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ON CROSSED HOMOMORPHISMS ON SYMPLECTIC MAPPING CLASS GROUPS

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Abstract. For a symplectic manifold (M, ω) with a relation Q between Chern classes of it and the cohomology class of the symplectic form ω, we construct a crossed homomorphism F_Q on the symplectomorphism group of (M, ω) with values in the cohomology group of M and show an application to the symplectic flux group. Moreover we see that F_Q descends to a crossed homomorphism on the symplectic mapping class group of (M, ω) and show a nontrivial example of it.

1. Introduction

There are a lot of works for the mapping class group $\mathcal{M}$ of an oriented surface Σ, which is the group of path components of the diffeomorphism group $Diff_+(\Sigma)$ of Σ with the C^∞ topology. If we take an area form ω_Σ on Σ, we also obtain the group $\mathcal{S}$ of path components of the group of area preserving diffeomorphisms. By Moser [6], $\mathcal{S}$ is isomorphic to $\mathcal{M}$. Since a symplectic manifold (M, ω) is a high-dimensional analogue to (Σ, ω_Σ), the group $S(M, \omega)$ of path components of the symplectomorphism group of (M, ω) is considered as a high-dimensional analogue to the mapping class group $\mathcal{M}$.

As works for $\mathcal{M}$, there are constructions of crossed homomorphisms from $\mathcal{M}$ to the first (co)homology group of Σ and investigations of them by Morita [7] and by Trapp [11]. In this paper, we carry out an analogy of them in terms of differential forms. For a symplectic manifold (M, ω) with a relation Q between Chern classes of (M, ω) and the cohomology class of the symplectic form ω, we construct a crossed homomorphism F_Q from the symplectomorphism group $Symp(M, \omega)$ of (M, ω) to the cohomology group of M using Chern-Simons forms or Bott homomorphisms. F_Q has a connection with the flux homomorphism which is a homomorphism from the identity component of $Symp(M, \omega)$ to $H^1(M; \mathbb{R})$. So we can show an application of F_Q to the flux group Γ_ω for (M, ω), which gives an extension of a result for Γ_ω by Kędra-Kotschick-Morita [2]. Moreover we show that F_Q descends to a crossed homomorphism $\mathcal{F}_Q$ from the symplectic mapping group $\mathcal{S}(M, \omega)$ to a quotient of the cohomology group of M and that $\mathcal{F}_Q$ is generally nontrivial by giving a nontrivial example.

This paper is organized as follows. In section 2, we define our crossed homomorphisms and state the main results. In sections 3 and 4, we introduce some tools for

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the proof of Theorem A: Bott homomorphisms and a derivation formula for a curve of pushforward connections. In section 5, we prove Theorem A. In section 6, we give a topological definition of our crossed homomorphisms, which is needed for the computation of a nontrivial example. In section 7, we recall Thurston's theorem for a symplectic fibration and consider diffeomorphisms on it with some properties. These are also needed for the construction of our example. In sections 8 and 9, we show a nontrivial example of our crossed homomorphisms on the symplectic mapping class group.

2. Definition of our crossed homomorphisms and main results

Let (M,ω) be a closed symplectic manifold of dimension $2n$. Let $p\colon \mathcal{J} \to M$ be the fiber bundle over M whose fiber at each $x \in M$ is the space of positive compatible complex structures on the symplectic vector space (T_xM, ω_x). Note that $\mathcal{J}$ is a smooth manifold. A section $J\colon M \to \mathcal{J}$ is an ω-positive compatible almost complex structure on M. Since the fibers of $\mathcal{J}$ are contractible, J is unique up to homotopy. Hence the Chern classes of (M,ω) are defined as those of the complex vector bundle (TM, J). Similarly Chern classes of a symplectic vector bundle $(E,\omega_E) \to N$ of $\mathrm{rank}_{\mathbb{R}} = 2n$ over a manifold N are defined. For each invariant symmetric multilinear function $q_j \in I_n^j := I^j(U(n))$ of degree j on the Lie algebra $u(n)$, the associated Chern class of (E,ω_E) in $H^{2j}(N;\mathbb{R})$ is denoted by $q_j(E,\omega_E)$. For a complex vector bundle ξ of $\mathrm{rank}_{\mathbb{C}} = n$, its Chern class associated with q_j is denoted by $q_j(\xi)$. Let $I_n[w]$ be the polynomial algebra of one variable w with coefficients in the invariant Weil algebra $I_n = \sum_{j\geqq 0} I_n^j$ of $U(n)$. Put $I_n[w]^k = \sum_{j=0}^k I_n^j \cdot w^{k-j}$. For each $Q = \sum_{j=0}^k q_j \cdot w^{k-j} \in I_n[w]^k$ and each $a \in H^2(N;\mathbb{R})$, we can consider the cohomology class $Q((E,\omega_E),a) = \sum_{j=0}^k q_j(E,\omega_E) \cup a^{k-j} \in H^{2k}(N;\mathbb{R})$. In particular, we put $Q(M,\omega) := Q((TM,\omega),[\omega]) \in H^{2k}(M;\mathbb{R})$ for the symplectic manifold (M,ω).

We assume that there exists $Q = \sum_{j=0}^k q_j \cdot w^{k-j} \in I_n[w]^k$ such that the equality

$$Q(M,\omega) = \sum_{j=0}^k q_j(TM,\omega) \cup [\omega]^{k-j} = 0$$

holds in $H^{2k}(M;\mathbb{R})$. We fix such a $Q \in I_n[w]^k$.

The pullback bundle p^*TM over $\mathcal{J}$ has a canonical hermitian structure which is given by the hermitian form $h_{J_x}(\ ,\ ) = \omega_x(\ , J_x\ ) - \sqrt{-1}\omega_x(\ ,\ )$ on the fiber $p^*TM|_{J_x} = T_xM$ at each $J_x \in \mathcal{J}$ with $p(J_x) = x$. Let $Symp(M,\omega)$ be the group of symplectomorphisms on (M,ω). The standard action of $Symp(M,\omega)$ on M lifts to ones on $\mathcal{J}$ and on p^*TM given, respectively, by

$$Symp(M,\omega) \times \mathcal{J} \ni (\varphi, J_x) \mapsto \varphi(J_x) := d\varphi_x \cdot J_x \cdot d\varphi^{-1}_{\varphi(x)} \in \mathcal{J}_{\varphi(x)} \subset \mathcal{J}$$

and

$$Symp(M,\omega) \times p^*TM \ni (\varphi, v) \mapsto d\varphi_x(v) \in T_{\varphi(x)}M = p^*TM|_{\varphi(J_x)} \subset p^*TM,$$

where $v \in p^*TM|_{J_x} = T_xM$. The action on p^*TM is one of $U(n)$ vector bundle isomorphisms.

The assumption $Q(M,\omega)=0$ implies $\sum_{j=0}^{k} q_j(p^*TM)\cup[p^*\omega]^{k-j}=0$ in $H^{2k}(\mathcal{J};\mathbb{R})$. This means that, if we take a $U(n)$-connection A on p^*TM with curvature form F_A, there exists a $2k-1$-form μ on $\mathcal{J}$ satisfying

$$\sum_{j=0}^{k} q_j(F_A^{(j)})\wedge p^*\omega^{k-j}+d\mu=0$$

since $q_j(p^*TM)$ is represented by the $2j$-form $q_j(F_A^{(j)})=q_j(F_A,F_A,\cdots,F_A)$ by Chern-Weil theory.

We define the map

$$\tilde{F}_Q\colon Symp(M,\omega)\to\Omega^{2k-1}(M)$$

by

$$\tilde{F}_Q(\varphi)=J^*\left\{\sum_{j=0}^{k} j\int_0^1 q_j(\alpha,F_{A_t}^{(j-1)})dt\wedge p^*\omega^{k-j}+\varphi_*\mu-\mu\right\},$$

where $\alpha=\varphi_*A-A$ and $A_t=A+t\alpha$. The integrals are relative Chern-Simons forms.

Put

$$Q'=\sum_{j=0}^{k}(k-j)q_j\cdot w^{k-j-1}\in I_n[w]^{k-1};$$

then we have $Q'(M,\omega)=\sum_{j=0}^{k}(k-j)q_j(TM,\omega)\cup[\omega]^{k-j-1}$ in $H^{2k-2}(M;\mathbb{R})$.

Theorem A. *For a closed symplectic $2n$-manifold (M,ω) with $Q(M,\omega)=0$ for some $Q\in I_n[w]^k$, the map $F_Q\colon Symp(M,\omega)\ni\varphi\mapsto[\tilde{F}_Q(\varphi)]\in H^{2k-1}(M;\mathbb{R})$ is a well-defined crossed homomorphism. Its class $[F_Q]$ in the group cohomology $H^1(Symp(M,\omega),H^{2k-1}(M;\mathbb{R}))$ is independent of the choice of connection A, $2k-1$ form μ and section J. Moreover the equality $\frac{d}{ds}F_Q(\varphi_s)=Q'(M,\omega)\cup[\iota(X_s)\omega]$ holds for any smooth path $\{\varphi_s\}_{s\in[0,1]}$ in $Symp(M,\omega)$ with family of vector fields X_s defined by $\frac{d}{ds}\varphi_s=X_s\circ\varphi_s$.*

The map F_Q is a crossed homomorphism if and only if it satisfies $F_Q(\varphi\psi)=F_Q(\varphi)+\varphi_*F_Q(\psi)$ for any $\varphi,\psi\in Symp(M,\omega)$, where the action of $Symp(M,\omega)$ on $H^*(M;\mathbb{R})$ is given by $(\varphi,a)\mapsto\varphi_*a:=(\varphi^{-1})^*a$ for each $(\varphi,a)\in Symp(M,\omega)\times H^*(M;\mathbb{R})$.

Let $Symp_0(M,\omega)$ be the identity component of $Symp(M,\omega)$. For any $\varphi\in Symp_0(M,\omega)$, choose a path $\varphi_s\in Symp_0(M,\omega)$ $(0\leqq s\leqq 1)$ from $\varphi_0=id$ to $\varphi_1=\varphi$. The flux homomorphism

$$Flux^\sim\colon Symp_0^\sim(M,\omega)\to H^1(M;\mathbb{R})$$

is defined by $Flux^\sim(\varphi)=[\int_0^1\iota(X_s)\omega ds]\in H^1(M;\mathbb{R})$, where $Symp_0^\sim(M,\omega)$ is the universal covering of $Symp_0(M,\omega)$. Let Γ_ω be the subgroup of $H^1(M;\mathbb{R})$ which is the image by $Flux^\sim$ of the fiber of the unversal covering at the identity. Γ_ω is called the flux group. $Flux^\sim$ descends to a homomorphism:

$$Flux\colon Symp_0(M,\omega)\to H^1(M;\mathbb{R})/\Gamma_\omega,$$

which is also called the flux homomorphism.

If we integrate the formula for a derivative of $F_Q(\varphi_s)$ in Theorem A in the case when φ_s is a path from the identity to φ, then we obtain $F_Q(\varphi) = Q'(M,\omega) \cup Flux^{\sim}([\varphi_s])$, where $[\varphi_s]$ denotes the element of $Symp_0^{\sim}(M,\omega)$ defined by the path φ_s. This implies $\Gamma_\omega \subset \ker[Q'(M,\omega)\cup\ ]$, where $Q'(M,\omega)\cup$ is the homomorphism from $H^1(M;\mathbb{R})$ to $H^{2k-1}(M;\mathbb{R})$ defined by the cup product with $Q'(M,\omega)$. This result has already been given by Kędra-Kotschick-Morita [2] in the case when $[\omega]^k$ is proportional to a product of fundamental Chern classes of (M,ω). In the case of $k=1$, namely in the case when $[\omega]$ is proportional to the first Chern class $c_1(M,\omega)$, Kotschick-Morita [3] proved $\Gamma_\omega = \{0\}$ and that $Flux$ extends to the whole group $Symp(M,\omega)$ as a crossed homomorphism. Our crossed homomorphism F_Q with $Q = w - \lambda c_1$ also gives an extension of $Flux$.

Corollary B. $\Gamma_\omega \subset \bigcap_Q \ker[Q'(M,\omega)\cup]$, *where* Q *runs over* $\bigcup_{k\geqq 0}\{Q \in I_n[w]^k \mid Q(M,\omega) = 0\}$.

Since $Flux$ is surjective and the restriction of F_Q to $Symp_0(M,\omega)$ factors through $H^1(M;\mathbb{R})/\Gamma_\omega$,

$$F_Q|_{Symp_0(M,\omega)} \colon Symp_0(M,\omega) \xrightarrow{Flux} H^1(M;\mathbb{R})/\Gamma_\omega \xrightarrow{Q'(M,\omega)\cup} H^{2k-1}(M;\mathbb{R}),$$

we have the following proposition.

Proposition C. *Under the assumption of Theorem A, if* $Q'(M,\omega)\cup \colon H^1(M;\mathbb{R}) \to H^{2k-1}(M;\mathbb{R})$ *is a nontrivial homomorphism, then the cohomology class* $[F_Q]$ *is nonzero in* $H^1(Symp(M,\omega), H^{2k-1}(M;\mathbb{R}))$.

Put

$$D_Q(M,\omega) = Q'(M,\omega) \cup H^1(M;\mathbb{R});$$

then it is a subgroup of $H^{2k-1}(M;\mathbb{R})$ invariant under the standard action by $Symp(M,\omega)$. Let $\mathcal{S}(M,\omega)$ be the quotient $Symp(M,\omega)/Symp_0(M,\omega)$, which is called the symplectic mapping class group of (M,ω).

Corollary D. F_Q *descends to a crossed homomorphism*

$$\mathcal{F}_Q \colon \mathcal{S}(M,\omega) \to H^{2k-1}(M;\mathbb{R})/D_Q(M,\omega),$$

whose cohomology class $[\mathcal{F}_Q] \in H^1(\mathcal{S}(M,\omega), H^{2k-1}(M;\mathbb{R})/D_Q(M,\omega))$ *is independent of the choice of connection* A, $2k-1$ *form* μ *and almost complex structure* J.

Next we give a nontrivial example of the crossed homomorphism $\mathcal{F}_Q$. Let $(B = T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}, \beta)$ be the standard symplectic torus of dimension $2n$ with coordinates $(x_i, y_i)_{i=1}^n$ and symplectic form $\beta = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$. Its cohomology ring is given by $H^*(T^{2n};\mathbb{Z}) = \bigwedge(dx_i, dy_i)$ with abuse of notation.

Let $L_i \to T^{2n}$ be a complex $U(1)$-line bundle over T^{2n} for $1 \leqq i \leqq n$ with $\gamma_i := c_1(L_i) = dx_i \wedge dy_i \in H^2(T^{2n};\mathbb{Z})$. Put $V = L_1 \oplus \cdots \oplus L_n$; then V is a complex $U(n)$-vector bundle of $\mathrm{rank}_{\mathbb{C}} V = n$. Let $\pi \colon M = P(V) \to T^{2n}$ be the projective bundle associated with V; then it is a symplectic fibration with fiber $\mathbb{C}P^{n-1}$ with the standard symplectic (Fubini-Study) form. By Thurston's theorem in [10] (see also [5] or Theorem 7.1), it is a symplectic manifold with symplectic form ω_K for any sufficiently large $K > 0$.

Let $\mathcal{L}$ be the Hopf line bundle associated with the projective bundle M. Put $\eta = -c_1(\mathcal{L})$; then we have $H^*(M;\mathbb{R}) = H^*(T^{2n};\mathbb{R})[\eta]/\sim$ with relation $\eta^n + c_1(V)\eta^{n-1} + \cdots + c_n(V) = 0$. Put

$$Q_{K,2} = \bar{c}_2 - \frac{1}{2(1-nK)^2}[nw^2 - 2\bar{c}_1 w + \{n(n-1)K^2 - 2(n-1)K + 1\}\bar{c}_1^2]$$

in $I_{2n-1}[w]^2$, where $\bar{c}_i \in I^i_{2n-1}$ $(i = 1, 2)$ is the form corresponding to the i-th Chern class. We can show $Q_{K,2}(M, \omega_K) = 0$, $Q'_{K,2} = -\frac{1}{(1-nK)^2}(nw - \bar{c}_1)$ and $D_{Q_{K,2}}(M, \omega_K) = (c_1 - n[\omega_K]) \cup H^1(M;\mathbb{R})$. By Corollary D, we have the crossed homomorphism

$$\mathcal{F}_{Q_{2,K}} \colon \mathcal{S}(M, \omega_K) \to H^3(M;\mathbb{R})/(c_1 - n[\omega_K]) \cup H^1(M;\mathbb{R}),$$

where $c_1 = \bar{c}_1(TM, \omega_K)$.

Let $\mathcal{S}_0(M, \omega_K)$ be the subgroup of $\mathcal{S}(M, \omega_K)$ consisting of all elements which act trivially on $H^*(M;\mathbb{R})$. The restriction of $\mathcal{F}_{Q_{K,2}}$ to $\mathcal{S}_0(M, \omega_K)$ is a homomorphism.

Theorem E. *The image of* $[\mathcal{F}_{Q_{K,2}}]$ *by the induced homomorphism* $H^1(\mathcal{S}(M, \omega_K), \mathcal{H}) \to H^1(\mathcal{S}_0(M, \omega_K), \mathcal{H})$ *by the restriction, where*

$$\mathcal{H} = H^3(M;\mathbb{R})/(c_1 - n[\omega_K]) \cup H^1(M;\mathbb{R}),$$

is nonzero, hence so is $[\mathcal{F}_{Q_{K,2}}]$.

3. Bott homomorphisms

In this section we shall recall Bott homomorphisms (see [12]) which are useful for our computation.

Let G be a Lie group, but we consider only $G = U(n)$ in this paper. Let $\pi \colon P \to M$ be a principal G-bundle over a manifold M. Let A_h $(h = 0, 1, \ldots, r)$ be $r+1$ connection forms on P and $\Delta^r := \{(t_0, t_1, \ldots, t_r) \in \mathbb{R}^{r+1} | t_h \geqq 0, \sum_{h=0}^r t_h = 1\}$ the standard r-simplex. Then we have the average connection $\tilde{A} = \sum_{h=0}^r t_h A_h$ on the product principal G-bundle $\pi \times id \colon P \times \Delta^r \to M \times \Delta^r$. Let $I^k(G)$ be the vector space of invariant, symmetric, multilinear functions on the k-th product $\mathfrak{g}^k$ of the Lie algebra $\mathfrak{g}$ of G with values in $\mathbb{R}$. For each $c \in I^k(G)$, put $\Delta(A_0, \ldots, A_r)c = (-1)^{[\frac{r+1}{2}]} \int_{\Delta^r} c(F^{(k)}_{\tilde{A}})$, where $F_{\tilde{A}}$ is the curvature form of $\tilde{A}$, $c(F^{(k)}_{\tilde{A}})$ denotes $c(F_{\tilde{A}}, F_{\tilde{A}}, \ldots, F_{\tilde{A}})$ and the orientation of Δ^r is given by $dt_1 \wedge \cdots \wedge dt_r$ $(t_0 = 1 - \sum_{h=1}^r t_h)$. Then we have the Bott homomorphism $\Delta(A_0, \ldots, A_r) \colon I^k(G) \to \Omega^{2k-r}(M)$. The Bott homomorphisms have the following properties:

(1) $d\Delta(A_0, \ldots, A_r)c = \sum\limits_{h=0}^{r} (-1)^h \Delta(A_0, \ldots, A_{h-1}, A_{h+1}, \ldots, A_r)c$;
in particular, $d\Delta(A_0)c = 0$ for $r = 0$.

(2) $\Delta(A_0) \colon I(G) \to \Omega^*(M)$ is the Chern-Weil homomorphism;
that is, the equality $\Delta(A_0)c = c(F^{(k)}_{A_0})$ holds.

(3) $\Delta(A_0, A_1)c = k \int_0^1 c(\alpha, F^{(k-1)}_{A_t}) dt$,
where $\alpha = A_1 - A_0$ and $A_t = A_0 + t\alpha$.

Let Q be another principal $U(n)$-bundle over M and $\tilde{\varphi} \colon Q \to P$ a $U(n)$-bundle isomorphism over a diffeomorphism φ of M. Then the pushforward connection $\tilde{\varphi}_* A = (\tilde{\varphi}^{-1})^* A$ can be considered.

(4) $\Delta(\tilde{\varphi}_* A_0, \tilde{\varphi}_* A_1, \ldots, \tilde{\varphi}_* A_r)c = \varphi_* \Delta(A_0, A_1, \ldots, A_r)c$,
where $\varphi_* = (\varphi^{-1})^*$.

4. The derivative of a curve of pushforward connections

In this section we shall derive a derivation formula for a curve of pushforward connections.

In the setting of Theorem A, let A be a $U(n)$-connection on p^*TM and $\nabla_A\colon \Omega^0(p^*TM) \to \Omega^1(p^*TM)$ the covariant derivative with respect to it. For any $\varphi \in Symp(M,\omega)$, $\varphi_* A$ denotes the pushforward connection of A by φ, which is the pullback connection $(\varphi^{-1})^*A$ of A by the inverse of φ.

Here and hereafter we use the same notation for the actions of $\varphi \in Symp(M,\omega)$ on M, on $\mathcal{J}$ and on p^*TM. In terms of a covariant derivative, the pushforward connection is given by $\nabla_{\varphi_* A} = \varphi_* \nabla_A = \varphi_*(\nabla_A((\varphi^{-1})_*(\,\cdot\,)))$.

Let J_x and K_y be two points of $\mathcal{J}$ with $p(J_x) = x$ and $p(K_y) = y$, respectively. Let (U, u^i) and (V, v^j) be local coordinate systems of $\mathcal{J}$ around J_x and K_y with orthonormal frames $e = (e_1, \dots, e_n)$ for $p^*TM|_U$ and $f = (f_1, \dots, f_n)$ for $p^*TM|_V$, respectively. We can put $\nabla_A e = e \otimes a$ for some $u(n)$-valued 1-form a on U.

For $\varphi \in Symp(M,\omega)$, we assume $\varphi(J_x) = K_y$ and consider it near J_x and K_y. Put $\psi = \varphi^{-1}$. We can set $\varphi^* f = e \otimes \Phi^{-1}$, where Φ is a $U(n)$-valued function on U near J_x. Similarly we can put $\psi^* e = f \otimes \Psi^{-1}$, where Ψ is a $U(n)$-valued function on V near K_y. Then we have $\psi^*\Phi = \Psi^{-1}$ and $\varphi^*\Psi = \Phi^{-1}$. Direct computations show the following lemma.

Lemma 4.1. *In the trivialization $p^*TM|_V \cong V \times \mathbb{C}^n$ near K_y with respect to the orthonormal frame f, the covariant derivative with respect to $\varphi_* A$ is given by*

$$\nabla_{\varphi_* A} = d + \psi^*(-d\Phi\Phi^{-1} + \Phi a \Phi^{-1}) = d + \Psi^{-1} d\Psi + \Psi^{-1}(\psi^* a)\Psi$$

and its curvature $F_{\varphi_* A}$ is given by

$$F_{\varphi_* A} = \psi^*(\Phi F_a \Phi^{-1}) = \Psi^{-1} \psi^* F_a \Psi,$$

where $F_a = da + a \wedge a$ is a local expression of the curvature of A near J_x.

Next we compute the derivative $\frac{d}{ds}(\varphi_{s*}A - A)$ of a smooth path φ_s in $Symp(M,\omega)$. Let $\tilde{X}_s$ be the smooth family of vector fields on $\mathcal{J}$ given by $\frac{d}{ds}\varphi_s = \tilde{X}_s \circ \varphi_s$. Hereafter we omit the parameter s if it is not particularly needed. In the trivialization $p^*TM|_V \cong V \times \mathbb{C}^n$ of Lemma 4.1, $\alpha = \varphi_* A - A$ is given by

$$\alpha = \Psi^{-1} d\Psi + \Psi^{-1}(\psi^* a)\Psi - b,$$

where b is the column vector valued 1-form defined by $\nabla_A f := (\nabla_A f_1, \dots, \nabla_A f_n) = f \otimes b$. A direct computation shows that

$$\dot{\alpha} = \frac{d}{ds}\alpha = \Psi^{-1}\{d(\dot{\Psi}\Psi^{-1}) + [\psi^* a, \dot{\Psi}\Psi^{-1}] + \frac{d}{ds}(\psi^* a)\}\Psi.$$

To begin with, we take $\varphi_0 = id$ and consider the derivative at $s = 0$. We can assume $e = f$, $b = a$ and $\Phi_0 = \Psi_0 = I$. Hence we have

$$\dot{\alpha}_0 = d\dot{\Psi} + [a, \dot{\Psi}] + \frac{d}{ds}(\psi^* a)|_{s=0} = d_a\{\dot{\Psi} + \iota(\tilde{Y}_0)a\} + \iota(\tilde{Y}_0)F_a,$$

where $d_a = d + [a,\ ]$ and $\tilde{Y}_s$ is defined by $\frac{d}{ds}\psi_s = \tilde{Y}_s \circ \psi_s$ on $\mathcal{J}$ with $\tilde{Y}_0 = \tilde{Y}_s|_{s=0}$. We have $\tilde{Y}_s = -\psi_{s*}\tilde{X}_s$.

We note that d_a and F_a define the global objects d_A, which are the covariant exterior differential and F_A, respectively. It is easy to see that $\beta := \dot{\Psi} + \iota(\tilde{Y}_0)a$ also defines a global section of the bundle $End(p^*TM)$ over $\mathcal{J}$, which can be denoted by $\beta_{\dot{\varphi}_0,A}$. Therefore we get

$$\dot{\alpha}_0 = d_A\beta_{\dot{\varphi}_0,A} + \iota(\tilde{Y}_0)F_A.$$

Next we consider the general case. For any smooth curve φ_s, put $\xi_u = \varphi_{u+s}\varphi_s^{-1}$; then we have $\xi_0 = id$. Since we have

$$\frac{d}{ds}(\varphi_{s*}A - A) = \frac{d}{du}(\varphi_{u+s}\varphi_s^{-1}\varphi_s)_*A|_{u=0} = \frac{d}{du}\xi_{u*}\varphi_{s*}A|_{u=0},$$

we obtain

$$\frac{d}{ds}(\varphi_{s*}A - A) = d_{\varphi_{s*}A}\beta_{\dot{\xi}_0,\varphi_{s*}A} + \iota(\tilde{V}_0)F_{\varphi_{s*}A},$$

where $\frac{d}{du}\xi_u^{-1} = \tilde{V}_u \circ \xi_u^{-1}$. The equality $\tilde{V}_0 = \frac{d}{du}\xi_u^{-1}|_{u=0} = -\tilde{X}_s$ implies the following lemma.

Lemma 4.2. *Let A be a $U(n)$-connection on p^*TM and φ_s a smooth curve in $Symp(M,\omega)$ with a smooth family of vector fields $\tilde{X}_s$ on $\mathcal{J}$ defined by $\frac{d}{ds}\varphi_s = \tilde{X}_s \circ \varphi_s$. Then the equality*

$$\frac{d}{ds}(\varphi_{s*}A - A) = d_{\varphi_{s*}A}\beta_{\dot{\xi}_0,\varphi_{s*}A} - \iota(\tilde{X}_s)F_{\varphi_{s*}A}$$

holds for some $\beta_{\dot{\xi}_0,\varphi_{s*}A} \in \Gamma(End(p^*TM))$, where $\xi_u = \varphi_{u+s}\circ\varphi_s^{-1}$ and $\dot{\xi}_0 = \frac{d\xi_u}{du}|_{u=0}$.

5. Proof of Theorem A

In this section we shall prove Theorem A.

Let (M,ω) be a closed symplectic manifold of dimension $2n$ and fix $Q = \sum_j q_j \cdot w^{k-j} \in I_n[w]^k$ satisfying $Q(M,\omega) = 0$ in $H^{2k}(M;\mathbb{R})$. As in section 2, we have the map $\tilde{F}_Q \colon Symp(M,\omega) \to \Omega^{2k-1}(M)$, which can be rewritten as

$$\tilde{F}_Q(\varphi) = J^*\left\{\sum_{j=0}^{k}\Delta(A,\varphi_*A)q_j \wedge p^*\omega^{k-j} + \varphi_*\mu - \mu\right\}$$

in terms of Bott homomorphisms. We note $\Delta(A,\varphi_*A)q_0 = 0$.

Put

$$\tilde{f}_Q(\varphi) = \sum_{j=0}^{k}\Delta(A,\varphi_*A)q_j \wedge p^*\omega^{k-j} + \varphi_*\mu - \mu;$$

then we have $\tilde{F}_Q = J^* \circ \tilde{f}_Q$.

Lemma 5.1. *The map $\tilde{f}_Q \colon Symp(M,\omega) \to \Omega^{2k-1}(\mathcal{J})$ induces a crossed homomorphism $f_Q \colon Symp(M,\omega) \to H^{2k-1}(\mathcal{J};\mathbb{R})$. Moreover the cohomology class $[f_Q]$ in $H^1(Symp(M,\omega), H^{2k-1}(\mathcal{J}))$ is independent of the choice of connection A and $2k-1$ form μ.*

Proof. Using the properties of Bott homomorphisms in section 3 and the definition of μ, we can easily show $d\tilde{f}_Q(\varphi) = 0$ and

$$\tilde{f}_Q(\varphi\psi) - \tilde{f}_Q(\varphi) - \varphi_*\tilde{f}_Q(\psi) = d\left\{-\sum_{j=0}^{k}\Delta(A,\varphi_*A,(\varphi\psi)_*A)q_j \wedge p^*\omega^{k-j}\right\}$$

for any $\varphi, \psi \in Symp(M,\omega)$. The map

$$f_Q\colon Symp(M,\omega) \ni \varphi \mapsto [\tilde{f}_Q(\varphi)] \in H^{2k-1}(\mathcal{J};\mathbb{R})$$

turns out to be a well-defined crossed homomorphism.

Let B and ν be another connection and $2k-1$ form satisfying

$$\sum_{j=0}^{k} \Delta(B) q_j \wedge p^*\omega^{k-j} + d\nu = 0.$$

Put

$$\alpha = \nu - \mu + \sum_{j=0}^{k} \Delta(A,B) q_j \wedge p^*\omega^{k-j};$$

then it is easy to see that α is a closed $2k-1$ form on $\mathcal{J}$.

Put

$$\tilde{g}_Q(\varphi) = \sum_{j=0}^{k} \Delta(B, \varphi_* B) q_j \wedge p^*\omega^{k-j} + \varphi_*\nu - \nu$$

for any $\varphi \in Symp(M,\omega)$; then it also induces a crossed homomorphism g_Q. A computation shows

$$\begin{aligned} &\tilde{g}_Q(\varphi) - \tilde{f}_Q(\varphi) \\ &= \varphi_*\alpha - \alpha + d\left[\sum_{j=1}^{k} \{-\Delta(B,\varphi_* A, \varphi_* B) q_j + \Delta(A,B,\varphi_* A) q_j\} \wedge p^*\omega^{k-j}\right]. \end{aligned}$$

Therefore $[g_Q(\varphi)] - [f_Q(\varphi)] = \varphi_*[\alpha] - [\alpha]$ holds in $H^{2k-1}(\mathcal{J};\mathbb{R})$. This shows $g_Q - f_Q = \delta\alpha$ as a 1-cochain of the group $Symp(M,\omega)$ with values in $H^{2k-1}(\mathcal{J};\mathbb{R})$; hence $[g_Q] = [f_Q]$ in $H^1(Symp(M,\omega), H^{2k-1}(\mathcal{J};\mathbb{R}))$. □

Since any ω-positive compatible almost complex structure $J\colon M \to \mathcal{J}$ induces the same isomorphism $H^*(\mathcal{J};\mathbb{R}) \xrightarrow{\cong} H^*(M;\mathbb{R})$, we have the first part of Theorem A.

Next we prove the second part of Theorem A. We compute the derivative $\frac{d}{ds} f_Q(\varphi_s)$ of a smooth curve φ_s in $Symp(M,\omega)$. There is the following derivation formula for Bott homomorphisms or Chern-Simons forms in the general setting.

Let $P \to M$ be a G-bundle and $q \in I^j(G)$ an invariant polynomial of degree j. For a connection $\bar{\theta}$ on the product G-bundle $P \times I \to M \times I$ with coordinate $s \in I = [0,1]$, we consider the corresponding family $\theta_s = \bar{\theta}|_{s=\mathrm{const}}$ of connections on P. Let $\bar{\theta}'$ and θ'_s be another pair. Then we have

$$\frac{\partial}{\partial s}\Delta(\theta_s, \theta'_s) q = j\{q(\frac{\partial \theta'_s}{\partial s}, F_{\theta'_s}^{(j-1)}) - q(\frac{\partial \theta_s}{\partial s}, F_{\theta_s}^{(j-1)})\} + \text{exact form} \tag{5.1}$$

(see [12]). We apply this formula to our situation. Take $\bar{\theta} = A$ and $\bar{\theta}' = \varphi_{s*}A$, which are connections on $p^*TM \times I$, where A is considered as a constant along I. Then we have $\theta_s = A$, which is independent of s, and $\theta'_s = \varphi_{s*}A$ with abuse of notation. Take $q = q_j$. In this case, the formula (5.1) and Lemma 4.2 imply

$$\begin{aligned}\frac{d}{ds}\Delta(A, \varphi_{s*}A)q_j &= jq_j(d_{\varphi_{s*}A}\beta_{\dot{\xi}_0,\varphi_{s*}A} - \iota(\tilde{X}_s)F_{\varphi_{s*}A}, F^{(j-1)}_{\varphi_{s*}A}) + \text{exact form}\\ &= jdq_j(\beta_{\dot{\xi}_0,\varphi_{s*}A}, F^{(j-1)}_{\varphi_{s*}A}) - \iota(\tilde{X}_s)q_j(F^{(j)}_{\varphi_{s*}A}) + \text{exact form}\\ &= -\iota(\tilde{X}_s)\Delta(\varphi_{s*}A)q_j + dR_j,\end{aligned}$$

where R_j is some $2j-2$-form on $\mathcal{J}$ for each j. On the other hand, we have

$$\frac{d}{ds}\psi_s^*\mu = \frac{d}{du}(\eta_u^*\psi_s^*\mu)|_{u=0} = -d\{\iota(\tilde{X}_s)\psi_s^*\mu\} - \iota(\tilde{X}_s)d(\psi_s^*\mu),$$

where $\eta_u = \xi_u^{-1} = \varphi_s\varphi_{u+s}^{-1} = \psi_s^{-1}\psi_{s+u}$ with $\frac{d}{du}\eta_u|_{u=0} = -\tilde{X}_s$.

Using these formulas, properties of Bott homomorphisms and the fact that the interior product is an anti-derivative of degree -1, we have

$$\begin{aligned}\frac{d}{ds}\tilde{f}_Q(\varphi_s) = &-\sum_{j=0}^{k}\{\iota(\tilde{X}_s)\Delta(\varphi_{s*}A)q_j - dR_j\}\wedge p^*\omega^{k-j} - d\{\iota(\tilde{X}_s)\psi_s^*\mu\} - \iota(\tilde{X}_s)d\psi_s^*\mu\\ &= -\sum_{j=0}^{k}\Big\{\iota(\tilde{X}_s)\Delta(\varphi_{s*}A)q_j\Big\}\wedge p^*\omega^{k-j}\\ &\qquad + \iota(\tilde{X}_s)\psi_s^*\left\{\sum_{j=0}^{k}\Delta(A)q_j\wedge p^*\omega^{k-j}\right\} + \text{exact form}\\ &= \sum_{j=0}^{k}\Delta(\varphi_{s*}A)q_j\wedge\iota(\tilde{X}_s)(p^*\omega^{k-j}) + \text{exact form}\\ &= \sum_{j=0}^{k}(k-j)\Delta(\varphi_{s*}A)q_j\wedge p^*\omega^{k-j-1}\wedge\iota(\tilde{X}_s)p^*\omega + \text{exact form.}\end{aligned}$$

Since $\varphi_{s*}A$ is also a $U(n)$-connection on p^*TM, we obtain

$$\frac{d}{ds}f_Q(\varphi_s) = \sum_{j=0}^{k}(k-j)q_j(p^*TM)\cup p^*[\omega]^{k-j-1}\cup[\iota(\tilde{X}_s)p^*\omega].$$

Let X_s be the family of vector fields on M defined by $\frac{d}{ds}\varphi_s = X_s\circ\varphi_s$; then we have $J^*(\iota(\tilde{X}_s)p^*\omega) = \iota(X_s)\omega$. Therefore $\frac{d}{ds}F_Q(\varphi_s) = J^*\frac{d}{ds}f_Q(\varphi_s)$ implies the following proposition.

Proposition 5.2. *The equality* $\frac{d}{ds}F_Q(\varphi_s) = \sum_{j=0}^{k}(k-j)q_j(M,\omega)\cup[\omega]^{k-j-1}\cup[\iota(X_s)\omega]$ *holds in* $H^{2k-1}(M;\mathbb{R})$.

This proposition is the second part of Theorem A, which concludes the proof of it.

6. Another definition of the crossed homomorphism

In this section we shall define another map $h_Q\colon Symp(M,\omega)\to H^{2k-1}(M;\mathbb{R})$ for the fixed A and μ in section 2 and show $h_Q = -F_Q$.

Let (M,ω) be a closed symplectic manifold of dimension $2n$. Fix $Q\in I_n[w]^k$ with $Q(M,\omega)=0$ in $H^{2k}(M;\mathbb{R})$ as before. For any $\varphi\in Symp(M,\omega)$, we consider the mapping tori $M_\varphi := [0,1]\times M/(1,x)\sim(0,\varphi^{-1}(x))$ of φ and $\mathcal{J}_\varphi$ of $\varphi\colon \mathcal{J}\to\mathcal{J}$. Then we have the fiber bundle $p_\varphi\colon \mathcal{J}_\varphi\to M_\varphi$. Note that $\mathcal{J}_\varphi$ and M_φ are homotopy equivalent.

Let $pr\colon(-\varepsilon,1+\varepsilon)\times M\to M_\varphi$ and $pr\colon(-\varepsilon,1+\varepsilon)\times\mathcal{J}\to\mathcal{J}_\varphi$ be the obvious projections for small $\varepsilon>0$. By considering ω as a closed 2-form on $(-\varepsilon,1+\varepsilon)\times M$ which is constant along $(-\varepsilon,1+\varepsilon)$, it descends to a closed 2-form ω_φ on M_φ. The pullback bundle $p_\varphi^*T^MM_\varphi$ of the tangent bundle T^MM_φ along the fibers also has the canonical hermitian structure. In particular it is a complex vector bundle.

We consider the class

$$\bar{Q}_\varphi := Q((p_\varphi^*T^MM_\varphi, p_\varphi^*\omega_\varphi),[p_\varphi^*\omega_\varphi]) = \sum_{j=0}^{k} q_j(p_\varphi^*T^MM_\varphi, p_\varphi^*\omega_\varphi)\cup[p_\varphi^*\omega_\varphi]^{k-j}$$

in $H^{2k}(\mathcal{J}_\varphi;\mathbb{R})$. We identify $\mathcal{J}$ with the fiber at 0 : $\mathcal{J}=\mathcal{J}\times 0\hookrightarrow\mathcal{J}_\varphi$. By the assumption $Q(M,\omega)=0$, we have $\bar{Q}_\varphi|_{\mathcal{J}}=0$.

Recall the notation in section 2: $p\colon\mathcal{J}\to M$ is the projection and A is a $U(n)$-connection on p^*TM. Put $Q(A,p^*\omega):=\sum_{j=0}^k q_j(F_A^{(j)})\wedge p^*\omega^{k-j}$; then it is an exact $2k$-form on $\mathcal{J}$. There exists $\mu\in\Omega^{2k-1}(\mathcal{J})$ such that $d\mu+Q(A,p^*\omega)=0$.

Take a $U(n)$-connection A_φ on $p_\varphi^*T^MM_\varphi$ and a $2k-1$ form μ_φ on $\mathcal{J}_\varphi$ satisfying $\iota_{(-\varepsilon,\varepsilon)}^*A_\varphi=A$ and $\iota_{(-\varepsilon,\varepsilon)}^*\mu_\varphi=\mu$, where $\iota_{(-\varepsilon,\varepsilon)}\colon(-\varepsilon,\varepsilon)\times\mathcal{J}\hookrightarrow\mathcal{J}_\varphi$ is the inclusion and A and μ are considered as a connection and as a $2k-1$ form on $(\varepsilon,\varepsilon)\times\mathcal{J}$ which are constant along $[0,1]$. Then we have $d\mu_\varphi+Q(A_\varphi,p_\varphi^*\omega_\varphi)=0$ near $\mathcal{J}$. Therefore we have $[Q(A_\varphi,p_\varphi^*\omega_\varphi)+d\mu_\varphi]\in H^{2k}(\mathcal{J}_\varphi,\mathcal{J};\mathbb{R})$ whose image in $H^{2k}(\mathcal{J}_\varphi;\mathbb{R})$ is $\bar{Q}_\varphi$.

We define the class $h_Q(\varphi)$ in $H^{2k-1}(\mathcal{J};\mathbb{R})$ as the image of $[Q(A_\varphi,p_\varphi^*\omega_\varphi)+d\mu_\varphi]$ by the composition of isomorphisms:

$$\begin{aligned} H^{2k}(\mathcal{J}_\varphi,\mathcal{J};\mathbb{R}) &\overset{\text{excision}}{\cong} H^{2k}([0,1]\times\mathcal{J},\{0,1\}\times\mathcal{J};\mathbb{R})\\ &\overset{\text{Künneth}}{\cong} H^1([0,1],\{0,1\};\mathbb{R})\otimes H^{2k-1}(\mathcal{J};\mathbb{R})\\ &\cong H^{2k-1}(\mathcal{J};\mathbb{R}).\end{aligned}$$

Lemma 6.1. *The map*

$$h_Q=h_{Q,A,\mu}\colon Symp(M,\omega)\ni\varphi\mapsto h_Q(\varphi)\in H^{2k-1}(\mathcal{J};\mathbb{R})$$

is well-defined, that is, it is independent of the choice of A_φ and μ_φ.

Proof. Let B_φ and ν_φ be another connection on $p_\varphi^*T^MM_\varphi$ and the $2k-1$ form on $\mathcal{J}_\varphi$ satisfying $\iota_{(-\varepsilon,\varepsilon)}^*B_\varphi=A$ and $\iota_{(-\varepsilon,\varepsilon)}^*\nu_\varphi=\mu$. Then we have $d\nu_\varphi+Q(B_\varphi,p_\varphi^*\omega_\varphi)=0$ near $\mathcal{J}$. We have only to show $[Q(B_\varphi,p_\varphi^*\omega_\varphi)+d\nu_\varphi]=[Q(A_\varphi,p_\varphi^*\omega_\varphi)+d\mu_\varphi]$ in $H^{2k}(\mathcal{J}_\varphi,\mathcal{J};\mathbb{R})$. But it follows from

$$\begin{aligned} Q(B_\varphi,p_\varphi^*\omega_\varphi)+d\nu_\varphi =& Q(A_\varphi,p_\varphi^*\omega_\varphi)+d\mu_\varphi\\ &+d\left\{\sum_{j=0}^k\Delta(A_\varphi,B_\varphi)q_j\wedge p_\varphi^*\omega_\varphi^{k-j}+\nu_\varphi-\mu_\varphi\right\}\end{aligned}$$

on $\mathcal{J}_\varphi$ and $\sum_{j=0}^k\Delta(A_\varphi,B_\varphi)q_j\wedge p_\varphi^*\omega_\varphi^{k-j}+\nu_\varphi-\mu_\varphi\equiv 0$ near $\mathcal{J}$. □

Proposition 6.2. $f_Q=-h_Q\colon Symp(M,\omega)\to H^{2k-1}(\mathcal{J};\mathbb{R})$.

Proof. For any sufficiently small $\varepsilon > 0$, let $\lambda_\varepsilon\colon (-\varepsilon, 1+\varepsilon) \to \mathbb{R}$ be a smooth function satisfying $|\lambda_\varepsilon(t)| < 2\varepsilon,\ |\lambda'_\varepsilon(t)| \leqq$ const. independent of ε and

$$\lambda_\varepsilon(t) = \begin{cases} -t, & t \in (-\varepsilon, \varepsilon], \\ 0, & t \in [2\varepsilon, 1-2\varepsilon], \\ 1-t, & t \in [1-\varepsilon, 1+\varepsilon). \end{cases}$$

Clearly there exists a family $\{\lambda_\varepsilon\}_\varepsilon$ of such functions.

Put $b = \varphi_* A - A$ on $\mathcal{J}$. The connection $\bar{A}_\varphi = A + (t + \nu_\varepsilon(t))b$ on $(-\varepsilon, 1+\varepsilon) \times p^*TM$ descends to a connection A_φ on $p^*_\varphi T^M M_\varphi \cong (-\varepsilon, 1+\varepsilon) \times p^*TM/\sim$. Similarly the 2-form $p^*\omega$ on $(-\varepsilon, 1+\varepsilon) \times \mathcal{J}$ descends to the closed 2-form $p^*_\varphi \omega_\varphi$. Therefore we can calculate forms on $\mathcal{J}_\varphi$ as those on $(-\varepsilon, 1+\varepsilon) \times \mathcal{J}$.

Let $\nabla_A \overset{\text{loc}}{=} d + a$ be a local expression of the covariant derivative associated with the connection A on $\mathcal{J}$ in some local trivialization of p^*TM; then we have

$$\nabla_{\bar{A}_\varphi} \overset{\text{loc}}{=} d^{\mathcal{J}_\varphi} + a + (t + \lambda_\varepsilon)b,$$

where $d^{\mathcal{J}_\varphi}$ denotes the exterior differential on $\mathcal{J}_\varphi$. The curvature $F_{\bar{A}_\varphi}$ of $\bar{A}_\varphi$ is given by

$$\begin{aligned} F_{\bar{A}_\varphi} &\overset{\text{loc}}{=} d^{\mathcal{J}_\varphi}\{a + (t+\lambda_\varepsilon)b\} + \{a + (t+\lambda_\varepsilon)b\} \wedge \{a + (t+\lambda_\varepsilon)b\} \\ &= F_t + dt \wedge (1 + \lambda'_\varepsilon)b + \lambda_\varepsilon B_t, \end{aligned}$$

where

$$F_t = F^{\mathcal{J}}_{A+tb} \overset{\text{loc}}{=} d^{\mathcal{J}}(a+tb) + (a+tb) \wedge (a+tb)$$

and

$$B_{t,\varepsilon} = d^{\mathcal{J}} b + (a+tb) \wedge b + b \wedge (a+tb) + \lambda_\varepsilon b \wedge b.$$

Here $d^{\mathcal{J}}$ denotes the exterior differential on $\mathcal{J}$. For any $q_j \in I^j_n$, we have

$$\begin{aligned} q_j(F^{(j)}_{\bar{A}_\varphi}) &= q_j((dt \wedge (1+\lambda'_\varepsilon)b + F_t + \lambda_\varepsilon B_{t,\varepsilon})^{(j)}) \\ &= j(1+\lambda'_\varepsilon) dt \wedge q_j(b, (F_t + \lambda_\varepsilon B_{t,\varepsilon})^{(j-1)}) + q((F_t + \lambda_\varepsilon B_{t,\varepsilon})^{(j)}). \end{aligned}$$

Put $\mu_\varphi = \xi_\varphi + dt \wedge \eta$; then we have $\xi_\varphi \equiv \mu$ near $0 \times \mathcal{J}$, $\xi_\varphi \equiv (\varphi^{-1})^*\mu$ near $1 \times \mathcal{J}$, $\eta \equiv 0$ near $\{0,1\} \times \mathcal{J}$ and $d^{\mathcal{J}_\varphi}\mu_\varphi = d^{\mathcal{J}}\xi_\varphi + dt \wedge \xi'_\varphi - dt \wedge d^{\mathcal{J}}\eta$. Put $Q(\bar{A}_\varphi, p^*_\varphi\omega_\varphi) = Q^{\mathcal{J}} + dt \wedge Q^I$. Since the equality $Q(\bar{A}_\varphi, p^*_\varphi\omega_\varphi) + d\mu_\varphi = 0$ holds on $(-\varepsilon, \varepsilon) \times \mathcal{J}$, we have $Q^{\mathcal{J}} + d^{\mathcal{J}}\xi_\varphi = 0$ and $Q^I + \xi'_\varphi - d^{\mathcal{J}}\eta = 0$ on $(-\varepsilon, \varepsilon) \times \mathcal{J}$.

Note that the integration is defined by $\int_I \alpha = \int_I \beta dt$ for $I = [0,1]$ and $\alpha = \beta \wedge dt$ by our convention.

A representative of $h_Q(\varphi)$ is given by

$$\begin{aligned} &\int_I (Q(\bar{A}_\varphi, p^*_\varphi\omega_\varphi) + d\mu_\varphi) = \int_I dt \wedge (Q^I + \xi'_\varphi - d^{\mathcal{J}}\eta) \\ &= (\int_I dt \wedge Q^I) - [\xi_\varphi]^1_0 - d^{\mathcal{J}} \int_I dt \wedge \eta \\ &= \int_I dt \wedge \left\{ \sum_{j=0}^k j(1+\lambda'_\varepsilon) q_j(b, (F_t + \lambda_\varepsilon B_{t,\varepsilon})^{(j-1)}) \wedge p^*\omega^{k-j} \right\} \\ &\qquad - (\varphi^{-1})^*\mu + \mu - d^{\mathcal{J}} \int_I dt \wedge \eta, \end{aligned}$$

which implies

$$h_Q(\varphi) = \left[\int_I dt \wedge \left\{ \sum_{j=0}^{k} j(1+\lambda'_\varepsilon) q_j(b, (F_t + \lambda_\varepsilon B_{t,\varepsilon})^{(j-1)}) \wedge p^*\omega^{k-j} \right\} - \varphi_*\mu + \mu \right].$$

For any sufficiently small $\varepsilon > 0$, we have $|\lambda'_\varepsilon| \leqq \text{const}, |\text{supp}(\lambda_\varepsilon)| \leqq 4\varepsilon$ and $|\lambda_\varepsilon| \leqq \varepsilon$, hence $|B_{t,\varepsilon}| \leqq \text{const}$. Take the limit as $\varepsilon \to 0$; then we have

$$\begin{aligned} h_Q(\varphi) &= \left[\int_I dt \wedge \sum_{j=0}^{k} j q_j(b, F_t^{(j-1)}) \wedge p^*\omega^{k-j} - \varphi_*\mu + \mu \right] \\ &= -\left[\sum_{j=0}^{k} \Delta(A, \varphi_* A) q_j \wedge p^*\omega^{k-j} + \varphi_*\mu - \mu \right] \\ &= -f_Q(\varphi). \end{aligned}$$

This finishes the proof of the proposition. □

Put $Q_\varphi := \sum_{j=0}^{k} q_j(T^M M_\varphi, \omega_\varphi) \cup [\omega_\varphi]^{k-j} \in H^{2k}(M_\varphi; \mathbb{R})$; then the image of it by the isomorphism $p_\varphi^* \colon H^*(M_\varphi; \mathbb{R}) \to H^*(\mathcal{J}_\varphi; \mathbb{R})$ is $\bar{Q}_\varphi$.

Here we recall the Leray-Hirsch theorem [9] over $\mathbb{R}$. Let $\pi \colon Y \to X$ be a fibration over an arcwise-connected CW-complex X with fiber F. Assume that the cohomology group $H^*(F; \mathbb{R})$ is finitely generated and that the induced homomorphism $\iota^* \colon H^*(Y; \mathbb{R}) \to H^*(F; \mathbb{R})$ by the inclusion $\iota \colon F \hookrightarrow Y$ is surjective. Then we can take a homomorphism $s \colon H^*(F; \mathbb{R}) \to H^*(Y; \mathbb{R})$ of graded modules of degree 0 such that $\iota^* \circ s = id$. We can define an $\mathbb{R}$-module isomorphism $\kappa_s \colon H^*(X; \mathbb{R}) \otimes H^*(F; \mathbb{R}) \to H^*(Y; \mathbb{R})$ by $\kappa_s(u \otimes v) = \pi^* u \cup s(v)$. We have the formula $\kappa_s(u \otimes v) = \pi^* u \cup \kappa_s(1 \otimes v)$.

Back to our situation, we consider the mapping torus M_φ of $\varphi \in Symp(M, \omega)$, which is a fiber bundle over S^1 with fiber M. In the case of $\varphi^* = id$ on $H^*(M; \mathbb{R})$, we can easily check that the induced homomorphism $\iota^* \colon H^*(M_\varphi; \mathbb{R}) \to H^*(M; \mathbb{R})$ of the inclusion $\iota \colon M \hookrightarrow M_\varphi$ is surjective. Hence we have an $\mathbb{R}$-module isomorphism

$$\kappa_\varphi = \kappa_{\varphi,s} \colon H^*(S^1; \mathbb{R}) \otimes H^*(M; \mathbb{R}) \xrightarrow{\cong} H^*(M_\varphi; \mathbb{R}). \tag{6.1}$$

The following lemma, which is a corollary of the proof of Proposition 6.2, is needed later.

Lemma 6.3. *If $\varphi_* = id$ on $H^*(M; \mathbb{R})$, then the equalities*

$$Q_\varphi = \kappa_\varphi(t^* \otimes H_Q(\varphi)) = \kappa_\varphi(t^* \otimes (-F_Q(\varphi)))$$

hold. Here $H_Q := J^* \circ h_Q \colon Symp(M, \omega) \to H^{2k-1}(M; \mathbb{R})$, and t^* is the generator of $H^1(S^1; \mathbb{Z}) \subset H^1(S^1 : \mathbb{R})$.

Proof. By the assumption, it is easy to see that the induced homomorphism

$$\iota^* \colon H^*(\mathcal{J}_\varphi; \mathbb{R}) \to H^*(\mathcal{J}; \mathbb{R})$$

of the inclusion $\iota \colon \mathcal{J} \hookrightarrow \mathcal{J}_\varphi$ as a fiber is surjective. Therefore we have the short exact sequences

$$0 \to H^*(\mathcal{J}_\varphi, \mathcal{J}; \mathbb{R}) \to H^*(\mathcal{J}_\varphi; \mathbb{R}) \to H^*(\mathcal{J}; \mathbb{R}) \to 0.$$

On the other hand, by the Leray-Hirsch theorem, we have the $\mathbb{R}$-module isomorphism
$$\bar{\kappa}_\varphi = \bar{\kappa}_{\varphi,s} \colon H^*(S^1;\mathbb{R}) \otimes H^*(\mathcal{J};\mathbb{R}) \to H^*(\mathcal{J}_\varphi;\mathbb{R}).$$
These imply isomorphisms
$$H^{2k}(\mathcal{J}_\varphi, \mathcal{J};\mathbb{R}) \cong H^1(S^1;\mathbb{R}) \otimes H^{2k-1}(\mathcal{J};\mathbb{R}) \overset{\bar{\kappa}_\varphi}{\cong} \ker[\iota^* \colon H^{2k}(\mathcal{J}_\varphi;\mathbb{R}) \to H^{2k}(\mathcal{J};\mathbb{R})].$$
Under these isomorphisms, we have $\bar{Q}_\varphi = \bar{\kappa}_\varphi(t^* \otimes h_Q(\varphi)) = \bar{\kappa}_\varphi(t^* \otimes (-f_Q(\varphi)))$ by the proof of Proposition 6.2. Since any section of the bundle $p_\varphi \colon \mathcal{J}_\varphi \to M_\varphi$ is homotopy equivalent, we obtain the lemma. $\square$

7. Thurston's theorem

In this section we shall recall Thurston's theorem [10] (see also [5]) for a symplectic fibration and consider some related diffeomorphisms.

Let $\pi \colon M \to B$ be a symplectic fibration whose typical fiber is a symplectic manifold (F, σ). By definition, there is an open covering $\{U_\alpha\}$ of B with trivialization
$$\begin{array}{ccc} \phi_\alpha \colon \pi^{-1}(U_\alpha) & \longrightarrow & U_\alpha \times F \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ U_\alpha & = & U_\alpha \end{array}$$
for each U_α. Moreover put $\phi_\alpha(b) = \mathrm{proj}_2 \circ \phi_\alpha|_{F_b = \pi^{-1}(b)} \colon F_b \to F$ and $\phi_{\beta\alpha}(b) = \phi_\beta(b) \circ \phi_\alpha(b)^{-1}$; then they define smooth maps $\phi_{\beta\alpha} \colon U_\alpha \cap U_\beta \to Symp(F, \sigma)$. For each $b \in B$, the canonical symplectic form σ_b is defined by $\sigma_b = \phi_\alpha(b)^* \sigma$. This form is independent of the choice of α.

A symplectic form ω on M is called compatible with the fibration π if each fiber (F_b, σ_b) is a symplectic submanifold of (M, ω), that is, the equality $\sigma_b = \iota_b^* \omega$ holds for each inclusion $\iota_b \colon F_b \hookrightarrow M$.

Theorem 7.1 (Thurston [10])**.** *Let $\pi \colon M \to B$ be a compact symplectic fibration with symplectic fiber (F, σ) and a connected symplectic base (B, β), and let $\sigma_b \in \Omega^2(F_b)$ be the canonical symplectic form on F_b. If there exists $a \in H^2(M)$ satisfying $\iota_b^* a = [\sigma_b]$ for some (and hence any) $b \in B$, then for all sufficiently large $K > 0$, there exists a symplectic form $\omega_K \in \Omega^2(M)$ such that it is compatible with π and represents the class $a + K[\pi^*\beta]$.*

The proof of his theorem constructs a closed 2-form τ on M with $[\tau] = a$ satisfying $\iota_b^* \tau = \sigma_b$ for all $b \in B$. The required symplectic form is given by $\omega_K = \tau + K\pi^*\beta$ for any sufficiently large $K > 0$.

We fix such a τ. Let $\hat{\phi} \colon M \to M$ be a diffeomorphism of M over a symplectomorphism $\phi \colon (B, \beta) \to (B, \beta)$ such that its restriction $\hat{\phi}_b \colon (F_b, \sigma_b) \to (F_{\phi(b)}, \sigma_{\phi(b)})$ to the fiber F_b at each $b \in B$ is a symplectomorphism:
$$\begin{array}{ccc} \hat{\phi} \colon M & \xrightarrow{\hat{\phi}} & M \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \phi \colon B & \xrightarrow{\phi} & B \end{array}$$

Moreover suppose that the induced homomorphism $\hat{\phi}^* \colon H^2(M) \to H^2(M)$ preserves the class $a = [\tau] \in H^2(M)$.

Let $Vert$ and Hor be the subbundles of TM whose fibers at $x \in M$ are given by $Vert_x := \ker d\pi_x$ and $Hor_x := (\ker d\pi_x)^\tau := \{\xi \in T_xM | \tau(\xi,\eta) = 0\ \forall \eta \in Vert_x\}$, respectively. Then we have the splitting $TM = Vert \oplus Hor$ and the isomorphism $d\pi_x : Hor_x \to T_{\pi(x)}B$ for each $x \in M$.

Put

$$\chi_{\hat{\phi}} : = \hat{\phi}^*\tau - \tau;$$

then we have

$$\hat{\phi}^*\omega_K = \omega_K + \chi_{\hat{\phi}}$$

and

$$\chi_{\hat{\phi}}|_{Vert} \equiv 0.$$

With respect to the splitting $TM = Vert \oplus Hor$, the 2-forms $\tau, \pi^*\beta$ and $\chi_{\hat{\phi}}$ can be represented by

$$\tau = \begin{pmatrix} A & O \\ O & B \end{pmatrix}, \qquad \pi^*\beta = \begin{pmatrix} O & O \\ O & C \end{pmatrix} \quad \text{and} \quad \chi_{\hat{\phi}} = \begin{pmatrix} O & E \\ F & G \end{pmatrix},$$

where A and C are nondegenerate. For each $t \in [0,1]$, put

$$\Omega_t := \omega_K + t\chi_{\hat{\phi}} = \begin{pmatrix} A & tE \\ tF & B + KC + tG \end{pmatrix};$$

then we have $\Omega_0 = \omega_K$ and $\Omega_1 = \hat{\phi}^*\omega_K$. Moreover for any sufficiently large $K > 0$, Ω_t is nondegenerate and has the same cohomology class $[\Omega_t] = [\omega_K]$ for all $t \in [0,1]$. Therefore we can apply Moser's method to them. There is a family of diffeomorphisms η_t on M satisfying $\eta_t^*\Omega_t = \Omega_0 = \omega_K$ and $\eta_0 = id_M$.

In particular we have

$$\omega_K = \eta_1^*\Omega_1 = \eta_1^*\hat{\phi}^*\omega_K = (\hat{\phi}\circ\eta_1)^*\omega_K,$$

which implies that the composition

$$\Phi := \hat{\phi}\circ\eta_1 \colon (M,\omega_K) \to (M,\omega_K)$$

is a symplectomorphism diffeotopic to $\hat{\phi}$. We have shown the following proposition.

Proposition 7.2. *Under the same assumption as Thurston's theorem, let $\hat{\phi}\colon M \to M$ be a fiberwise symplectomorphism over a symplectomorphism ϕ on (B,β) with $\hat{\phi}^*a = a$. Then for any sufficiently large $K > 0$, $\hat{\phi}$ is diffeotopic to a symplectomorphism $\Phi\colon (M,\omega_K) \to (M,\omega_K)$, where ω_K is the symplectic form on M in Thurston's theorem.*

For the symplectomorphism Φ on (M,ω_K) in this proposition, we consider the mapping torus $M_\Phi = [0,1]\times M/(0,m) \sim (1,\Phi(m))$ of Φ. The closed 2-form ω_K defines a fiberwise symplectic form $\omega_{K,\Phi}$ on M_Φ. Let $T^M M_\Phi$ be the tangent bundle along the fibers M; then the symplectic vector bundle $(T^M M_\Phi, \omega_{K,\Phi})$ over M_Φ is obtained. In order to calculate the Chern classes of it, we consider a family of the mapping tori $\{M_{\hat{\phi}\circ\eta_s}\}_{s\in[0,1]}$ of $\{\hat{\phi}\circ\eta_s\}_{s\in[0,1]}$. These mapping tori are diffeomorphic to each other as fiber bundles over S^1. In particular we have the family of diffeomorphisms

$$\xi_s \colon M_{\hat{\phi}} \to M_{\hat{\phi}\circ\eta_s}$$

induced by

$$[0,1]\times M \ni (t,m) \mapsto (t, \hat{\phi}\circ\eta_{s\nu(t)}\circ\hat{\phi}^{-1}(m)) \in [0,1]\times M,$$

where $\nu(t)$ is a monotonically increasing function on $[0,1]$ with $\nu(t)\equiv 0$ near $t=0$ and $\nu(t)\equiv 1$ near $t=1$. Put $\xi=\xi_1\colon M_{\hat{\phi}}\to M_{\hat{\phi}\circ\eta_1}=M_\Phi$.

For each $s\in[0,1]$, the family of 2-forms

$$\bar{\Omega}_s := \{(\hat{\phi}^{-1})^*\Omega_{1+t(s-1)}\}_{t\in[0,1]}$$

on M parameterized by $t\in[0,1]$ defines a fiberwise symplectic form $\bar{\Omega}_s$ (we use the same symbol) on $M_{\hat{\phi}\circ\eta_s}$ because of

$$(\hat{\phi}\circ\eta_s)^*(\hat{\phi}^{-1})^*\Omega_{1+1(s-1)}=\eta_s^*\Omega_s=\omega_K=(\hat{\phi}^{-1})^*\Omega_1.$$

Let $T^M M_{\hat{\phi}\circ\eta_s}$ be the tangent bundle of $M_{\hat{\phi}\circ\eta_s}$ along the fibers M as a fiber bundle over S^1; then the family $\{(T^M M_{\hat{\phi}\circ\eta_s},\bar{\Omega}_s)\}_{s\in[0,1]}$ of symplectic vector bundles is obtained. The Chern classes of the symplectic vector bundles $(T^M M_{\hat{\phi}},\bar{\Omega}_0)=(T^M M_{\hat{\phi}\circ\eta_0},\bar{\Omega}_0)$ and $(T^M M_\Phi,\omega_{K,\Phi})=(T^M M_{\hat{\phi}\circ\eta_1},\bar{\Omega}_1)$ agree under the isomorphism $\xi^*\colon H^*(M_\Phi;\mathbb{R})\cong H^*(M_{\hat{\phi}};\mathbb{R})$.

Next we consider $(T^M M_{\hat{\phi}},\bar{\Omega}_0)$. For each $t\in[0,1]$, we have

$$\bar{\Omega}_0|_t=(\hat{\phi}^{-1})^*\Omega_{1-t}=\tau-t(\hat{\phi}^{-1})^*\chi_{\hat{\phi}}+K\pi^*\beta.$$

For each $\hat{m}=(t,m)\in[0,1]\times M/\sim=M_{\hat{\phi}}$, put

$$T^{MV}_{\hat{m}}M_{\hat{\phi}}:=Vert_m$$

and

$$T^{MH}_{\hat{m}}M_{\hat{\phi}}=\{\xi\in T^M_{\hat{m}}M_{\hat{\phi}}|(\tau-t(\hat{\phi}^{-1})^*\chi_{\hat{\phi}})(\xi,\eta)=0\text{ for }\forall\eta\in T^{MV}_{\hat{m}}M_{\hat{\phi}}\}.$$

Since $\hat{\phi}$ is a fiber mapping, we have $(\hat{\phi}^{-1})^*Vert\subset Vert$. The 2-form $\tau-t(\hat{\phi}^{-1})^*\chi_{\hat{\phi}}$ is nondegenerate on $Vert$ for each t because of $\chi_{\hat{\phi}}|Vert\equiv 0$. Therefore we have the splitting

$$T^M M_{\hat{\phi}}=T^{MV}M_{\hat{\phi}}\oplus T^{MH}M_{\hat{\phi}},$$

where $T^{MV}M_{\hat{\phi}}=\coprod_{\hat{m}\in M_{\hat{\phi}}}T^{MV}_{\hat{m}}M_{\hat{\phi}}$ and $T^{MH}M_{\hat{\phi}}=\coprod_{\hat{m}\in M_{\hat{\phi}}}T^{MH}_{\hat{m}}M_{\hat{\phi}}$ are subbundles of $T^M M_{\hat{\phi}}$.

With respect to the above splitting, we can take matrix expressions

$$\tau-t(\hat{\phi}^{-1})^*\chi_{\hat{\phi}}=\begin{pmatrix}A' & O\\ O & B'\end{pmatrix}\text{ and }\pi^*\beta=\begin{pmatrix}O & O\\ O & C'\end{pmatrix},$$

where A' and C' are nondegenerate; we then have

$$\bar{\Omega}_0=\tau-t(\hat{\phi}^{-1})^*\chi_{\hat{\phi}}+K\pi^*\beta=\begin{pmatrix}A' & O\\ O & B'+KC'\end{pmatrix}.$$

Note that $T^{MV}M_{\hat{\phi}}$ is a symplectic vector bundle over $M_{\hat{\phi}}$ with symplectic structure $(\tau-t(\hat{\phi}^{-1})^*\chi_{\hat{\phi}})|_{T^{MV}M_{\hat{\phi}}}=\tau|_{T^{MV}M_{\hat{\phi}}}$, which is nothing but the canonical symplectic structures $\{\sigma_b\}_{b\in B}$ and is denoted by $\sigma_{\hat{\phi}}$. This symplectic vector bundle is denoted by $(T^{MV}M_{\hat{\phi}},\sigma_{\hat{\phi}})$. The mapping torus B_ϕ of ϕ is also a symplectic fiber bundle over S^1 with fiber (B,β). Its tangent bundle along the fibers is also a symplectic vector bundle $(T^B B_\phi,\beta_\phi)$. Let $\sigma_{\hat{\phi}}\oplus\pi^*\beta_\phi$ be the symplectic structure on the vector bundle $T^M M_{\hat{\phi}}$ induced from $\sigma_{\hat{\phi}}$ and β_ϕ by the isomorphism $T^M M_{\hat{\phi}}\cong T^{MV}M_{\hat{\phi}}\oplus\pi^*T^B B_\phi$; then we have a symplectic vector bundle $(T^M M_{\hat{\phi}},\sigma_{\hat{\phi}}\oplus\pi^*\beta_\phi)$. Let J^{MV} be a positive compatible complex structure of $(T^{MV}M_{\hat{\phi}},\sigma_{\hat{\phi}})$ and let $J^B_{B_\phi}$ be a fiberwise positive compatible complex structure of (B_ϕ, β_ϕ); then a complex structure J of $T^M M_{\hat\phi}$ is induced from $J^{MV} \oplus \pi^* J^B_{B_\phi}$ by $T^M M_{\hat\phi} \cong T^{MV} M_{\hat\phi} \oplus \pi^* T^B B_\phi$. J is compatible to the symplectic form $\sigma_{\hat\phi} \oplus \pi^* \beta_\phi$.

Lemma 7.3. *For any sufficiently large $K > 0$, the complex structure J on $T^M M_{\hat\phi}$ is $\bar\Omega_0$-tame, i.e. $\bar\Omega_0(\ ,J\ ) > 0$.*

Proof. From the matrix representation of $\bar\Omega_0$ above, we have

$$K >> 0 \Rightarrow (B' + KC')(\ \cdot\ , \pi^* J^B_{B_\phi}(\ \cdot\ )) > 0 \Leftrightarrow \bar\Omega_0(\ \cdot\ , J(\ \cdot\ )) > 0.$$

This implies the lemma. □

By this lemma, the Chern classes of $(T^M M_{\hat\phi}, \bar\Omega_0)$ agree with those of $(T^M M_{\hat\phi}, \sigma_{\hat\phi} \oplus \pi^* \beta_\phi)$. Since the Chern classes of $(T^M M_\Phi, \omega_{K,\Phi})$ and $(T^M M_{\hat\phi}, \bar\Omega_0)$ also agree under the isomorphism ξ^*, we have the following corollary.

Corollary 7.4. *The Chern classes of $(T^M M_\Phi, \omega_{K,\Phi})$ agree with those of $(T^M M_{\hat\phi}, \sigma_{\hat\phi} \oplus \pi^* \beta_\phi)$ under the isomorphism $\xi^* \colon H^*(M_\Phi; \mathbb{R}) \cong H^*(M_{\hat\phi}; \mathbb{R})$ for any sufficiently large $K > 0$.*

Next we assume $\hat\phi = id$ on $H^*(M;\mathbb{R})$. In this case, we have isomorphisms (6.1) $\kappa_\varphi \colon H^*(S^1;\mathbb{R}) \otimes H^*(M;\mathbb{R}) \xrightarrow{\cong} H^*(M_\varphi;\mathbb{R})$ $(\varphi = \hat\phi, \Phi)$. Put $\Xi = \kappa_{\hat\phi}^{-1} \circ \xi^* \circ \kappa_\Phi$; then we have the following commutative diagram:

$$\begin{tikzcd}
H^*(M_\Phi;\mathbb{R}) \arrow[r, "\xi^*"] & H^*(M_{\hat\phi};\mathbb{R}) \\
H^*(S^1;\mathbb{R}) \otimes H^*(M;\mathbb{R}) \arrow[u, "\kappa_\Phi"', "\cong"] \arrow[r, "\Xi"] & H^*(S^1;\mathbb{R}) \otimes H^*(M;\mathbb{R}). \arrow[u, "\kappa_{\hat\phi}"', "\cong"]
\end{tikzcd}$$

Since the restriction of ξ to the fiber $M = 0 \times M$ at $0 \in S^1$ is the identity, the matrix representation of Ξ of j-dimension with respect to the splitting $H^j(M;\mathbb{R}) \oplus H^1(S^1;\mathbb{R}) \otimes H^{j-1}(M;\mathbb{R})$ (whose elements are considered as column vectors) is given by

$$\Xi_j = \begin{pmatrix} id & 0 \\ * & id \end{pmatrix}.$$

Therefore, by Lemma 6.3, we have

$$\xi^* Q_\Phi = \xi^* \kappa_\Phi(t^* \otimes (-F_Q(\Phi))) = \kappa_{\hat\phi} \Xi(t^* \otimes (-F_Q(\Phi))) = \kappa_{\hat\phi}\left(t^* \otimes (-F_Q(\Phi))\right).$$

On the other hand, it is easy to see that we can put $\kappa_{\hat\phi}^{-1} \xi^* [\omega_{K,\Phi}] = [\omega_K] + t^* \otimes a$ for some $a \in H^1(M;\mathbb{R})$, where $t^* \in H^1(S^1;\mathbb{Z}) \subset H^1(S^1;\mathbb{R})$ is the generator. We have

$$\xi^*[\omega_{K,\Phi}] = \kappa_{\hat\phi}([\omega_K] + t^* \otimes a) = \kappa_{\hat\phi}[\omega_K] + \pi^* t^* \cup \kappa_{\hat\phi} a,$$

hence

$$(\xi^*[\omega_{K,\Phi}])^{k-j} = (\kappa_{\hat\phi}[\omega_K])^{k-j} + (k-j)(\kappa_{\hat\phi}[\omega_K])^{k-j-1} \cup \pi^* t^* \cup \kappa_{\hat\phi} a.$$

By Corollary 7.4, we have

$$\begin{aligned}
\xi^* Q_\Phi &= \xi^* \left(\sum_{j=0}^{k} q_j(T^M M_\Phi, \omega_{K,\Phi}) \cup [\omega_{K,\Phi}]^{k-j} \right) \\
&= \sum_{j=0}^{k} q_j(T^M M_{\hat\phi}, \sigma_{\hat\phi} \oplus \pi^*\beta_\phi) \cup (\xi^*[\omega_{K,\Phi}])^{k-j} \\
&= \sum_{j=0}^{k} q_j(T^M M_{\hat\phi}, \sigma_{\hat\phi} \oplus \pi^*\beta_\phi) \cup (\kappa_{\hat\phi}[\omega_K])^{k-j} \\
&\qquad + \pi^* t^* \cup \sum_{j=0}^{k} (k-j) q_j(T^M M_{\hat\phi}, \sigma_{\hat\phi} \oplus \pi^*\beta_\phi) \cup (\kappa_{\hat\phi}[\omega_K])^{k-j-1} \cup \kappa_{\hat\phi} a.
\end{aligned}$$

In general, the equality $\pi^* t^* \cup x = \kappa_{\hat\phi}(t^* \otimes \iota^* x)$ holds for any $x \in H^j(M_{\hat\phi}; \mathbb{R})$, where $\iota\colon M = M \times 0 \hookrightarrow M_{\hat\phi}$ is the inclusion.

Using this fact, we have

$$\begin{aligned}
&\pi^* t^* \cup \sum_{j=0}^{k} (k-j) q_j(T^M M_{\hat\phi}, \sigma_{\hat\phi} \oplus \pi^*\beta_\phi) \cup (\kappa_{\hat\phi}[\omega_K])^{k-j-1} \cup \kappa_{\hat\phi} a \\
=&\kappa_{\hat\phi} \left(t^* \otimes \iota^* \left(\sum_{j=0}^{k} (k-j) q_j(T^M M_{\hat\phi}, \sigma_{\hat\phi} \oplus \pi^*\beta_\phi) \cup (\kappa_{\hat\phi}[\omega_K])^{k-j-1} \cup \kappa_{\hat\phi} a \right) \right) \\
=&\kappa_{\hat\phi} \left(t^* \otimes \left(\sum_{j=0}^{k} (k-j) q_j(TM, \omega_K) \cup [\omega_K]^{k-j-1} \cup a \right) \right).
\end{aligned}$$

Here we note $\iota^* q_j(T^M M_{\hat\phi}, \sigma_{\hat\phi} \oplus \pi^*\beta_\phi) = q_j(TM, \omega_1) = q_j(TM, \omega_K)$ for all $K > 0$ since we can take the same almost complex structure compatible to ω_K for all $K > 0$. Recall our notation:

$$Q((T^M M_{\hat\phi}, \sigma_{\hat\phi} \oplus \pi^*\beta_\phi), \kappa_{\hat\phi}[\omega_K]) = \sum_{j=0}^{k} q_j(T^M M_{\hat\phi}, \sigma_{\hat\phi} \oplus \pi^*\beta_\phi) \cup (\kappa_{\hat\phi}[\omega_K])^{k-j}.$$

Putting these equalities together, we have

$$\begin{aligned}
&\kappa_{\hat\phi}(t^* \otimes (-F_Q(\Phi))) = \xi^* Q_\Phi \\
&= Q((T^M M_{\hat\phi}, \sigma_{\hat\phi} \oplus \pi^*\beta_\phi), \kappa_{\hat\phi}[\omega_K]) \\
&\qquad + \kappa_{\hat\phi} \left(t^* \otimes \left(\sum_{j=0}^{k} (k-j) q_j(TM, \omega_K) \cup [\omega_K]^{k-j-1} \cup a \right) \right),
\end{aligned}$$

which implies the following proposition.

Proposition 7.5. *$\kappa_{\hat\phi}(t^* \otimes (-F_Q(\Phi))) \equiv Q((T^M M_{\hat\phi}, \sigma_{\hat\phi} \oplus \pi^*\beta_\phi), \kappa_{\hat\phi}[\omega_K])$ in $H^1(S^1; \mathbb{R}) \otimes H^{2k-1}(M; \mathbb{R})$ modulo $\kappa_{\hat\phi}(t^* \otimes D_Q(M, \omega_K))$.*

8. The first Chern class of a line bundle over a torus

In this section we shall consider the first Chern class of a complex $U(1)$-line bundle over a torus in preparation for our example.

Let L be a complex $U(1)$-line bundle over a torus T^b of dimension b. Let $h^U\colon T^b \to U(1)$ be a smooth map; then it defines the $U(1)$-automorphism

$$h\colon L \supset L_x \ni u \mapsto h^U(x)u \in L_x \subset L$$

of L over the identity. Let L_h be the mapping torus of h; then it is a complex $U(1)$-bundle over $S^1 \times T^b = T^{b+1}$. Let $t^* \in H^1(S^1;\mathbb{Z})$ be the generator. Let

$$\kappa\colon H^*(S^1;\mathbb{R}) \otimes H^*(T^b;\mathbb{R}) \to H^*(S^1 \times T^b;\mathbb{R})$$

be the isomorphism in Künneth's theorem, which can also be considered as the isomorphism (6.1) with $\varphi = id$ and $s = p^*\colon H^*(T^b;\mathbb{R}) \to H^*(S^1 \times T^b;\mathbb{R})$, where $p\colon S^1 \times T^b \to T^b$ is the projection.

For simplicity, we consider elements of $H^*(S^1;\mathbb{R})$ and $H^*(T^b;\mathbb{R})$ as sitting in $H^*(S^1 \times T^b;\mathbb{R})$ in the obvious way. In the case of $b = 1$, we have

$$c_1(L_h) = \kappa(t^* \otimes (h^U)^*u) = t^* \cup (h^U)^*u,$$

where $u \in H^1(U(1);\mathbb{Z})$ is the generator. Using this fact, it is easy to see the following lemma for any $b \geqq 1$.

Lemma 8.1.

$$c_1(L_h) = \kappa(t^* \otimes (h^U)^*u + 1 \otimes c_1(L)) = t^* \cup (h^U)^*u + c_1(L) \in H^2(S^1 \times T^b;\mathbb{R}).$$

Next we consider $U(1)$-line bundles $L_i \to T^b$ $(1 \leqq i \leqq n)$ and their direct sum $V = L_1 \oplus \cdots \oplus L_n$ over T^b. Let $h_i\colon L_i \to L_i$ be the $U(1)$-automorphism defined as above by a smooth map $h_i^U\colon T^b \to U(1)$ for $1 \leqq i \leqq n$. Then we have the $U(n)$-bundle automorphism $h = h_1 \oplus \cdots \oplus h_n$ of $V = L_1 \oplus \cdots \oplus L_n$. The mapping torus V_h of h is identified with the direct sum $V_h = (L_1)_{h_1} \oplus \cdots \oplus (L_n)_{h_n}$ of the mapping tori $(L_i)_{h_i}$.

Let $\pi\colon P(V) \to T^b$ and $\pi_h\colon P(V_h) \to S^1 \times T^b$ be the projective bundles associated with V and V_h, respectively. Let $\mathcal{L} \to P(V)$ and $\tilde{\mathcal{L}}_h \to P(V_h)$ be the Hopf line bundles. The automorphism h on V induces a fiber map $\hat{h}$ on $P(V)$ and a $U(1)$-automorphism $\tilde{h}$ on $\mathcal{L}$ over $\hat{h}$. The mapping torus $\mathcal{L}_{\tilde{h}}$ of $\tilde{h}$ is a line bundle over the mapping torus $P(V)_{\hat{h}}$ of $\hat{h}$.

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "\tilde{h}"] \arrow[d] & \mathcal{L} \arrow[d] \\
P(V) \arrow[r, "\hat{h}"] \arrow[d] & P(V) \arrow[d] \\
T^b \arrow[r, "id"] & T^b,
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{\mathcal{L}}_h \arrow[d] \\
P(V_h) \arrow[d, "\pi_h"] \\
S^1 \times T^b,
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{L}_{\tilde{h}} \arrow[d] \\
P(V)_{\hat{h}} \arrow[d, "\pi_{\hat{h}}"] \\
S^1 \times T^b.
\end{tikzcd}$$

Clearly the following lemma is true.

Lemma 8.2. *$P(V)_{\hat{h}} \cong P(V_h)$ as $\mathbb{C}P^{n-1}$-bundles and $\mathcal{L}_{\tilde{h}} \cong \tilde{\mathcal{L}}_h$ as line bundles.*

9. An example

In this section we shall give the nontriviality of our crossed homomorphisms on the symplectic mapping class group, which is the proof of Theorem E.

Let $B = T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ be the standard $2n$-dimensional torus with the standard coordinates $(x_i, y_i)_{i=1}^n$ and the standard symplectic structure $\beta = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$ for $n \geqq 3$. Its cohomology ring is given by $H^*(T^{2n};\mathbb{Z}) = \bigwedge(dx_i, dy_i)$ with abuse of notation.

Let $L_i \to T^{2n}$ be a complex $U(1)$-line bundle over T^{2n} with first Chern class $\gamma_i := c_1(L_i) = dx_i \wedge dy_i \in H^2(T^{2n};\mathbb{Z})$ for $1 \leqq i \leqq n$. Then the direct sum $V = L_1 \oplus \cdots \oplus L_n \to T^{2n}$ is a complex vector bundle with total Chern class

$$c(V) = \prod_{i=1}^n c(L_i) = \prod_{i=1}^n (1+\gamma_i) = e^\beta,$$

where β also denotes the cohomology class of β. Hereafter we use the same symbol for a cohomology class and its representative differential form and omit the symbol $\cup$ of the cup product in cohomological computation. Let $\pi\colon M = P(V) \to T^{2n}$ be the projective bundle associated with V; then it is a symplectic fibration with fiber $\mathbb{C}P^{n-1}$ with the Fubini-Study form. Let $\mathcal{L} \to M$ be the Hopf line bundle associated with the projective bundle M. Put

$$\eta = c_1(\mathcal{L}^*) = -c_1(\mathcal{L}) \in H^2(M;\mathbb{Z});$$

then the cohomology group of M is the $H^*(T^{2n};\mathbb{R})$-module

$$H^*(M;\mathbb{R}) = H^*(T^{2n};\mathbb{R})[\eta]/\sim$$

with relation

$$\eta^n + c_1(V)\eta^{n-1} + \cdots + c_n(V) = 0.$$

The tangent bundle $T^{\mathbb{C}P}M$ along the fibers is obtained by the exact sequence

$$0 \to \underline{\mathbb{C}} \to \mathcal{L}^* \otimes \pi^*V \to T^{\mathbb{C}P}M \to 0$$

of vector bundles, which implies

$$c_1(T^{\mathbb{C}P}M) = n\eta + \pi^*\beta;$$

hence

$$\iota_b^*\eta = \frac{1}{n}\iota_b^* c_1(T^{\mathbb{C}P}M)$$

for any $b \in B$. Therefore $\eta \in H^2(M;\mathbb{R})$ can be taken as the class a in Thurston's theorem (Theorem 7.1) for the symplectic fibration M (if necessary the symplectic forms on fibers are multiplied by a common constant). By the proof of Thurston's theorem, there exists a closed 2-form τ on M satisfying $[\tau] = a = \eta$ and $\iota_b^*\tau = \sigma_b$ for all $b \in B$. Moreover for any sufficiently large $K > 0$, $\omega_K = \tau + K\pi^*\beta$ is a symplectic form on M. Because of $c(T^{2n}, \beta) = 1$, the total Chern class of (M, ω_K)

is computed as

$$\begin{aligned} c(M,\omega_K) = c(T^{\mathbb{C}P}M) &= c(\bigoplus_{i=1}^{n} \mathcal{L}\otimes \pi^* L_i) \\ &= \prod_{i=1}^{n}(1+\eta+\gamma_i) \\ &= (1+\eta)^n + (1+\eta)^{n-1}\sigma_1 + (1+\eta)^{n-2}\sigma_2 + \cdots + \sigma_n, \end{aligned}$$

where σ_i is the i-th fundamental symmetric polynomial of $\gamma_1, \dots, \gamma_n$ for $1 \leqq i \leqq n$.

Put $s_j = \sum_{i=1}^{n} \gamma_i^j$; then we have

$$s_1 = \sigma_1 = \beta, \quad s_2 = s_3 = \cdots = 0$$

because of $\gamma_i^2 = 0$ for all i. By using Newton's formulas which are given by

$$\begin{cases} s_j - \sigma_1 s_{j-1} + \sigma_2 s_{j-2} - \cdots + (-1)^{j-1}\sigma_{j-1}s_1 + (-1)^j j\sigma_j = 0 & (j = 1, \dots, n) \\ s_j - \sigma_1 s_{j-1} + \cdots + (-1)^j \sigma_n s_{j-n} = 0 & (j = n+1, n+2, \dots), \end{cases}$$

we obtain

$$\sigma_{j-1}s_1 - j\sigma_j = 0 \quad (j = 1, \dots, n),$$

hence

$$\sigma_j = \frac{1}{j!}\sigma_1^j = \frac{1}{j!}\beta^j \quad (j = 1, \dots, n).$$

Since we have $\beta^j = 0$ for all $j \geqq n+1$, we obtain

$$c(M,\omega_K) = \sum_{j=0}^{n}(1+\eta)^{n-j}\frac{1}{j!}\beta^j = (1+\eta)^n \exp\frac{\beta}{1+\eta}. \tag{9.1}$$

Since $\pi^* \colon H^*(T^{2n};\mathbb{R}) \to H^*(M;\mathbb{R})$ is injective, hereafter we shall omit π^* and write β instead of $\pi^*\beta$, etc., so we have

$$\omega_K = \eta + K\beta$$

and

$$c_1 := c_1(M,\omega_K) = n\eta + \sum_{i=1}^{n}\gamma_i = n\eta + \beta,$$

hence

$$\eta = \frac{\omega_K - Kc_1}{1-nK} \quad \text{and} \quad \beta = \frac{-n\omega_K + c_1}{1-nK}$$

in $H^*(M;\mathbb{R})$. We can eliminate η and β from the formula (9.1):

$$c(M,\omega_K) = \left(1 + \frac{\omega_K - Kc_1}{1-nK}\right)^n \exp\frac{c_1 - n\omega_K}{1-nK+\omega_K - Kc_1}.$$

Put

$$\begin{aligned} Q_K := \sum_{k\geqq 0} Q_{K,k} &:= \sum_{k\geqq 0}\bar{c}_k - (1+\bar{\eta})^n \exp\frac{\bar{\beta}}{1+\bar{\eta}} \\ &= \sum_{k\geqq 0}\bar{c}_k - \left(1+\frac{w-K\bar{c}_1}{1-nK}\right)^n \exp\frac{\bar{c}_1 - nw}{1-nK+w-K\bar{c}_1} \end{aligned}$$

in $\sum_{k\geqq 0} I_{2n-1}[w]^k$, where $\bar{c}_i$ $(i \geqq 0)$ is the corresponding multilinear form to the i-th Chern class, $\bar{\eta} = \dfrac{w - K\bar{c}_1}{1-nK}$, $\bar{\beta} = \dfrac{-nw + \bar{c}_1}{1-nK}$ and $Q_{K,k} \in I_{2n-1}[w]^k$ for all $k \geqq 0$. Then we have $Q_{K,0} = Q_{K,1} = 0$ and

$$\begin{aligned} Q'_K &:= \sum_{k\geqq 0} Q'_{K,k} := \frac{\bar{\beta}}{1-nK}(1+\bar{\eta})^{n-2}\exp\frac{\bar{\beta}}{1+\bar{\eta}} \\ &= \frac{\bar{\beta}}{1-nK}\left[1 + \{(n-2)\bar{\eta} + \bar{\beta}\} + \left\{\frac{(n-2)(n-3)}{2!}\bar{\eta}^2 + (n-3)\bar{\eta}\bar{\beta} + \frac{1}{2!}\bar{\beta}^2\right\} + \cdots\right]. \end{aligned}$$

Consequently, for any sufficiently large $K > 0$, we have the symplectic manifold (M, ω_K) with the relations $Q_{K,k}(M, \omega_K) = 0$ $(k = 2, \dots, 2n-1)$ and the crossed homomorphisms

$$\mathcal{F}_{Q_{K,k}} \colon \mathcal{S}(M, \omega_K) \to H^{2k-1}(M; \mathbb{R})/D_{Q_{K,k}}(M, \omega_K)$$

by Theorem A.

Next we show that these crossed homomorphisms are nontrivial for large $K > 0$. As in section 8, let $h\colon V = L_1 \oplus \cdots \oplus L_n \to V$ be the $U(n)$-bundle isomorphism over the identify of T^{2n} corresponding to a smooth map

$$h^U = \begin{pmatrix} h_1^U & & O \\ & \ddots & \\ O & & h_n^U \end{pmatrix} : T^{2n} \to U(n)$$

with $h_i^U \colon T^{2n} \to U(1)$ $(1 \leqq i \leqq n)$. The induced fiber map on $M = P(V)$ is denoted by

$$\hat{h}\colon M \to M,$$

whose restriction to each fiber is a symplectomorphism.

Lemma 9.1. *$\hat{h}^* = id$ on $H^*(M; \mathbb{R})$.*

Proof. The cohomology group of M is given by $H^*(M;\mathbb{R}) = H^*(T^{2n};\mathbb{R})[\eta]/\sim$. Since the bundle isomorphism h on V naturally induces a $U(1)$-bundle isomorphism $\tilde{h}\colon \mathcal{L} \to \mathcal{L}$ over $\hat{h}$, we have an isomorphism $\tilde{h}^*\mathcal{L} \cong \mathcal{L}$ over the identity on M, which implies $\hat{h}^*\eta = \eta$. Since $\hat{h}$ is a fiber preserving map over the identity of T^{2n}, we have $\hat{h}^* = id$. □

By Proposition 7.2, the map $\hat{h}$ is diffeotopic to a symplectomorphism Φ on (M, ω_K):

$$\hat{h} \simeq \Phi\colon (M, \omega_K) \to (M, \omega_K)$$

for any sufficiently large $K > 0$.

Next we compute $\mathcal{F}_{Q_{K,k}}(\Phi)$ using Proposition 7.5. Our setting is $\hat{\phi} = \hat{h}$, $\phi = id_{T^{2n}}$ in Proposition 7.5. Put $y_i = h_i^{U*}u$ and $Y = \sum_{i=1}^n y_i$ in $H^1(T^{2n};\mathbb{R})$. Let $\tilde{\eta} = c_1(\mathcal{L}_{\tilde{h}}^*) \in H^2(M_{\hat{h}};\mathbb{R})$ be the first Chern class of the dual line bundle to $\mathcal{L}_{\tilde{h}}$. We consider the $\mathbb{R}$-isomorphism (6.1) $\kappa_{\hat{h}}\colon H^*(S^1;\mathbb{R}) \otimes H^*(M;\mathbb{R}) \xrightarrow{\cong} H^*(M_{\hat{h}};\mathbb{R})$ with the homomorphism

$$\begin{aligned} s\colon H^*(M;\mathbb{R}) = H^*(T^{2n};\mathbb{R})[\eta]/\sim \ni a\cdot\eta^n& \\ \mapsto (p^*a)\cdot\tilde{\eta}^n \in H^*(M_{\hat{h}};\mathbb{R}) &= H^*(S^1 \times T^{2n};\mathbb{R})[\tilde{\eta}]/\sim, \end{aligned}$$

where $p\colon S^1 \times T^{2n} \to T^{2n}$ is the projection. We have the following commutative diagram:

$$\begin{array}{ccc}
H^*(S^1\mathbb{R}) \otimes H^*(M;\mathbb{R}) & \xrightarrow{\kappa_{\hat{h}}} & H^*(M_{\hat{h}};\mathbb{R}) = H^*(S^1 \times T^{2n};\mathbb{R})[\tilde{\eta}]/\sim \\
{\scriptstyle 1\otimes(\pi|_M)^*}\uparrow & & \uparrow{\scriptstyle \pi^*} \\
H^*(S^1;\mathbb{R}) \otimes H^*(T^{2n};\mathbb{R}) & \xrightarrow{\kappa} & H^*(S^1 \times T^{2n};\mathbb{R}).
\end{array}$$

Similarly as before, elements of $H^*(S^1;\mathbb{R})$ and $H^*(T^{2n};\mathbb{R})$ are considered as sitting in $H^*(M_{\hat{h}};\mathbb{R})$ in the obvious way, so we have

$$H^*(S^1;\mathbb{R}) \otimes H^*(T^{2n};\mathbb{R}) \ni t^* \otimes a \mapsto t^*a (= t^* \cup a) \in H^*(M_{\hat{h}};\mathbb{R})$$

in the diagram above.

Put $\hat{\omega}_K := \kappa_{\hat{h}}(\omega_K) \in H^2(M_{\hat{h}};\mathbb{R})$; then we have

$$\hat{\omega}_K = \tilde{\eta} + K\beta$$

because of $\omega_K = \eta + K\beta$ in $H^2(M;\mathbb{R})$.

Next we compute

$$\begin{aligned}
&Q_K((T^M M_{\hat{h}}, \sigma_{\hat{h}} \oplus \pi^*\beta_{id}), \hat{\omega}_K) \\
&= c(T^M M_{\hat{h}}, \sigma_{\hat{h}} \oplus \pi^*\beta_{id}) - \left(1 + \frac{\hat{\omega}_K - K\hat{c}_1}{1 - nK}\right)^n \exp \frac{\hat{c}_1 - n\hat{\omega}_K}{1 - nK + \hat{\omega}_K - K\hat{c}_1},
\end{aligned}$$

where $\hat{c}_1 = c_1(T^M M_{\hat{h}}, \sigma_{\hat{h}} \oplus \pi^*\beta_{id})$. Using Lemma 8.1 and an isomorphism $T^M M_{\hat{h}} \cong T^{\mathbb{C}P} M_{\hat{h}} \oplus \pi^* T^B B_{id}$, where $T^B B_{id} = (TT^{2n}) \times S^1$ is considered as a trivial $U(n)$-bundle, we have

$$\begin{aligned}
c(T^M M_{\hat{h}}, \sigma_{\hat{h}} \oplus \pi^*\beta_{id}) &= c(T^{\mathbb{C}P} M_{\hat{h}} \oplus \pi^* T^B B_{id}) = c(T^{\mathbb{C}P} M_{\hat{h}}) \\
&= c(\mathcal{L}_{\tilde{h}} \otimes \pi_{\hat{h}}^* V_h) = c\Big(\bigoplus_{i=1}^n \mathcal{L}_{\tilde{h}} \otimes \pi_{\hat{h}}^* (L_i)_{h_i}\Big) \\
&= \prod_{i=1}^n \left\{1 + \tilde{\eta} + t^* \cup h_i^{U*} u + c_1(L_i)\right\} \\
&= \prod_{i=1}^n (1 + \tilde{\eta} + t^* y_i + \gamma_i)
\end{aligned}$$

in $H^*(M_{\hat{h}};\mathbb{R}) = H^*(S^1 \times T^{2n};\mathbb{R})[\tilde{\eta}]/\sim$. In particular, we have

$$\hat{c}_1 = n\tilde{\eta} + t^* \cup Y + \beta.$$

Put

$$\zeta_i = t^* y_i + \gamma_i;$$

then we have

$$\zeta_i^2 = (t^* y_i + \gamma_i)^2 = 2t^* y_i \gamma_i$$

and

$$\zeta_i^3 = \cdots = \zeta_i^j = \cdots = 0 \ (j \geqq 3).$$

Let $\bar{\sigma}_i = \bar{\sigma}_i(\zeta_1, \ldots, \zeta_n)$ be the i-th fundamental symmetric polynomial of ζ_j's for $1 \leqq i \leqq n$. Put $\bar{s}_\nu = \sum_{j=1}^n \zeta_j^\nu$; then Newton's formulas imply

$$\bar{\sigma}_{\nu-2}\bar{s}_2 - \bar{\sigma}_{\nu-1}\bar{s}_1 + \nu\bar{\sigma}_\nu = 0$$

for $\nu = 1, \dots, n$, where $\bar{\sigma}_{-1} = 0$ and $\bar{\sigma}_0 = 1$. This equality also holds for $\nu = n+1, n+2, \dots$ in $H^*(T^{2n}; \mathbb{R})$.

Put

$$f(t) = \sum_{\nu=0}^{\infty} \bar{\sigma}_\nu t^\nu = \bar{\sigma}_0 + \bar{\sigma}_1 t + \cdots + \bar{\sigma}_\nu t^\nu + \cdots, \ f(0) = \bar{\sigma}_0 = 1;$$

then we have

$$f'(t) - \bar{s}_1 f(t) + \bar{s}_2 t f(t) = \sum_{\nu=1}^{\infty} (\bar{\sigma}_\nu \nu - \bar{s}_1 \bar{\sigma}_{\nu-1} + \bar{s}_2 \bar{\sigma}_{\nu-2}) t^{\nu-1} = 0.$$

This differential equation with $f(0) = 1$ is solved as

$$f(t) = \exp\left(\bar{s}_1 t - \bar{s}_2 \frac{t^2}{2}\right).$$

Since we have

$$\begin{aligned} \bar{s}_1 &= \sum_{i=1}^{n} \zeta_i = \sum_{i=1}^{n} (t^* y_i + \gamma_i) = t^* Y + \beta, \\ \bar{s}_2 &= \sum_{i=1}^{n} \zeta_i^2 = \sum_{i=1}^{n} 2 t^* y_i \gamma_i = 2 t^* Z, \end{aligned}$$

where

$$Z = \sum_{i=1}^{n} y_i \gamma_i,$$

we obtain

$$\begin{aligned} &\prod_{i=1}^{n} (1 + \tilde{\eta} + t^* y_i + \gamma_i) = \prod_{i=1}^{n} (1 + \tilde{\eta} + \zeta_i) \\ &= (1+\tilde{\eta})^n + (1+\tilde{\eta})^{n-1} \bar{\sigma}_1 + \cdots + (1+\tilde{\eta}) \bar{\sigma}_{n-1} + \bar{\sigma}_n \\ &= (1+\tilde{\eta})^n f\left(\frac{1}{1+\tilde{\eta}}\right) \\ &= (1+\tilde{\eta})^n \exp\left(\frac{t^* Y + \beta}{1+\tilde{\eta}} - \frac{t^* Z}{(1+\tilde{\eta})^2}\right) \\ &= (1+\tilde{\eta})^{n-2} \exp \frac{\beta}{1+\tilde{\eta}} \left[(1+\tilde{\eta})^2 + t^* \left\{(1+\tilde{\eta}) Y - Z\right\}\right]. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\left(1 + \frac{\hat{\omega}_K - K\hat{c}_1}{1 - nK}\right)^n \exp \frac{\hat{c}_1 - n\hat{\omega}_K}{1 - nK + \hat{\omega}_K - K\hat{c}_1} \\ &= \left(1 + \tilde{\eta} - \frac{K}{1-nK} t^* Y\right)^n \exp \frac{t^* Y + (1-nK)\beta}{1 - nK + (1-nK)\tilde{\eta} - K t^* Y} \\ &= (1+\tilde{\eta})^{n-2} \exp \frac{\beta}{1+\tilde{\eta}} \cdot \left((1+\tilde{\eta})^2 + \frac{K\beta t^* Y}{1-nK} + (1+\tilde{\eta}) t^* Y\right), \end{aligned}$$

where we particularly used $(t^*)^2 = 0$. Summing up the above computation, we have

$$\begin{aligned}
&Q_K((T^M M_{\hat{h}}, \sigma_{\hat{h}} \oplus \pi^*\beta_{id}), \hat{\omega}_K) \\
&= (1+\tilde{\eta})^{n-2} \exp \frac{\beta}{1+\tilde{\eta}} \left[(1+\tilde{\eta})^2 + t^* \left\{ (1+\tilde{\eta})Y - Z \right\} \right] \\
&\quad - (1+\tilde{\eta})^{n-2} \exp \frac{\beta}{1+\tilde{\eta}} \cdot \left((1+\tilde{\eta})^2 + \frac{K\beta t^* Y}{1-nK} + (1+\tilde{\eta}) t^* Y \right) \\
&= -t^*(1+\tilde{\eta})^{n-2} \exp \frac{\beta}{1+\tilde{\eta}} \cdot (Z + \frac{K}{1-nK}\beta Y)
\end{aligned}$$

in $H^*(M_{\hat{h}}; \mathbb{R})$.

By Proposition 7.5 we obtain

$$\begin{aligned}
-\sum_{k \geqq 0} F_{Q_{K,k}}(\Phi) &\equiv \iota^* \left\{ -(1+\tilde{\eta})^{n-2} \exp \frac{\beta}{1+\tilde{\eta}} \cdot (Z + \frac{K}{1-nK}\beta Y) \right\} \\
&\qquad \bmod \bigoplus_{k \geqq 0} D_{Q_{K,k}}(M, \omega_K) \\
&= -(1+\eta)^{n-2} \exp \frac{\beta}{1+\eta} \cdot (Z + \frac{K}{1-nK}\beta Y).
\end{aligned}$$

Put

$$(1+\eta)^{n-2} \exp \frac{\beta}{1+\eta} = a_0 + a_2 + a_4 + \cdots,$$

where $a_{2k} \in H^{2k}(M; \mathbb{R})$; then we have

$$\mathcal{F}_{Q_{K,k}}([\Phi]) = \left[a_{2k-4}(Z + \frac{K}{1-nK}\beta Y) \right] = [a_{2k-4} Z]$$

in $H^{2k-1}(M; \mathbb{R}) / D_{Q_{K,k}}(M, \omega_K) = H^{2k-1}(M; \mathbb{R}) / a_{2k-4}\beta \cdot H^1(M; \mathbb{R})$. In the case of $k = 2$, it is given by

$$\begin{aligned}
\mathcal{F}_{Q_{K,2}} \colon \mathcal{S}(M, \omega) &\to H^3(M; \mathbb{R}) / \beta \cdot H^1(M; \mathbb{R}) \\
&\cong \left\{ H^3(T^{2n}; \mathbb{R}) / \beta \cdot H^1(T^{2n}; \mathbb{R}) \right\} \oplus H^1(T^{2n}; \mathbb{R}) \cdot \eta
\end{aligned}$$

with

$$\mathcal{F}_{Q_{K,2}}([\Phi]) = [Z] = \left[\sum_{i=1}^n y_i \gamma_i \right] = \left[\sum_{i=1}^n (h_i^U)^* u \gamma_i \right]. \tag{9.2}$$

Moreover we have $\mathcal{F}_{Q_{K,k}} = a_{2k-4} \mathcal{F}_{Q_{K,2}}$ for $k \geqq 2$, where a_{2k-4} is considered as a well-defined homomorphism from $H^3(M; \mathbb{R}) / D_{Q_{K,2}}(M, \omega_K)$ to $H^{2k-1}(M; \mathbb{R}) / D_{Q_{K,k}}(M, \omega_K)$. Since there is no condition for the maps $h_i^U (1 \leqq i \leqq n)$ and the corresponding class $[\Phi]$ belongs to $\mathcal{S}_0(M, \omega)$ by Lemma 9.1, it is easy to see that the restriction of $\mathcal{F}_{Q_{K,2}}$ to $\mathcal{S}_0(M, \omega)$ is a nontrivial homomorphism by (9.2). Therefore we have $\iota^*[\mathcal{F}_{Q_{K,2}}] \neq 0$ in $H^1(\mathcal{S}_0(M, \omega_K), H^3(M; \mathbb{R}) / D_{Q_{K,2}}(M, \omega_K))$, hence $[\mathcal{F}_{Q_{K,2}}] \neq 0$ in $H^1(\mathcal{S}(M, \omega_K), H^3(M; \mathbb{R}) / D_{Q_{K,2}}(M, \omega_K))$. Consequently we have Theorem E.

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