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GENERALIZED ARTIN AND BRAUER INDUCTION FOR COMPACT LIE GROUPS

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Abstract. Let G be a compact Lie group. We present two induction theorems for certain generalized G-equivariant cohomology theories. The theory applies to G-equivariant K-theory K_G, and to the Borel cohomology associated with any complex oriented cohomology theory. The coefficient ring of K_G is the representation ring $R(G)$ of G. When G is a finite group the induction theorems for K_G coincide with the classical Artin and Brauer induction theorems for $R(G)$.

1. Introduction

The Artin induction theorem, also called Artin's theorem on induced characters, says that for any finite group G, the unit element in the representation ring $R(G)$, multiplied by the order of G, is an integral linear combination of elements induced from $R(C)$, for cyclic subgroups C of G. Similarly, the Brauer induction theorem, or Brauer's theorem on induced characters, says that the unit element in $R(G)$ is an integral linear combination of elements induced from $R(H)$, for subgroups H of G which are extensions of cyclic subgroups by p-groups. These theorems in combination with the double coset formula and the Frobenius reciprocity law allow $R(G)$ to be reconstructed integrally from $R(H)$, for subgroups H which are extensions of cyclic groups by p-groups, and the restriction maps between them, and rationally from the $R(C)$, for cyclic subgroups C, and the restriction maps between them. These reconstruction results are called restriction theorems since they are conveniently formulated using the restriction maps.

We present a generalization of the Artin and Brauer induction and restriction theorems for the representation ring of a finite group G. The generalization is in three directions. First, we give an induction theory for a general class of equivariant cohomology theories; the induction theorems apply to the cohomology groups of arbitrary spectra, not just the coefficients of the cohomology theory. Second, we extend the induction theory from finite groups to compact Lie groups. Third, we allow induction from more general classes of subgroups than the cyclic subgroups. We use the following classes of abelian subgroups of G characterized by the number of generators allowed: The class of the maximal tori ($n = 0$), and for each $n \geq 1$ the class of all closed abelian subgroups A of G with finite index in its normalizer and with a dense subgroup generated by n or fewer elements.

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In section 2 we collect some needed facts on compact Lie groups. In section 3 we describe the induction and restriction maps in homology and in cohomology. The induction theory makes use of the Burnside ring module structure on equivariant cohomology theories. The Burnside ring is isomorphic to the ring of homotopy classes of stable self maps of the unit object $\Sigma_G^\infty S^0$ in the G-equivariant stable homotopy category. In section 4 we recall some alternative descriptions of the Burnside ring of a compact Lie group G and discuss some of their properties.

The following condition on a cohomology theory suffices to give the induction theorems: We say that a ring spectrum E has the 0-induction property if the unit map $\eta\colon \Sigma_G^\infty S^0 \to E$ pre-composed with a map $f\colon \Sigma_G^\infty S^0 \to \Sigma_G^\infty S^0$ is null in the stable G-equivariant homotopy category whenever the underlying nonequivariant map $f\colon \Sigma^\infty S^0 \to \Sigma^\infty S^0$ is null. We say that a ring spectrum E has the n-induction property, for some $n \geq 1$, if the unit map $\eta\colon \Sigma_G^\infty S^0 \to E$ pre-composed with a map $f\colon \Sigma_G^\infty S^0 \to \Sigma_G^\infty S^0$ is null in the stable G-equivariant homotopy category whenever the degree of f^A is 0 for all abelian subgroups A of G, that is, the closure of a subgroup generated by n or fewer elements. For example, singular Borel cohomology has the 0-induction property, equivariant K-theory has the 1-induction property, and the Borel cohomology associated with a suitable height n-complex oriented ring spectrum (such as E_n) has the n-induction property.

Let E_G be a G-ring spectrum satisfying the n-induction property. Pick one subgroup A_i in each conjugacy class of the abelian subgroups of G with finite index in its normalizer and with a dense subgroup generated by n or fewer elements. Let $|G|_n$ be the least common multiple of the order of the Weyl groups of these abelian groups $\{A_i\}$ (there are only finitely many such subgroups by Corollary 2.2). In section 5 we prove the following Artin induction theorem.

Theorem 1.1. *The integer $|G|_n$ times the unit element in E_G^* is in the image of the induction map*

$$\bigoplus_i \operatorname{ind}_{A_i}^G\colon \bigoplus_i E_{A_i}^0 \to E_G^0.$$

Let M_G be an E_G-module spectrum, and let X be an arbitrary G-spectrum. There is a restriction map

$$\operatorname{res}\colon M_G^\alpha(X) \to \operatorname{Eq}[\prod_i M_{A_i}^{\operatorname{res}\alpha}(X) \rightrightarrows \prod_{i,j,g} M_{A_i \cap g A_j g^{-1}}^{\operatorname{res}\alpha}(X)].$$

Here Eq denotes the equalizer (the kernel of the difference of the two parallel maps) and α denotes the grading by a formal difference of two finite dimensional real G-representations. The second product is over i, j and over $g \in G$. The maps in the equalizer are the two restriction (composed with conjugation) maps. The Artin induction theorem implies the following Artin restriction theorem.

Theorem 1.2. *There exists a map*

$$\psi\colon \operatorname{Eq}[\prod_i M_{A_i}^{\operatorname{res}\alpha}(X) \rightrightarrows \prod_{i,j,g} M_{A_i \cap g A_j g^{-1}}^{\operatorname{res}\alpha}(X)] \to M_G^\alpha(X)$$

such that both the composites $\operatorname{res} \circ \psi$ *and* $\psi \circ \operatorname{res}$ *are $|G|_n$ times the identity map.*

The Brauer induction theorem is analogous to the Artin induction theorem. At the expense of using a larger class, $\{H_j\}$, of subgroups of G than those used for Artin induction, we get that the unit element of E_G^* is in the image of the induction map from $\bigoplus_j E_{H_j}^*$. As a consequence the corresponding restriction map, res is an isomorphism. The exact statements are given in section 6.

The G-equivariant K-theory $K_G(X_+)$ of a compact G-CW-complex X is the Grothendieck construction on the set of isomorphism classes of finite dimensional complex G-bundles on X. In particular, when X is a point we get that $K_G(S^0)$ is isomorphic to the complex representation ring $R(G)$. The induction and restriction maps for the equivariant cohomology theory K_G give the usual induction and restriction maps for the representation ring $K_G(S^0) \cong R(G)$. The ring spectrum K_G satisfies the 1-induction property. The resulting induction theorems for $R(G)$ are the classical Artin and Brauer induction theorems. The details of this example are given in section 9.

This work is inspired by an Artin induction theorem used by Hopkins, Kuhn, and Ravenel [HKR00]. They calculated the Borel cohomology associated with certain complex oriented cohomology theories for finite abelian groups; furthermore they used an Artin restriction theorem to describe the Borel cohomology, rationally, for general finite groups. We discuss induction and restriction theorems for the Borel cohomology associated with complex oriented cohomology theories when G is a compact Lie group in section 7. Singular Borel cohomology is discussed in section 8.

A Brauer induction theorem for the representation ring of a compact Lie group was first given by G. Segal [Seg68a]. Induction theories for G-equivariant cohomology theories, when G is a compact Lie group, have also been studied by G. Lewis [Lew96, sec.6]. He develops a Dress induction theory for Mackey functors.

The idea to use the Burnside ring module structure to prove induction theorems goes back to Conlon and Solomon [Con68, Sol67] [Ben95, chap. 5].

2. Compact Lie groups

In this section we recall some facts about compact Lie groups and provide a few new observations. We say that a subgroup H of G is topologically generated by n elements (or fewer) if there is a dense subgroup of H generated by n elements; e.g., any torus is topologically generated by one element. By a subgroup of a compact Lie group G we mean a closed subgroup of G unless otherwise stated. It is convenient to give the set of conjugacy classes of (closed) subgroups of G a topology [tD79, 5.6.1]. Let d be a metric on G so that the metric topology is equal to the topology on G. We give the space of all closed subgroups of G the Hausdorff topology from the metric

$$d_H(A,B) = \sup_{a\in A}\inf_{b\in B} d(a,b) + \sup_{b\in B}\inf_{a\in A} d(a,b).$$

The metric topology on the space of closed subgroups is independent of the choice of metric. Let ΨG denote the space of conjugacy classes of closed subgroups of G given the quotient topology from the space of closed subgroups of G. The space ΨG is a compact space with a metric given by

$$d_\Psi(A,B) = \inf_{g\in G} d_H(A, gBg^{-1}).$$

The Weyl group $W_G H$ of a subgroup H in G is $N_G H/H$. Let ΦG denote the subspace of ΨG consisting of conjugacy classes of subgroups of G with finite Weyl group. We have that ΦG is a closed subspace of ΨG [tD79, 5.6.1].

We denote the conjugacy class of a subgroup H in G by (H), leaving G to be understood from the context. Conjugacy classes of subgroups of G form a partially ordered set; $(K) \leq (H)$ means that K is conjugate in G to a subgroup of H.

Note that a subgroup of a compact Lie group G cannot be conjugate to a proper subgroup of itself. (There are no properly contained closed n-manifolds of a closed connected n-manifold.)

A theorem of Montgomery and Zippin says that for any subgroup H of G there is an open neighborhood U of the identity element in G such that all subgroups of HU are subconjugate to H [Bre72, II.5.6], [MZ42].

Let $K \leq H$ be subgroups of G. The normalizer $N_G K$ acts from the left on $(G/H)^K$. Montgomery and Zippin's theorem implies that the coset $(G/H)^K/N_G K$ is finite [Bre72, II.5.7]. In particular, if $W_G K$ is finite, then $(G/H)^K$ is finite. The Weyl group $W_G H$ acts freely on $(G/H)^K$ from the right by $gH \cdot nH = gnH$, where $gH \in (G/H)^K$ and $nH \in W_G H$. So $|W_G H|$ divides $|(G/H)^K|$. The following consequence of Montgomery and Zippin's theorem is important for this paper. Let G° denote the unit component of the group G.

Lemma 2.1. *Let G be a compact Lie group. The conjugacy class of any abelian subgroup A of G with finite Weyl group is an open point in ΦG.*

Proof. Fix a metric on G. By Montgomery and Zippin's theorem there is an $\epsilon > 0$ such that if K is a subgroup of G and $d_\Psi((A),(K)) < \epsilon$, then K is conjugated in G to a subgroup of A that meets all the components of A. Let K be such a subgroup and assume in addition that it has finite Weyl group. Then $K^\circ = A^\circ$ since $A < N_G K$ and $|W_G K|$ is finite. Thus we have that $(K) = (A)$. Hence (A) is an open point in ΦG. □

Since ΦG is compact we get the following.

Corollary 2.2. *There are only finitely many conjugacy classes of abelian subgroups of G with finite Weyl group.*

Given a subgroup H of G we can extend H by tori until we get a subgroup K with $W_G K$ finite. This extension of H is unique up to conjugation. We denote the conjugacy class by $\omega(H)$. The conjugacy class $\omega(H)$ does only depend on the conjugacy class of H. Hence we get a well defined map $\omega\colon \Psi G \to \Phi G$. This map is continuous [FO05, 1.2]. We say that $\omega(H)$ is the G subgroup conjugacy class with finite Weyl group associated with H. One can also show that the conjugacy class $\omega(H)$ is the conjugacy class (HT) where T is a maximal torus in $C_G H$ [FO05, 2.2]. This result implies the following.

Lemma 2.3. *The map $\omega\colon \Psi G \to \Phi G$ sends conjugacy classes of abelian groups to conjugacy classes of abelian groups.*

We now define the classes of abelian groups used in the Artin induction theory.

Definition 2.4. Let $\mathcal{A}G$ denote the set of all conjugacy classes of abelian subgroups of G with finite Weyl group. Let $\mathcal{A}_n G$ denote the set of conjugacy classes of abelian subgroups A of G that are topologically generated by n or fewer elements and that have a finite Weyl group. We let $\mathcal{A}_0 G$ be the conjugacy class of the maximal torus in G.

We often suppress G from the notation of $\mathcal{A}_n G$ and write $\mathcal{A}_n$. We have that $\mathcal{A}_n G = \mathcal{A}G$ for some n by Corollary 2.2.

Example 2.5. The topologically cyclic subgroups of G are well understood. They were called Cartan subgroups and studied by G. Segal in [Seg68a]. The following is a summary of some of his results: All elements of G are contained in a Cartan subgroup. An element g in G is called regular if the closure of the cyclic subgroup generated by g has finite Weyl group. The regular elements of G are dense in G. Two regular elements in the same component of G generate conjugate Cartan subgroups. The map $S \mapsto G^\circ S/G^\circ$ gives a bijection between conjugacy classes of Cartan subgroups and conjugacy classes of cyclic subgroups of the group of components G/G°. In particular, if G is connected, then the Cartan subgroups are precisely the maximal tori. The order $|S/S^\circ|$ is divided by $|S/G^\circ|$ and divides $|S/G^\circ|^2$ [Seg68a, p. 117]. For example the nontrivial semidirect product $S^1 \rtimes \mathbb{Z}/2$ has Cartan subgroups S^1 and (conjugates of) $0 \rtimes \mathbb{Z}/2$.

Lemma 2.6. *If A is a compact abelian Lie group, then it splits as*

$$A \cong A^\circ \times \pi_0(A).$$

Proof. Since A is compact we have that $\pi_0(A) \cong \bigoplus_i \mathbb{Z}/p_i^{n_i}$. The unit component A° is a torus. We construct an explicit splitting of $A \to \pi_0(A)$. Let $a_i \in A$ be an element such that a_i maps to a fixed generator in $\mathbb{Z}/p_i^{n_i}$ and to zero in $\mathbb{Z}/p_k^{n_k}$ for all $k \neq i$. Then a_i raised to the $p_i^{n_i}$ power maps to zero in $\pi_0(A)$, hence is in the torus A°. There is an element $b_i \in A^\circ$ such that

$$a_i^{p_i^{n_i}} = b_i^{p_i^{n_i}}.$$

Set $\bar{a}_i = a_i b_i^{-1}$ and define the splitting $\pi_0(A) \to A$ by sending the fixed generator of $\mathbb{Z}/p_i^{n_i}$ in $\pi_0(A)$ to $\bar{a}_i$. Since A is commutative this gives a well defined group homomorphism. □

The splitting in Lemma 2.6 is not natural.

Lemma 2.7. *Let G be a compact Lie group, and let $A \leq B$ be abelian subgroups of G such that $\omega(B)$ is in $\mathcal{A}_n$. Then $\omega(A)$ is in $\mathcal{A}_n$.*

Proof. The minimal number of topological generators of an abelian group A is equal to the minimal number of generators of the group of components of A by Lemma 2.6. We assume without loss of generality that B has finite Weyl group. The unit component B° of B is contained in the normalizer of A. Hence the component group of a representative for the conjugacy class $\omega(A)$ is isomorphic to a quotient of a subgroup of $\pi_0(B)$. The result then follows. □

We introduce several different orders for compact Lie groups. T. tom Dieck has proved that for any given compact Lie group G there is an integer n_G so that the order of the group of components of the Weyl group $W_G H$ is less than or equal to n_G for all closed subgroups H of G [tD77].

Definition 2.8. The order $|G|$ of a compact Lie group G is the least common multiple of the orders $|W_G H|$ for all $(H) \in \Phi G$. For any nonnegative integer n let $|G|_n$ be the least common multiple of $|W_G A|$ for all $(A) \in \mathcal{A}_n G$.

When G is a finite group all these orders coincide and are equal to the number of elements in G.

Remark 2.9. Let T be a maximal torus in G. Then we have that

$$N_GT/(G^\circ \cap N_GT) \cong G/G^\circ$$

since all maximal tori of G are conjugated by elements in G°. Hence the number of components $|G/G^\circ|$ of G divides the smallest order $|G|_0 = |N_GT/T|$. The order $|G|_m$ divides $|G|_n$ for $0 \le m \le n$.

Example 2.10. For compact Lie groups the various orders might be different. An example is given by $SO(3)$. The only conjugacy classes of abelian subgroups of $SO(3)$ with finite Weyl group are represented by the subgroups $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ and S^1 of $SO(3)$. The normalizers of these subgroups are $(\mathbb{Z}/2 \oplus \mathbb{Z}/2) \rtimes \Sigma_3$ and $S^1 \rtimes \mathbb{Z}/2$, respectively [tD79, 5.14]. So the Weyl groups have orders 6 and 2, respectively. Hence $|SO(3)|_n = 2$ for $n = 0, 1$ and $|SO(3)|_n = 6$ for $n \ge 2$. By taking cartesian products of copies of $SO(3)$ we get a connected compact Lie group with many different orders. The order $|SO(3)^{\times N}|_{2m}$ is 2^N3^m for $m \le N$ and 6^N for $m \ge N$. The abelian subgroups of $SO(3)^{\times N}$ with finite Weyl group are product subgroups obtained from all conjugates of S^1 and $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, and furthermore all subgroups of these product subgroups so that each of the n canonical projections to $SO(3)$ are conjugate to either S^1 or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ in $SO(3)$.

3. G-equivariant cohomology theories

We work in the homotopy category of G-spectra indexed on a complete G-universe. Most of the results used are from [LMS86]. If X is a based G-space the suspension spectrum, $\Sigma_G^\infty X$, is simply denoted by X.

We recall the definition of homology and cohomology theories associated with a G-equivariant spectrum M_G. Let X and Y be G-spectra. Let $\{X, Y\}_G$ denote the group of stable (weak) G-homotopy classes of maps from X to Y. We grade our theories by formal differences of G-representations. For brevity let α denote the formal difference $V - W$ of two finite dimensional real G-representations V and W. Let S^V denote the one point compactification of V. Let S_G^α denote the spectrum $S^{-W}\Sigma_G^\infty S^V$. The homology is

$$M_\alpha^G(X) = \{S_G^\alpha, M_G \wedge X\}_G \cong \{S_G^V, S_G^W \wedge M_G \wedge X\}_G.$$

The cohomology is

$$M_G^\alpha(X) = \{S_G^{-\alpha} \wedge X, M_G\}_G \cong \{S_G^W \wedge X, S_G^V \wedge M_G\}_G.$$

In this paper a ring spectrum E is a spectrum together with a multiplication $\mu\colon E \wedge E \to E$ and a left unit $\eta\colon S_G \to E$ for the multiplication in the stable homotopy category. We do not need to assume that E is associative nor commutative. An E-module spectrum M is a spectrum with an action $E \wedge M \to M$ by E that respects the unit and multiplication. Let E_G be a G-equivariant ring spectrum. The coefficients $E_G^\alpha = E_{-\alpha}^G$ have a bilinear multiplication that is $RO(G)$-graded and have a left unit element. Let M_G be an E_G-module spectrum. We have that $M_G^*(X)$ is naturally an E_G^*-module, and $M_*^G(X)$ is naturally an E_*^G-module for any G-spectrum X.

Let M_G be a spectrum indexed on a G-universe $\mathcal{U}$. For a closed subgroup H in G let M_H denote M_G regarded as an H-spectrum indexed on $\mathcal{U}$ now considered as an H-universe. The forgetful functor from G-spectra to H-spectra respects the smash product. A complete G-universe $\mathcal{U}$ is also a complete H-universe for all closed subgroups H of G. For lack of a reference we include an argument proving this well-known result.

Let V be an H-representation. The manifold $G \times V$ has a smooth and free H-action given by $h \cdot (g, v) = (gh^{-1}, hv)$. It also has a smooth G-action by letting G act from the right on G in $G \times V$. The H-quotient $G \times_H V$ is a smooth G-manifold [Bre72, VI.2.5]. Now consider the tangent G-representation W at $(1, 0) \in G \times_H V$. This can be arranged so that W is an orthogonal G-representation by using a G-invariant Riemannian metric on G. Compactness of H gives that V is a summand of W regarded as an H-representation.

We have the following isomorphisms for any α and any G-spectrum X [May96, XVI.4]:

$$M^H_{\mathrm{res}^G_H \alpha}(X) \cong \{G/H_+ \wedge S^\alpha_G, M_G \wedge X\}_G,$$
$$M_H^{\mathrm{res}^G_H \alpha}(X) \cong \{G/H_+ \wedge S^{-\alpha}_G \wedge X, M_G\}_G.$$

The forgetful functor from G-spectra to H-spectra respects the smash product.

We now consider induction and restriction maps. The collapse map $c\colon G/H_+ \to S_G$ is the stable map associated with the G-map that sends the disjoint basepoint $+$ to the basepoint 0, and G/H to 1 in $S^0 = \{0, 1\}$. Let $\tau\colon S_G \to G/H_+$ be the transfer map [LMS86, IV.2]. We recall a construction of τ after Proposition 3.1.

There is an induction map natural in the G-spectra X and M_G:

$$\mathrm{ind}^G_H\colon M^H_{\mathrm{res}^G_H \alpha}(X) \to M^G_\alpha(X).$$

It is defined by pre-composing with the transfer map $S_G \xrightarrow{\tau} G/H_+$ as follows:

$$M^H_{\mathrm{res}^G_H \alpha}(X) \cong \{G/H_+ \wedge S^\alpha_G, M_G \wedge X\}_G \xrightarrow{\tau^*} M^G_\alpha(X).$$

There is a restriction map natural in the G-spectra X and M_G:

$$\mathrm{res}^G_H\colon M^G_\alpha(X) \to M^H_{\mathrm{res}^G_H \alpha}(X).$$

It is defined by pre-composing with the collapse map $G/H_+ \xrightarrow{c} S_G$. The definition is analogous for cohomology. Alternatively, we can describe the induction map in cohomology as follows:

$$M^\alpha_G(\tau \wedge 1_X)\colon M^\alpha_G(G/H_+ \wedge X) \to M^\alpha_G(X)$$

and the restriction map as

$$M^\alpha_G(c \wedge 1_X)\colon M^\alpha_G(X) \to M^\alpha_G(G/H_+ \wedge X)$$

composed with the isomorphism

$$\{S^{-\alpha}_G \wedge G/H_+ \wedge X, M_G\}_G \xrightarrow{(k \wedge 1_X)^*} \{G/H_+ \wedge S^{-\alpha}_G \wedge X, M_G\}_G \cong M_H^{\mathrm{res}^G_H \alpha}(X)$$

where $k\colon G/H_+ \wedge S^{-\alpha}_G \cong S^{-\alpha}_G \wedge G/H_+$.

The classical Frobenius reciprocity law says that the induction map $R(H) \to R(G)$ between representation rings is linear as an $R(G)$-module, where $R(H)$ is given the $R(G)$-module structure via the restriction map. In our more general context the Frobenius reciprocity law says that the induction map $M_H^{\mathrm{res}^G_H\alpha}(X) \to M_G^\alpha(X)$ is linear as a map of $E_G^*(S_G)$-modules (via the restriction map). We need the following slightly different version.

Proposition 3.1. *Let M_G be a module over a ring spectrum E_G. Let $e \in E^H_{\mathrm{res}^H_G\alpha}$ and $m \in M^G_\beta(X)$. Then we have that*

$$\mathrm{ind}^G_H(e) \cdot m = \mathrm{ind}^G_H(e \cdot \mathrm{res}^G_H m)$$

in $M^G_{\alpha+\beta}(X)$. The same result applies to cohomology.

Proof. Let $e\colon G/H_+ \wedge S^\alpha_G \to E_G$ represent the element e in $E^H_{\mathrm{res}^H_G\alpha}$, and let $m\colon S^\beta_G \to M_G \wedge X$ represent the element $m \in M^G_\beta(X)$.

We get that both products are

$$S^\alpha_G \wedge S^\beta_G \xrightarrow{\tau\wedge 1} G/H_+ \wedge S^\alpha_G \wedge S^\beta_G \xrightarrow{e\wedge m} E_G \wedge M_G \wedge X$$

composed with the $E_G \wedge M_G \to M_G$. The proof for cohomology is similar. □

We now describe the induction and restriction maps for homology theories in more detail. This is used in section 7. We have that

$$\{S^\alpha_G \wedge G/H_+, E \wedge X\}_G \cong \{S^\alpha_G, D(G/H_+) \wedge E \wedge X\}_G$$

where $D(G/H_+)$ is the Spanier-Whitehead dual of G/H_+. Using the equivalences $S_G^{-\alpha} \wedge G/H_+ \cong G/H_+ \wedge S_G^{-\alpha}$ and $D(G/H_+) \wedge E \cong E \wedge D(G/H_+)$ we get an isomorphism

$$E^H_{\mathrm{res}^G_H\alpha}(X) \cong E^G_\alpha(D(G/H_+) \wedge X).$$

Under this isomorphism the induction map is given by $E^G_\alpha(D(\tau) \wedge 1_X)$, and the restriction map as $E^G_\alpha(D(c) \wedge 1_X)$. In the rest of this section we recall a description of the transfer map [LMS86, IV.2.3] and the Spanier-Whitehead dual of the collapse and transfer maps [LMS86, IV.2.4]. Let M be a smooth compact manifold without boundary. In our case $M = G/H$. There is an embedding of M into some finite dimensional real G-representation V [Bre72, VI.4.2]. The normal bundle νM of M in V can be embedded into an open neighborhood of M in V by the equivariant tubular neighborhood theorem. The Thom construction $\mathrm{Th}(\xi)$ of a bundle ξ on a compact manifold is equivalent to the one point compactification of the bundle ξ. We get a map

$$t'\colon S^V \to \mathrm{Th}(\nu M)$$

by mapping everything outside of the tubular neighborhood of M to the point at infinity.

The Thom construction of the inclusion map $\nu M \to \nu M \oplus TM \cong V \times M$ gives $s'\colon \mathrm{Th}(\nu M) \to S^V \wedge M_+$. Let the pretransfer $t\colon S_G \to S^{-V} \wedge \mathrm{Th}(\nu M)$ be $S^{-V} \wedge t'$ pre-composed with $S_G \cong S^{-V} \wedge S^V$ and let $s\colon S^{-V} \wedge \mathrm{Th}(\nu M) \to M_+$ be the composite of $S^{-V} \wedge s'$ with $S^{-V} \wedge S^V \wedge M_+ \cong M_+$. The transfer map τ is defined to be the composite map

$$s \circ t\colon S_G \to M_+.$$

We now let M be the G-manifold G/H. Atiyah duality gives that the Spanier-Whitehead dual of G/H_+ is equivalent to $S^{-V}\mathrm{Th}(\nu G/H)$. When G is finite this is just G/H_+ itself. The proof of the equivariant Atiyah duality theorem [LMS86, III.5.2] gives that $D(c) \simeq t$. It is easy to see that $D(s) \simeq s$. Hence we get

$$D(\tau) \simeq c \circ s \quad \text{and} \quad D(c) \simeq t.$$

The discussion above gives the following.

Lemma 3.2. *Let $c\colon G/H_+ \to S_G$, $s\colon S^{-V} \wedge \mathrm{Th}(\nu G/H) \to G/H_+$, and $t\colon S_G \to S^{-V} \wedge \mathrm{Th}(\nu G/H)$ be as above. Then the restriction map in homology is*

$$(t \wedge 1_X)_\alpha \colon E^G_\alpha(X) \to E^G_\alpha(S^{-V}\mathrm{Th}(\nu G/H) \wedge X)$$

and the induction map in homology is

$$((c \circ s) \wedge 1_X)_\alpha \colon E^G_\alpha(S^{-V}\mathrm{Th}(\nu G/H) \wedge X) \to E^G_\alpha(X)$$

composed with the isomorphism $E^G_\alpha(S^{-V}\mathrm{Th}(\nu G/H) \wedge X) \cong E^H_{\mathrm{res}^H_G \alpha}(X)$.

If G is a finite group, then s is the identity map. Hence the induction map is the induced map from the collapse map c, and the restriction map is the induced map from the transfer map τ, composed with the isomorphism $E^G_\alpha(G/H_+ \wedge X) \cong E^H_{\mathrm{res}^H_G \alpha}(X)$.

4. The Burnside ring

The stable homotopy classes of maps between two G-spectra are naturally modules over the Burnside ring of G. We use this Burnside ring module structure to prove our induction theorems.

We recall the following description of the Burnside ring $A(G)$ of a compact Lie group G from [tD75, tD79, LMS86]. Let $a(G)$ be the semiring of isomorphism classes of compact G-CW-complexes with disjoint union as sum, cartesian product as product, and the point as the multiplicative unit object. Let $C(\Phi G;\mathbb{Z})$, or $C(G)$ for short, be the ring of continuous functions from the space ΦG of conjugacy classes of closed subgroups of G with finite Weyl group to the integers $\mathbb{Z}$. Let $\chi_{\mathrm{Eu}}(X)$ denote the Euler characteristic of a space X. We define a semi-ring homomorphism from $a(G)$ to $C(G)$ by sending X to the function $(H) \mapsto \chi_{\mathrm{Eu}}(X^H)$ [tD79, 5.6.4]. This map extends to a ring homomorphism ϕ' from the Grothendieck construction $b(G)$ of $a(G)$ to $C(G)$. The Burnside ring $A(G)$ is defined as $b(G)/\ker\phi'$. We get an injective ring map $\phi\colon A(G) \to C(G)$. The image of ϕ is generated by $\phi(G/H_+)$ for $H \in \Phi G$. One can show that $C(G)$ is freely generated by $|W_G H|^{-1}\phi(G/H_+)$ for $H \in \Phi G$ [LMS86, V.2.11]. We have that $|G|C(G) \subset A(G)$ where $|G|$ is the order of G [tD77, thm.2]. We denote the class $\chi(X)$ in $A(G)$ corresponding to a finite G-CW-complex X by $[X]$. Define a ring homomorphism $d\colon \pi^G_0(S_G) \to C(G)$ by sending a stable map $f\colon S_G \to S_G$ to the function that sends (H) in ΦG to the degree of the fixed point map f^H. It follows from Montgomery and Zippin's theorem that the maps ϕ' and d take values in continuous functions from ΦG to $\mathbb{Z}$.

There is a map $\chi\colon A(G) \to \pi^G_0(S_G)$ given by sending the class of a compact G-CW-complex X to the composite of the transfer and the collapse map

$$S_G \to X_+ \to S_G.$$

The map χ is the categorical Euler characteristic [LMS86, V.1]. It turns out to be a ring homomorphism [LMS86, V.1]. It has the property that the degree of the H-fixed point of a map in the homotopy class $\chi(X)$ is equal to $\chi_{\mathrm{Eu}}(X^H)$ (the ordinary Euler characteristic of the fixed point space X^H) [LMS86, V.1.7]. A proof is given in Lemma 4.2. We have the following commutative triangle:

$$\begin{tikzcd} A(G) \arrow[rr, "\chi"] \arrow[dr, "\phi"'] & & \pi_0^G(S_G) \arrow[dl, "d"] \\ & C(\Phi G; \mathbb{Z}). & \end{tikzcd}$$

A theorem, due to Segal when G is a finite group and to tom Dieck when G is a compact Lie group, says the map χ is an isomorphism [LMS86, V.2.11]. This allows us to use the following three different descriptions of elements in the Burnside ring:

(1) Formal differences of equivalence classes of compact G-CW-complexes.
(2) Stable homotopy classes of self maps of S_G.
(3) Certain continuous functions from ΦG to the integers.

Since the Burnside ring $A(G)$ is isomorphic to $\{S_G, S_G\}_G$ we have that G-equivariant cohomology and homology theories naturally take values in the category of modules over $A(G)$.

In the rest of this section we prove that the degree of the H-fixed points of a stable map $f\colon S_G \to S_G$ is the same as the degree of the $\omega(H)$-fixed point of f for any closed subgroup H of G. The following is well known.

Lemma 4.1. *Let X be any space with an action by a torus T. If both $\chi_{\mathrm{Eu}}(X)$ and $\chi_{\mathrm{Eu}}(X^T)$ exist, then $\chi_{\mathrm{Eu}}(X) = \chi_{\mathrm{Eu}}(X^T)$.*

Proof. Replace X by a weakly equivalent T-CW-complex. We get that the quotient complex X/X^T is built out of one single point $*$ and cells $D^n \wedge T/A$ for a proper subgroup A of T. All nontrivial cosets of T are tori (of positive dimension). Hence all the cells have Euler characteristic equal to 0 except for the point which has Euler characteristic 1. The claim follows by the long exact sequence in homology and the assumptions that both $\chi_{\mathrm{Eu}}(X)$ and $\chi_{\mathrm{Eu}}(X^T)$ exist. □

The following is a generalization of [LMS86, V.1.7]. They consider closed subgroups with finite Weyl group.

Lemma 4.2. *Let X be a compact G-CW-complex and let $f\colon S_G \to S_G$ be a stable G-map in the stable homotopy class $\chi(X) \in \pi_0^G(S_G)$. Then $\deg(f^L) = \chi_{\mathrm{Eu}}(X^L)$ for any closed subgroup L of G.*

Proof. The geometric fixed point functor Φ^L is a strong monoidal functor from the stable homotopy category of G-spectra to the stable homotopy category of W_GL-spectra [LMS86, II.9.12]. We also have that $\Phi^L(\Sigma_G^\infty X_+) \cong \Sigma_{W_GL}^\infty X_+^L$ [May96, XVI.6]. The forgetful functor from the stable W_GH-homotopy category to the nonequivariant stable homotopy category is also strong monoidal. The categorical Euler characteristic respects strong monoidal functors [May01, 3.2]. Hence we get that $\deg(f^L)$ is equal to the degree of the categorical Euler characteristic of the spectrum $\Phi^L(\Sigma_G^\infty X)$ regarded as a nonequivariant spectrum. This is $\chi_{\mathrm{Eu}}(X^L)$. □

Proposition 4.3. *Let $f\colon S_G \to S_G$ be a stable G-map. Let H be a closed subgroup of G and let $\omega(H)$ be the associated conjugacy class of subgroups with finite Weyl group. Then we have that*

$$\deg(f^H) = \deg(f^{\omega(H)}).$$

Proof. Let K be a subgroup in the conjugacy class $\omega(H)$ so that $H < K$ and K/H is a torus. Let X and Y be finite G-CW-complexes such that f is in the homotopy class $\chi(X) - \chi(Y)$. Since

$$\chi_{\mathrm{Eu}}(X^K) = \chi_{\mathrm{Eu}}((X^H)^{K/H}) \quad \text{and} \quad \chi_{\mathrm{Eu}}(Y^K) = \chi_{\mathrm{Eu}}((Y^H)^{K/H})$$

the previous two lemmas give that

$$\deg(f^K) = \chi_{\mathrm{Eu}}(X^K) - \chi_{\mathrm{Eu}}(Y^K) = \chi_{\mathrm{Eu}}(X^H) - \chi_{\mathrm{Eu}}(Y^H) = \deg(f^H). \qquad \square$$

We need the following corollary in section 7.

Corollary 4.4. *Let $f\colon S_G \to S_G$ be a map such that $\deg(f^A) = 0$ for all abelian subgroups of G with finite Weyl group. Then f is null homotopic restricted to the K-equivariant stable homotopy category for any abelian subgroup K of G.*

Proof. Proposition 4.3 together with Lemma 2.3 give that the degree $\deg(f^A)$ is 0 for all abelian subgroups A of G. The claim follows since a self map of S^0_K is null homotopic if and only if the degrees of all its fixed point maps are 0 [tD79, 8.4.1]. $\square$

5. Artin induction

We first introduce some conditions on ring spectra and then prove the Artin induction and restriction theorems. Recall Definition 2.4. Let J_n be the $A(G)$-ideal consisting of all elements $\beta \in A(G)$ such that $\deg(\beta^A) = 0$ for all $(A) \in \mathcal{A}_n$. Let J be the intersection of all J_n.

Definition 5.1. We say that a G-equivariant ring spectrum E_G satisfies the n-induction property if $J_n E^0_G = 0$. We say that E_G satisfies the induction property if $JE^0_G = 0$.

Let $\eta\colon S_G \to E_G$ be the unit map of the ring spectrum E_G. Then E_G satisfies the n-induction property if and only if the ideal J_n is in the kernel of the unit map

$$\eta\colon A(G) \to E^0_G.$$

If E_G satisfies the n-induction property and E'_G is an E_G-algebra, then E'_G also satisfies the n-induction property.

Let $e_H\colon \Phi G \to \mathbb{Z}$ be the function defined by letting $e_H(H) = 1$ and $e_H(K) = 0$ for $(K) \neq (H)$. We have that e_A is a continuous function for every $(A) \in \mathcal{A}$ since (A) is an open-closed point in ΦG by Lemma 2.1. Since $|G|C(G) \subset \phi A(G)$ we have that $|G|e_A \in \phi A(G)$ for all $A \in \mathcal{A}$. When G is a compact Lie group it turns out that we can sharpen this result. Recall Definition 2.8.

Proposition 5.2. *Let (K) be an element in $\mathcal{A}_n$. Then $|G|_n e_K$ is an element in $\phi A(G)$. Moreover, the element can be written as*

$$|G|_n e_K = \sum_{(A) \leq (K)} c_A \phi(G/A)$$

where $c_A \in \mathbb{Z}$ and $(A) \in \mathcal{A}_n$.

Proof. Let $\mathcal{S}(K)$ denote the subset of ΦG consisting of all $(A) \leq (K)$ in ΦG. Lemmas 2.1 and 2.7 imply that if K is in $\mathcal{A}_n$, then $\mathcal{S}(K)$ is a finite subset of $\mathcal{A}_n$ consisting of open-closed points in ΦG. We prove the proposition by induction on the length of chains (totally ordered subsets) in the partially ordered set $\mathcal{S}(K)$. If

(K) is minimal in $\mathcal{A}_n$, then $\phi(G/K) = |W_G K| e_K$ and the claim is true. Assume the claim is true for all (A) such that all chains in $\mathcal{S}(A)$ have length $l-1$ or less. If all chains of subgroups in $\mathcal{S}(K)$ have length l or less, then we get

$$\frac{|G|_n}{|W_G K|}\,\phi(G/K) = |G|_n e_K + \sum_{(A)\lneqq(K)} |G|_n m_A e_A$$

where $m_A = |(G/K)^A|/|W_G K|$ are integers (as explained earlier in the beginning of section 2). By the induction hypothesis we get that $|G|_n e_K$ is in $\phi A(G)$, and $|G|_n e_K = \sum c_A \phi(G/A)$ where the sum is over $(A) \le (K)$. □

We conclude that there is a stable map $\alpha_n \colon S_G \to S_G$ whose degree function $d(\alpha_n) \in C(G)$ is $\sum_{(A)\in\mathcal{A}_n} |G|_n e^G_A$. The degree of $(|G|_n - \alpha_n)^A$ is 0 for all $(A) \in \mathcal{A}_n$. Hence if E_G satisfies the n-induction property, then $(|G|_n - \alpha_n)E^*_G = 0$. So $|G|_n E^*_G$ is equal to $\alpha_n E^*_G$.

The element $[G/H]$ in $A(G)$ corresponds, via the isomorphism χ, to

$$S_G \xrightarrow{\tau} G/H_+ \xrightarrow{c} S_G$$

in $\pi^G_0(S_G)$. Here τ is the transfer map and c is the collapse map. Hence we have that

$$[G/H] = \mathrm{ind}^G_H[*].$$

The isomorphism class of a point $[*] \in A(H)$ corresponds to the identity map in $\pi^H_0(S^0_H)$.

The next result is the Artin induction theorem.

Theorem 5.3 (Artin induction theorem). *Assume that E_G is a ring spectrum satisfying the n-induction property. Then the integer $|G|_n$ times the unit element in E^*_G is in the image of the induction map*

$$\bigoplus_{(A)\in\mathcal{A}_n} \mathrm{ind}^G_A \colon \bigoplus_{(A)\in\mathcal{A}_n} E^0_A \to E^0_G$$

where the sum is over representatives for each conjugacy class $(A) \in \mathcal{A}_n$.

Proof. We have that $|G|_n 1 = \alpha_n 1$ in E^α_G. The theorem follows from the Frobenius reciprocity law 3.1 and Lemma 5.2. More precisely, let $f \colon S_G \to E_G$ represent an element in E^0_G. Then $[G/H]\cdot f = \mathrm{ind}^G_H[c \wedge f]$, where $c \wedge f \colon G/H_+ \to E_G$ represents an element in $E^0_H \cong E^0_G(G/H_+)$. □

As a consequence of the Artin induction theorem we can reconstruct $E_G(X)$, rationally, from all the $E_A(X)$ with $(A) \in \mathcal{A}_n$ and the restriction and conjugation maps. To do this we need the double coset formula for compact Lie groups. The double coset formula was first proved by M. Feshbach [Fes79]. We follow the presentation given in [LMS86, IV.6]. To state the double coset formula it is convenient to express the induction and restriction maps between E_H and E_K for subgroups H and K of G by maps in the G-stable homotopy category. Let $H \le K$ be subgroups of G. There is a collapse map

$$c^K_H \colon G/H_+ \to G/K_+$$

and a transfer map

$$\tau^K_H \colon G/K_+ \to G/H_+,$$

which induce restriction and induction maps [LMS86, p. 204]. Let g be an element in G. Right multiplication by g induces an equivalence of G-manifolds

$$\beta_g\colon G/H_+ \to G/(g^{-1}Hg)_+.$$

Consider G/H as a left K-space. The space is a compact differentiable K-manifold so it has finitely many orbit types [tD79, 5.9.1]. The orbit type of an element x is the K-isomorphism class of the homogeneous space Kx. The stabilizer of the element gH is $K \cap gHg^{-1}$. The left K-quotients of subspaces of G/H consisting of all points of a fixed orbit type are manifolds [Bre72, IV.3.3]. These manifolds are called the orbit type manifolds of $K\backslash G/H$. We decompose the double coset space $K\backslash G/H$ as a disjoint union of the connected components M_i of all the orbit type manifolds. We are now ready to state the double coset formula.

Theorem 5.4 (Double coset formula)**.** *Let G be a compact Lie group and H and K be closed subgroups of G. Then we have*

$$\tau_H^G \circ c_K^G \simeq \textstyle\sum_{M_i} z(M_i)\, \beta_g \circ c_{K\cap gHg^{-1}}^{gHg^{-1}} \circ \tau_{K\cap gHg^{-1}}^{K}$$

where the sum is over orbit-type manifold components M_i and $g \in G$ is a representative of each M_i. The integer $z(M_i)$ is the internal Euler characteristic. It is the Euler characteristic of the closure of M_i in $K\backslash G/H$ subtracting the Euler characteristic of its boundary.

Recall that the transfer map

$$\tau_H^K\colon G/K_+ \to G/H_+$$

is trivial if the Weyl group $W_K H$ is infinite. In particular, we have the following [Fes79, II.17], [LMS86, IV.6.7].

Lemma 5.5. *Assume $H = K$ is a maximal torus T in G. Then the sum in the double coset formula, Theorem* 5.4*, simplifies to the sum over elements $g \in G$ representing each gT in the Weyl group $W_G T$ of T.*

Let E_G be a G-equivariant ring spectrum that satisfy the n-induction property, and let M_G be a module over E_G. There is a restriction map

$$M_G^\alpha(X) \to \mathrm{Eq}[\textstyle\prod_A M_A^{\mathrm{res}\,\alpha}(X) \rightrightarrows \prod_{K,L,g} M_{K\cap gLg^{-1}}^{\mathrm{res}\,\alpha}(X)].$$

The first product in the equalizer is over representatives for conjugacy classes of $\mathcal{A}_n$, and the second product is over pairs K, L of these subgroup representatives and over representatives g for each of the orbit-type manifold components of $K\backslash G/L$. The maps in the equalizer are the two restriction (and conjugation) maps.

Definition 5.6. Let r be an integer. We say that a pair of maps

$$f\colon A \to B \quad\text{and}\quad g\colon B \to A$$

between abelian groups is an r-isomorphism pair if $f \circ g = r$ and $g \circ f = r$. A map is an r-isomorphism if it is a map belonging to an r-isomorphism pair.

The Artin induction theorem implies the following Artin restriction theorem.

Theorem 5.7. *Let E_G be a G-ring spectrum satisfying the n-induction property. Let M_G be an E_G-module spectrum. Then there exists a homomorphism*

$$\psi\colon \mathrm{Eq}[\textstyle\prod_A M_A^{\mathrm{res}\,\alpha}(X) \rightrightarrows \prod_{K,L,g} M_{K\cap gLg^{-1}}^{\mathrm{res}\,\alpha}(X)] \to M_G^\alpha(X)$$

such that the restriction map and ψ form a $|G|_n$-isomorphism pair. The first product in the equalizer is over representatives for conjugacy classes of $\mathcal{A}_n$, and the second product is over pairs K, L of these subgroup representatives and over representatives g for each of the orbit-type manifold components of $K\backslash G/L$.

An analogous result holds for homology.

Proof. The following argument is standard. Our proof is close to [McC86, 2.1]. We prove the result in the following generality. Consider an element $r \in E^0_G$ in the image of the induction maps from $E^0_{H_i}$ for a set of subgroups H_i of G. In our case $r = |G|_n$ and the subgroups are representatives for the conjugacy classes $\mathcal{A}_n$ by the Artin induction theorem 5.3.

Let $r = \sum_{i=1}^k \mathrm{ind}_{H_i}^G r_i$ where $r_i \in E_{H_i}$. Define ψ by setting

$$\psi(\textstyle\prod_{H_i} m_{H_i}) = \sum_{i=1}^k \mathrm{ind}_{H_i}^G (r_i\, m_{H_i}).$$

We have that

$$\psi \circ \mathrm{res}(m) = \textstyle\sum_{i=1}^k \mathrm{ind}_{H_i}^G (r_i\, \mathrm{res}_{H_i}^G m) = \sum_{i=1}^k \mathrm{ind}_{H_i}^G (r_i) m = rm.$$

The second equality follows from the Frobenius reciprocity law 3.1.

We now consider the projection of $\mathrm{res} \circ \psi(\prod m_{H_i})$ to M_K. It is

$$\textstyle\sum_i \mathrm{res}_K^G \mathrm{ind}_{H_i}^G (r_i m_{H_i}).$$

By the double coset formula, Theorem 5.4, this equals

$$\textstyle\sum_i \sum_{KgH_i} z_i\ \mathrm{ind}^K_{gH_ig^{-1}\cap K} \mathrm{res}^{gH_ig^{-1}}_{gH_ig^{-1}\cap K} \beta_g(r_i\, m_{H_i})$$

where for each i the sum is over representatives KgH_i of components of orbit-type manifolds of the double coset $K\backslash G/H_i$ and z_i is an integer. By our assumptions we have that

$$\mathrm{res}^{gH_ig^{-1}}_{gH_ig^{-1}\cap K}\ m_{gH_ig^{-1}} = \mathrm{res}^K_{gH_ig^{-1}\cap K}\ m_K.$$

So by Frobenius reciprocity we get

$$\textstyle\sum_i \left(\sum_{KgH_i} z_i\ \mathrm{ind}^K_{gH_ig^{-1}\cap K} \mathrm{res}^{gH_ig^{-1}}_{gH_ig^{-1}\cap K} \beta_g r_i\right) m_K.$$

This equals

$$\textstyle\sum_i (\mathrm{res}_K^G \mathrm{ind}_{H_i}^G (r_i)) m_K = \mathrm{res}_K^G(r) m_K. \qquad \square$$

For a fixed compact Lie group G both the restriction map and the map Ψ are natural in M_G and X.

6. Brauer induction

We present an integral induction theorem for cohomology theories satisfying the n-induction property. We first discuss some classes of subgroups.

Definition 6.1. A subgroup H of G is n-hyper if it has finite Weyl group and there is an extension

$$0 \to A \to H \to P \to 1$$

such that:

(1) P is a finite p-group for some prime number p,
(2) A is an abelian subgroup of G, such that $\omega(A)$ is topologically generated by n or fewer elements, and $|A/A^\circ|$ is relatively prime to p.

Lemma 6.2. *Let H be an n-hyper subgroup of G (for the prime p) and let K be a subgroup of H. Then K is an n-hyper subgroup of G (for the prime p) if K has finite Weyl group in G.*

Proof. Let H be as in Definition 6.1. Let Q be the image of K in P under the homomorphism $H \to P$. Then Q is a p-group. Let B be the kernel of the homomorphism $K \to Q$. Then B is a subgroup of A. The abelian group B is a direct sum of subgroups C and D, such that D is a finite p-group and $|C/C^\circ|$ is relatively prime to p (see Lemma 2.6). Lemma 2.7 implies that $\omega(C)$ is in $\mathcal{A}_n$ since C is a subgroup of A, which is in $\mathcal{A}_n$. The subgroup C is normal in K and the quotient group K/C is an extension of D by Q, hence a p-group. $\square$

We next describe the idempotent elements in the Burnside ring $A(G)$ localized at a rational prime. First we need some definitions. A group H is said to be p-perfect if it does not have a nontrivial (finite) quotient p-group. The maximal p-perfect subgroup H'_p of H is the preimage in H of the maximal p-perfect subgroup of the group of components H/H°. Let H be a subgroup of a fixed compact Lie group G. Let H_p denote the conjugacy class $\omega(H'_p)$ of subgroups of G with finite Weyl group associated with H'_p. Let $\Phi_p G$ denote the subspace of ΦG consisting of conjugacy classes of all p-perfect subgroups of G with finite Weyl group in G. Let $N_{(p)}$ denote the largest factor of an integer N that is relatively prime to p. The next result is proved for finite groups in [tD79, 7.8], and for compact Lie groups in [FO05, 3.4].

Theorem 6.3. *Let G be a compact Lie group. Let H be a p-perfect subgroup such that $(H) \in \Phi_p G$ is an open-closed point in $\Phi_p G$. Let N be an integer such that Ne_H is in $\phi A(G)$. Then there exists an idempotent element $I_{H,p} \in C(\Phi G, \mathbb{Z})$ such that $N_{(p)} I_{H,p} \in \phi A(G)$, and $I_{H,p}$ evaluated at (K) is 1 if $K_p = (H)$ and zero otherwise. In particular, $I_{H,p}$ is an idempotent element in the localized ring $\phi A(G)_{(p)}$.*

An abelian group A is p-perfect if and only if $|A/A^\circ|$ is relatively prime to p by Lemma 2.6. We can apply Theorem 6.3 to p-perfect abelian groups in $\mathcal{A}_n$ with $N = |G|_n$ by Lemma 2.1 and Proposition 5.2. Let $I_{(p,n)}$ be $(|G|_n)_{(p)} \sum_{(A)} I_{A,p}$ where the sum is over all $(A) \in \mathcal{A}_n$ such that $|A/A^\circ|$ is relatively prime to p. The element $I_{(p,n)} \in C(G)$ is in $\phi A(G)$. The function $I_{(p,n)}$ has the value $(|G|_n)_{(p)}$ at each conjugacy class (H) of the form

$$0 \to S \to H \to P \to 1$$

where P is a p-group and S is abelian with $|S/S^\circ|$ relatively prime to p and $\omega(S) \in \mathcal{A}_n$, and $I_{(p,n)}$ has the value 0 at all other elements of ΦG. In particular, $I_{(p,n)}$ has the value $(|G|_n)_{(p)}$ at each $A \in \mathcal{A}_n$ by Lemma 2.6. The greatest common divisor of $\{(|G|_n)_{(p)}\}$, where p runs over primes p dividing $|G|_n$, is 1. Hence there is a set of integers z_p such that $\sum_{p|\,|G|} z_p |G|_{(p)} = 1$. Let I_n be the element $\sum_p z_p I_{(p,n)}$ where the sum is over primes p dividing $|G|$. The element I_n is not an idempotent element in $C(G)$. The function $I_n \colon \Phi G \to \mathbb{Z}$ has the value 1 for all $(A) \in \mathcal{A}_n$. Since I_n is in the image of $\phi\colon A(G) \to C(G)$, there is a map $\beta_n \colon S_G \to S_G$ so that the degree $d(\beta_n)$ is I_n. The degree of $(1-\beta_n)^A$ is zero for all $(A) \in \mathcal{A}_n$. Assume E_G is a G-equivariant ring spectrum satisfying the n-induction property. We get that

$$E^*_G(S_G) = \beta_n \cdot E^*_G(S_G).$$

Lemma 6.4. *The element $I_n \in A(G)$ can be written as*

$$I_n = \phi(\textstyle\sum_{H_i} k_{H_i} [G/H_i])$$

where the subgroups H_i are n-hyper subgroups of G for primes dividing $|G|_n$, and k_{H_i} are integers.

Proof. We know that I_n can be written in the above form for some subgroups H_i of G with finite Weyl groups. Let H_j be a maximal subgroup of G in the sum describing I_n. The value of I_n at (H_j) is $k_{H_j}|W_G H_j|$. This value is nonzero; hence the maximal subgroups H_j are n-hyper subgroups of G. By Lemma 6.2 all H_i are n-hyper subgroups of G. $\square$

The next result is the Brauer induction theorem.

Theorem 6.5 (Brauer induction theorem)**.** *Assume E_G is a ring spectrum satisfying the n-induction property. Then the unit element 1 in E^*_G is in the image of the induction map*

$$\bigoplus_{(H)} \mathrm{ind}_H^G(X)\colon \bigoplus_{(H)} E_H^0 \to E_G^0$$

where the sum is over n-hyper subgroups of G for primes p dividing $|G|_n$.

Proof. We get as in Theorem 5.3 that the unit element 1 is in the image of the induction maps from all the subgroups H_i in Lemma 6.4. $\square$

As a consequence we get the following Brauer restriction theorem.

Theorem 6.6 (Brauer restriction theorem)**.** *Let E_G be a G-equivariant ring spectrum satisfying the n-induction property, and let M_G be a module over E_G. Then the restriction map*

$$M_G^\alpha(X) \to \mathrm{Eq}[\textstyle\prod_H M_H^{\mathrm{res}\,\alpha}(X) \rightrightarrows \prod_{K,L,g} M_{K\cap gLg^{-1}}^{\mathrm{res}\,\alpha}(X)]$$

is an isomorphism. The first product in the equalizer is over representatives for conjugacy classes of n-hyper subgroups of G for primes dividing $|G|_n$, and the second product is over pairs K, L of these subgroup representatives and over representatives g for each of the orbit-type manifold components of $K\backslash G/L$.

Proof. This follows from the proof of Theorem 5.7 and the Brauer induction theorem 6.5. $\square$

An analogous result holds for homology.

7. Induction theory for Borel cohomology

Let k be a nonequivariant spectrum. The Borel cohomology and Borel homology on the category of based G-spaces are

$$k^*(X \wedge_G EG_+) \text{ and } k_*(\Sigma^{\mathrm{Ad}(G)} X \wedge_G EG_+).$$

The adjoint representation $\mathrm{Ad}(G)$ of G is the tangent vector space at the unit element of G with G-action induced by the conjugation action by G on itself. If G is a finite group, then $\mathrm{Ad}(G) = 0$. Borel homology and cohomology can be extended to an $RO(G)$-graded cohomology theory defined on the stable equivariant homotopy category. We follow Greenlees and May [GM95].

Let M_G be a G-spectrum. The geometric completion of M_G is defined to be

$$c(M_G) = F(EG_+, M_G),$$

where F denotes the internal hom functor in the G-equivariant stable homotopy category. Let $f(M_G)$ be $M_G \wedge EG_+$. The Tate spectrum of M_G is defined to be

$$t(M_G) = F(EG_+, M_G) \wedge \tilde{E}G.$$

The space $\tilde{E}G$ is the cofiber of the collapse map $EG_+ \to S^0$.

If E_G is a ring spectrum, then $c(E_G)$ and $t(E_G)$ are also ring spectra [GM95, 3.5]. More precisely, $c(E_G)$ is an algebra over E_G, and $t(E_G)$ is an algebra over $c(E_G)$. The product on the spectrum $f(E_G)$ is not unital in general; however, it is a $c(E_G)$ module spectrum. Hence we have the following.

Proposition 7.1. *Let E_G be a G-ring spectrum satisfying the n-induction property. Then $c(E_G)$ and $t(E_G)$ are G-equivariant ring spectra satisfying the n-induction property. The spectrum $f(E_G)$ is a $c(E_G)$-module spectrum.*

Let $i\colon \mathcal{U}^G \to \mathcal{U}$ be the inclusion of the universe $\mathcal{U}^G$ into a complete G-universe $\mathcal{U}$. Let i_*k denote the G-spectrum obtained by building in suspensions by G-representations. If k is a ring spectrum, then i_*k is a G-equivariant ring spectrum. The following result is proved in [GM95, 2.1,3.7].

Proposition 7.2. *Let k be a spectrum. Let X be a naive G-spectrum (indexed on $\mathcal{U}^G$). Then we have isomorphisms*

$$(c(i_*k))^n_G(X) \cong k^n(X \wedge_G EG_+) \text{ and } (f(i_*k))^G_n(X) \cong k_n(\Sigma^{\mathrm{Ad}(G)} X \wedge_G EG_+)$$

where $\mathrm{Ad}(G)$ is the adjoint representation of G.

We next show that $c(k)$ has the induction property (see Definition 5.1) when k is a complex oriented spectrum. For every compact Lie group G, there is a finite dimensional unitary faithful G-representation V. Hence G is a subgroup of the unitary group $U(V)$. Let the flag manifold F associated with the G-representation V be $U(V)/T$ where T is some fixed maximal torus of $U(V)$. Let G act on F via the embedding $G \leq U(V)$. The following result is well known [HKR00, 2.6].

Proposition 7.3. *Let k be a complex oriented spectrum. Then the map*

$$k^*(BG_+) \to k^*(F_+ \wedge_G EG_+),$$

induced by the collapse map $F \to *$, is injective.

Theorem 7.4. *Given a compact Lie group G let N be an integer so that $\mathcal{A}_N = \mathcal{A}$. Let k be a complex oriented spectrum. Then there exists an integer d so that the Borel cohomology G-spectrum $c(i_*k)$ satisfies the following: $(|G|_N - \alpha_N)^d c(i_*k)_* = 0$ and $(1-\beta_N)^d c(i_*k)_* = 0$.*

The class α_N is defined after Proposition 5.2 and the class β_N is defined before Lemma 6.4.

Proof. An easy induction gives the following: Let Y be a d-dimensional G-CW complex, and let E_G be a cohomology theory. Assume that an element $r \in A(G)$ kills the E^*_G-cohomology of all the cells in Y. Then r^d kills $E^*_G(Y)$.

By Proposition 7.3 it suffices to show that $r^d(i_*k)^*(U/T) = 0$ when r is $|G|_N - \alpha_N$ or $1 - \beta_N$. We have that U/T is a finite G-CW complex with orbit types $G/(G \cap gTg^{-1})$ for $g \in G$. Since both $|G|_N - \alpha_N$ and $1 - \beta_N$ restricted to any abelian group are 0 by Corollary 4.4, we get that they kill the $c(i_*k)$-cohomology of all the cells in $Y = U/T$. We can take d to be the dimension of the G-CW complex U/T. $\square$

The Propositions 7.3 and 7.4 give Artin and Brauer restriction theorems for $c(k)$ and $t(k)$ where k is a complex oriented spectrum. We state the theorems only for $c(k)$ applied to naive G-spectra.

Theorem 7.5. *Let k be a complex oriented cohomology theory. Then for any naive G-spectrum X the restriction map from $k^*(X \wedge_G EG_+)$ to the equalizer of*

$$\prod_{(A)\in\mathcal{A}} k^*(X \wedge_A EG_+) \rightrightarrows \prod_{(K),(L)\in\mathcal{A} \text{ and } KgL\in K\backslash G/L} k^*(X \wedge_{K\cap gLg^{-1}} EG_+)$$

is a natural isomorphism after inverting $|G|$. The first product is over representatives for conjugacy classes of $\mathcal{A}$, and the second product is over pairs K, L of these subgroup representatives and over representatives g for each of the orbit-type manifold components of $K\backslash G/L$.

Theorem 7.6. *Let k be a complex oriented cohomology theory. Then for any naive G-spectrum X the restriction map from $k^*(X \wedge_G EG_+)$ to the equalizer of*

$$\prod_H k^*(X \wedge_H EG_+) \rightrightarrows \prod_{K,L,g} k^*(X \wedge_{K\cap gLg^{-1}} EG_+)$$

is a natural isomorphism. The first product is over representatives for conjugacy classes of hyper subgroups of G for primes dividing $|G|$, and the second product is over pairs K, L of these subgroup representatives and over representatives g for each of the orbit-type manifold components of $K\backslash G/L$.

Remark 7.7. See also [Fes81] and [LMS86, IV.6.10].

Since $f(i_*k)$ is a $c(i_*k)$ module spectrum we get Artin and Brauer restriction theorems for the Borel homology $k_*(EG \wedge_G \Sigma^{\mathrm{Ad}(G)} X)$ when k is complex oriented.

Theorem 7.8. *Let k be a complex oriented cohomology theory. Then for any naive G-spectrum X the restriction map from $k_*(\Sigma^{\mathrm{Ad}(G)} X \wedge_G EG_+)$ to the equalizer of*

$$\prod_A k_*(\Sigma^{\mathrm{Ad}(A)} X \wedge_A EG_+) \rightrightarrows \prod_{K,L,g} k_*(\Sigma^{\mathrm{Ad}(K\cap gLg^{-1})} X \wedge_{K\cap gLg^{-1}} EG_+)$$

is a natural isomorphism after inverting $|G|$. The first product in the equalizer is over representatives for conjugacy classes of $\mathcal{A}$, and the second product is over pairs K, L of these subgroup representatives and over representatives g for each of the orbit-type manifold components of $K\backslash G/L$.

Theorem 7.9. *Let k be a complex oriented cohomology theory. Then for any naive G-spectrum X the restriction map from $k_*(\Sigma^{\mathrm{Ad}(G)} X \wedge_G EG_+)$ to the equalizer of*

$$\prod_H k_*(\Sigma^{\mathrm{Ad}(H)} X \wedge_H EG_+) \rightrightarrows \prod_{K,L,g} k_*(\Sigma^{\mathrm{Ad}(K\cap gLg^{-1})} X \wedge_{K\cap gLg^{-1}} EG_+)$$

is a natural isomorphism. The first product is over representatives for conjugacy classes of hyper subgroups of G for primes dividing $|G|$, and the second product is over pairs K, L of these subgroup representatives and over representatives g for each of the orbit-type manifold components of $K\backslash G/L$.

It is immediate from the definition that the induction and restriction maps in Borel cohomology are given by the transfer and the collapse maps

$$\tau\colon S_G \to G/H_+ \quad \text{and} \quad c\colon G/H_+ \to S_G$$

as follows:

$$k^*(X \wedge_H EG_+) \cong k^*((G/H_+ \wedge X) \wedge_G EG_+) \xrightarrow{\tau^*} k^*(X \wedge_G EG_+)$$

and

$$k^*(X \wedge_G EG_+) \xrightarrow{c_*} k^*((G/H_+ \wedge X) \wedge_G EG_+) \cong k^*(X \wedge_H EG_+).$$

When G is a finite group the induction map in Borel homology is induced from the collapse map c and the restriction map is induced from the transfer map τ. This is more complicated for compact Lie groups. The Spanier-Whitehead dual $S^{-V}\mathrm{Th}(\nu G/H)$ of G/H_+ (in the statement of Lemma 3.2) is equivalent to $G \ltimes_H S^{-L(H)}$, where $\ltimes$ is the half smash product [May96, XVI.4] and $L(H)$ is the H-representation on the tangent space at eH in G/H induced by the action $h, gH \mapsto hgH$. By considering

$$H \to G \to G/H,$$

we see that $\mathrm{Ad}(G)$ is isomorphic to $\mathrm{Ad}(H) \oplus L(H)$ as H-representations. By properties of the half smash product we have that

$$\left(G \ltimes_H S^{-L(H)}\right) \wedge S^{\mathrm{Ad}(G)} \cong G \ltimes_H S^{\mathrm{Ad}(G)-L(H)} \cong G \ltimes_H S^{\mathrm{Ad}(H)}$$

and

$$(G \ltimes_H S^{\mathrm{Ad}(H)} \wedge X) \wedge_G EG_+ \cong (S^{\mathrm{Ad}(H)} \wedge X) \wedge_H EG_+.$$

Combined with Lemma 3.2 this give a description of induction and restriction maps.

Let G be a finite group. We state some results from [HKR00] to show that under some hypotheses a local complex oriented cohomology theory satisfies n-induction. Let k^* be a local and complete graded ring with residue field of characteristic $p > 0$. Assume that

$$k^0(BG) \to p^{-1}k^0(BG)$$

is injective and that the formal group law of k^* modulo the maximal ideal has height n.

In [HKR00] the authors show that the restriction maps from

$$p^{-1}k^*(BG)$$

into the product of all $p^{-1}k^*(BA)$ for all p-groups $(A) \in \mathcal{A}_n$ are injective.

Hence if β is an element in the Burnside ring $A(G)$ of G such that $\deg(\beta^A) = 0$ for all A in $\mathcal{A}_n$, then we have that

$$\beta k^*(BG) = 0.$$

So the Borel cohomology $k^*(X \wedge_G EG_+)$ satisfies the n-induction property. The authors also show that $k^*(X \wedge_G EG_+)$ does not satisfy the $(n-1)$-induction property.

8. 0-induction, singular Borel cohomology

We consider induction theorems for ordinary singular Borel cohomology. Let M be a G/G°-module. The Borel cohomology of an unbased G-space X with coefficients in M is the singular cohomology of the Borel construction of X,

$$H^*(X \times_G EG; M),$$

with local coefficients via

$$\Pi(X \times_G EG) \to \Pi(* \times_G EG) \to \pi_1(* \times_G EG) \cong G/G^\circ$$

where Π is the fundamental groupoid. Since BG is path connected the fundamental groupoid of BG is noncanonically isomorphic to the one-object category $\pi_1(BG)$, which is the group of components of G. Borel cohomology is an equivariant cohomology theory.

Lemma 8.1. *The Borel cohomology of an unbased G-space X with coefficients in a G/G°-module M is represented in the stable G-equivariant homotopy category by the geometric completion of an Eilenberg-Mac Lane spectrum $H\tilde{M}$, where $\tilde{M}$ is a Mackey functor so that $\tilde{M}(G/1)$ is isomorphic to M as a G-module.*

Pre-composing with the G-equivariant suspension spectrum functor is implicit in the statement that the Borel cohomology is represented.

Proof. The Borel cohomology is isomorphic to the cohomology of the cochain complex of G-homomorphisms from the G-cellular complex of $X \times EG$ to the G-module M [Hat02, 3.H]. Assume that $\tilde{M}$ is a Mackey functor so that $\tilde{M}(G/1)$ is isomorphic to M as a G-module. By the cell complex description of Bredon cohomology we get that the Borel cohomology group is isomorphic to the G-Bredon cohomology of $X \times EG$ with coefficients in $\tilde{M}$. It follows from [GM95, 6.1] that there exist Mackey functors of the requested form. □

The relation between the geometric completion of Eilenberg-Mac Lane spectra and classical Borel cohomology theory is also treated in [GM95, §6,§7]. From now on we consider the Borel cohomology defined on G-spectra.

The restriction map on the zeroth coefficient groups of the geometric completion of HM is described by the following commutative diagram:

$$\begin{tikzcd}
H^0(BG_+;M) \arrow[r, "\cong"] \arrow[d, "\mathrm{res}"'] & M^G \arrow[d] \\
H^0(B*_+;M) \arrow[r, "\cong"] & M.
\end{tikzcd}$$

We have that $\mathrm{res}(\beta)\mathrm{res}(m) = \deg(\beta)\mathrm{res}(m)$ for any $\beta \in A(G)$ and $m \in M^G$. Since the restriction map is injective we get that $\beta m = \deg(\beta)m$. So if $\deg(\beta) = 0$, then $\beta H^0(BG_+;M) = 0$. We have that $\deg(\beta) = \deg(\beta^T)$ for any torus T in G by Proposition 4.3. So singular Borel cohomology satisfies the 0-induction property. The argument above applies more generally to show that Bredon homology and cohomology with Mackey functor coefficients M such that $M(G/G) \to M(G/e)$ is injective (or, alternatively, $M(G/e) \to M(G/G)$ is surjective) satisfies 0-induction.

The Artin restriction theorem gives a refinement of Borel's description of rational Borel cohomology. Recall Lemma 5.5. See also [Fes81, II.3].

Theorem 8.2. *Let G be a compact Lie group, X a G-spectrum, and M a G/G°-module. Then the restriction map*

$$H^*(X \wedge_G EG_+;M) \to H^*(X \wedge_T EG_+;M)^{W_G T}$$

is a $|W_G T|$-isomorphism.

In particular, with $X = S^0$ we get that

$$H^*(BG_+;M) \to H^*(BT_+;M)^{W_G T}$$

is a $|W_G T|$-isomorphism. When G is a finite group this says that $H^0(BG_+;M) \to M^G$ is a $|G|$-isomorphism and $|G|\, H^k(BG_+;M) = 0$, for $k > 0$.

We next give the Brauer restriction theorem.

Theorem 8.3. *Let G be a compact Lie group, X a G-spectrum and M a G/G°-module. Fix a maximal torus T in G. Then the restriction map*

$$H^*(X \wedge_G EG_+; M) \to \lim_K H^*(X \wedge_K EG_+; M)$$

is an isomorphism. The limit is over all subgroups K of G with finite Weyl group that have a normal abelian subgroup A of K such that $A \le T$ and K/A is a p-group for some prime p dividing $|W_G T|$. The maps in the limit are restriction maps and conjugation maps.

We use singular homology with local coefficients.

Theorem 8.4. *Let G be a compact Lie group, X a G-spectrum, and M a G/G°-module. Then the restriction map (induced by t)*

$$H_*(S^{\mathrm{Ad}(G)} X \wedge_G EG_+; M) \to H_*(S^{\mathrm{Ad}(T)} X \wedge_T EG_+; M)^{W_G T}$$

is a $|W_G T|$-isomorphism.

There is also a Brauer restriction theorem for homology.

Theorem 8.5. *Let G be a compact Lie group, X a G-spectrum and M a G/G°-module. Fix a maximal torus T in G. Then the restriction map*

$$H_*(S^{\mathrm{Ad}(G)} X \wedge_G EG_+; M) \to \lim_K H_*(S^{\mathrm{Ad}(K)} X \wedge_K EG_+; M)$$

is an isomorphism. The limit is over all the subgroups K of G with finite Weyl group that have a normal abelian subgroup A of K such that $A \le T$ and K/A is a p-group for some prime p dividing $|W_G T|$. The maps in the limit are restriction and conjugation maps.

9. 1-induction, K-theory

In this section we consider equivariant K-theory K_G. For details on K_G, see [Seg68b]. An element $g \in G$ is said to be regular if the closure of the cyclic subgroup generated by g has finite Weyl group in G.

Definition 9.1. Let ρG denote the space of conjugacy classes of regular elements in G. Define $r\colon \rho G \to \Phi G$ by sending a regular element g in G to the closure of the cyclic subgroup generated by g.

The space ρG is given the quotient topology from the subspace of regular elements in G. The map r is continuous since two regular elements in the same component of G generate conjugate cyclic subgroups [Seg68a, 1.3]. Let $C(X, R)$ denote the ring of continuous functions from a space X into a topological ring R. The map r, together with the inclusion of $\mathbb{Z}$ in $\mathbb{C}$, induces a ring homomorphism

$$r^*\colon C(\Phi G, \mathbb{Z}) \to C(\rho G, \mathbb{C}).$$

The ring of class functions on G is a subring of $C(\rho G, \mathbb{C})$ since the regular elements of G are dense in G. Let $R(G)$ denote the (complex) representation ring of G. Let $\chi\colon R(G) \to C(\rho G, \mathbb{C})$ be the character map. The map χ is an injective ring map. Let V be a G-representation. The value of $\chi(V)$ at (a regular element) $g \in G$ is the trace of $g\colon V \to V$. This only depends on the isomorphism class of V.

We now give a description of the induction map $\mathrm{ind}_H^G\colon R(H) \to R(G)$ for the coefficient ring of equivariant K-theory [Oli98, Seg68a]. Let ξ be an H-character. On a regular element g in G the induction map is given by

$$\mathrm{ind}_H^G \xi(g) = \textstyle\sum_{kH} \xi(k^{-1}gk),$$

where the sum is over the finite fixed set $(G/H)^g$. If $x \in R(H)$, then $\mathrm{ind}_H^G \xi(x)$ is in the image of $\xi\colon R(G) \to C(\rho G, \mathbb{C})$ [Oli98, 2.5]. So we get a well defined map $\mathrm{ind}_H^G\colon R(H) \to R(G)$. This agrees with the induction map defined in section 3 by [Nis78, 5.2] and [Seg68a, §2].

The unit map in G-equivariant K-theory induces a map $\sigma\colon A(G) \to R(G)$. It is the generalized permutation representation map

$$\sigma([X]) = \textstyle\sum_i (-1)^i [H^i(X;\mathbb{C})],$$

where $[H^i(X;\mathbb{C})]$ is the isomorphism class of the G-representation $H^i(X;\mathbb{C})$ [tD75, 7]. As pointed out in [tD75, 7], see also [tD79, 5.3.11], the following diagram commutes:

$$\begin{tikzcd} A(G) \arrow[r, "\phi"] \arrow[d, "\sigma"] & C(\Phi G, \mathbb{Z}) \arrow[d, "r^*"] \\ R(G) \arrow[r, "\chi"] & C(\rho G, \mathbb{C}). \end{tikzcd}$$

Lemma 9.2. *Equivariant K-theory K_G satisfies the 1-induction property.*

Proof. Since χ is injective it suffices to note that the map $r\colon \rho G \to \Phi G$ factors through $\mathcal{A}_1$. □

The Artin restriction theorem for equivariant K-theory follows from Theorem 5.7.

Theorem 9.3. *For every G-spectrum X the restriction map*

$$K_G(X) \to \mathrm{Eq}[\textstyle\prod_A K_A(X) \rightrightarrows \prod_{H,L,HgL} K_{H\cap gLg^{-1}}(X)]$$

is a $|G|_1$-isomorphism. The first product in the equalizer is over representatives for conjugacy classes of topologically cyclic subgroups of G, and the second product is over pairs of these subgroups and elements $HgL \in H\backslash G/L$.

It suffices to pick a representative for each path component of the H-orbit space of each submanifold of G/L consisting of points with a fixed orbit type under the H-action.

When G is connected, the maximal torus T is the only conjugacy class of subgroups of G with a dense subgroup generated by one element and with finite Weyl group by Example 2.5. So when G is connected the restriction map

$$K_G(X) \to K_T(X)^{W_G T}$$

is a $|W_G T|$-isomorphism.

We give an explicit description of $R(G)$ up to $|G|_1$-isomorphism using the Artin restriction theorem. Let $A^* = \hom(A, S^1)$ denote the Pontrjagin dual of A. The elements of A^* are the one dimensional unitary representations of A. All irreducible complex representations of a compact abelian Lie group are one dimensional. We verify that when A is a compact abelian Lie group the canonical map

$$\mathbb{Z}[A^*] \to R(A)$$

is an isomorphism. A subgroup inclusion $f\colon H \to L$ induces a restriction map $f^*\colon L^* \to H^*$ of representations. By the Artin restriction theorem there is an (injective) $|W_G T|$-isomorphism

$$R(G) \to \mathrm{Eq}[\textstyle\prod_A \mathbb{Z}[A^*] \rightrightarrows \prod_{H,L,HgL} \mathbb{Z}[(H \cap gLg^{-1})^*]\,],$$

where A, H, and L are representatives for conjugacy classes of topologically cyclic subgroups of G, and HgL are representatives for each of the orbit-type manifold components of the coset $H\backslash G/L$.

The Brauer induction theorem 6.5 and Lemma 9.2 applied to K_G give that the identity element in $R(G)$ can be induced up from the 1-hyper subgroups of G. This was first proved by G. Segal [Seg68a, 3.11]. We get the following Brauer restriction theorem from Theorem 6.6 and Lemma 9.2.

Theorem 9.4. *For every G-spectrum X the restriction map*

$$K_G(X) \to \mathrm{Eq}[\textstyle\prod_H K_H(X) \rightrightarrows \prod_{H,L,g} K_{H\cap gLg^{-1}}(X)]$$

is an isomorphism. The first product in the equalizer is over representatives for conjugacy classes of 1*-hyper subgroups of* G*, and the second product is over pairs of these subgroups and representatives* g *for each of the orbit-type manifold components of the coset* $H\backslash G/L$.

There is another equivariant orthogonal K-theory KO_G obtained by using real instead of complex G-bundles. It is a ring spectrum and the unit map of K_G factors as

$$S_G \to KO_G \to K_G,$$

where the last map is induced by tensoring the real G-bundles by $\mathbb{C}$. The coefficient ring of KO_G is $RO(G)$ in degree 0. Since $RO(G)$ injects into $R(G)$ we get that KO_G satisfies the 1-induction property. Another example of a ring spectrum that satisfies 1-induction is Greenlees' equivariant connective K-theory [Gre04].

The restriction map $R(G) \to \prod R(C)$ is injective for all compact Lie groups G, where the product is over topological cyclic subgroups of G. When G is connected we even have that $R(G) \to R(T)^{W_G T}$ is an isomorphism. Sometimes this can be used to prove stronger induction and detection type results about K_G than the results we get from our theory [Jac77, McC86]. See also Proposition 3.3 in [AS69]. An F-isomorphism theorem for equivariant K-theory has been proved by Bojanowska [Boj83].

Remark 9.5. It is reasonable to ask if tom Dieck's equivariant complex cobordism spectrum satisfies the induction property [tD70]. Brun has shown that this is not the case. In fact the unit map $A(G) \to MU_G^0$ is injective when G is a finite group [Bru04].

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