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ABELIAN STRICT APPROXIMATION IN AW^*-ALGEBRAS AND WEYL-VON NEUMANN TYPE THEOREMS

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Dedicated to Professor E. Effros on his 70^{th} *birthday*

Abstract. In this paper, for a C^*-algebra A with $M = M(A)$ an AW^*-algebra, or equivalently, for an essential, norm-closed, two-sided ideal A of an AW^*-algebra M, we investigate the strict approximability of the elements of M from commutative C^*-subalgebras of A. In the relevant case of the norm-closed linear span A of all finite projections in a semi-finite AW^*-algebra M we shall give a complete description of the strict closure in M of any maximal abelian self-adjoint subalgebra (masa) of A. We shall see that the situation is completely different for discrete, respectively continuous, M:

In the discrete case, for any masa C of A, the strict closure of C is equal to the relative commutant $C' \cap M$, while in the continuous case, under certain conditions concerning the center valued quasitrace of the finite reduced algebras of M (satisfied by all von Neumann algebras), C is already strictly closed. Thus in the continuous case no elements of M which are not already belonging to A can be strictly approximated from commutative C^*-subalgebras of A.

In spite of this pathology of the strict topology in the case of the norm-closed linear span of all finite projections of a continuous semi-finite AW^*-algebra, we shall prove that in general situations also including this case, any normal $y \in M$ is equal modulo A to some $x \in M$ which belongs to an order theoretical closure of an appropriate commutative C^*-subalgebra of A. In other words, if we replace the strict topology with order theoretical approximation, Weyl-von Neumann-Berg-Sikonia type theorems will hold in substantially greater generality.

Introduction

Let A be a C^*-algebra. The *multiplier algebra* of A is the C^*-subalgebra

$$\{x \in A^{**};\ xa,\ ax \in A \text{ for all } a \in A\}$$

of the second dual A^{**} (see [25], Section 3.12, or [30], Chapter 2). A natural locally convex vector space topology on $M(A)$, called the *strict topology* β, is defined by the seminorms

$$x \mapsto \|xa\| \text{ and } x \mapsto \|ax\|, \qquad a \in A.$$

It is complete and compatible with the duality between $M(A)$ and A^*. Hence the strict topology is weaker than the norm-topology on $M(A)$, but stronger than the

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restriction to $M(A)$ of the weak * topology of A^{**}. In particular, A is strictly dense in $M(A)$.

We notice that for A the C^*-algebra $K(H)$ of all compact linear operators on a complex Hilbert space H, $M(A)$ can be identified with the C^*-algebra $B(H)$ of all bounded linear operators on H, and on every bounded subset of $B(H)$ the strict topology coincides with the s^*-topology.

More generally, if M is an AW^*-algebra (see [17], or [6], §4, or [28], §9) and A is an essential, norm-closed, two-sided ideal of M, then, by a theorem of B. E. Johnson, M can be identified with $M(A)$ (see [13] or [26]). Thus the pairs $\big(A, M(A)\big)$, where A is a C^*-algebra such that $M(A)$ is an AW^*-algebra, are exactly the pairs $\big(A, M\big)$, where M is an AW^*-algebra and A is an essential, norm-closed, two-sided ideal of M.

A relevant case of an essential, norm-closed, two-sided ideal of an AW^*-algebra is the norm-closed linear subspace A generated by all finite projections of a semi-finite AW^*-algebra M. Then there are central projections p_1, p_2, p_3 of M with $p_1 + p_2 + p_3 = 1_M$ such that Mp_1 is finite, Mp_2 is properly infinite and discrete, while Mp_3 is properly infinite and continuous (see [6], §15, Theorem 1). Since $Ap_1 = Mp_1$, the non-trivial cases are Ap_2 and Ap_3, with $M(Ap_2) = Mp_2$ properly infinite and discrete and $M(Ap_3) = Mp_3$ properly infinite and continuous.

Throughout the C^*-algebra theory the possibility to reduce certain verifications to the case of commutative C^*-algebras is an important issue. Concerning the strict approximation of the multipliers of a C^*-algebra A with elements of A, such a reduction would be clearly possible if every normal $x \in M(A)$ would belong to the strict closure of some commutative C^*-subalgebra of A, hence to the strict closure of some maximal abelian self-adjoint subalgebra (*masa*) C_x of A, in which case we say that x belongs to the *abelian strict closure* of A. Unfortunately, this is not true even in the case of $A = K(H)$ with infinite-dimensional H.

However, by the classical Weyl-von Neumann-Berg-Sikonia (WNBS) Theorem, if $A = K(H)$, where H is a separable complex Hilbert space, then every normal element of $M(A) = B(H)$ is of the form $a + x$ with $a \in A$ and x in the strict closure of some masa of A. Extensions of this result to general C^*-algebras A, which are σ-unital (that is, the unit of $M(A)$ is the strict limit of a sequence in A, which of course can be chosen to belong to a commutative C^*-subalgebra of A) and with $M(A)$ of real rank zero (see [9]), were obtained in [23] and [31] (see also [12], [19], [20], [21]).

It is natural to ask to which extent generalizations of the WNBS Theorem hold if we renounce to one of the two assumptions above. For general σ-unital C^*-algebras the only result we know is contained in our previous paper [10]. There we proved a partial extension of the WNBS Theorem for an arbitrary σ-unital C^*-algebra A (see [10], Theorem 1), which implies that each element $y \in M(A)$ is of the form $a + x_1 + x_2$, where $a \in A, x_1 \in B_1, x_2 \in B_2$ with B_1, B_2 separable C^*-subalgebras of $M(A)$ such that every normal element of B_j, $j = 1, 2$, belongs to the abelian strict closure of A. Moreover, if y is self-adjoint, then x_1, x_2 can be chosen self-adjoint, so in this situation x_1, x_2 themselves belong to the abelian strict closure of A. Though unsatisfactory, this result still allows the reduction of the proof of an important result of L. G. Brown (concerning the non-existence of non-zero separable hereditary C^*-subalgebras of the corona algebra of a σ-unital C^*-algebra; see [8]) to the commutative case.

In the present paper we discuss abelian strict approximability for a C^*-algebra A which is the norm-closed linear subspace generated by all finite projections of some semi-finite AW^*-algebra M. We recall that any AW^*-algebra is of real rank zero, so in this case $M(A) = M$ is of real rank zero. Since the abelian strict closure of A is the union of all $\overline{C}^{\beta}$ with C a masa of A, we are interested in describing $\overline{C}^{\beta}$ for any masa C of A. We shall see that the situation is completely different for discrete, respectively continuous, M:

In the discrete case $\overline{C}^{\beta}$ is equal to the relative commutant $C' \cap M(A)$ (Theorem 1.2), while in the continuous case, under a certain condition on the centre valued quasitrace of the reduced AW^*-subalgebras of $M(A)$ by finite projections (always satisfied if $M(A)$ is a von Neumann algebra), C is already strictly closed (Theorem 2.6).

Consequently, if M is a properly infinite, continuous, semi-finite ($= \mathrm{II}_\infty$) AW^*-algebra satisfying the above mentioned condition and A is the norm-closed linear span of all finite projections of M, then the unit of $M(A) = M$ does not belong to the abelian strict closure of A, that is, there is no approximate unit for A contained in a commutative $*$-subalgebra of A. In particular, in this case A is not σ-unital. We notice that it was already shown in [1], Proposition 4.5, that the norm-closed linear span of all finite projections of a type II_∞ factor is a non-σ-unital C^*-algebra. Nevertheless, also in this case WNBS type theorems can be proved. Indeed, if A is the norm-closed linear subspace generated by all finite projections of some countably decomposable semi-finite W^*-factor M, then, according to [32], Theorem 3.1, every normal $y \in M(A) = M$ is of the form $a + x$ with $a \in A$ and x in the s^*-closure in M of some masa C of A. Since the s^*-closure of a commutative $*$-subalgebra of a W^*-algebra is equal to its monotone order closure (cf. [14] and [24]), it is natural to expect that for extensions of the WNBS Theorem to non-σ-unital C^*-algebras the strict closure should be replaced by an order theoretical closure. Along this line we prove several WNBS type theorems in a general setting which includes the case of the norm-closed linear span of all finite projections of a countably decomposable semi-finite AW^*-algebra.

More precisely, we prove that if $\mathcal{J}$ is a norm-closed two-sided ideal of a (unital) Rickart C^*-algebra M (a Rickart C^*-algebra is a C^*-algebra in which every positive element has a support projection, in particular it is of real rank zero), which has a countable "order theoretical approximate unit", then any normal $y \in M$ is of the form $y = a + x$, where $a \in A$ is of arbitrarily small norm and x belongs to the order theoretical closure of some masa of $\mathcal{J}$ (Theorem 3.2 and the subsequent remark). Moreover, the above x can be chosen as a particular infinite linear combination of a sequence of mutually orthogonal projections from $\mathcal{J}$ (Theorems 3.5 and 3.6).

Since only little of the specific properties of Rickart C^*-algebras is used, we are left with the question as to which extent the above mentioned WNBS type theorems hold if M is assumed to be only a C^*-algebra of real rank zero.

1. Abelian strict closure in discrete AW*-algebras

First we prove a general result concerning a masa C of a C^*-algebra A, whose multiplier algebra is an AW^*-algebra; that is, according to the theorem of B. E. Johnson quoted in the Introduction (see [13] or [26]), a masa C of an essential, norm-closed, two-sided ideal A of some AW^*-algebra. We notice that a part of this result holds for a masa of an essential, norm-closed, two-sided ideal of any Rickart C^*-algebra. We shall restrict ourselves to unital Rickart C^*-algebras, because adjoining a unit to a non-unital Rickart C^*-algebra M, we obtain a unital Rickart C^*-algebra $\widetilde{M}$ (see [6], §5, Theorem 1, or [28], 9.11.(1)), and it is easy to see that every essential, norm-closed, two-sided ideal of M is an essential, norm-closed, two-sided ideal also of $\widetilde{M}$.

Any essential, two-sided ideal $\mathcal{J}$ of a C^*-algebra M induces a strict topology $\beta_{\mathcal{J}}$ on M, defined by the seminorms

$$M \ni x \mapsto \|xa\| \text{ and } x \mapsto \|ax\|, \qquad a \in \mathcal{J}.$$

With this definition, the usual strict topology on the multiplier algebra of a C^*-algebra A is β_A.

For the basic facts concerning Rickart C^*-algebras and AW^*-algebras see [6], §§ 3, 4 and 5, or [28], §9.

Lemma 1.1. *Let M be a unital C^*-algebra, $\mathcal{J}$ an essential, norm-closed, two-sided ideal of M, and C a masa of $\mathcal{J}$. By the strict topology on M we shall understand $\beta_{\mathcal{J}}$, which of course is the usual strict topology when M is an AW^*-algebra and so can be identified with the multiplier algebra $M(\mathcal{J})$. Then*

(i) *every $x \geq 0$ in the strict closure of C in M belongs to the strict closure of $\{b \in C\,; 0 \leq b \leq x\}$ in M.*

Let us next assume that M is a Rickart C^*-algebra. Then

(ii) *for every $0 \leq b \in C$ and every $\delta > 0$ there is a projection $f_\delta \in C$ such that*
$$bf_\delta \geq \delta f_\delta, \quad b(1_M - f_\delta) \leq \delta(1_M - f_\delta),$$
so C is the norm-closed linear span of its projections;

(iii) *any projection e in the strict closure of C in M belongs to the strict closure of $\{f \in C\,; f \leq e \text{ projection}\}$ in M;*

(iv) *any projection e in the relative commutant $C' \cap M$ is the least upper bound of $\{f \in C\,; f \leq e \text{ projection}\}$ in the projection lattice of M, in particular $C' \cap M$ is a masa of M.*

Finally, assuming M to be an AW^*-algebra,

(v) *the relative commutant $C' \cap M$ is the AW^*-subalgebra of M generated by C, so $C' \cap M$ can be identified with $M(C)$;*

(vi) *the strict closure of C in M coincides with $C' \cap M$ if and only if C contains a two-sided approximate unit for $\mathcal{J}$, in which case the strict topology of $M(C) = C' \cap M$ is the restriction of the strict topology of $M(\mathcal{J}) = M$.*

Proof. The strict closure $\overline{C}^{\beta_{\mathcal{J}}}$ of C being an abelian C^*-subalgebra of $M(A)$, we have for every $b \in C$

$$(x-b)^*(x-b) \geq (x - \text{Re } b)^2 \geq \big(x - (\text{Re } b)_+\big)^2 \geq (x - b_o)^2,$$

where

$$b_o = \frac{1}{2}\Big(x + (\text{Re } b)_+ - |x - (\text{Re } b)_+|\Big)$$

denotes the greatest lower bound of x and $(\text{Re } b)_+$ in the Hermitian part of $\overline{C}^{\beta_{\mathcal{J}}}$. Since

$$0 \leq b_o \leq (\text{Re } b)_+ \in C \subset \mathcal{J},$$

by [25], Proposition 1.4.5, we have $b_o \in \mathcal{J}$, so

$$b_o \in C' \cap \mathcal{J} = C.$$

Thus, for every $a \in \mathcal{J}$ and $b \in C$ we have $\|(x-b)a\| \geq \|(x-b_o)a\|$ for some $0 \leq b_o \leq x$ in C and (i) follows.

For (ii) put

$$f_\delta = \text{ support of } (b-\delta 1_M)_+ \text{ in } M\,.$$

Then f_δ commutes with every element of C and

$$bf_\delta \geq \delta f_\delta, \quad b(1_M - f_\delta) \leq \delta(1_M - f_\delta)\,.$$

In particular, $f_\delta \leq \frac{1}{\delta} b \in A$ and [25], Proposition 1.4.5, yields $f_\delta \in \mathcal{J}$. Consequently $f_\delta \in C' \cap A = C$.

For (iii) let $0 \neq a \in \mathcal{J}$ and $\varepsilon > 0$ be arbitrary. According to (i) there exists $0 \leq b \leq e$ in C such that

$$\|(e-b)a\| < \frac{\varepsilon}{2}\,.$$

Further, by (ii) there is a projection $f \in C$ with

$$bf \geq \frac{\varepsilon}{2\|a\|} f, \qquad b(1_M - f) \leq \frac{\varepsilon}{2\|a\|} \cdot (1_M - f)\,.$$

Then $f \leq e$ and $e - f \leq (e - bf)^2$, so

$$\begin{aligned}\|(e-f)a\| &= \|a^*(e-f)a\|^{1/2} \\ &\leq \|a^*(e-bf)^2 a\|^{1/2} = \|(e-bf)e\| \\ &\leq \|(e-b)e\| + \|b(1_M - f)e\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2\|a\|}\|a\| = \varepsilon.\end{aligned}$$

For (iv) we have to show that if a projection $g \in M$ majorizes all projections $C \ni f \leq e$, then $g \geq e$, that is, e is equal to the greatest lower bound $e \wedge g$ of e and g in the projection lattice of M. Let us assume that

$$e_o = e - e \wedge g \neq 0\,.$$

Since $\mathcal{J}$ is essential ideal in M, there exists $a \in \mathcal{J}$ with $ae_o \neq 0$. Choosing some $0 < \delta < \|e_o a^* a e_o\|$ and putting

$$e_1 = \text{ support of } (e_o a^* a e_o - \delta 1_M)_+ \text{ in } M\,,$$

we have

$$0 \neq e_1 \leq \frac{1}{\delta} e_o a^* a e_o \in \mathcal{J}\,.$$

Clearly, $e_1 \leq e_o$ and [25], Proposition 1.4.5, yields also $e_1 \in \mathcal{J}$. Furthermore, for every projection $f \in C$ we get successively

$$\begin{aligned} fe \ &\in C' \cap \mathcal{J} = C \text{ and } fe \leq e, \\ fe \ &\leq e \wedge g, \text{ hence } fe_o = (fe)e_o = 0, \\ fe_1 &= (fe_o)e_1 = 0\,. \end{aligned}$$

Taking into account (ii), it follows that

$$be_1 = 0 \text{ for all } b \in C\,,$$

in particular

$$e_1 \in C' \cap \mathcal{J} = C\,.$$

But then $e_1 \leq e_o \leq e$ implies $e_1 \leq e \wedge g$, which contradicts $0 \neq e_1 \leq e_o = e - e \wedge g$.

In particular, $C' \cap M$ is commutative. For the proof we notice that, since $C' \cap M$ is a Rickart C^*-subalgebra of M (see [6], §5, Proposition 5, or [28], 9.12.(1)), it is the norm-closed linear span of its projections (see e.g. [28], 9.4) and therefore it is enough to show that any two projections $e_1, e_2 \in C' \cap M$ commute. But the $*$-automorphism $M \ni x \longmapsto (2\,e_2 - 1_M)x(2\,e_2 - 1_M) \in M$ leaves fixed C, hence also the least upper bound of any projection family in C in the projection lattice of M. Therefore it leaves fixed e_1, that is, $e_1 e_2 = e_2 e_1$.

Moreover, $C' \cap M$ is a masa of M. Indeed, if $C_o \supset C' \cap M$ is a commutative subalgebra of M, then $C_o \supset C$ and thus we also have $C_o \subset {C_o}' \cap M \subset C' \cap M$.

For (v) we first notice that $C' \cap M$ is an AW^*-subalgebra of M containing C (see [6], §4, Proposition 8, or [28], 9.24.(1)). Now let N be any AW^*-subalgebra of M containing C. By (iv) N contains all projections from $C' \cap M$, hence $N \supset C' \cap M$. Consequently $C' \cap M$ is the AW^*-subalgebra of M generated by C.

Further, C is a two-sided ideal of $C' \cap M$:

$$b \in C \text{ and } y \in C' \cap M \implies by \in C' \cap \mathcal{J} = C\,.$$

Moreover, it is essential, because a projection $e \in C' \cap M$ with $Ce = \{0\}$ belongs to the AW^*-subalgebra of $C' \cap M$ generated by C only if $e = 0$. Hence we can identify $C' \cap M$ with $M(C)$ (see [13] or [26]).

Finally we prove (vi). If the strict closure of C in M is $C' \cap M \ni 1_M$, then there exists a net $(u_\iota)_\iota$ in C with $u_\iota \to 1_M$ strictly in M, that is,

$$\|a - u_\iota a\| \to 0 \text{ and } \|a - u_\iota a\| \to 0 \text{ for all } a \in \mathcal{J}\,.$$

Conversely, let us assume that C contains a two-sided approximate unit $(u_\iota)_\iota$ for $\mathcal{J}$. Then the strict topology β_C of $M(C) = C' \cap M$ agrees with the strict topology $\beta_{\mathcal{J}}$ of $M(\mathcal{J}) = M$ on every norm bounded subset of $C' \cap M$. Indeed, if $(y_\lambda)_\lambda$ is a norm bounded net in $C' \cap M$, convergent to 0 with respect to β_C, and $0 \neq a \in \mathcal{J}$, $\varepsilon > 0$ are arbitrary, then there exists ι_o such that

$$\|y_\lambda\| \cdot \|a - u_{\iota_o} a\| < \frac{\varepsilon}{2} \text{ for all } \lambda\,,$$

and then there exists some λ_o with

$$\|y_\lambda u_{\iota_o}\| < \frac{\varepsilon}{2\|a\|} \text{ for every } \lambda \geq \lambda_o\,.$$

It follows for every $\lambda \geq \lambda_o$:

$$\|y_\lambda a\| \leq \|y_\lambda (a - u_{\iota_o} a)\| + \|y_\lambda u_{\iota_o} a\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2\|a\|}\|a\| = \varepsilon\,.$$

But β_C is the finest locally convex vector space topology on $C' \cap M$ that agrees with β_C on every norm bounded subset of $C' \cap M$ (see [29], Cor. 2.7). Thus the restriction of $\beta_{\mathcal{J}}$ to $C' \cap M$, which is plainly finer than β_C, is actually equal to β_C. In particular, the β_C-density of C in $M(C)$ implies the $\beta_{\mathcal{J}}$-density of C in $C' \cap M$. □

It is well known that every commutative AW^*-algebra Z is monotone complete (see e.g. [28], 9.26, Proposition 1). If M is an arbitrary AW^*-algebra, we call

$$\Phi : \big\{e \in M; e \text{ projection}\big\} \to Z^+$$

completely additive whenever, for every family $(e_\iota)_\iota$ of mutually orthogonal projections in M, we have

$$\Phi\Big(\bigvee_\iota e_\iota\Big) = \sum_\iota \Phi(e_\iota)\,,$$

where the sum stands for the least upper bound in Z^+ of all finite sums of $\Phi(e_\iota)$.

Now we describe the strict closure of a masa of the norm-closed two-sided ideal generated by the finite projections of a discrete semi-finite AW^*-algebra :

Theorem 1.2 (on the abelian strict closure in discrete AW^*-algebras)**.** *Let M be a discrete AW^*-algebra, A the norm-closed linear span of all finite projections of M, and C a masa of A. Then the strict closure of C in $M(A) = M$ is equal to $C' \cap M$.*

Proof. According to Lemma 1.1 (vi), we have to show that C contains a two-sided approximate unit for A. Without loss of generality we may assume that $A \neq \{0\}$, hence $C \neq \{0\}$.

Let $(e_\iota)_{\iota \in I}$ be a maximal family of mutually orthogonal non-zero projections in C. Then

$$\bigvee_\iota e_\iota = 1_M .$$

Indeed, $e_o = 1_M - \bigvee_\iota e_\iota$ belongs to $C' \cap M$, so Lemma 1.1 (iv) yields

$$e_o = \bigvee \{ f \in C ; f \leq e_o \text{ projection} \} .$$

Thus $e_o \neq 0$ would imply the existence of some projection $0 \neq f \leq e_o$ in C, contradicting the maximality of $(e_\iota)_{\iota \in I}$.

Denoting by Z the centre of M, we call central partition of 1_M any set of mutually orthogonal projections in Z with least upper bound 1_M. The projections

$$\bigvee_{p \in \mathcal{P}} \Big(\sum_{\iota \in I_p} e_\iota \Big) p , \quad \mathcal{P} \text{ a central partition of } 1_M , \quad I_p \subset I \text{ finite for any } p \in \mathcal{P}$$

belong to $C' \cap M$ and are finite (see [6], §15, Proposition 8), hence they belong to $C' \cap A = C$. We show that their family is an (increasing positive) approximate unit for A. For we have to prove that every finite projection e in M has the property

$$\text{(P)} \qquad \begin{cases} \text{for every } \varepsilon > 0 \text{ there are } \mathcal{P} \text{ and } I_p , p \in \mathcal{P}, \text{ with} \\ \Big\| \Big(1_M - \bigvee_{p \in \mathcal{P}} \big(\sum_{\iota \in I_p} e_\iota \big) p \Big) e \Big\| \leq \varepsilon . \end{cases}$$

But standard arguments show that every finite projection e in M is of the form

$$e = \bigvee_{n \geq 1} (e_{n,1} + \cdots + e_{n,n}) p_n ,$$

where $p_n, n \geq 1$ are mutually orthogonal projections in Z and, for every $n \geq 1, e_{n,1}, \ldots , e_{n,n}$ are mutually orthogonal abelian projections of central support p_n (use [6], §18, Exercises 3, 4, and Proposition 1), so it is enough to prove (P) for every abelian projection e in M. Moreover, since every abelian projection is majorized by an abelian projection of central support 1_M, without loss of generality we can restrict ourselves to the case of an abelian projection e of central support 1_M.

For every $x \in M$ there exists a unique $\Phi_e(x) \in Z$ such that

$$exe = \Phi_e(x) e$$

(see [6], §15, Proposition 6, and §5). Clearly, $\Phi_e : M \to Z$ is a conditional expectation and, according to [18], Lemma 7, it is completely additive on the projection lattice of M. Furthermore, $Z \ni z \mapsto ze \in Ze$ being $*$-isomorphism, we have

$$\| xe \|^2 = \| ex^* xe \| = \| \Phi_e(x^* x) e \| = \| \Phi_e(x^* x) \| , \quad x \in M .$$

Now, by the complete additivity of Φ_e,

$$\sum_{\iota} \Phi_e(e_\iota) = \Phi_e(1_M) = 1_M \,.$$

Thus, according to [18], Lemma 5, for every $e > 0$ there exist a central partition $\mathcal{P}$ of 1_M and finite sets $I_p \subset I, p \in \mathcal{P}$ such that

$$\Big\| \Big(1_M - \sum_{\iota \in I_p} \Phi_e(e_\iota) \Big) p \Big\| \leq \varepsilon^2 \text{ for all } p \in \mathcal{P} \,.$$

But then we have for every $p \in \mathcal{P}$

$$\begin{aligned} \Big\| \Big(1_M - \sum_{\iota \in I_p} e_\iota \Big) pe \Big\|^2 &= \Big\| \Phi_e\Big(1_M - \sum_{\iota \in Ip} e_\iota \Big) p \Big\| \\ &= \Big\| \Big(1_M - \sum_{\iota \in Ip} \Phi_e(e_\iota) \Big) p \Big\| \leq \varepsilon^2, \end{aligned}$$

so, taking into account [17], Lemma 2.5,

$$\Big\| \Big(1_M - \bigvee_{p \in \mathcal{P}} \big(\sum_{\iota \in I_p} e_\iota \big) p \Big) e \Big\| = \sup_{p \in \mathcal{P}} \Big\| \Big(1_M - \sum_{\iota \in I_p} e_\iota \Big) pe \Big\| \leq \varepsilon \,. \qquad \square$$

2. Abelian strict closure in continuous AW*-algebras

For the treatment of the case of continuous M we need several lemmas on AW^*-algebras, which could be of interest for themselves. First we extend [32], Lemma 2.2, concerning a Darboux property of normal functionals on von Neumann algebras without minimal projections, to the case of centre valued completely additive maps on the projection lattice of a continuous AW^*-algebra (similar results can be found in [5] and, for tracial maps, in [15], Proposition 3.13, [16], Proposition 27).

Lemma 2.1. *Let M be a continuous AW^*-algebra, Z its centre, C a masa of M, and*

$$\Phi : \{ e \in M ; e \text{ projection} \} \to Z^+$$

a completely additive map such that

$$\Phi(ep) = \Phi(e)p, \quad e \in M \text{ and } p \in Z \text{ projections.}$$

Then, for every projection $e \in C$,

$$\{ z \in Z ; 0 \leq z \leq \Phi(e) \} = \{ \Phi(f) ; e \geq f \in C \text{ projection} \} \,.$$

Proof. a) First we prove that for every projection $0 \neq g \in C$ there exists a projection $0 \neq h \leq g$ in C such that

$$\Phi(h) \leq \frac{1}{2} \Phi(g) \,.$$

The case $\Phi(g) = 0$ being trivial, we can assume without loss of generality that $\Phi(g) \neq 0$.

Let $(g_\iota)_\iota$ be a maximal family of mutually orthogonal projections in Cg such that $\Phi(g_\iota) = 0$ for every ι. Put $g_1 = g - \bigvee_\iota g_\iota \in C$. Then

$$\Phi(g_1) = \Phi(g) - \sum_{\iota} \Phi(g_\iota) = \Phi(g) \neq 0 \,,$$

so $g_1 \neq 0$. By the maximality of $(g_\iota)_\iota$, for no projection $0 \neq g' \leq g_1$ in C can hold $\Phi(g') = 0$.

Now there exists a projection $g_2 \leq g_1$ in C such that $g_2 \notin Zg_1$. Let us assume the contrary, that is, that

$$C = Zg_1 + C(1_M - g_1)\,.$$

There exist projections $h_1, h_2 \in M$ such that $g_1 = h_1 + h_2$ and $h_1 \sim h_2$ ([6], §19, Theorem 1), and then

$$C \subset Zh_1 + Zh_2 + C(1_M - g_1)$$

and the maximal abelianness of C imply that

$$C = Zh_1 + Zh_2 + C(1_M - g_1)\,.$$

Thus

$$h_1, h_2 \in Cg_1 = Zg_1\,.$$

But, denoting by $z(g_1)$ the central support of g_1,

$$Zz(g_1) \ni z \mapsto zg_1 \in Zg_1$$

is a $*$-isomorphism and it follows that h_1 and h_2 have orthogonal central supports, in contradiction to $h_1 \sim h_2 \neq 0$.

We claim that $\Phi(g_2)\Phi(g_1 - g_2) \neq 0$. Indeed, otherwise there would exist a projection $p \in Z$ such that

$$\Phi(g_2) = \Phi(g_2)p \text{ and } \Phi(g_1 - g_2)p = 0$$

and it would follow successively that

$$\begin{gathered}\Phi(g_2(1_M - p)) = 0 \text{ and } \Phi((g_1 - g_2)p) = 0\,,\\ (g_2(1_M - p)) = 0 \text{ and } (g_1 - g_2)p = 0\,,\\ g_2 = g_2 p = g_1 p \in Zg_1\,.\end{gathered}$$

Let $q \in Z$ denote the support projection of $(\Phi(g_1) - 2\Phi(g_2))_+$. Then

$$\begin{aligned}\Phi(g_1 q) - 2\Phi(g_2 q) &= (\Phi(g_1) - 2\Phi(g_2))_+ \geq 0,\\ \Phi(g_2 q) \leq \frac{1}{2}\Phi(g_1 q) &\leq \frac{1}{2}\Phi(g_1) \leq \frac{1}{2}\Phi(g)\,.\end{aligned}$$

Similarly,

$$\Phi((g_1 - g_2)(1_M - q)) \leq \frac{1}{2}\Phi(g)\,.$$

But we cannot simultaneously have

$$\Phi(g_2 q) = 0 \text{ and } \Phi((g_1 - g_2)(1_M - q)) = 0\,,$$

because this would imply

$$\Phi(g_2)\Phi(g_1 - g_2) = \Phi(g_2 q)\Phi(g_1 - g_2) + \Phi(g_2)\Phi((1_M - q)(g_1 - g_2)) = 0\,.$$

Therefore, putting $h = g_2 q$ if $\Phi(g_2 q) \neq 0$ and $h = (g_1 - g_2)(1_M - q)$ otherwise, h is a non-zero projection in C, majorized by g, such that $\Phi(h) \leq \frac{1}{2}\Phi(g)$.

b) Now let $e \in C$ be a projection and let $x \in Z$, $0 \leq z \leq \Phi(e)$ be arbitrary. Choose a maximal family $(f_\iota)_\iota$ of mutually orthogonal projections in Ce satisfying

$$\sum_\iota \Phi(f_\iota) \leq z\,.$$

Then the projection $f = \bigvee_\iota f_\iota \leq e$ belongs to C and

$$\Phi(f) = \sum_\iota \Phi(f_\iota) \leq z\,.$$

We claim that actually $\Phi(f) = z$.

For let us assume the contrary. Then there exist a projection $0 \neq p \in Z$ and $\varepsilon > 0$ such that

$$(z - \Phi(f))p \geq \varepsilon p\,.$$

The projection $g = (e - f)p \in C$ is not zero, because otherwise it would follow

$$0 = (\Phi(e) - \Phi(f))p \geq (z - \Phi(f))p \geq \varepsilon p\,,$$

contradicting $p \neq 0, \varepsilon > 0$. Choosing an integer $n \geq 1$ with $2^{-n}\|\Phi(e - f)\| \leq \varepsilon$, the n-fold application of a) yields the existence of a projection $0 \neq h \leq g$ in C such that

$$\Phi(h) \leq 2^{-n}\Phi((e - f)p) \leq \varepsilon p\,.$$

Since $0 \neq h \in Ce$ is orthogonal to every f_ι and

$$\Phi(h) + \sum_\iota \Phi(f_\iota) = \Phi(h) + \Phi(f) \leq \varepsilon p + \Phi(f) \leq z\,,$$

the maximality of $(f_\iota)_\iota$ is contradicted. □

It is well known that if the projection family $(e_\iota)_\iota$ in a finite AW^*-algebra M is upward directed and, for some projection $f \in M$, $e_\iota \prec f$ for all ι , then $\bigvee_\iota e_\iota \prec f$ (see [6], §33, Exercise 1). The above statement actually holds in any AW^*-algebra M under the only assumption of the finiteness of f (see Appendix, Corollary 1). Here we give a proof for this, assuming additionally that the projections e_ι are the finite partial sums of a family of mutually orthogonal projections in M :

Lemma 2.2. *Let M be an AW^*-algebra, $f \in M$ a finite projection, and $(e_\iota)_{\iota \in I}$ a family of mutually orthogonal projections in M such that*

$$\sum_{\iota \in F} e_\iota \prec f \text{ for every finite } F \subset I\,.$$

Then

$$\bigvee_{\iota \in I} e_\iota \prec f\,.$$

Proof. According to the theory of Murray-von Neumann equivalence for projections in AW^*-algebras, we can assume without loss of generality that either fMf is of type I_n for some natural number $n \geq 1$, or that it is continuous (see [6], §15, Theorem 1, §18, Theorem 2, and §6, Corollary 2 of Proposition 4).

Let us first assume that fMf is of type I_n. By the Zorn Lemma there exists a maximal set $\mathcal{P}$ of mutually orthogonal central projections in M such that

$$\text{card}\,\{\iota \in I; pe_\iota \neq 0\} \leq n \text{ for every } p \in \mathcal{P}\,.$$

We claim that $\bigvee \mathcal{P} = 1_M$. Let us assume that $p_o = \bigvee \mathcal{P} \neq 1_M$. Then we can recursively find $n + 1$ indices $\iota_1, \dots, \iota_{n+1} \in I$ such that

$$p_1 = (1_M - p_o)z(e_{\iota_1}) \dots z(e_{\iota_{n+1}}) \neq 0\,,$$

where $z(e_\iota)$ denotes the central support of e_ι . By the assumption of the lemma there exist mutually orthogonal projections $f_{\iota_1}, \dots, f_{\iota_{n+1}} \leq f$ in M such that $e_{\iota_j} \sim f_{\iota_j}$ for every $1 \leq j \leq n + 1$. For every $1 \leq j \leq n + 1$, the central support of $p_1 f_{\iota_j}$

is p_1, there exists an abelian projection $g_j \leq p_1 f_{\iota_j}$ of central support p_1 (see [6], §18, exercise 4). But then $g_1, \ldots, g_{n+1}$ are mutually orthogonal, equivalent, non-zero projections in fMf (see [6], §18, Proposition 1), which contradicts [6], §18, Proposition 4.

By the very orthogonal additivity of equivalence in AW^*-algebras (see [6], §11, Proposition 2) we conclude that

$$\bigvee_{\iota \in I} e_\iota = \bigvee \left\{ \sum_{pe_\iota \neq 0} pe_{\iota_j} \,;\, p \in \mathcal{P} \right\} \prec \bigvee \{ pf \,;\, p \in \mathcal{P} \} = f\,.$$

Let us next assume that fMf is continuous and let $x \mapsto x^\natural$ denote the centre valued dimension function of the finite AW^*-algebra fMf (see [6], Ch.6).

For every $\iota \in I$ there exists a projection $e'_\iota \leq f$ in M such that $e_\iota \sim e'_\iota$. Since $(e'_\iota)^\natural$ does not depend on the choice of e'_ι, we can put

$$e_\iota^\natural = (e'_\iota)^\natural\,.$$

By the assumption of the lemma, for every finite $F \subset I$ we can choose the projections e'_ι, $\iota \in F$, mutually orthogonal and then

$$\sum_{\iota \in F} e_\iota^\natural = \sum_{\iota \in F} (e'_\iota)^\natural = \left(\sum_{\iota \in F} e'_\iota \right)^\natural \leq f\,.$$

It follows that all sums

$$\sum_{\iota \in J} e_\iota^\natural \leq f, \quad J \subset I,$$

exist in the monotone complete centre of fMf.

Now let us consider the set of all families of mutually orthogonal projections in fMf

$$(f_\iota)_{\iota \in J} \text{ with } J \subset I\,,$$

for which $f_\iota \sim e_\iota$ for every $\iota \in J$. We can endow this set with the partial order

$$(f_\iota)_{\iota \in J} \leq (f'_\iota)_{\iota \in J'} \iff J \subset J' \text{ and } f_\iota = f'_\iota \text{ for all } \iota \in J\,.$$

By the Zorn Lemma there exists a maximal element $(f_\iota)_{\iota \in J}$ of the above partially ordered set. We claim that then $J = I$. Let us assume the existence of some $\iota_o \in I \backslash J$. Since

$$e_{\iota_o}^\natural + \left(\bigvee_{\iota \in J} f_\iota \right)^\natural = e_{\iota_o}^\natural + \sum_{\iota \in J} f_\iota^\natural \leq \sum_{\iota \in I} e_\iota^\natural \leq f\,,$$

that is,

$$e_{\iota_o}^\natural \leq \left(f - \bigvee_{\iota \in J} f_\iota \right)^\natural,$$

by [6], §33, Theorem 3 (a particular case of the above Lemma 2.1) there exists a projection $f_{\iota_o} \leq f - \bigvee_{\iota \in J} f_\iota$ in M such that $f_{\iota_o}^\natural = e_{\iota_o}^\natural = \left(e'_{\iota_o}\right)^\natural$, hence $f_{\iota_o} \sim e'_{\iota_o} \sim e_{\iota_o}$. But this contradicts the maximality of $(f_\iota)_{\iota \in J}$.

By the general additivity of equivalence in AW^*-algebras (see [6], §20, Theorem 1) we can conclude also in this case that

$$\bigvee_{\iota \in I} e_\iota \sim \bigvee_{\iota \in I} f_\iota \leq f\,. \qquad \square$$

Let M be a semi-finite AW^*-algebra, and A the norm-closed linear span of all finite projections of M. We then recall that $M = M(A)$.

Let us call a masa $\tilde{C}$ of M M-semi-finite if $\tilde{C} \cap A$ is an essential ideal of $\tilde{C}$ or, equivalently, if every non-zero projection in $\tilde{C}$ majorizes a non-zero projection in $\tilde{C} \cap A$ (cf. with [16], Definition 1). For $\tilde{C} \subset M$ are equivalent:

1) $\tilde{C}$ is an M-semi-finite masa of M;
2) $\tilde{C} = C' \cap M$ for some masa C of A.

Indeed, 2) implies 1) by Lemma 1.1 (iv), while 1) $\Rightarrow$ 2) follows by noticing that, according to the M-semi-finiteness of $\tilde{C}$, every projection in $\tilde{C}$ is the least upper bound of a family of mutually orthogonal projections from $C = \tilde{C} \cap A$, and so $C' \cap M = \tilde{C}' \cap M = \tilde{C}$, $C' \cap A = (C' \cap M) \cap A = \tilde{C} \cap A = C$.

The following result extends [15], Theorem 3.18, and [16], Corollary 31, in the case of an M-semi-finite masa :

Theorem 2.3 (on labeling Murray-von Neumann equivalence classes)**.** *Let M be a semi-finite AW^*-algebra, A the norm closed linear span of all finite projections of M, and C a masa of A. Then*

- (i) *for any projections $M \ni f \leq e \in C' \cap M$ there exists a projection $f \sim g \leq e$ in $C' \cap M$;*
- (ii) *for any projections $M \ni f \leq e \in C' \cap M$ of equal central supports, f finite and e properly infinite, there is a set $\mathcal{P}$ of mutually orthogonal central projections in M with $\bigvee \mathcal{P} = 1_M$ such that, for every $p \in \mathcal{P}$, ep is the least upper bound in the projection lattice of M of some family of mutually orthogonal projections from C, each one of which is equivalent in M to fp.*

Proof. (a) First we prove (i) in the case $e \in C$. Similarly as in the proof of Lemma 2.2, we can assume without loss of generality that either $eMe = eAe$ is of type I_n for some natural number $n \geq 1$, or it is continuous.

If eMe is of type I_n, by [15], Lemma 3.7, there exist mutually orthogonal projections $e_1, \dots, e_n \in C$ with $\sum_{j=1}^{n} e_j = e$, such that each e_j is abelian in M and has the same central support in M as e (actually [15], Lemma 3.7, is proved only for von Neumann algebras, but an inspection of the proof shows that it also works without any change in the realm of the AW^*-algebras). On the other hand, using [6], §18, Exercise 4 and Proposition 4, it is easy to see that there exist mutually orthogonal abelian projections $f_1, \dots, f_n \in M$ with $\sum_{j=1}^{n} f_j = f$ and central supports $z(f) = z(f_1) \geq \dots \geq z(f_n)$. By [6], §18, Proposition 1, it follows that $f_j \sim e_j z(f_j)$ for all $1 \leq j \leq n$, so f is equivalent to $C \ni \sum_{j=1}^{n} e_j z(f_j) \leq e$.

Now let us assume that eMe is continuous and let $x \mapsto x^\natural$ denote the centre valued dimension function of the finite AW^*-algebra eMe. Then Lemma 2.1 yields the existence of a projection $C \ni g \leq e$ such that $g^\natural = f^\natural$, hence $g \sim f$.

(b) Next we prove (i) in the case $f \in A$.

By Lemma 1.1 (iv) there exists a family $(e_\iota)_{\iota \in I}$ of mutually orthogonal projections in C such that

$$e = \bigvee_{\iota \in I} e_\iota .$$

Let $\mathcal{P}$ be a maximal set of mutually orthogonal central projections in M such that, for every $p \in \mathcal{P}$, there is a finite set $F_p \subset I$ with

$$fp \prec p \sum_{\iota \in F_p} e_\iota \in C\,.$$

By the above part (a) of the proof, for every $p \in \mathcal{P}$ there exists a projection

$$g(p) \in C \text{ with } fp \sim g(p) \leq p \sum_{\iota \in F_p} e_\iota\,.$$

If $\bigvee \mathcal{P} = 1_M$, then $f = \bigvee\{fp\,;\, p \in \mathcal{P}\}$ is equivalent to $C' \cap M \ni \bigvee\{g(p)\,;\, p \in \mathcal{P}\} \leq e$, so let us assume in the sequel that $p_o = 1_M - \bigvee \mathcal{P} \neq 0$.

By the maximality of $\mathcal{P}$ and by the comparison theorem (see [6], §14, Corollary 1 of Proposition 7) we have

$$p_o \sum_{\iota \in F} e_\iota \prec f \text{ for every finite } F \subset I\,.$$

According to Lemma 2.2 it follows that

$$p_o e = \bigvee_{\iota \in I} p_o e_\iota \prec f\,,$$

so by the Schröder-Bernstein theorem (see [6], §12) we have

$$f p_o \sim e p_o\,.$$

Consequently $f = fp_o + \bigvee\{fp\,;\, p \in \mathcal{P}\}$ is equivalent to

$$C' \cap M \ni ep_o + \bigvee\{g(p)\,;\, p \in \mathcal{P}\} \leq e\,.$$

(c) Now we prove (ii).

Let $\mathcal{P}$ be a maximal set of mutually orthogonal central projections in M such that, for every $p \in \mathcal{P}$, ep is the least upper bound in the projection lattice of M of some family of mutually orthogonal projections from C, each one of which is equivalent in M to fp. We claim that then $\bigvee \mathcal{P} = 1_M$.

Let us assume that $p_o = 1_M - \bigvee \mathcal{P} \neq 0$. We notice that $fp \neq 0$ for any central projection $0 \neq p \leq p_o$ in M : indeed, otherwise p would be orthogonal to the common central support of f and e, so $ep = 0$ would be equal to $fp = 0 \in C$, in contradiction with the maximality of $\mathcal{P}$.

Let $(e_\iota)_{\iota \in I}$ be a maximal family of mutually orthogonal projections in C such that $fp_o \sim e_\iota \leq ep_o$ for all $\iota \in I$. By the comparison theorem there exists a central projection $p_1 \leq p_o$ in M such that

$$\Big(ep_o - \bigvee_{\iota \in I} e_\iota\Big)p_1 \prec fp_1\,,$$

$$\Big(ep_o - \bigvee_{\iota \in I} e_\iota\Big)(p_o - p_1) \succ f(p_o - p_1)\,.$$

Then $p_1 \neq 0$: indeed, $p_1 = 0$ would imply

$$A \ni fp_o \prec ep_o - \bigvee_{\iota \in I} e_\iota \in C' \cap M$$

and, by the above proved (b), there would exist a projection $fp_o \sim e' \leq ep_o - \bigvee_{\iota\in I} e_\iota$ in $(C'\cap M)\cap A = C$, contradicting the maximality of $(e_\iota)_{\iota\in I}$. Put

$$e_o = ep_1 - \bigvee_{\iota\in I} e_\iota p_1 \prec fp_1 .$$

Then e_o is finite and belongs to $C'\cap M$, so it belongs to $C'\cap A = C$. On the other hand, the proper infiniteness of e and $ep_1 \neq 0$ imply that $ep_1 = e_o + \bigvee_{\iota\in I} e_\iota p_1$ is properly infinite. It follows that the set I is necessarily infinite, hence containing an infinite sequence $\iota_1, \iota_2, \ldots$.

For every $j \geq 1$, $e_o \prec fp_1 \sim e_{\iota_j} p_1 \in C$ and the above proved a) yield the existence of some projection $e_o \sim e^{(1)}_{\iota_j} \leq e_{\iota_j} p_1$ in C. In particular, all projections $e^{(1)}_{\iota_j}$ are equivalent, hence, the projections $e_\iota p_1$ being finite, the projections $e^{(2)}_{\iota_j} = e_{\iota_j} p_1 - e^{(1)}_{\iota_j}$ are also all equivalent (see [6], §17, Exercise 3). Consequently, the projections from C

$$e'_{\iota_1} = e_o + e^{(2)}_{\iota_1} \text{ and } e'_{\iota_j} = e^{(1)}_{\iota_{j-1}} + e^{(2)}_{\iota_j}, \quad j \geq 2,$$

are all equivalent in M to $e^{(1)}_{\iota_1} + e^{(2)}_{\iota_1} = e_{\iota_1} p_1 \sim fp_1$. Clearly, they are mutually orthogonal and

$$\bigvee_{j\geq 1} e'_{\iota_j} = e_o \vee \bigvee_{j\geq 1} e^{(1)}_{\iota_j} \vee \bigvee_{j\geq 1} e^{(2)}_{\iota_j} = e_o \vee \bigvee_{j\geq 1} e_{\iota_j} p_1 .$$

Letting

$$e'_\iota = e_\iota p_1 \text{ for } \iota \in I \setminus \{\iota_1, \iota_2, \ldots\},$$

we conclude that all projections e'_ι, $\iota \in I$, belong to C and are equivalent in M to fp_1. Moreover, they are mutually orthogonal and

$$\bigvee_{\iota\in I} e'_\iota = \bigvee_{j\geq 1} e'_{\iota_j} \vee \bigvee_{\iota\neq\iota_j} e'_\iota = e_o \vee \bigvee_{j\geq 1} e_{\iota_j} p_1 \vee \bigvee_{\iota\neq\iota_j} e_\iota p_1 = e_o \vee \bigvee_{\iota\in I} e_\iota p_1 = ep_1 .$$

But this contradicts the maximality of $\mathcal{P}$.

(d) Finally we prove (i) in full generality.

We can assume without loss of generality that either f is finite, or it is properly infinite. The case of finite f was already settled in (b), so it remains to consider only the case of properly infinite f.

Choose some finite projection $M \ni f_o \leq f$ of the same central support as f (see [6], §17, Exercise 19 iii)). According to the above proved (c), we can assume without loss of generality that there are families $(e_\iota)_{\iota\in I}$ and $(f_\kappa)_{\kappa\in K}$ of mutually orthogonal projections in M such that

$$e_\iota \sim f_o \sim f_\kappa \text{ for all } \iota\in I \text{ and } \kappa\in K,$$
$$\bigvee_{\iota\in I} e_\iota = e, \quad \bigvee_{\kappa\in K} f_\kappa = f .$$

If card $K \leq$ card I, that is, if there exists an injective map $K \ni \kappa \mapsto \iota(\kappa) \in I$, then the projection $g = \bigvee_{\kappa\in K} e_{\iota(\kappa)} \leq e$ belongs to $C'\cap M$ and is equivalent to $\bigvee_{\kappa\in K} f_\kappa = f$. On the other hand, if card $I \leq$ card K, then $e = \bigvee_{\iota\in I} e_\iota \prec \bigvee_{\kappa\in K} f_\kappa = f \leq e$ and the Schröder-Bernstein theorem imply that $e \sim f$. □

Let us now prove the statement of [15], Theorem 3.18, and [16], Corollary 31, in the case of an M-semi-finite masa of an arbitrary semi-finite AW^*-algebra M :

Corollary 2.4. *Let M be a semi-finite AW^*-algebra, A the norm-closed linear span of all finite projections of M, and C a masa of A. If $e \in C' \cap M$ is a projection and $1 \leq n \leq \aleph_o$ is a cardinal number such that e is the least upper bound of n mutually orthogonal, equivalent projections from M, then there exist n mutually orthogonal projections in $C' \cap M$, all equivalent in M, whose least upper bound is e.*

Proof. It is enough to separately treat the case of finite, resp. properly infinite, e.

If e is finite, n can be only a natural number. Let $f_1, \dots f_n$ be mutually orthogonal, equivalent projections in M with $\sum_{j=1}^{n} f_j = e$. By (i) in the above theorem there exists a projection $f_1 \sim e_1 \leq e$ in C. Since e is finite, it follows that $\sum_{j=2}^{n} f_j \sim e - e_1$, so we can again apply (i) in the above theorem to get a projection $f_2 \sim e_2 \leq e - e_1$ in C. By induction we obtain n mutually orthogonal projections $e_1, \dots, e_n \in C$ such that $f_j \sim e_j$ for all j and $\sum_{j=1}^{n} e_j = e$.

Now let us assume that e is properly infinite and consider a set I of cardinality n. Choosing a finite projection $M \ni f \leq e$ of the same central support as e (see [6], §17, Exercise 19 iii)), (ii) in the above theorem entails the existence of a set $\mathcal{P}$ of mutually orthogonal central projections in M with $\bigvee \mathcal{P} = 1_M$ such that, for every $p \in \mathcal{P}$, ep is the least upper bound of some set $\mathcal{E}_p$ of mutually orthogonal projections from C, each one of which is equivalent in M to fp. If $ep \neq 0$, then $\mathcal{E}_p$ must be infinite, so there exists a partition $(\mathcal{E}_{p,\iota})_{\iota \in I}$ of $\mathcal{E}_p$ in n sets of equal cardinality. Then the projections $e_\iota = \bigvee_{ep \neq 0} \bigvee \mathcal{E}_{p,\iota}$, $\iota \in I$, belong to $C' \cap M$, are mutually orthogonal and equivalent in M, and $\bigvee_{\iota \in I} e_\iota = e$. □

Let M be a finite AW^*-algebra with centre Z and let $x \mapsto x^\natural$ denote its centre valued dimension function (see [6], Ch. 6). It is known (see [7], II, 1) that $\natural$ can be uniquely extended to a centre valued quasitrace on M, that is, to a map $\Phi : M \to Z$ such that

- Φ is linear on commutative $*$-subalgebras of M,
- $\Phi(a + ib) = \Phi(a) + i\Phi(b)$ for all selfadjoint $a, b \in M$,
- Φ acts identically on Z,
- $0 \leqslant \Phi(x^*x) = \Phi(xx^*)$ for all $x \in M$,

and then

- $\Phi(a) \leqslant \Phi(b)$ whenever $a \leqslant b$ are selfadjoint elements of M,
- Φ is norm continuous, more precisely, $\|\Phi(a) - \Phi(b)\| \leqslant \|a - b\|$ for all selfadjoint $a, b \in M$.

We shall also use the symbol $\natural$ to denote the above Φ.

According to classical results of F.J. Murray and J. von Neumann, the centre valued quasitrace of every finite W^*-algebra is additive, hence linear.

It is an open question, raised by I. Kaplansky, whether the centre valued quasitrace of every finite AW^*-algebra is additive. Recently U. Haagerup has proven that the answer to Kaplansky's question is positive for any finite AW^*-algebra which is generated (as an AW^*-algebra) by an exact C^*-subalgebra (see [11], Theorem 5.11, Proposition 3.12 and Lemma 3.7 (4)).

We notice that if M is a finite AW^*-algebra and $n \geq 1$ is an integer, then the $*$-algebra $\mathrm{Mat}_n(M)$ of all $n \times n$ matrices over M is again a finite AW^*-algebra (see [6], §62). Denoting by $\natural$ and $\natural_n$ the respective centre valued quasitraces, it is easily seen that

$$n \cdot \begin{pmatrix} x & 0 & & 0 \\ 0 & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix}^{\natural_n} = \begin{pmatrix} x^\natural & 0 & & 0 \\ 0 & x^\natural & & \\ & & \ddots & \\ 0 & & & x^\natural \end{pmatrix}, \quad x \in M.$$

Moreover the additivity of $\natural$ is equivalent to the validity of

$$2 \cdot \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}^{\natural_2} = \begin{pmatrix} x_{11}^\natural + x_{22}^\natural & 0 \\ 0 & x_{11}^\natural + x_{22}^\natural \end{pmatrix}.$$

Indeed, using the above equality, we get for all $0 \leq a, b \in M$

$$\begin{aligned} \begin{pmatrix} (a+b)^\natural & 0 \\ 0 & (a+b)^\natural \end{pmatrix} &= 2 \cdot \begin{pmatrix} a+b & 0 \\ 0 & 0 \end{pmatrix}^{\natural_2} \\ &= 2 \cdot \left[\begin{pmatrix} a^{1/2} & b^{1/2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a^{1/2} & 0 \\ b^{1/2} & 0 \end{pmatrix} \right]^{\natural_2} \\ &= 2 \cdot \left[\begin{pmatrix} a^{1/2} & 0 \\ b^{1/2} & 0 \end{pmatrix} \begin{pmatrix} a^{1/2} & b^{1/2} \\ 0 & 0 \end{pmatrix} \right]^{\natural_2} \\ &= 2 \cdot \begin{pmatrix} a & a^{1/2}b^{1/2} \\ b^{1/2}a^{1/2} & b \end{pmatrix}^{\natural_2} \\ &= \begin{pmatrix} a^\natural + b^\natural & 0 \\ 0 & a^\natural + b^\natural \end{pmatrix}. \end{aligned}$$

Conversely, assuming that $\natural$ is additive, it is easy to verify that

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \longmapsto \frac{1}{2} \begin{pmatrix} x_{11}^\natural + x_{22}^\natural & 0 \\ 0 & x_{11}^\natural + x_{22}^\natural \end{pmatrix}$$

is a centre valued quasitrace on $\mathrm{Mat}_2(M)$.

For a given $\delta > 0$, we say that the centre valued quasitrace $\natural$ of a finite AW^*-algebra M is δ-subadditive (resp. δ-superadditive) if the map $M_+ \ni a \mapsto (a^\natural)^\delta$ is subadditive (resp. superadditive). Clearly, δ-subadditivity (δ-superadditivity) of $\natural$ implies its δ'-subadditivity (δ'-superaddivity) whenever $\delta' < \delta (\delta' > \delta)$. It was proven by U. Haagerup that $\natural$ is always $\frac{1}{2}$-subadditive (see [11], Lemma 3.5 (1)) and it seems reasonable to conjecture that it is also always 2-superadditive (or, at least, k-superadditive for some $k \geqslant 1$).

We notice as a curiosity that, for any two projections p, q in a finite AW^*-algebra M with centre valued quasitrace $\natural$,

$$(p+q)^\natural = p^\natural + q^\natural.$$

Indeed, since

$$\begin{pmatrix} p+q & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} p & \pm q \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p & 0 \\ \pm q & 0 \end{pmatrix},$$

$$\begin{pmatrix} p & \pm pq \\ \pm qp & q \end{pmatrix} = \begin{pmatrix} p & 0 \\ \pm q & 0 \end{pmatrix} \begin{pmatrix} p & \pm q \\ 0 & 0 \end{pmatrix},$$

and $\begin{pmatrix} p & pq \\ qp & q \end{pmatrix}$, $\begin{pmatrix} p & -pq \\ -qp & q \end{pmatrix}$ commute, we have

$$\begin{aligned} \begin{pmatrix} (p+q)^\natural & 0 \\ 0 & (p+q)^\natural \end{pmatrix} &= 2\begin{pmatrix} p+q & 0 \\ 0 & 0 \end{pmatrix}^{\natural_2} \\ &= \begin{pmatrix} p & pq \\ qp & q \end{pmatrix}^{\natural_2} + \begin{pmatrix} p & -pq \\ -qp & q \end{pmatrix}^{\natural_2} \\ &= 2\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}^{\natural_2} \\ &= 2\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}^{\natural_2} + 2\begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}^{\natural_2} \\ &= \begin{pmatrix} p^\natural & 0 \\ 0 & p^\natural \end{pmatrix} + \begin{pmatrix} q^\natural & 0 \\ 0 & q^\natural \end{pmatrix} \\ &= \begin{pmatrix} p^\natural + q^\natural & 0 \\ 0 & p^\natural + q^\natural \end{pmatrix} . \end{aligned}$$

This can also be deduced from Haagerup's result, taking into account that the C^*-algebra generated by two projections is of type I , hence nuclear, hence exact.

Lemma 2.5. *Let M be a finite AW^*-algebra, whose centre valued quasitrace $\natural$ is k-superadditive for some $k \geq 1$. Further, let $e_1 , \dots , e_n \in M$ be mutually equivalent projections with $\sum\limits_{j=1}^{n} e_j = 1_M$. Then there exists a projection $e_1 \sim p \in M$ such that, for every projection $f \in \{e_1 , \dots , e_n\}' \cap M$,*

$$f^\natural \geq (1 - \|(1_M - f)p\|^2) n^{\frac{1}{k}-1} 1_M .$$

Proof. Let $v_1, \dots v_n \in M$ be partial isometries such that

$$v_j^* v_j = e_1, \quad v_j v_j^* = e_j, \quad 1 \leq j \leq n .$$

Since

$$\left(\frac{1}{\sqrt{n}} \sum_{j=1}^{n} v_j\right)^* \frac{1}{\sqrt{n}} \sum_{j=1}^{n} v_j = \frac{1}{n} \sum_{j_1,j_2=1}^{n} v_{j_1}^* v_{j_2} = \frac{1}{n} \sum_{j=1}^{n} v_j^* v_j = e_1 ,$$

$p = \dfrac{1}{n} \sum\limits_{j_1,j_2=1}^{n} v_{j_1} v_{j_2}^* = \dfrac{1}{\sqrt{n}} \sum\limits_{j=1}^{n} v_j \left(\dfrac{1}{\sqrt{n}} \sum\limits_{j=1}^{n} v_j\right)^*$ is a projection in M equivalent to e_1 .

Now let the projection

$$f \in \{e_1 , \dots , e_n\}' \cap M$$

be arbitrary and set $\delta = \|(1_M - f)p\|$. Since the case $\delta = 1$ is trivial, we can assume without loss of generality that $\delta < 1$. Then

$$\|p - pfp\| = \|(1_M - f)p\|^2 = \delta^2 < 1 ,$$

so $pfp \geq (1 - \delta^2)p$ is invertible in pMp . Thus the polar decomposition $fp = w \cdot |fp|$ exists in the C^*-algebra generated by p and f, and we have

$$w^* w = p, fpf = w(pfp)w^* \geq (1 - \delta^2) ww^* .$$

Let us denote $\zeta = e^{i\frac{n}{\pi}}$. Then

$$u = \sum_{j=1}^{n} \zeta^j e_j \in \{e_1, \dots, e_n, f\}' \cap M$$

is unitary. Since

$$\begin{aligned} u^m p u^{-m} &= \frac{1}{n} \sum_{j, j_1, j_2, j'=1}^{n} \zeta^{mj} e_j v_{j_1} v_{j_2}^* \zeta^{-mj'} e_{j'} \\ &= \frac{1}{n} \sum_{j_1, j_2=1}^{n} \zeta^{m(j_1 - j_2)} v_{j_1} v_{j_2}^* \end{aligned}$$

and

$$\sum_{m=1}^{n} \zeta^{mj} = 0 \text{ for every } 1 \leq j \leq n-1\,,$$

we have

$$\begin{aligned} \sum_{m=1}^{n} u^m p u^{-m} &= \frac{1}{n} \sum_{j_1, j_2=1}^{n} \left(\sum_{m=1}^{n} \zeta^{m(j_1 - j_2)} \right) v_{j_1} v_{j_2}^* \\ &= \frac{1}{n} \sum_{j_1=1}^{n} n v_{j_1} v_{j_1}^* = 1_M. \end{aligned}$$

Therefore

$$f = f \sum_{m=1}^{n} u^m p u^{-m} f = \sum_{m=1}^{n} u^m (fpf) u^{-m} \geq (1-\delta^2) \sum_{m=1}^{n} u^m w w^* u^{-m}$$

and, using the superadditivity of $\natural$, we get

$$\begin{aligned} f^\natural &\geq (1-\delta^2) \left(\sum_{m=1}^{n} u^m w w^* u^{-m} \right)^\natural \\ &\geq (1-\delta^2) \left(\sum_{m=1}^{n} \left((u^m w w^* u^{-m})^\natural \right)^k \right)^{\frac{1}{k}} \\ &= (1-\delta^2) \Big(n \big((w^* w)^\natural \big)^k \Big)^{\frac{1}{k}} = (1-\delta^2) n^{\frac{1}{k}} p^\natural\,. \end{aligned}$$

But $p^\natural = e_j^\natural$ for all $1 \leq j \leq n$, so

$$n p^\natural = \sum_{j=1}^{n} e_j^\natural = \left(\sum_{j=1}^{n} e_j \right)^\natural = 1_M,$$

and we conclude that $f^\natural \geq (1-\delta^2) n^{\frac{1}{k}-1} 1_M$. □

Now we are ready to prove the following

Theorem 2.6 (the abelian strict closure in continuous semi-finite AW^*-algebras)**.** *Let M be a continuous semi-finite AW^*-algebra such that, for some finite projection $e_o \in M$ of central support 1_M and some $k \geq 1$, the centre valued quasitrace of $e_o M e_o$ is k-superadditive. Further, let A denote the norm-closed linear span of all finite projections of M , and C a masa of A . Then the strict closure of C in $M = M(A)$ is C .*

Proof. Let us assume that the strict closure $\overline{C}^{\beta} \subset C' \cap M$ of C contains some $0 \leq x \notin C$.

(a) First we prove that $\overline{C}^{\beta}$ contains some projection $e \notin C$.

Let e_δ denote, for every $\delta > 0$, the support of $(x - \delta 1_M)_+$ in the AW^*-subalgebra $C' \cap M$ of M. Then

$$x e_\delta \geq \delta e_\delta, \quad x(1_M - e_\delta) \leq \delta(1_M - e_\delta).$$

In particular, there exists $0 \leq y \in C' \cap M$ with $yx = e_\delta$. Moreover, $e_\delta \in \overline{C}^{\beta}$. Indeed, by Lemma 1.1 (i) there is a net $(b_\iota)_\iota$ in C with

$$0 \leq b_\iota \leq x \text{ for all } \iota, \quad b_\iota \to x \text{ strictly.}$$

Then $0 \leq y b_\iota \in C' \cap A = C$ for all ι and

$$\|(e_\delta - y b_\iota)a\| = \|y(x - b_\iota)a\| \leq \|y\| \cdot \|(x - b_\iota)a\| \to 0$$

for every $a \in A$.

(b) Next we prove the existence of an infinite sequence of mutually orthogonal projections $0 \neq e_1, e_2, \ldots \in C$, all equivalent in M to $e_o q_o$ for some projection q_o in the centre Z of M, such that $\bigvee_{n \geq 1} e_n \in \overline{C}^{\beta}$.

Let e be a projection as in (a). Then e is not finite, so there exists a projection $q \in Z$ such that eq is properly infinite. But then, by the comparison theorem, there exists a projection $0 \neq q_o \in Z$ such that $e_o q_o \prec eq$. Since the central support of e_o is 1_M, we have $q_o \leq q$.

Now, according to (ii) in Theorem 2.3 (on labeling Murray-von Neumann equivalence classes), there exists a family $(e_\iota)_{\iota \in I}$ of mutually orthogonal projections in C, all equivalent in M to $e_o q_o \neq 0$, such that $\bigvee_{\iota \in I} e_\iota = e q_o$. I must be infinite, so it contains an infinite sequence $\iota_1, \iota_2, \ldots$. Put

$$e_n = e_{\iota_n}, \quad n \geq 1.$$

Then $\bigvee_{n \geq 1} e_n$ belongs to $\overline{C}^{\beta}$. Indeed, since $\bigvee_{n \geq 1} e_n \in C' \cap M$, if $(b_\kappa)_\kappa$ is a net in C which converges strictly to e, then the net $\big(b_\kappa \bigvee_{n \geq 1} e_n\big)_\kappa$ is contained in C and converges clearly to $e \bigvee_{n \geq 1} e_n = \bigvee_{n \geq 1} e_n$ in the strict topology of M.

(c) Finally we prove that the statement in (b) leads to a contradiction.

Let us denote by $\natural$ the map $\bigcup_{n \geq 1} e_n M e_n \to Z q_o$ such that, for every $n \geq 1$, $e_n M e_n \ni x \longmapsto x^\natural e_n$ is the centre valued quasitrace of $e_n M e_n$. It is easy to see that $\natural$ takes the same value in two projections from $\bigcup_{n \geq 1} e_n M e_n$ if and only if they are equivalent in M.

Let $n \geq 1$ be arbitrary and let $j_n = \big[n^{\frac{k+1}{k}}\big] \geq 1$ denote the integer part of $n^{\frac{k+1}{k}}$. According to Corollary 2.4, there exist projections

$$e_{n,1}, \ldots, e_{n,j_n} \in C, \quad \sum_{j=1}^{j_n} e_{n,j} = e_n,$$

such that

$$e_{n,j}^\natural = \frac{1}{j_n} q_o \text{ for all } 1 \leq j \leq j_n.$$

Since $e_n \sim e_o q_o$, the centre valued quasitrace of $e_n M e_n$ is k-superadditive and Lemma 2.5 yields the existence of a projection $p_n \in e_n M e_n$ with $p_n^\natural = \frac{1}{j_n} q_o$ such that, for every projection $g \in \{e_{n,1}, \ldots, e_{n,j_n}\}' \cap e_n M e_n$,

$$g^\natural \geq \big(1 - \|(e_n - g)p_n\|^2\big) \frac{1}{j_n^{\frac{k-1}{k}}} q_o \geq \big(1 - \|(e_n - g)p_n\|^2\big) \frac{1}{n} q_o .$$

Now put $p = \bigvee_{n\geq 1} p_n$. Since $p_n^\natural = \frac{1}{j_n} q_o$ and $\sum_{n\geq 1} \frac{1}{j_n} < +\infty$, using Lemma 2.1 it is easy to verify that p is equivalent to a subprojection of the sum of finitely many e_n's. In particular, p is finite, that is, $p \in A$. Therefore, $\bigvee_{n\geq 1} e_n$ being in $\overline{C}^\beta$, Lemma 1.1 (iii) yields the existence of a projection $\bigvee_{n\geq 1} e_n \geq f \in C$ with

$$\Big\| \Big(\bigvee_{n\geq 1} e_n - f \Big) p \Big\| \leq \frac{1}{\sqrt{2}} .$$

But then, for every $n \geq 1$, fe_n is a projection in $C \cap e_n M e_n \subset \{e_{n,1}, \ldots, e_{n,j_n}\}' \cap e_n M e_n$ and the above yield

$$(fe_n)^\natural \geq \big(1 - \|(e_n - fe_n)p_n\|^2\big) \frac{1}{n} q_o \geq \frac{1}{2n} q_o .$$

Since $\sum_{n\geq 1} \frac{1}{2n} = +\infty$, again using Lemma 2.1, it is easily seen that $f = \bigvee_{n\geq 1}(fe_n)$ is equivalent to $\bigvee_{n\geq 1} e_n$. In particular, f is properly infinite, in contradiction with $f \in C \subset A$. □

3. Weyl-von Neumann-Berg-Sikonia type theorems

We recall that any Rickart C^*-algebra M is σ-normal, which means that, for every increasing sequence $(e_k)_{k\geq 1}$ of projections in M, the least upper bound of $(e_k)_{k\geq 1}$ in the projection lattice of M is actually its least upper bound in the ordered space M_h of all self-adjoint elements of M (see [4] or [27]). Therefore we shall speak in the sequel simply about the least upper bound of increasing sequences of projections in M.

Let us first prove a lemma about the sequential approximability of a projection in a Rickart C^*-algebra from a two-sided ideal :

Lemma 3.1. *Let M be a unital Rickart C^*-algebra, $\mathcal{J}$ a two-sided ideal of M, and $f \in M$ a projection. Then the following statements are equivalent* :

(a) *there exists a sequence $(b_k)_{k\geq 1}$ of positive elements in $\mathcal{J}$ such that $b_k \leq f$ for all $k \geq 1$ and every projection $e \in M$ with $b_k \leq e$, $k \geq 1$ satisfies $f \leq e$;*

(b) *there exists an increasing sequence $(f_k)_{k\geq 1}$ of projections in $\mathcal{J}$, whose least upper bound in M is f.*

Proof. Let us assume that (a) holds and put

$$f_{k,l} = \text{ support of } \Big(b_k - \frac{1}{l} 1_M\Big)_+ \leq f , \qquad k,l \geq 1 ,$$

$$f_n = \bigvee_{1\leq k,l\leq n} f_{k,l} \text{ in the projection lattice of } M \leq f , \qquad n \geq 1 .$$

Since $b_k f_{k,l} \geq \dfrac{1}{l} f_{k,l}$, and so $f_{k,l}$ can be factorized by $b_k \leq f$, we have $f \geq f_{k,l} \in \mathcal{J}$ for all k and l. Further, using the validity of the Parallelogramm Law in all Rickart C^*-algebras (see [6], §13, Th. 1), we also obtain $f \geq f_n \in \mathcal{J}$, $n \geq 1$.

Now $(f_n)_{n\geq 1}$ is an increasing sequence, whose least upper bound in the projection lattice of M is f . Indeed, if $e \in M$ is a projection which majorizes every f_n , hence every $f_{k,l}$, then we have for all k and l

$$b_k^{\frac{1}{2}}(1_M - e)b_k^{\frac{1}{2}} \leq b_k^{\frac{1}{2}}(1_M - f_{k,l})b_k^{\frac{1}{2}} \leq \frac{1}{l}(1_M - f_{k,l}) ,$$
$$\|(1_M - e)b_k^{\frac{1}{2}}\|^2 \leq \frac{1}{l} .$$

Thus

$$b_k = e\, b_k e \leq e \text{ for all } k \geq 1$$

and it follows that $f \leq e$.

Conversely, (b) obviously implies (a) with $b_k = f_k$. □

For unital Rickart C^*-algebras we have the following Weyl-von Neumann-Berg-Sikonia type result (cf. with [32], Theorem 3.1, and [1], §4) :

Theorem 3.2. *Let M be a unital Rickart C^*-algebra, and $\mathcal{J}$ a norm-closed two-sided ideal of M , which contains a sequence of positive elements such that 1_M is the only projection in M majorizing the sequence. Then, for any normal $y \in M$ and every $\varepsilon > 0$, there exist a masa C of $\mathcal{J}$ and an element x of the masa $C' \cap M$ of M , such that*

1) *C contains an increasing sequence of projections, whose least upper bound in M is 1_M ,*
2) *$y - x \in \mathcal{J}$ and $\|y - x\| \leq \varepsilon$.*

Remark 3.3. We notice that in Theorem 3.2 $C' \cap M$ is the sequentially monotone closure of C in the following sense : every $0 \leq a \in C' \cap M$ is the least upper bound in M_h of some increasing sequence of positive elements from $\mathcal{J}$.

Indeed, if $\left(e_k\right)_{k\geq 1}$ is an increasing sequence of projections in C , whose least upper bound in M is 1_M , then $\left(a^{1/2}e_k a^{1/2}\right)_{k\geq 1}$ is an increasing sequence of positive elements from $\mathcal{J}$, whose least upper bound in A_h is $a^{1/2}1_M a^{1/2} = a$ (see [28], 9.14, the remark after Proposition 3). □

For the proof of Theorem 3.2 we need the next result on quasi-central approximate units, implicitly contained in [32], Proposition 1.2 :

Lemma 3.4. *Let M be a unital Rickart C^*-algebra, $\mathcal{J}$ an essential, norm-closed, two-sided ideal of M , and $B \subset M$ a commutative C^*-subalgebra. Then the upward directed set of all projections of $\mathcal{J}$ contains a subnet $(e_\iota)_{\iota\in I}$ which, besides being automatically approximate unit for $\mathcal{J}$, is quasi-central for B , that is,*

$$\lim_{\iota} \| e_\iota b - b\, e_\iota\| = 0 \textit{ for all } b \in B .$$

Proof. Passing to the Rickart C^*-subalgebra of M generated by B and 1_M (see e.g. [28], 9.11 (3)), we can assume without loss of generality that B is a Rickart C^*-subalgebra of M containing 1_M .

Let $\mathcal{P}$ denote the set of all finite sets P of projections from B satisfying the equality $\sum_{p\in P} p = 1_M$ and set

$$I = \{f \in \mathcal{J}\, ; f \text{ projection}\} \times \mathcal{P} .$$

We endow I with a partial order by putting $(f_1, P_1) \leq (f_2, P_2)$ whenever $f_1 \leq f_2$ and the C^*-algebra $C^*(P_1)$ generated by P_1 is contained in $C^*(P_2)$ (that is, the partition P_2 is a refinement of P_1). Clearly, in this way I becomes an upward directed ordered set.

Let $\iota = (f, P) \in I$ be arbitrary. For every $p \in P$, the right support $\mathbf{r}(fp)$ of fp is equivalent in M to the left support $\mathbf{l}(fp) \leq f \in \mathcal{J}$ (see [2] or [3]), so it belongs to $\mathcal{J}$. Thus

$$e_\iota = \sum_{p\in P} \mathbf{r}(fp)$$

is a projection in $\mathcal{J}$. Since every $\mathbf{r}(fp) \leq p$ belongs to the commutant P', then also $e_\iota \in P'$. Furthermore,

$$f \leq e_\iota .$$

Indeed, for every $q \in P$,

$$fq = fq\,\mathbf{r}(fq) = \sum_{p\in P} fq\,\mathbf{r}(fp) = fq\,e_\iota ,$$

so

$$f = f\sum_{q\in P} q = \sum_{q\in P} fqe_\iota = fe_\iota \leq e_\iota .$$

It is easily seen that

$$\iota_1 \leq \iota_2 \Rightarrow e_{\iota_1} \leq e_{\iota_2} ,$$

so $(e_\iota)_{\iota\in I}$ is a subnet of the upward directed set of all projections of $\mathcal{J}$.

Now, the upward directed set of all projections f of $\mathcal{J}$ is an increasing approximate unit for $\mathcal{J}$. Indeed, $\big\{x \in \mathcal{J}\,;\, \lim\limits_f \|x(1_M - f)\| = 0\big\}$ is a norm-closed linear subspace of $\mathcal{J}$ containing all projections from $\mathcal{J}$, hence it is equal to $\mathcal{J}$. Thus also the subnet $(e_\iota)_{\iota\in I}$ is an approximate unit for $\mathcal{J}$.

On the other hand, the norm-closed linear subspace $\big\{b \in B; \lim\limits_\iota \|e_\iota b - be_\iota\| = 0\big\}$ contains every projection from B: for any projection $p \in B$ and every $\iota = (f, P)$ with $p \in C^*(P)$ we have $e_\iota \in P' \cap A = C^*(P)' \cap \mathcal{J}$, so $e_\iota p - pe_\iota = 0$. Consequently the above subspace of B is actually equal to B. □

Proof of Theorem 3.2. Put $y_1 = \dfrac{1}{2}(y + y^*)$, $y_2 = \dfrac{1}{2i}(y - y^*)$ and

$$p_j(\lambda) = \text{ support of } (y_j - \lambda 1_M) \text{ in } M , \quad \lambda \in \mathbb{R} .$$

Further, let $\{\lambda_1, \lambda_2, \dots\}$ be the countable set of all rational numbers. Then

$$a = \sum_{k=1}^{\infty} 3^{-(2k-1)}(2p_1(\lambda_k) - 1_{A^{**}}) + \sum_{k=1}^{\infty} 3^{-2k}(2p_2(\lambda_k) - 1_{A^{**}}) + \frac{1}{2}1_{A^{**}} \in M ,$$
$$0 \leq a \leq 1_M$$

and it is easy to see that the C^*-subalgebra of M generated by a and 1_M contains all projections $p_j(\lambda)$, $j = 1, 2$, $\lambda \in \mathbb{Q}$, hence also $y = y_1 + iy_2$. Therefore there exists a continuous function $f : [0, +\infty) \to \mathbb{C}$ such that $y = f(a)$. Furthermore, by a well known continuity property of the functional calculus (see e.g. [28], 1.18 (5)), there exists some $\delta > 0$ such that

$$0 \leq b \in M ,\ \|a - b\| \leq \delta \implies \|f(a) - f(b)\| \leq \varepsilon .$$

Now, by Lemma 3.1, there exists an increasing sequence $\big(f_k\big)_{k\geq 1}$ of projections in $\mathcal{J}$, whose least upper bound in M is 1_M . Using Lemma 3.4, we can then construct by induction a sequence $0 = e_o \leq e_1 \leq e_2 \leq \dots$ of projections in $\mathcal{J}$ such that

$$f_k \leq e_k\,, \quad \|e_k a - a e_k\| \leq 2^{-k-1}\delta\,.$$

Since the elements e_k and $(e_k - e_{k-1})a(e_k - e_{k-1})$ of $\mathcal{J}$ are mutually commuting, there exists a masa C of $\mathcal{J}$ containing all of them. Then C contains the increasing projection sequence $(e_k)_{k\geq 1}$, whose least upper bound in M is 1_M .

Let us denote

$$\begin{aligned} b_o &= a\,, \\ b_n &= \sum_{k=1}^{n}(e_k - e_{k-1})a(e_k - e_{k-1}) + (1_M - e_n)a(1_M - e_n)\,, \quad n \geq 1\,. \end{aligned}$$

Then, for every $n \geq 1$,

$$\begin{aligned} & b_{n-1} - b_n \\ &= (1_M - e_{n-1})\big(a - (1_M - e_n)a(1_M - e_n) - e_n a e_n\big)(1_M - e_{n-1}) \\ &= (1_M - e_{n-1}) \cdot [e_n\,,\, e_n a - a e_n] \cdot (1_M - e_{n-1})\,, \\ & \|b_{n-1} - b_n\| \leq 2\|e_n a - a e_n\| \leq 2^{-n}\delta\,. \end{aligned}$$

It follows that $\sum\limits_{n=1}^{\infty} \|b_{n-1} - b_n\| \leq \delta$, so the sequence $(b_n)_{n\geq 1}$ is norm convergent to some $b \in M(A)^+$ and

$$\|a - b\| = \lim_{n\to\infty} \|b_o - b_n\| \leq \delta\,.$$

Put $x = f(b)$.

We claim that $b \in C' \cap M$, hence also $x \in C' \cap M$. Since $C' \cap M$ is a masa of M (see Lemma 1.1 (iv)), it is enough to prove that b is commuting with all elements $a' \in C' \cap M \subset \big\{e_k\,, (e_k - e_{k-1})a(e_k - e_{k-1})\,;\, k \geq 1\big\}' \cap M$. We notice that, for every $n \geq 1$,

$$b_n a' - a' b_n = (1_M - e_n)(a a' - a' a)(1_M - e_n)\,,$$

hence

$$\begin{aligned} |b_n a' - a' b_n|^2 &\leq (1_M - e_n)|a a' - a' a|^2(1_M - e_n) \\ &\leq \|a a' - a' a\|^2(1_M - e_n)\,. \end{aligned}$$

Therefore

$$|b_n a' - a' b_n|^2 \leq \|a a' - a' a\|^2(1_M - e_k)\,, \quad n \geq k \geq 1,$$

and, passing to the limit for $n \to \infty$, we get for every $k \geq 1$

$$|b a' - a' b|^2 \leq \|a a' - a' a\|^2(1_M - e_k)\,,$$
$$\text{support of } |b a' - a' b|^2 \text{ in } M \text{ is} \leq 1_M - e_k\,.$$

Since the least upper bound of $(e_k)_{k\geq 1}$ in M is 1_M , it follows that $b a' - a' b = 0$.

Finally, according to the choice of δ, $\|a-b\| \le \delta$ implies that

$$\|y-x\| = \|f(a)-f(b)\| \le \varepsilon .$$

On the other hand,

$$\begin{aligned} a - b_n &= \sum_{k=1}^{n} (b_{k-1}-b_k) \\ &= \sum_{k=1}^{n} (1_M - e_{k-1}) \cdot [e_k \, , \, e_k a - a e_k] \cdot (1_M - e_{k-1}) \in \mathcal{J} \end{aligned}$$

implies by passing to the limit for $n \to \infty$ that $a-b \in \mathcal{J}$. Using the Weierstrass Approximation Theorem, we infer that $y-x = f(a)-f(b) \in \mathcal{J}$. □

We shall prove that in Theorem 3.2 the element x can be found under the form of an "infinite linear combination" of a sequence of mutually orthogonal projections from $\mathcal{J}$. To this aim we need an appropriate understanding of the summation of series in Rickart C^*-algebras.

We recall that every commutative Rickart C^*-algebra C is sequentially monotone complete (see e.g. [28], 9.16, Proposition 1). Thus, if $(a_k)_{k\ge 1}$ is a sequence in C^+ such that the partial sums $\sum_{k=1}^{n} a_k$, $n \ge 1$, are bounded, then there exists the least upper bound in C_h,

$$\sum_{k=1}^{\infty} a_k = \sup \left\{ \sum_{k=1}^{n} a_k \, ; \, n \ge 1 \right\} \in C^+ .$$

Next let M be an arbitrary Rickart C^*-algebra, $(a_k)_{k\ge 1}$ a bounded sequence in M^+ such that the supports $\mathbf{s}(a_k)$, $k \ge 1$, are mutually orthogonal, and $(e_k)_{k\ge 1}$ a sequence of mutually orthogonal projections in M, for which $\mathbf{s}(a_k) \le e_k$, $k \ge 1$ (we can take, for example, $e_k = \mathbf{s}(a_k)$). Then $\{a_k \, ; \, k \ge 1\} \cup \{e_k \, ; \, k \ge 1\}$ generates a commutative Rickart C^*-subalgebra C of M, so there exists $a = \sum_{k=1}^{\infty} a_k \in C^+$. Moreover, a is the least upper bound of the partial sums $\{ \sum_{k=1}^{n} a_k \, ; \, n \ge 1 \}$ even in M_h . Indeed, by the σ-normality of the Rickart C^*-algebras, $\bigvee_{k=1}^{\infty} e_k$ is the least upper bound in M_h of the sequence $\left(\bigvee_{k=1}^{n} e_k \right)_{n\ge 1}$, and it follows that

$$a = a^{1/2} \left(\bigvee_{k=1}^{\infty} e_k \right) a^{1/2} \text{ is the least upper bound in } M_h \text{ of}$$

$$\text{the increasing sequence } a^{1/2} \left(\bigvee_{k=1}^{n} e_k \right) a^{1/2} = \sum_{k=1}^{n} a_k \, , \, n \ge 1$$

(see [28], 9.14, the remark after Proposition 3). In particular, a is the only element of M_h satisfying the conditions

$$a\, e_k = a_k \, , \, k \ge 1 \, , \qquad \mathbf{s}(a) \le \bigvee_{k=1}^{\infty} e_k \, .$$

For the sake of completeness we notice that, by the above characterization, if $\big(e_k\big)_{k\geq 1}$ is a sequence of mutually orthogonal projections in M, then $\sum_{k=1}^{\infty} e_k = \bigvee_{k=1}^{\infty} e_k$.

Now let $\big(x_k\big)_{k\geq 1}$ be a bounded sequence in M such that, denoting by $\mathbf{l}(x_k)$ the left support of x_k and by $\mathbf{r}(x_k)$ the right one, the projections $\mathbf{l}(x_k)\vee\mathbf{r}(x_k)$, $k\geq 1$, are mutually orthogonal. Then we can define

$$\sum_{k=1}^{\infty} x_k = \Big(\sum_{k=1}^{\infty}(\mathrm{Re}\ x_k)_+ - \sum_{k=1}^{\infty}(\mathrm{Re}\ x_k)_-\Big) + i\Big(\sum_{k=1}^{\infty}(\mathrm{Im}\ x_k)_+ - \sum_{k=1}^{\infty}(\mathrm{Im}\ x_k)_-\Big).$$

It is easy to see that, if $\big(e_k\big)_{k\geq 1}$ is any sequence of mutually orthogonal projections in M such that $\mathbf{l}(x_k)\vee\mathbf{r}(x_k)\leq e_k$, $k\geq 1$, then $\sum_{k=1}^{\infty} x_k$ is the only element $x\in M$ for which

$$x\,e_k = e_k x = x_k\,,\ k\geq 1\,, \qquad \mathbf{l}(x)\vee\mathbf{r}(x)\leq \bigvee_{k=1}^{\infty} e_k\,. \tag{3.1}$$

By the above, considering the direct product C^*-algebra

$$\bigoplus_{k=1}^{\infty} e_k M e_k = \Big\{ (y_k)_{k\geq 1}\in \prod_{k=1}^{\infty} e_k M e_k\,;\ \sup_{k\geq 1}\|y_k\| < +\infty \Big\},$$

the mapping

$$\bigoplus_{k=1}^{\infty} e_k M e_k \ni (y_k)_{k\geq 1} \longmapsto \sum_{k=1}^{\infty} y_k \in M$$

is well defined and it is an injective $*$-homomorphism. Consequently

$$\Big\| \sum_{k=1}^{\infty} x_k \Big\| = \sup_{k\geq 1}\|x_k\|\,. \tag{3.2}$$

Finally, let $\big(e_k\big)_{k\geq 1}$ be a sequence of mutually orthogonal projections in M, and

$$(x_k)_{k\geq 1}\,,\ (y_k)_{k\geq 1} \in \bigoplus_{k=1}^{\infty} e_k M e_k\,.$$

Denoting by $\overline{\mathrm{lin}}\big\{x_k - y_k\,;\, k\geq 1\big\}$ the norm-closed linear subspace of M generated by $\big\{x_k - y_k\,;\, k\geq 1\big\}$, we have

$$\sum_{k=1}^{\infty} x_k - \sum_{k=1}^{\infty} y_k \in \overline{\mathrm{lin}}\big\{x_k - y_k\,;\, k\geq 1\big\}\ \text{ if }\ \|x_k - y_k\| \longrightarrow 0\,. \tag{3.3}$$

Indeed, according to (3.2), we have :

$$\Big\| \sum_{k=1}^{\infty} x_k - \sum_{k=1}^{\infty} y_k - \underbrace{\sum_{k=1}^{n}(x_k - y_k)}_{\in\,\overline{\mathrm{lin}}\{x_k - y_k\,;\,k\geq 1\}} \Big\| = \sup_{k\geq n+1}\|x_k - y_k\| \xrightarrow{n\to\infty} 0\,.$$

A slight modification of the proof of Theorem 3.2 yields the following Weyl-von Neumann-Berg-Sikonia type result, which is much closer to [32], Theorem 3.1, than Theorem 3.2.

Theorem 3.5. *Let M be a unital Rickart C^*-algebra, and $\mathcal{J}$ a norm-closed two-sided ideal of M which contains a sequence of positive elements such that 1_M is the only projection in M majorizing the sequence. Then, for any normal $y \in M$ and every $\varepsilon > 0$, there are*

- a sequence $(p_k)_{k\geq 1}$ of mutually orthogonal projections in $\mathcal{J}$,
- a sequence $(\lambda_k)_{k\geq 1}$ in the spectrum $\sigma(y)$ of y,

such that

1) *the least upper bound of $(p_n)_{n\geq 1}$ in M is 1_M,*
2) $y - \sum_{k=1}^{\infty} \lambda_k \, p_k \in \mathcal{J}$ *and* $\left\| y - \sum_{k=1}^{\infty} \lambda_k \, p_k \right\| \leq \varepsilon$.

Proof. Repeating word for word the arguments from the first paragraph of the proof of Theorem 3.2, we get $a \in M$ with $0 \leq a \leq 1_M$, a continuous function $f : [0, +\infty) \to \mathbb{C}$ and $\delta > 0$, such that $y = f(a)$ and

$$0 \leq b \in M \, , \; \|a - b\| \leq \delta \implies \|f(a) - f(b)\| \leq \varepsilon \, . \tag{3.4}$$

Subtracting from a an appropriate positive multiple of 1_M and modifying f correspondingly, if necessary, we can assume that $0 \in \sigma(a)$.

Choose a sequence $\delta/3 = \delta_1 > \delta_2 > \ldots > 0$ which converges to 0. According to the upper semicontinuity of the spectrum, there exist

$$\begin{array}{ccccccc} & & \eta_1 & > & \eta_2 & > & \ldots \; > \; 0 \\ & & \wedge & & \wedge & & \\ \delta/3 & = & \delta_1 & > & \delta_2 & > & \ldots \end{array}$$

such that the spectrum of every $b \in M$ with $\|a - b\| \leq \eta_k$ is contained in

$$U_{\delta_k}\big(\sigma(a)\big) = \big\{\mu \in \mathbb{C} \, ; \, |\mu - \lambda(\mu)| < \delta_k \; \text{ for some } \; \lambda(\mu) \in \sigma(a)\big\} \, .$$

Arguing again as in the proof of Theorem 3.2, we can construct a sequence $0 = e_o \leq e_1 \leq e_2 \leq \ldots$ of projections in $\mathcal{J}$, whose least upper bound in M is 1_M, such that

$$\|e_k a - a e_k\| \leq 2^{-k-1} \eta_{k+1} \; \text{ for all } \; k \geq 1 \, .$$

Then setting

$$b_o = a \, ,$$
$$b_n = \sum_{k=1}^{n} (e_k - e_{k-1}) a (e_k - e_{k-1}) + (1_M - e_n) a (1_M - e_n) \, , \quad n \geq 1 \, ,$$

we have

$$b_{n-1} - b_n = (1_M - e_{n-1}) \cdot [e_n \, , \, e_n a - a e_n] \cdot (1_M - e_{n-1}) \, , \quad n \geq 1 \, ,$$

so $\|b_{n-1} - b_n\| \leq 2^{-n} \eta_{n+1} \leq 2^{-n} \delta/3$ and $b_{n-1} - b_n \in \mathcal{J}$. Therefore the sequence $(b_n)_{n\geq 1}$ is norm convergent to some $b_\infty \in M^+$, for which $\|a - b_\infty\| \leq \delta/3$ and $a - b_\infty \in \mathcal{J}$.

We claim that

$$b_\infty = \sum_{k=1}^{\infty} (e_k - e_{k-1}) a (e_k - e_{k-1}) \, .$$

Indeed, since

$$b_n \, (e_k - e_{k-1}) = (e_k - e_{k-1}) \, b_n = (e_k - e_{k-1}) a (e_k - e_{k-1}) \, , \qquad n \geq k \geq 1 \, ,$$

by passing to the limit for $n \to \infty$ we get

$$b_\infty\,(e_k - e_{k-1}) = (e_k - e_{k-1})\,b_\infty = (e_k - e_{k-1})a(e_k - e_{k-1})\,, \qquad k \geq 1\,.$$

Thus, taking into account that $\bigvee_{k=1}^{\infty}(e_k - e_{k-1}) = \bigvee_{k=1}^{\infty} e_k = 1_M$, the description (3.1) yields the desired equality.

We notice that, for every $k \geq 1$,

$$\sigma\big((e_k - e_{k-1})a(e_k - e_{k-1})\big) \subset U_{\delta_k}\big(\sigma(a)\big)\,. \tag{3.5}$$

Indeed, since the norm of

$$\begin{aligned} &a - \Big((e_k - e_{k-1})a(e_k - e_{k-1}) + \big(1_{A^{**}} - (e_k - e_{k-1})\big)a\big(1_{A^{**}} - (e_k - e_{k-1})\big)\Big) \\ &= \Big[[e_k - e_{k-1}, a]\,, 1_{A^{**}} - (e_k - e_{k-1})\Big] \end{aligned}$$

is majorized by $2\left(\|e_k a - a e_k\| + \|e_{k-1} a - a e_{k-1}\|\right) \leq 2\left(2^{-k-2}\eta_{k+1} + 2^{-k-1}\eta_k\right) < \eta_k$, by the choice of η_k we have

$$\begin{aligned} &\sigma\big((e_k - e_{k-1})a(e_k - e_{k-1})\big) \\ &\subset \sigma\Big((e_k - e_{k-1})a(e_k - e_{k-1}) + \big(1_{A^{**}} - (e_k - e_{k-1})\big)a\big(1_{A^{**}} - (e_k - e_{k-1})\big)\Big) \cup \{0\} \\ &\subset U_{\delta_k}\big(\sigma(a)\big)\,. \end{aligned}$$

For any $k \geq 1$, let $[r_1^{(k)}, r_2^{(k)}]$ denote the smallest compact interval in $\mathbb{R}$ containing the spectrum $\sigma\big((e_k - e_{k-1})a(e_k - e_{k-1})\big)$. Choose

$$r_1^{(k)} = \mu_1^{(k)} < \,\ldots\, < \mu_j^{(k)} < \,\ldots\, < \mu_{j_k}^{(k)} = r_2^{(k)}$$

in $\sigma\big((e_k - e_{k-1})a(e_k - e_{k-1})\big)$ such that $|\mu_j^{(k)} - \mu_{j-1}^{(k)}| \leq \eta_k$ for all $2 \leq j \leq j_k$. Then there exist mutually orthogonal projections $\big(p_j^{(k)}\big)_{1 \leq j \leq j_k}$ in $\mathcal{J}$ such that

$$\sum_{j=1}^{j_k} p_j^{(k)} = e_k - e_{k-1} \;\text{ and }\; \Big\|(e_k - e_{k-1})a(e_k - e_{k-1}) - \sum_{j=1}^{j_k} \mu_j^{(k)} p_j^{(k)}\Big\| \leq \eta_k\,.$$

For example, we can set $p_j^{(k)} = e_j^{(k)} - e_{j+1}^{(k)}$, $1 \leq j \leq j_k$, where

$$e_j^{(k)} = \mathbf{s}\Big(\big((e_k - e_{k-1})a(e_k - e_{k-1}) - \mu_j^{(k)}\,(e_k - e_{k-1})\big)_+\Big)\,, \qquad 1 \leq j \leq j_k,$$

and $e_{j_k+1}^{(k)} = 0$ (see e.g. [28], 9.9, Proposition 1). Using (3.5), we can find for every $\mu_j^{(k)}$ some $\lambda_j^{(k)} \in \sigma(a)$ with $|\lambda_j^{(k)} - \mu_j^{(k)}| < \delta_k$ and then

$$\Big\|(e_k - e_{k-1})a(e_k - e_{k-1}) - \sum_{j=1}^{j_k} \lambda_j^{(k)} p_j^{(k)}\Big\| \leq \eta_k + \delta_k < 2\,\delta_k \leq 2\,\delta/3\,.$$

Now $\bigcup_{k=1}^{\infty}\big\{p_j^{(k)}\,;\, 1 \leq j \leq j_k\big\}$ consists of mutually orthogonal projections in M , whose least upper bound in M is 1_M , while $\bigcup_{k=1}^{\infty}\big\{\lambda_j^{(k)}\,;\, 1 \leq j \leq j_k\big\} \subset \sigma(a)$. Set

$b = \sum_{k=1}^{\infty} \sum_{j=1}^{j_k} \lambda_j^{(k)} p_j^{(k)} \in M^+$. Then (3.2) yields

$$\|b_\infty - b\| = \Big\| \sum_{k=1}^{\infty} (e_k - e_{k-1}) a (e_k - e_{k-1}) - \sum_{k=1}^{\infty} \sum_{j=1}^{j_k} \lambda_j^{(k)} p_j^{(k)} \Big\|$$
$$= \sup_{k \geq 1} \Big\| (e_k - e_{k-1}) a (e_k - e_{k-1}) - \sum_{j=1}^{j_k} \lambda_j^{(k)} p_j^{(k)} \Big\| \leq 2\,\delta/3\,,$$

so $\|a - b\| \leq \|a - b_\infty\| + \|b_\infty - b\| \leq \delta/3 + 2\,\delta/3 = \delta$. On the other hand, since

$$\Big\| \underbrace{(e_k - e_{k-1}) a (e_k - e_{k-1}) - \sum_{j=1}^{j_k} \lambda_j^{(k)} p_j^{(k)}}_{\in \mathcal{J}} \Big\| < 2\,\delta_k \longrightarrow 0\,,$$

(3.3) implies that $b_\infty - b \in \mathcal{J}$, hence $a - b = (a - b_\infty) + (b_\infty - b) \in \mathcal{J}$.

Using the characterization (3.1), it is easy to deduce that

$$f(b) = \sum_{k=1}^{\infty} \sum_{j=1}^{j_k} f(\lambda_j^{(k)})\, p_j^{(k)}\,,$$

where, by the Spectral Mapping Theorem, $\bigcup_{k=1}^{\infty} \big\{ f(\lambda_j^{(k)})\,;\, 1 \leq j \leq j_k \big\}$ is contained in $f\big(\sigma(a)\big) = \sigma\big(f(a)\big) = \sigma(y)$. On the other hand, (3.4) yields the norm estimation $\|y - f(b)\| = \|f(a) - f(b)\| \leq \varepsilon$. Finally, using $a - b \in \mathcal{J}$ and the Weierstrass Approximation Theorem, we infer also that $y - f(b) = f(a) - f(b) \in \mathcal{J}$. □

If in the above theorem we are not requiring the norm estimation in 2), then the coefficients λ_k can be chosen even in the essential spectrum of y modulo $\mathcal{J}$:

Theorem 3.6. *Let M be a unital Rickart C^*-algebra, and $\mathcal{J}$ a norm-closed two-sided ideal of M, which contains a sequence of positive elements such that 1_M is the only projection in M majorizing the sequence. For any normal $y \in M$ there are*

- *a sequence $(p_k)_{k \geq 1}$ of mutually orthogonal projections in $\mathcal{J}$,*
- *a sequence $(\lambda_k)_{k \geq 1}$ in the spectrum $\sigma_{\mathcal{J}}(y)$ of the canonical image of y in the quotient C^*-algebra $M/\mathcal{J}$*

such that

1) *the least upper bound of $(p_n)_{n \geq 1}$ in M is 1_M ,*
2) $y - \sum_{k=1}^{\infty} \lambda_k\, p_k \in \mathcal{J}$.

For the proof we need the next lifting result, which is essentially [32], Proposition 2.1 :

Lemma 3.7. *Let M be a unital Rickart C^*-algebra, and $\mathcal{J}$ a norm-closed two-sided ideal of M . For any self-adjoint $a \in M$ there exists a self-adjoint $b \in M$ such that $\sigma(b) = \sigma_{\mathcal{J}}(b)$ and $a - b \in \mathcal{J}$.*

Proof. A moment's reflection shows that the proof of [32], Proposition 2.1, works for M unital Rickart C^*-algebra instead of W^*-algebra. □

Proof of Theorem 3.6. Again repeating the arguments from the first paragraph of the proof of Theorem 3.2, we get some $a \in M$ with $0 \leq a \leq 1_M$ and a continuous function $f : [0, +\infty) \to \mathbb{C}$ such that $y = f(a)$. Now, according to Lemma 3.7, there exists a self-adjoint $b \in M$ such that $\sigma(b) = \sigma_{\mathcal{J}}(b)$ and $a - b \in \mathcal{J}$. In particular, $\sigma(b) = \sigma_{\mathcal{J}}(a) \subset [0,1]$, and so $0 \leq b \leq 1_M$.

Let x denote the normal element $f(b)$. Using the Weierstrass Approximation Theorem, we infer that $y - x \in \mathcal{J}$; hence, by the Spectral Mapping Theorem, $\sigma(x) = f\big(\sigma(b)\big) = f\big(\sigma_{\mathcal{J}}(a)\big) = \sigma_{\mathcal{J}}(y)$. Now Theorem 3.5 yields the existence of

- a sequence $(p_k)_{k\geq 1}$ of mutually orthogonal projections in $\mathcal{J}$,
- a sequence $(\lambda_k)_{k\geq 1}$ in $\sigma(x) = \sigma_{\mathcal{J}}(y)$,

such that the least upper bound of $(p_n)_{n\geq 1}$ in M is 1_M and $x - \sum_{k=1}^{\infty} \lambda_k\, p_k \in \mathcal{J}$. Then $y - \sum_{k=1}^{\infty} \lambda_k\, p_k = (y - x) + \big(x - \sum_{k=1}^{\infty} \lambda_k\, p_k\big) \in \mathcal{J}$. □

Let us say that a C^*-algebra A is σ-subunital if there exists a sequence $(b_n)_{n\geq 1}$ in A^+, whose least upper bound in $M(A)_h$ is $1_{A^{**}}$. Clearly, if A is σ-unital, then it is σ-subunital. For commutative A the two notions coincide. However, if M is a countably decomposable type II_∞-factor and A is the norm-closed linear span of all finite projections of M, then A is not σ-unital (see [1], Proposition 4.5), but it is easily seen that it is σ-subunital.

We remark that the sequence $(b_n)_{n\geq 1}$ in the definition of the σ-subunitalness can be considered a kind of "approximate unit with respect to the order structure". Indeed, according to [28], 9.14, the remark after Proposition 3, if the least upper bound of $(b_n)_{n\geq 1}$ in $M(A)_h$ is $1_{A^{**}}$ and $x \in M(A)$, then the least upper bound of the sequence $\big(x^* b_n x\big)_{n\geq 1}$ in $M(A)_h$ is $x^* x$.

By Theorems 3.5 and 3.6 we have :

Corollary 3.8. *Let A be a σ-subunital C^*-algebra, whose multiplier algebra $M(A)$ is a Rickart C^*-algebra. For any normal $y \in M(A)$ and any $\varepsilon > 0$ there exist*

- *a sequence $(p_k)_{k\geq 1}$ of mutually orthogonal projections in A,*
- *a sequence $(\lambda_k)_{k\geq 1}$ in the spectrum $\sigma(y)$ of y,*

such that

1) *the least upper bound of $(p_n)_{n\geq 1}$ in $M(A)_h$ is $1_{A^{**}}$,*
2) $y - \sum_{k=1}^{\infty} \lambda_k\, p_k \in A$ *and* $\Big\| y - \sum_{k=1}^{\infty} \lambda_k\, p_k \Big\| \leq \varepsilon$.

Moreover, if we don't require the second inequality in 2)*, then the sequence $(\lambda_k)_{k\geq 1}$ can be chosen even in the spectrum of the canonical image of y in the corona algebra $C(A) = M(A)/A$.* □

In particular, the above corollary can be applied to $A = K(H)$, where H is a separable complex Hilbert space, in which case the series $\sum_{k=1}^{\infty} \lambda_k\, p_k$ converges even with respect to the strict topology of $M(A) = B(H)$. This is the statement of the classical Weyl-von Neumann-Berg-Sikonia Theorem, but convergence with respect to the strict topology is also used in its subsequent extensions to σ-unital C^*-algebras with real rank zero multiplier algebra (see e.g. [23], [31], [12], [19], [20], [21]).

On the other hand, in the early extension from [32] of the Weyl-von Neumann-Berg-Sikonia Theorem to the norm-closed linear span A of all finite projections of an arbitrary semi-finite W^*-factor M, which for M of type II_∞ turns out not to be σ-unital, the series $\sum_{k=1}^{\infty} \lambda_k\, p_k$ is proved to converge only with respect to the s^*-topology. The reason why here a weaker topology than the strict topology should be used, is given by Theorem 2.6: if M is a type II_∞ W^*-factor and we assume that a sum $\sum_{k=1}^{\infty} \lambda_k \, p_k$ with $p_k \in A$ is strictly convergent, then, according to Theorem 2.6, we must have $\sum_{k=1}^{\infty} \lambda_k \, p_k \in A$.

Appendix

We give here, for the convenience of the reader, a treatment of a set-theoretical result of T. Iwamura (see [22], Appendix II) and two applications to the theory of AW^*-algebras.

Proposition. *Let $I, \leq$ be an upward directed partially ordered uncountable set. Then, there exist a well order $\preccurlyeq$ on I and a family $(I_\iota)_{\iota \in I}$ of subsets of I such that*

- *I_ι is upward directed for every $\iota \in I$,*
- *$\mathrm{card}\, I_\iota < \mathrm{card}\, I$, $\quad \iota \in I$,*
- *$I_{\iota_1} \subset I_{\iota_2}$ whenever $\iota_1 \prec \iota_2$,*
- *$\bigcup_{\iota \in I} I_\iota = I$.*

Proof. By Zermelo's theorem there exists a well order $\preccurlyeq$ on I. We can choose it such that

$$(\star) \qquad \mathrm{card}\, \{\iota' \in I \,;\, \iota' \prec \iota\} < \ \mathrm{card}\, I \text{ for every } \iota \in I .$$

Indeed, if there exists some $\iota \in I$ such that

$$\mathrm{card}\, \{\iota' \in I \,;\, \iota' \prec \iota\} = \mathrm{card}\, I ,$$

then there exists a smallest ι with respect to $\preccurlyeq$, having the above property. Choose for this ι a bijection

$$\Phi : I \to \{\iota' \in I \,;\, \iota' \prec \iota\}$$

and replace $\preccurlyeq$ by the well order, according to which ι_1 less than or equal to ι_2 means $\Phi(\iota_1) \preccurlyeq \Phi(\iota_2)$.

We notice that, I being infinite, $(\star)$ implies that I does not contain a largest element with respect to $\preccurlyeq$.

Let us denote

$$J_\iota = \{\iota' \in I;\ \iota' \prec i\}, \quad \iota \in I .$$

Then

$$\mathrm{card}\, J_\iota < \mathrm{card}\, I , \quad \iota \in I ,$$
$$J_{\iota_1} \subset J_{\iota_2} \text{ whenever } \iota_1 \prec \iota_2,$$
$$\bigcup_{\iota \in I} J_\iota = I .$$

On the other hand, $I, \leq$ being upward directed, we can choose for each finite $F \subset I$ some $\iota(F) \in I$ such that

$$\iota \leq \iota(F) \text{ for all } \iota \in F .$$

Denote for every $J \subset I$

$$D_1(J) = J \cup \{\iota(F) \,;\, F \subset J \text{ finite}\} .$$

We notice that

$$D_1(J) \text{ is finite for } J \text{ finite},$$
$$\mathrm{card}\, D_1(J) = \mathrm{card}\, J \text{ for } J \text{ infinite}$$

and

$$D_1(J_1) \subset D_1(J_2) \text{ whenever } J_1 \subset J_2\,.$$

Now we define by recursion

$$D_{n+1}(J) = D_1(D_n(J)) \supset D_n(J),\ n \geq 1 \text{ integer},$$
$$D_\omega(J) = \bigcup_{n\geq 1} D_n(J)\,.$$

Then

$$D_\omega(J) \text{ is countable for } J \text{ finite},$$
$$\text{card } D_\omega(J) = \text{ card } J \text{ for } J \text{ infinite}$$

and

$$D_\omega(J_1) \subset D_\omega(J_2) \text{ whenever } J_1 \subset J_2\,.$$

Moreover, $D_\omega(J), \leq$ is upward directed for every $J \subset I$.

Now, putting

$$I_\iota = D_\omega(J_\iota), \quad \iota \in I\,,$$

it is easy to see that all conditions from the statement are satisfied. □

The first corollary extends Lemma 2.2 (compare with [6], §33, Exercise 1):

Corollary 1. *Let M be an AW^*-algebra, $f \in M$ a finite projection, and $(e_\iota)_{\iota\in I}$ an upward directed family of projections in M such that*

$$e_\iota \prec f \textit{ for all } \iota \in I\,.$$

Then

$$\bigvee_{\iota\in I} e_\iota \prec f\,.$$

Proof. The case of countable I can easily be reduced to Lemma 2.2. Indeed, choosing a cofinal sequence $\iota_1 \leq \iota_2 \leq \ldots$ in I, we have

$$\bigvee_{\iota\in I} e_\iota = \bigvee_{n\geq 1} e_{\iota_n} = e_{\iota_1} \vee \bigvee_{n\geq 1} (e_{\iota_{n+1}} - e_{\iota_n})$$

and we can apply Lemma 2.2 to f and the family $e_{\iota_1}\,, e_{\iota_2} - e_{\iota_1}\,, e_{\iota_3} - e_{\iota_2}\,, \ldots$.

For the proof in the general case let $f \in M$ be a finite projection and let us assume the existence of some upward directed family $(e_\iota)_{\iota\in I}$ of projections in M such that

$$e_\iota \prec f \text{ for all } \iota \in I\,, \text{ but } \bigvee_{\iota\in I} e_\iota \not\prec f\,.$$

Choose among all such families one with I of the smallest cardinality. By the first part of the proof I is then uncountable.

Let the well order $\preccurlyeq$ on I and the family $(I_\iota)_{\iota\in I}$ of subsets of I be as in the above proposition.

According to the minimality property of card I, we have

$$p_\iota = \bigvee_{\iota'\in I_\iota} e_{\iota'} \prec f, \quad \iota \in I\,.$$

On the other hand,

$$p_{\iota_1} \le p_{\iota_2} \text{ whenever } \iota_1 \prec \iota_2 ,$$
$$\bigvee_{\iota\in I} p_\iota = \bigvee_{\iota\in I} e_\iota .$$

Consequently, denoting

$$q_\iota = p_\iota - \bigvee_{\iota'\prec\iota} p_{\iota'} \le p_\iota , \quad \iota \in I ,$$

the projections $(q_\iota)_{\iota\in I}$ are mutually orthogonal and

$$\sum_{\iota\in F} q_\iota \prec f \text{ for any finite } F \subset I .$$

By Lemma 2.2 it follows that

$$\bigvee_{\iota\in I} q_\iota \prec f .$$

But

$$\bigvee_{\iota\in I} q_\iota = \bigvee_{\iota\in I} p_\iota = \bigvee_{\iota\in I} e_\iota .$$

Indeed, otherwise there would exist a smallest $\iota \in I$ with respect to $\preccurlyeq$ such that

$$(\star\star) \qquad p_\iota \not\le \bigvee_{\iota'\in I} q_{\iota'} .$$

But then we would have

$$\bigvee_{\iota''\prec\iota} p_{\iota''} \le \bigvee_{\iota'\in I} q_{\iota'} ,$$

which contradicts $(\star\star)$. □

For M an arbitrary AW^*-algebra and Z a commutative AW^*-algebra we call

$$\Phi : \{e \in M \, ; \, e \text{ projection}\} \to Z^+$$

normal if, for every upward directed family $(e_\iota)_\iota$ of projections in M , we have

$$\Phi\Big(\bigvee_\iota e_\iota\Big) = \sup \Phi(e_\iota) ,$$

where sup denotes the least upper bound in Z^+ . Clearly,

$$\Phi \text{ normal } \Longrightarrow \Phi \text{ completely additive,}$$

but, using the above proposition similarly as in the proof of Corollary 1, we also get the converse implication (which should be known, but for which we have no reference):

Corollary 2. *Let M , Z be AW^*-algebras, Z commutative, and $\Phi : \{e \in M \, ; \, e$ projection$\} \to Z^+$. Then*

$$\Phi \textit{ normal } \Longleftrightarrow \Phi \textit{ completely additive.}$$

In particular, the centre valued dimension function of a finite AW^*-algebra is normal (see [6], §33, Exercise 4). Also, if M is a discrete AW^*-algebra and $e \in M$ is an abelian projection of central support 1_M, then the map Φ_e considered in the proof of Theorem 1.2 (on the abelian strict closure in discrete AW^*-algebras) is normal on the projection lattice of M .

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