_1,u^+(t_1))$ and v_2 is the (nonnegative) solution of (3.1), (3.2) on $G \times [t_1,\infty)$ with

$$v_2(x,t_1) = (\omega_1\, u(x,t_1;0,u^+(0)) - u^+(x,t_1))^- \quad (x \in G).$$

The inequality (7.24) with $t = t_1$ implies $||v_2(t_1)||_{L^\infty(G)} \leq ||\tilde{u}^-(t_1)||_{L^\infty(G)}$ and by the maximum principle $||v_2(t)||_{L^\infty(G)} \leq e^{\beta_0}\, ||v_2(t_1)||_{L^\infty(G)}$ for any $t \in [t_1, t_1 + 1]$. These inequalities and (7.26) with $t = t_1$ imply that for any $t \in [t_1, t_1 + 1]$ we have

$$(7.30) \qquad ||v_2(t)||_{L^\infty(G)} \leq 2\,C\,e^{\beta_0}\,e^{(-k+\log c_\Omega)(t_1-\widetilde{T}_0)}\,||u(t_1;0,u^+(0))||_{L^\infty(G)}.$$

Define finally the function

$$(7.31) \quad \tilde{v}(x,t) = (\omega_1 + C_{12}/8)\,u(x,t;0,u^+(0)) - u(x,t;t_1,u^+(t_1)) \quad (x \in G,\, t \geq t_1).$$

Our next goal is to prove, via Theorem 3.6, that $\tilde{v} > 0$ on $D_2 \times [t_1,\infty)$ if t_1 is suitably chosen. By uniqueness of the solutions of the initial value problems, using the definitions of v, v_1, v_2 and $\tilde{v}$, we see that for any $x \in G$ and $t \geq t_1$ one in fact has

$$(7.32) \qquad \tilde{v}(x,t) = (C_{12}/8)\,u(x,t;0,u^+(0)) + v(x,t) - v_2(x,t).$$

The nonnegativity of v and (7.30) with $t = t_1$ then imply
$$(7.33)$$
$$||\tilde{v}^-(t_1)||_{L^\infty(G)} \leq ||v_2(t_1)||_{L^\infty(G)} \leq 2Ce^{\beta_0}e^{(-k+\log c_\Omega)(t_1-\widetilde{T}_0)}\,||u(t_1;0,u^+(0))||_{L^\infty(G)}.$$

Next, using Lemma 7.1 (with $i = j = 1$) and Corollary 6.5 (with $s = t_1$), we get
$$(7.34)$$
$$(C_{12}/8)\,u(x,t;0,u^+(0)) \geq \bar{C}\,||u(t_1;0,u^+(0))||_{L^\infty(G)} \quad (x \in D_2,\, t \in [t_1, t_1+1]),$$

where $\bar{C} > 0$ depends only on $\mathrm{inrad}(\Omega)$, $\mathrm{diam}(\Omega)$, D_0, d_0, D_1, d_1, D_2, d and the structure of L. Finally, the nonnegativity of v and the inequalities (7.34) and (7.30) imply that for all $x \in \bar{D}_2$ and $t \in [t_1, t_1 + 1]$,

$$(7.35) \qquad \tilde{v}(x,t) \geq (\bar{C} - 2Ce^{\beta_0}e^{(-k+\log c_\Omega)(t_1-\widetilde{T}_0)})\,||u(t_1;0,u^+(0))||_{L^\infty(G)}.$$

Let $\mu = \mu(N, \mathrm{diam}(\Omega), \alpha_0, \beta_0, d/2, 1/8, \mathrm{inrad}(\Omega)/4)$ be as in Theorem 3.6. It is now easy to see, using $-k + \log c_\Omega < -\log 2$ (see (7.21)), (7.33) and (7.35), that if $t_1 = T_1 > \widetilde{T}_0 + 1$ is sufficiently large (depending only on $\mathrm{inrad}(\Omega)$, $\mathrm{diam}(\Omega)$, D_0, d_0, D_1, d_1, D_2, d and the structure of L), then the assumptions (3.12) and (3.13) in Theorem 3.6 with $D = D_2$ are satisfied. The remaining assumptions (3.10) and (3.11) are also satisfied since $D_1 \subset D_2$ (see the choice of D_1 in the paragraph following Remark 6.2). Therefore, by Theorem 3.6, $\tilde{v}(x,t) > 0$ for all $x \in \bar{D}_2$ and $t \geq T_1$.

The positivity of $\tilde{v}$ on $\bar{D}_2 \times [T_1,\infty)$ allows us to use similar arguments as those leading to the inequality (7.27). We will only point out the necessary modifications. In the text (and all formulas) appearing between (and including) (7.20) and (7.27) one has to replace everywhere $\tilde{u}$ by $\tilde{v}$, $\widetilde{T}_0$ by T_1, and ω_1 by $\omega_1 + C_{12}/8$. With such a replacement, the inequalities (7.20), (7.22) and (7.23) remain valid. In (7.24) we in addition need to replace $u^+(x,t)$ by $u(x,t;T_1,u^+(T_1))$ and in (7.25) and (7.27) the function $u^+(t)$ is now replaced by $u(t;T_1,u^+(T_1))$. With these modifications in place, the inequalities (7.20)–(7.27) continue to hold. Using in particular the

(modified) inequality (7.27), we get for any $t \geq T_1$,

$$||u(t; T_1, u^+(T_1))||_{L^\infty(G)}$$

(7.36)
$$\leq (\omega_1 + C_{12}/8 + 2\,C\,e^{(-k+\log c_\Omega)(t-T_1)})\,||u(t; 0, u^+(0))||_{L^\infty(G)}.$$

Obviously, making T_1 so large that $2\,C\,e^{(-k+\log c_\Omega)T_1} < C_{12}/8$, the inequality (7.36) implies that for any $t \geq 2T_1$ we have

$$||u(t; T_1, u^+(T_1))||_{L^\infty(G)}$$
$$\leq (\omega_1 + C_{12}/8 + C_{12}/8)\,||u(t; 0, u^+(0))||_{L^\infty(G)} = \sigma_0\,||u(t; 0, u^+(0))||_{L^\infty(G)},$$

as claimed in (7.14). The proof of (7.15) is completely analogous. $\qquad\square$

Next, we prove the following lemma.

Lemma 7.3. *Let G, d be as in Lemma* 7.1 *and let u be a solution of* (3.4) *with $u_0 \in C_0(\bar{G})$ and such that $u(\cdot, t)$ vanishes somewhere in G for all $t > 0$. There exists a constant $T_2 > 0$, depending only on* $\mathrm{inrad}(\Omega)$, $\mathrm{diam}(\Omega)$, D_0, d_0, D_1, d_1, D_2, d *and the structure of L, such that we have*

(7.37)
$$\frac{||u(T_2)||_{L^\infty(G)}}{||\varphi_G(T_2)||_{L^\infty(G)}} \leq \frac{1}{2}\,\frac{||u_0||_{L^\infty(G)}}{||\varphi_G(0)||_{L^\infty(G)}}.$$

Proof. Let m be an integer so large that

(7.38)
$$\sigma_0^{m-1} < \frac{1}{2\,\tilde{C}_{10}},$$

where σ_0 is as in Lemma 7.2 and $\tilde{C}_{10}$ is as in (7.2). In view of the dependences of σ_0 and $\tilde{C}_{10}$, m can be chosen depending only on $\mathrm{inrad}(\Omega)$, $\mathrm{diam}(\Omega)$, D_0, d_0, D_1, d_1, D_2, d and the structure of L. We will show that the conclusion of Lemma 7.3 holds with

(7.39)
$$T_2 = mT_1 + 5,$$

where T_1 is as in Lemma 7.2. We may assume in the remainder of this proof that $u(\cdot, t) \not\equiv 0$ for all $t \in [0, T_2]$ (for otherwise (7.37) is trivially satisfied). We then have the following (mutually exclusive) possibilities:

(i)

(7.40)
$$\frac{||u(t + \tau)||_{L^\infty(G)}}{||u(t)||_{L^\infty(G)}} \geq \frac{(\tilde{C}_2)^5}{2\,(\tilde{C}_{10})^2} \quad (\tau \in [0, 5],\, t \in [0, mT_1]),$$

or

(ii) for some $t_0 \in [0, mT_1]$ and some $\tau_0 \in [0, 5]$,

(7.41)
$$\frac{||u(t_0 + \tau_0)||_{L^\infty(G)}}{||u(t_0)||_{L^\infty(G)}} < \frac{(\tilde{C}_2)^5}{2\,(\tilde{C}_{10})^2},$$

where $\tilde{C}_2$ and $\tilde{C}_{10}$ are as in (6.3) and (7.2), respectively.

Let us first consider the possibility (i). Writing

$$u(x, t) = u(x, t; (m-1)T_1, u^+((m-1)T_1)) - u(x, t; (m-1)T_1, u^-((m-1)T_1))$$

for $(x, t) \in G \times [(m-1)T_1, \infty)$, we see that

$$||u^+(T_2)||_{L^\infty(G)} \leq ||u(T_2; (m-1)T_1, u^+((m-1)T_1))||_{L^\infty(G)}.$$

The assumption (i) allows us to use (7.14) $m-1$ times. We get

$$||u(T_2; (m-1)T_1, u^+((m-1)T_1))||_{L^\infty(G)}$$
$$\leq \sigma_0 ||u(T_2; (m-2)T_1, u^+((m-2)T_1))||_{L^\infty(G)}$$
$$\leq \cdots \leq \sigma_0^{m-1}||u(T_2; 0, u^+(0))||_{L^\infty(G)}.$$

Further, by Corollary 6.9,

$$||u(T_2; 0, u^+(0))||_{L^\infty(G)} \leq \tilde{C}_{10} ||u^+(0)||_{L^\infty(G)} \frac{||\varphi_G(T_2)||_{L^\infty(G)}}{||\varphi_G(0)||_{L^\infty(G)}}.$$

Combining the above inequalities and using (7.38), we obtain

$$||u^+(T_2)||_{L^\infty(G)} < \frac{1}{2} ||u^+(0)||_{L^\infty(G)} \frac{||\varphi_G(T_2)||_{L^\infty(G)}}{||\varphi_G(0)||_{L^\infty(G)}}.$$

The same estimate holds if we replace "u^+" by "u^-". We therefore have

$$||u(T_2)||_{L^\infty(G)} \leq \max\{||u^+(T_2)||_{L^\infty(G)}, ||u^-(T_2)||_{L^\infty(G)}\}$$
$$< \frac{1}{2} \max\{||u^+(0)||_{L^\infty(G)}, ||u^-(0)||_{L^\infty(G)}\} \frac{||\varphi_G(T_2)||_{L^\infty(G)}}{||\varphi_G(0)||_{L^\infty(G)}}$$
$$= \frac{1}{2} ||u_0||_{L^\infty(G)} \frac{||\varphi_G(T_2)||_{L^\infty(G)}}{||\varphi_G(0)||_{L^\infty(G)}}.$$

It remains to prove (7.37) under the assumption (ii) above. In that case, using (ii), Corollary 6.9 and Corollary 5.3, we derive

$$\frac{||u(T_2)||_{L^\infty(G)}}{||u_0||_{L^\infty(G)}} = \frac{||u(T_2)||_{L^\infty(G)}}{||u(t_0+\tau_0)||_{L^\infty(G)}} \frac{||u(t_0+\tau_0)||_{L^\infty(G)}}{||u(t_0)||_{L^\infty(G)}} \frac{||u(t_0)||_{L^\infty(G)}}{||u_0||_{L^\infty(G)}}$$
$$\leq \tilde{C}_{10} \frac{||\varphi_G(T_2)||_{L^\infty(G)}}{||\varphi_G(t_0+\tau_0)||_{L^\infty(G)}} \frac{(\tilde{C}_2)^5}{2(\tilde{C}_{10})^2} \tilde{C}_{10} \frac{||\varphi_G(t_0)||_{L^\infty(G)}}{||\varphi_G(0)||_{L^\infty(G)}}$$
$$\leq \frac{(\tilde{C}_2)^5}{2} \frac{||\varphi_G(t_0)||_{L^\infty(G)}}{||\varphi_G(t_0+\tau_0)||_{L^\infty(G)}} \frac{||\varphi_G(T_2)||_{L^\infty(G)}}{||\varphi_G(0)||_{L^\infty(G)}}$$
$$\leq \frac{1}{2} \frac{||\varphi_G(T_2)||_{L^\infty(G)}}{||\varphi_G(0)||_{L^\infty(G)}},$$

finishing the proof of Lemma 7.3. $\square$

Lemma 7.3 implies the following theorem.

Theorem 7.4. *Let G be a smooth domain such that $D_2 \subset\subset G \subseteq \Omega$ and $\mathrm{dist}(\bar{D}_2, \partial G) \geq d > 0$. Let u be a solution of (3.1), (3.2) on $G \times [s_0, \infty)$ with $u(\cdot, s_0) \in C_0(\bar{G})$ and such that $u(\cdot, t)$ vanishes somewhere in G for all $t > s_0$. There exist positive constants C, γ, depending only on $\mathrm{inrad}(\Omega)$, $\mathrm{diam}(\Omega)$, D_0, d_0, D_1, d_1, D_2, d and the structure of L, such that the following holds:*

$$(7.42) \qquad \frac{||u(t)||_{L^\infty(G)}}{||\varphi_G(t)||_{L^\infty(G)}} \leq C\, e^{-\gamma(t-s)} \frac{||u(s)||_{L^\infty(G)}}{||\varphi_G(s)||_{L^\infty(G)}} \quad (t \geq s \geq s_0).$$

Proof. Let T_2 be as in Lemma 7.3 and $s \geq s_0$. First, by the maximum principle, we have $||u(t)||_{L^\infty(G)} \leq e^{\beta_0 T_2}||u(s)||_{L^\infty(G)}$ for any $t \in [s, s+T_2]$. Further, by Corollary 5.3 with $D = D_0$ we also have $||\varphi_G(t)||_{L^\infty(G)} \geq (\tilde{C}_2)^{T_2+1}||\varphi_G(s)||_{L^\infty(G)}$ for all

$t \in [s, s + T_2]$, where $\tilde{C}_2$ depends only on $\mathrm{inrad}(\Omega)$, D_0, d_0 and the structure of L. These inequalities give

$$\frac{||u(t)||_{L^\infty(G)}}{||\varphi_G(t)||_{L^\infty(G)}} \leq \frac{e^{\beta_0 T_2}}{(\tilde{C}_2)^{T_2+1}} \frac{||u(s)||_{L^\infty(G)}}{||\varphi_G(s)||_{L^\infty(G)}} \quad (t \in [s, s+T_2],\ s \geq s_0).$$

For any $t \geq s \geq s_0$ there are a unique integer j and $a \in [0, T_2)$ such that $t = s + jT_2 + a$. The above estimate and Lemma 7.3 then imply

$$\frac{||u(t)||_{L^\infty(G)}}{||\varphi_G(t)||_{L^\infty(G)}} \leq \frac{e^{\beta_0 T_2}}{(\tilde{C}_2)^{T_2+1}} \frac{||u(s+jT_2)||_{L^\infty(G)}}{||\varphi_G(s+jT_2)||_{L^\infty(G)}} \leq \frac{e^{\beta_0 T_2}}{2^j (\tilde{C}_2)^{T_2+1}} \frac{||u(s)||_{L^\infty(G)}}{||\varphi_G(s)||_{L^\infty(G)}},$$

which can be written as in (7.42) for appropriate C, $\gamma > 0$ depending only on $\mathrm{inrad}(\Omega)$, $\mathrm{diam}(\Omega)$, D_0, d_0, D_1, d_1, D_2, d and the structure of L. $\square$

8. PROOF OF THEOREM 2.2 AND COROLLARY 2.4

In this section we fix φ_Ω, a positive entire solution of (1.1) given by Theorem 2.1. We defer the proof of uniqueness of φ_Ω to the next section and we keep this solution fixed throughout this section. Also, in the following text, D_0, d_0, D_1, d_1, and D_2 are as chosen in the previous sections. Let us first prove the following lemma.

Lemma 8.1. *Let G, d be as in Lemma 7.1. For each $u_0 \in C_0(\bar{G})$ and $s \in \mathbb{R}$ there exist a unique $q_0 \in \mathbb{R}$ and $\zeta_0 \in C_0(\bar{G})$ such that (with φ_G as in Theorem 3.8)*

$$(8.1) \qquad u_0(x) = q_0 \frac{\varphi_G(x, s)}{||\varphi_G(s)||_{L^\infty(G)}} + \zeta_0(x) \quad (x \in G),$$

and such that the solution $u(\cdot, t; s, \zeta_0)$ of (3.1), (3.2) on $G \times [s, \infty)$ vanishes somewhere in G for all $t > s$. Moreover, we have the following estimate:

$$(8.2) \qquad |q_0| \leq C_{13} ||u_0||_{L^\infty(G)},$$

and the constant $C_{13} > 0$ depends only on $\mathrm{inrad}(\Omega)$, $\mathrm{diam}(\Omega)$, D_0, d_0, D_1, d_1, D_2, d and the structure of L.

Proof. The existence and uniqueness of the decomposition (8.1) follows from [17, Theorem 2.6]. We thus only need to prove the estimate (8.2). Define the solution

$$(8.3) \qquad \tilde{u}(x, t) = u(x, t; s, u_0) + C \frac{\varphi_G(x, t)}{||\varphi_G(s)||_{L^\infty(G)}} \quad (x \in G,\ t \geq s).$$

Set $\kappa = C_1(D_2, d)$, where $C_1(D_2, d)$ is as in Lemma 5.2. With this κ we apply Lemma 6.6 with $t_0 = s$ to $v(\cdot, t) = C\varphi_G(\cdot, t)/||\varphi_G(s)||_{L^\infty(G)}$ and $u(\cdot, t) = u(\cdot, t; s, u_0)$. We obtain that if $C > 0$ exceeds $C_{13} ||u_0||_{L^\infty(G)}$, where $C_{13} = (C_1(D_2, d)C_9(D_2, d))^{-1}$ depends only on $\mathrm{inrad}(\Omega)$, $\mathrm{diam}(\Omega)$, D_0, d_0, D_1, d_1, D_2, d and the structure of L, then $\tilde{u}(x, t) > 0$ for all $x \in \bar{D}_2$ and all $t > s + 1$. Hence, by Proposition 6.11, $\tilde{u}(x, t) > 0$ for all $(x, t) \in G \times [t_G(\tilde{u}), \infty)$, and the properties of the decomposition (8.1) then imply that $\tilde{u}(x, t) \not\equiv u(x, t; s, \zeta_0)$. Thus, necessarily $-q_0 < C_{13} ||u_0||_{L^\infty(G)}$. Replacing "+" in (8.3) by "−", we also get $q_0 < C_{13} ||u_0||_{L^\infty(G)}$, finishing the proof. $\square$

Define the set

$$(8.4) \quad \tilde{X}^2(s) = \{u_0 \in X(s) : \frac{||u(\cdot, t; s, u_0)||_{L^\infty(\Omega)}}{||\varphi_\Omega(t)||_{L^\infty(\Omega)}} e^{\gamma(t-s)} \text{ is bounded for } t \geq s\},$$

where $X(s)$ is as in (4.24) and $\gamma > 0$ is as in Theorem 7.4 with $d = d_2/2$, where

$$(8.5) \qquad d_2 = \operatorname{dist}(\bar{D}_2, \partial\Omega).$$

From (8.4) and the linear dependence of $u(\cdot, t; s, u_0)$ on u_0 it follows that $\tilde{X}^2(s)$ is a subspace of $X(s)$. We will now show that

$$(8.6) \qquad X(s) = \operatorname{span}\{\varphi_\Omega(\cdot, s)\} \oplus \tilde{X}^2(s) \quad (s \in \mathbb{R}).$$

Let $u_0 \in X(s)$. Let $\Omega_j \subset\subset \Omega_{j+1} \subset\subset \Omega$ be a sequence of smooth subdomains such that $\bigcup_{j=1}^\infty \Omega_j = \Omega$ and assume without loss of generality that $\operatorname{dist}(\bar{D}_2, \partial\Omega_j) \geq d_2/2$ for all $j \geq 1$. Furthermore, let $\xi_j u_0$ be the approximating initial conditions as defined in Proposition 4.13. For each $j \geq 1$ we can write by Lemma 8.1,

$$(8.7) \qquad \xi_j(x)\, u_0(x) = q_j\, \frac{\varphi_{\Omega_j}(x, s)}{||\varphi_{\Omega_j}(s)||_{L^\infty(\Omega_j)}} + \zeta_{j0}(x) \quad (x \in \Omega_j),$$

and $|q_j| \leq C_{13}||\xi_j\, u_0||_{L^\infty(\Omega_j)} \leq C_{13}||u_0||_{L^\infty(\Omega)}$ and $u(\cdot, t; s, \zeta_{j0})$ is a sign-changing solution on $\Omega_j \times (s, \infty)$. We also note that our assumption on Ω_j implies that C_{13} depends only on $\operatorname{inrad}(\Omega)$, $\operatorname{diam}(\Omega)$, D_0, d_0, D_1, d_1, D_2, d_2 and the structure of L. By the boundedness of $|q_j|$ we can choose a subsequence (keeping the same notation) such that $q_j \to q_0$ for some $q_0 \in \mathbb{R}$. The proof of Theorem 5.5 shows that (a subsequence of) the functions $\varphi_{\Omega_j}(\cdot, s)/||\varphi_{\Omega_j}(s)||_{L^\infty(\Omega_j)}$ converge locally uniformly in Ω to $\varphi_\Omega(\cdot, s)/||\varphi_\Omega(s)||_{L^\infty(\Omega)}$. Therefore

$$(8.8) \qquad \xi_j\, u_0 - q_j\, \frac{\varphi_{\Omega_j}(s)}{||\varphi_{\Omega_j}(s)||_{L^\infty(\Omega_j)}} = \zeta_{j0} \to u_0 - q_0\, \frac{\varphi_\Omega(s)}{||\varphi_\Omega(s)||_{L^\infty(\Omega)}} =: \zeta_0$$

locally uniformly in Ω. Since $u(\cdot, t; s, \zeta_{j0})$ is a sign-changing solution on $\Omega_j \times (s, \infty)$ and $\operatorname{dist}(\bar{D}_2, \partial\Omega_j) \geq d_2/2$, we may apply Theorem 7.4 with $G = \Omega_j$ and $d = d_2/2$ to get

$$\frac{||u(\cdot, t; s, \zeta_{j0})||_{L^\infty(\Omega_j)}}{||\varphi_{\Omega_j}(t)||_{L^\infty(\Omega_j)}} \leq C\, e^{-\gamma(t-s)}\, \frac{||\zeta_{j0}||_{L^\infty(\Omega_j)}}{||\varphi_{\Omega_j}(s)||_{L^\infty(\Omega_j)}} \quad (t \geq s,\, j \geq 1),$$

where C and $\gamma > 0$ depend only on $\operatorname{inrad}(\Omega)$, $\operatorname{diam}(\Omega)$, D_0, d_0, D_1, d_1, D_2, d_2 and the structure of L. In particular, these constants do not depend on j, and we note that the γ here is the same as in (8.4). Letting $j \to \infty$ and using Proposition 4.13 (note that $\zeta_0 \in X(s)$) and (8.8), we obtain

$$(8.9) \qquad \frac{||u(\cdot, t; s, \zeta_0)||_{L^\infty(\Omega)}}{||\varphi_\Omega(t)||_{L^\infty(\Omega)}} \leq C\, e^{-\gamma(t-s)}\, \frac{||\zeta_0||_{L^\infty(\Omega)}}{||\varphi_\Omega(s)||_{L^\infty(\Omega)}} \quad (t \geq s).$$

This in particular gives that $\zeta_0 \in \tilde{X}^2(s)$. Therefore any $u_0 \in X(s)$ can be written as $u_0 = q_0 \varphi_\Omega(s)/||\varphi_\Omega(s)||_{L^\infty(\Omega)} + \zeta_0$ for some $q_0 \in \mathbb{R}$ and $\zeta_0 \in \tilde{X}^2(s)$ and the solution $u(\cdot, t; s, \zeta_0)$ satisfies (8.9). The uniqueness of this decomposition clearly follows from the definition of $\tilde{X}^2(s)$, and the proof of (8.6) is finished.

We add that the above arguments also prove that (8.9) holds for all $\zeta_0 \in \tilde{X}^2(s)$. This implies that $\tilde{X}^2(s)$ is closed in $X(s)$. Indeed, consider a sequence $u_j \in \tilde{X}^2(s)$ approaching some $u_0 \in X(s)$. It follows from (8.9) that the expression in (8.4) with u_0 replaced by u_j is bounded by a constant independent of j. Taking the limit we obtain that the expression with u_0 is bounded; hence $u_0 \in \tilde{X}^2(s)$.

We would like to prove now that $\tilde{X}^2(s) = X^2(s)$, $s \in \mathbb{R}$, where $X^2(s)$ is defined as in Theorem 2.2, i.e.,

$$X^2(s) = \{u_0 \in X(s) : \text{for all } T > s \text{ the function } u(\cdot, t; s, u_0)$$

(8.10)   has a zero in $\bar{D}_2$ for some $t \geq T\}$.

Let us first prove the following implication:

$$(8.11) \qquad\qquad u_0 \notin X^2(s) \Rightarrow u_0 \notin \tilde{X}^2(s).$$

Let $u_0 \in X(s)$ be such that $u_0 \notin X^2(s)$, i.e., without loss of generality,

$$(8.12) \qquad\qquad u(x, t; s, u_0) > 0 \qquad (x \in \bar{D}_2, \, t \geq t_0),$$

where t_0 is some sufficiently large number. Since both sets $X^2(s)$ and $\tilde{X}^2(s)$ are clearly invariant with respect to (1.1) ($u_0 \in X^2(s)$ implies $u(\cdot, t; s, u_0) \in X^2(t)$ for all $t \geq s$), we may assume $t_0 = s$ and for simplicity of notation we will take $s = 0$. Let u_j be the approximating solutions considered in Proposition 4.13 and let Ω_j, $j \geq 1$ be as specified above. The local uniform convergence of these solutions to u along with (8.12) (with $t_0 = s = 0$) implies that for each $T > 1$ there exists an integer $j_0(T)$ such that for all $j \geq j_0(T)$ we have

$$(8.13) \qquad\qquad u_j(x, t) > 0 \qquad (x \in \bar{D}_2, \, t \in [0, T+1]),$$

and with x_Ω and c_Ω as in (5.1) we have

$$(8.14) \qquad u_j(x_\Omega, t) > \frac{u(x_\Omega, t)}{2} \geq e^{-(\log c_\Omega)(t-1)} \frac{u(x_\Omega, 1)}{2} \quad (t \in [2, T+1]),$$

for all $j \geq j_0(T)$ where we have used the Harnack inequality in the second inequality above. Moreover, by Lemma 3.5 (with "$U = \Omega_j \setminus \bar{D}_2$") we have for all $j \geq j_0(T)$,

$$\|u_j^-(t)\|_{L^\infty(\Omega_j \setminus \bar{D}_2)} \leq 2\, e^{-kt} \|u_j^-(0)\|_{L^\infty(\Omega_j \setminus \bar{D}_2)}$$

$$(8.15) \qquad\qquad\qquad \leq 2\, e^{-kt} \|u_0\|_{L^\infty(\Omega)} \quad (t \in [0, T+1]),$$

where k is as in (7.3), and the second inequality above follows from the definition of u_j. Since $k > \log c_\Omega$ and $\operatorname{dist}(\bar{D}_2, \partial \Omega_j) \geq d_2/2$, the inequalities (8.14), (8.15) imply that we can choose $T > 1$ so large (depending on u) that for any $j \geq j_0(T)$ all the assumptions in Theorem 3.6 are satisfied with $\rho = \operatorname{inrad}(\Omega)/4$, $d = d_2/4$, $\theta = 1/8$, $D = D_2$, $G = \Omega_j$, $v = u_j$ and $\tau = T$. Therefore, for all $j \geq j_0(T)$, we have

$$(8.16) \qquad\qquad u_j(x, t) > 0 \qquad (x \in \bar{D}_2, \, t \in [T, \infty)).$$

Note that this inequality (and (8.13)) implies that (8.15) holds for all $t \geq 0$. Observe now that by the same arguments as those leading to (8.16), possibly increasing T if necessary, one proves that (8.16) remains valid if we replace $u_j(\cdot, t)$ by $u_j(\cdot, t) - \alpha \varphi_{\Omega_j}(\cdot, t)/\|\varphi_{\Omega_j}(T)\|_{L^\infty(\Omega_j)}$ for some small $\alpha > 0$ depending on u but not on j as long as $j \geq j_0(T)$. Therefore

$$(8.17) \qquad u_j(x, t) \geq \alpha \frac{\varphi_{\Omega_j}(x, t)}{\|\varphi_{\Omega_j}(T)\|_{L^\infty(\Omega_j)}} \quad (x \in \bar{D}_2, \, t \geq T, j \geq j_0(T)).$$

Using Lemma 5.2 (with $D = D_2$ and $G = \Omega_j$) and taking the limit $j \to \infty$ in (8.17), we obtain for some $C > 0$,

$$(8.18) \qquad \frac{\|u(t)\|_{L^\infty(\Omega)}}{\|\varphi_\Omega(t)\|_{L^\infty(\Omega)}} \geq \alpha \frac{C}{\|\varphi_\Omega(T)\|_{L^\infty(\Omega)}} \quad (t \geq T),$$

which implies that $u_0 \notin \tilde{X}^2(s)$. Therefore (8.11) holds and hence $\tilde{X}^2(s) \subseteq X^2(s)$.

Let us now prove the opposite inclusion. Let $u_0 \in X(s)$. By (8.6) we can write $u_0 = q_0 \varphi_\Omega(s) + \zeta_0$ for some $q_0 \in \mathbb{R}$ and $\zeta_0 \in \tilde{X}^2(s)$. Suppose $q_0 > 0$, for example. The estimate (8.9), which has been shown to hold for all $\zeta_0 \in \tilde{X}^2(s)$, and Theorem 5.5 with $D = D_2$ imply that $u(x,t;s,u_0) > 0$ for all $x \in \bar{D}_2$ and $t \geq t_0 > s$, where t_0 is a sufficiently large number. Therefore $u_0 \notin X^2(s)$, showing that for $u_0 \in X^2(s)$ we must have $q_0 = 0$ and thus $X^2(s) \subseteq \tilde{X}^2(s)$. Therefore $X^2(s) = \tilde{X}^2(s)$ for all $s \in \mathbb{R}$. This equality and (8.6) and (8.9) yield both (i) and (ii) in Theorem 2.2.

Proof of Corollary 2.4. For any $s \in \mathbb{R}$ define the set

$$Z(s) = \{u_0 \in X(s) : \text{for all } T > s \text{ the function } u(\cdot,t;s,u_0)$$

(8.19)
$$\text{has a zero in } \bar{D} \text{ for some } t \geq T\}.$$

Since $D_2 \subseteq D$ we obviously have $X^2(s) \subseteq Z(s)$ $(s \in \mathbb{R})$. Let $u_0 \in X(s)$. By (2.2) we can write $u_0 = q_0 \varphi_\Omega(s) + \zeta_0$ for some $q_0 \in \mathbb{R}$ and $\zeta_0 \in X^2(s)$. Suppose $q_0 > 0$, for example. The estimate (2.3), applied to "$u_0 = \zeta_0$", and Theorem 5.5 imply that $u(x,t;s,u_0) > 0$ for all $x \in \bar{D}$ and $t \geq t_0 > s$, where t_0 is a sufficiently large number. Therefore $u_0 \notin Z(s)$, showing that for $u_0 \in Z(s)$ we must have $q_0 = 0$ and thus $Z(s) \subseteq X^2(s)$. Hence $Z(s) = X^2(s)$ for all $s \in \mathbb{R}$. $\square$

9. Proof of Proposition 2.5

It is sufficient to show that any positive entire solution φ_1 of (1.1), which satisfies (2.5), is a constant multiple of φ_Ω, where φ_Ω is as chosen at the beginning of the previous section. Indeed, the solution φ_Ω satisfies (2.5) for some $c > 0$. This is a simple consequence of Theorem 5.5, Corollary 6.5 (see also Corollary 6.10) and the maximum principle (Corollary 4.11). From now on we may therefore assume that (2.5) holds for both φ_Ω and φ_1 with the same $c > 0$. With x_Ω as in (5.1), we shall also assume

(9.1)
$$\varphi_1(x_\Omega, 0) = \varphi_\Omega(x_\Omega, 0) = 1,$$

and prove that necessarily $\varphi_1 = \varphi_\Omega$. Note that this will also prove the uniqueness statement at the beginning of Theorem 2.2.

For any smooth subdomain D in Ω such that $D_2 \subset\subset D \subset\subset \Omega$ and $\text{dist}(\bar{D}_2, \partial D) \geq d > 0$ we claim the following is true. There exists a constant $C(D) > 0$, depending only on D, Ω, d and the structure of L, such that

(9.2)
$$C(D)\varphi_\Omega(x,t) - \varphi_1(x,t) > 0 \qquad ((x,t) \in \bar{D} \times (-\infty, -2]).$$

Indeed, suppose that for some $x_0 \in \bar{D}$, $t_0 \leq -2$ and $C_0 > 0$ we had $\varphi_1(x_0, t_0) \geq C_0 \varphi_\Omega(x_0, t_0)$. It would then follow from Theorem 5.5 that $\varphi_1(x_0, t_0) \geq C_0 C_4(D) \|\varphi_\Omega(t_0)\|_{L^\infty(\Omega)}$. Proposition 4.13 implies that Lemma 6.6 holds with $G = \Omega$ if we replace $C_9(D,d)$ by, say, $C_9(D, d/2)/2$. Using Lemma 6.6 with

(9.3)
$$G = \Omega, \ \kappa = \frac{\|\varphi_1(t)\|_{L^\infty(D)}}{\|\varphi_1(t)\|_{L^\infty(\Omega)}}, \ v = \varphi_1, \ u = \varphi_\Omega,$$

we get that if $C_0 C_4(D) \geq 2C_9^{-1}(D, d/2)$, then $\varphi_1(x,t) > \varphi_\Omega(x,t)$ for all $x \in \bar{D}$ and $t \in [t_0 + 1, \infty)$; in particular we would get $\varphi_1(x_\Omega, 0) > \varphi_\Omega(x_\Omega, 0)$, contradicting (9.1). Thus $C_0 < 2C_4^{-1}(D)C_9^{-1}(D, d/2)$, which shows that (9.2) holds with $C(D) = 2C_4^{-1}(D)C_9^{-1}(D, d/2)$.

We next claim that if $|\Omega \setminus \bar{D}|$ is sufficiently small, then the inequality (9.2) extends to $\Omega \times \mathbb{R}$. To prove this, we define the function $v_D(x,t) = C(D)\varphi_\Omega(x,t) - \varphi_1(x,t)$ for $(x,t) \in \Omega \times \mathbb{R}$. Then v_D is a solution of (1.1) on $\Omega \times \mathbb{R}$ and it satisfies

$$(9.4) \quad v_D > 0 \text{ on } \bar{D} \times (-\infty, -2] \quad \text{and} \quad |v_D(x,t)| \leq (C(D)+1) \, c \, e^{c\,|t|} \text{ on } \Omega \times \mathbb{R},$$

where $c > 0$ is as in (2.5). The first inequality in (9.4) follows from (9.2) and the second from the discussion in the first paragraph of this section. Let $k > c$ and let $\delta > 0$ be the corresponding number given by Lemma 3.5. Suppose, in addition to $D_2 \subset\subset D \subset\subset \Omega$, that D is smooth and such that $|\Omega \setminus \bar{D}| < \delta$. Again using approximation by solutions on smooth subdomains via Proposition 4.13, in view of the first inequality in (9.4) and Lemma 3.5, one obtains the following estimate:

$$(9.5) \qquad ||v_D^-(t)||_{L^\infty(\Omega)} \leq 2 \, e^{-k\,(t-\tau)} \, ||v_D^-(\tau)||_{L^\infty(\Omega)} \quad (\tau \leq t \leq -2).$$

It follows from (9.4), (9.5) and the fact that $k > c$ that necessarily $v_D^-(x,t) = 0$ for all $(x,t) \in \Omega \times (-\infty, -2]$. Hence, by Proposition 4.9, $v_D \geq 0$ on $\Omega \times \mathbb{R}$, which means $\varphi_1 \leq C(D)\varphi_\Omega$ on $\Omega \times \mathbb{R}$.

By (i) in Theorem 2.2 for each $s \in \mathbb{R}$ there exist a unique $q(s) \in \mathbb{R}$ and $\zeta_0(\cdot,s) \in X^2(s)$ such that

$$(9.6) \qquad 0 \leq C(D)\varphi_\Omega(x,s) - \varphi_1(x,s) = q(s)\varphi_\Omega(x,s) + \zeta_0(x,s) \quad (x \in \Omega).$$

Proposition 4.13 implies that Lemma 8.1 holds with $G = \Omega$. Therefore

$$(9.7) \qquad |q(s)| \leq C_{13} \, \frac{||C(D)\varphi_\Omega(s) - \varphi_1(s)||_{L^\infty(\Omega)}}{||\varphi_\Omega(s)||_{L^\infty(\Omega)}} \leq C_{13} \, C(D),$$

where C_{13} is as in Lemma 8.1 with $d = \text{dist}(\bar{D}_2, \partial D)/2$.

Now fix $t, s \in \mathbb{R}$ with $t \geq s$. Applying (2.3) to $u(\cdot,t) = (C(D) - q(s)) \, \varphi_\Omega(\cdot,t) - \varphi_1(\cdot,t)$ and using (9.6) and (9.7), we get the following estimate:

$$\frac{||u(t)||_{L^\infty(\Omega)}}{||\varphi_\Omega(t)||_{L^\infty(\Omega)}} \leq C \, e^{-\gamma(t-s)} \, \frac{||\zeta_0(s)||_{L^\infty(\Omega)}}{||\varphi_\Omega(s)||_{L^\infty(\Omega)}}$$

$$\leq C \, e^{-\gamma(t-s)} \, \frac{||C(D)(1 + C_{13})\varphi_\Omega(s)||_{L^\infty(\Omega)}}{||\varphi_\Omega(s)||_{L^\infty(\Omega)}}$$

$$= C \, C(D)(1 + C_{13}) \, e^{-\gamma(t-s)}.$$

This inequality holds for any $t \geq s$, $t, s \in \mathbb{R}$. By (9.7) we can choose a sequence $s_j \to -\infty$ such that $q(s_j) \to q$ for some $q \in \mathbb{R}$. Taking $s = s_j \to -\infty$ in the above estimate and recalling our definition of u, we obtain $\varphi_1(x,t) = (C(D) - q)\varphi_\Omega(x,t)$ for all $(x,t) \in \Omega \times \mathbb{R}$. Finally, by (9.1) we have $C(D) - q = 1$, finishing the proof.

10. Proof of Corollary 2.6

Let us first prove (i). We will use the fact that u is a solution of $u_t - Lu = 0$ on $\Omega \times \mathbb{R}$ if and only if $v(x,t) = u(x,t) \, e^{\lambda t}$ is a solution of $u_t - Lu = \lambda u$ on $\Omega \times \mathbb{R}$. Let φ_Ω be as given by Theorem 2.2. For future use, we note that by Theorem 2.1, Corollary 6.5 (see also Corollary 6.10), and Corollary 4.11 we have

$$(10.1) \qquad c \, e^{-\log c_\Omega \, (t-s)} \leq \frac{||\varphi_\Omega(t)||_{L^\infty(\Omega)}}{||\varphi_\Omega(s)||_{L^\infty(\Omega)}} \leq e^{\beta_0(t-s)} \quad (t \geq s),$$

where c_Ω is as in (5.1) and $c > 0$ depends only on Ω and the structure of L.

By the T-periodicity of the coefficients of L, the function $u(x,t) = \varphi_\Omega(x, t+T)$ is also a positive entire solution of (1.1). Proposition 2.5 (and (10.1)) imply that $\varphi_\Omega(x, t+T) = q\,\varphi_\Omega(x,t)$ for some $q > 0$ and all $(x,t) \in \Omega \times \mathbb{R}$. Set $\lambda_1 = -(\ln q)/T$. Then $\varphi_1(x,t) = \varphi_\Omega(x,t)\,e^{\lambda_1 t}$ is a positive solution of $u_t - Lu = \lambda_1 u$ on $\Omega \times \mathbb{R}$ and it is easy to check that $\varphi_1(x,t) = \varphi_1(x, t+T)$ for all $(x,t) \in \Omega \times \mathbb{R}$. This proves the existence of λ_1. Moreover, by Proposition 2.5 the eigenfunction φ_1 is unique and, by its definition and Theorem 2.1, it satisfies (2.1). The proof of (i) is complete.

It follows from the construction of λ_1 above and from (10.1) that $-\beta_0 \leq \lambda_1 \leq \log c_\Omega$, proving (ii).

A straightforward modification of [17, Proof of Corollary 2.5] yields (iii), and we omit the details.

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