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BENNEQUIN'S INEQUALITY AND THE POSITIVITY OF THE SIGNATURE

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ABSTRACT. We use an algorithm for special diagrams to prove a Bennequin type inequality for the signature of an arbitrary link diagram, related to its Murasugi sum decomposition. We apply this inequality to show that the signature of a non-trivial positive 3-braid knot is greater than its genus, and that the signature of a positive braid link is minorated by an increasing function of its negated Euler characteristic. The latter property is conjectured to extend to positive links.

1. Introduction and motivation

1.1. **The Signature Growth conjecture.** A braid positive link is a link which can be represented as the closure of a positive braid. Braid positive links occur in several contexts, e.g., the theory of dynamical systems [FW], singularity theory [BW], and (in some vague and yet-to-be understood way) 4-dimensional QFTs. They are a subclass of the class of positive links [Cr2, O, Yo], which have diagrams with all crossings positive.

Knot-theoretically, one is interested how positivity can be detected by the examination of link invariants. One of the most classical such invariants is the signature σ. The positivity of the signature on positive links (or subclasses thereof) has been a theme occurring throughout the literature over a long period.

The first related result was already established by Murasugi in his initial study of the signature [Mu2]. If a link bounds no disconnected Seifert surface (any non-split positive or alternating link satisfies this condition), then $\sigma \leq 1-\chi$, where χ is the maximal Euler characteristic of a Seifert surface. Murasugi showed that this natural upper bound is exact for a special alternating link, i.e. an alternating (and simultaneously) positive link. Simple examples illustrate that this is not true for general positive, or braid positive links.

Motivated by their study in dynamical systems, in [Ru] Lee Rudolph showed that (non-trivial) braid positive links have (at least) strictly positive[1] signature σ. This result was subsequently extended to positive links by Cochran-Gompf [CG, corollary 3.4]. A different proof, proposed by Traczyk [Tr], unfortunately has a gap

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[1]There is often confusion about the choice of sign in the definition of σ. Here (following [Ru], rather than [Tr] or [CG]) we use the more natural seeming convention that positive links have positive, and not negative σ.

and breaks down at least partly. (It still applies for braid positive links, the special case previously settled by Rudolph.) Przytycki observed the result (also for almost positive knots) to be a consequence of Taniyama's work [Ta], but a draft with an account on the subject was not finished. A related proof was written down in [St9].

Since $1-\chi$ gives a natural upper bound on the signature, it is suggestive to ask how the signature of positive links behaves with regard to $1-\chi$. One should believe in an increase of σ, in the range between the maximal value in Murasugi's result and the mere positivity property. So we are led to a natural seeming conjecture, which we state as follows. (See Section 5 for some more discussion of evidence.)

Conjecture 1.1 (Signature Growth conjecture). *Let $\chi'(L) := n'(L) - \chi(L)$, where $n'(L)$ is the number of split components of a link L, and $\chi(L)$ the maximal Euler characteristic of a spanning Seifert surface of L. Then*

$$\liminf_{n\to\infty} \min\{\,\sigma(L) \,:\, L \text{ positive link},\ \chi'(L)=n\,\} \,=\, \infty.$$

Alternatively speaking, one asks whether

$$\Sigma_\sigma \,=\, \{\,\chi'(L) \,:\, L \text{ positive link},\ \sigma(L)=\sigma\,\}$$

is finite for every σ.

This conjecture, although suggestive, is by no means obvious, or easily approachable. Although σ is easily calculated for any specific link, it has turned out to be difficult to make general statements about it on large link classes. This situation is a bit opposite to $1-\chi$, for which much more general formulas are available, but whose calculation for specific (other) links may be very complicated.

1.2. **Statement of results.** In this paper we shall prove an important partial result towards Conjecture 1.1. Beside the special alternating links, for which it follows directly from Murasugi's original work, we will show as our main result the conjecture for braid positive links.

Theorem 1.1 (Main result).

$$\liminf_{n\to\infty} \min\{\,\sigma(L) \,:\, L \text{ braid positive link},\ \chi'(L)=n\,\} \,=\, \infty.$$

Note that, while the mere positivity of the signature has now been established in a variety of ways, this is qualitatively all we know so far about it for braid positive links, as a non-constant lower bound for it has not been known for any reasonably general class of links, except for the initial Murasugi result [Mu2] on the special alternating links. The lack of such a result for braid positive links should also be contrasted with the fact that it has been well-known for a while how to determine the invariants of such a link estimated by the signature: the genus, smooth 4-ball genus and unknotting/unlinking number (see the beginning of §3.2). For the signature no such formulas seem to be known, except for the few very restricted cases in [H, GLM] and [Mu]. While the signature can at least be calculated case-by-case, we can say even less on the topological 4-ball genus.

Rudolph's positivity result follows at once from Theorem 1.1 by considering the iterated connected sums of a braid positive link. The following further direct consequence seems new, in contrast, though it is clear in the smooth category (because only finitely many such knots have the same genus; see, e.g., [Cr2]):

Corollary 1.1. *Among braid positive knots only finitely many are topologically concordant.* □

It is tempting to conjecture that this feature holds (at least smoothly) for general positive knots. We will address this issue at a separate place.

Although the established behaviour of the signature may be expected, one should be cautioned that in general σ can grow slower than χ'. For example it follows from the signature formulas for torus knots [H] (see therein equation (3) in §2, and the remark at the very end of the paper) that on the sequence of $(n, n+2)$-torus knots (n odd) σ grows like $\chi'/2$. This should explain that some subtlety is needed in our arguments. Our proof, which is in the presented form indirect, can be made more explicit to show a lower bound for σ proportional to $\sqrt[3]{\chi'}$ (see Corollary 4.1).

Our first step towards the proof of Theorem 1.1 will be to give an interpretation of the slower growth of σ compared to χ'. It measures the way how a positive diagram decomposes as a Murasugi sum of special alternating diagrams. This way σ acquires some new application, after it was majorated as a 4-genus estimate by the slice Bennequin inequality for many (incl. positive) links. We prove an inequality for the signature of a positive link diagram related to its Murasugi sum decomposition, which yields a 'Bennequin type' inequality for general link diagrams (Theorem 3.1). Thus we have an answer to Bennequin's problem on how to modify his inequality so as to also apply to the signature. This answer appears reasonable, as the correction term is easily defined, and the inequality is asymptotically sharp in infinitely many cases. The inequality for σ originates from an algorithm, first found by Hirasawa [Hr], to make any link diagram into a special one without altering the canonical genus.

Before we prove Theorem 1.1, we will first use the signature inequality to extend and make more explicit the positivity result for σ on closed positive braids of 3 and 4 strands. The 3-braid theorem will then play a decisive part in the proof for general positive braids. The idea we use for the extension is adapted from (the correct part of) Traczyk's arguments [Tr]. Note also that the proof of Theorem 1.1 uses only (combinatorial) knot theory, rather than contact geometry [Be] or appeals to gauge theory [Ru2]. In any case, the fact that σ had not previously been involved in an inequality of Bennequin type provided a main motivation for the present result.

2. General preliminaries

Here we recall a few basic facts and notation.

2.1. **Miscellanea.**

First, we fix some general (mathematical and linguistic) terminology.

By $\lfloor n \rfloor$ we will mean the greatest integer not greater than n. By $\lceil n \rceil$ we will mean the smallest integer not smaller than n.

For a set S, the expressions $|S|$ and $\#S$ are equivalent and both denote the cardinality of S. In the sequel the symbol '$\subset$' denotes a not necessarily proper inclusion.

For two sequences of positive integers $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ we say that $a_n = O(b_n)$ iff $\limsup\limits_{n\to\infty} a_n/b_n < \infty$; $a_n = O^{\succeq}(b_n)$ iff $\liminf\limits_{n\to\infty} a_n/b_n > 0$; and $a_n = O^{\asymp}(b_n)$ iff a_n is both $O(b_n)$ and $O^{\succeq}(b_n)$. We write $a_n \sim b_n$ if $\lim\limits_{n\to\infty} a_n/b_n = 1$.

'W.l.o.g.' will abbreviate 'without loss of generality' and 'r.h.s' (resp. 'l.h.s') 'right hand-side' (resp. 'left hand-side').

2.2. **Braids.** The n-strand *braid group* B_n is generated by the elementary *(Artin) generators* σ_i for $1 \le i \le n-1$ with relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, henceforth called *Yang-Baxter* (YB) relations, and $[\sigma_i, \sigma_j] = 1$ for $|i-j| > 1$ (the brackets denoting the commutator), called *commutativity relations*.

We introduce an alternative notation for braid(word)s by replacing the σ_i by their subscripts and their inverses by the negated subscripts, and putting the result into angle brackets, e.g., $(\sigma_1\sigma_2^2)^5\sigma_1^{-1}\sigma_2 = \big\langle (12^2)^5 - 12 \big\rangle$.

A braid (word) is called *alternating* if it contains no generators σ_i and σ_j occurring with powers of the same sign, such that $i-j$ is odd. A braid word is called *positive* if it contains no generators occurring with negative powers.

By $[\beta]$ we denote the *exponent sum* of a braid $\beta \in B_n$, that is, the image of β under the homomorphism $B_n \to \mathbb{Z}$ given by $[\sigma_i] := 1$ for any i; by $n(\beta) = n$ we write the *strand number* of β. By $[\beta]_i$ we denote the exponent sum of the generator σ_i in the braid *word* β, which is clearly not invariant under the YB relation. That is, $[\,.\,]_i$ is a homomorphism of the free group in the σ_j given by $[\sigma_j]_i := \delta_{ij}$ (where δ is the Kronecker delta). We call a positive braid β *non-split*, if $[\beta]_i > 0$ for any $1 \le i \le n-1$. (This property is independent on the word representation of β, although $[\beta]_i$ may vary.)

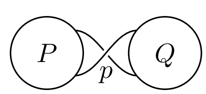
In this paper we shall be particularly concerned with the 3-strand braid group B_3 and its distinguished element $\Delta = \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, whose square Δ^2 is central in B_3 (and in fact generates its center).

There are canonical inclusions of the braid groups $B_n \hookrightarrow B_{n+1}$. By these inclusions, a braid (word) in $B_3 \hookrightarrow B_n$ is alternating if it is a word only in σ_1 and σ_2^{-1} (resp. σ_2 and σ_1^{-1}), and positive if it is a word only in σ_1 and σ_2.

By $\hat\alpha$ we denote the link, which is the braid closure of α. We call α also a *braid representation* of $\hat\alpha$.

2.3. **Link diagrams.**

Definition 2.1. A crossing p in a link diagram D is called *reducible* (or *nugatory*) if D can be represented in the form



D is called reducible if it has a reducible crossing, otherwise it is called *reduced*.

Definition 2.2. The diagram on the right of Figure 1 is called the *connected sum* $A\#B$ of the diagrams A and B. If a diagram D can be represented as the connected sum of diagrams A and B, such that both A and B have at least one crossing, then D is called *composite*, otherwise it is called *prime*. We call A and B *factors* of D.

Alternatively, if $D = A\#B$, then there is a closed curve β in the plane intersecting D in two points (and doing so transversely), such that A and B are contained in the in/exterior of β.

Note in particular that prime diagrams are reduced.

Definition 2.3. If there is a closed curve β in the plane intersecting D nowhere and containing at least one component of D in both its interior and exterior, we say that D is *split* or *disconnected*; otherwise D is *non-split* or *connected*. A *split component* of a link L is a maximal set S of components of L with the property

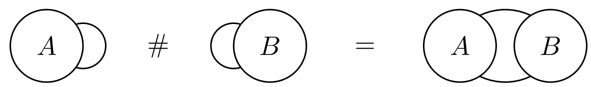
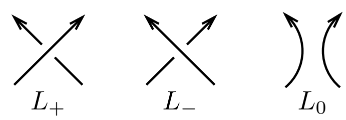


FIGURE 1

that if $a, b \in S$, then in any split diagram D of L with β as a splitting curve, a and b are contained in the same of the two regions of $\mathbb{R}^2 \setminus \beta$. A link is *split* if it has a split diagram, or equivalently, if it has more than one split component. A

(1)



positive, resp. *negative*, crossing is the fragment of L_+, resp. L_-, shown in (1). Replacing any of these fragments by the fragment of L_0 in (1) is called *smoothing out* the crossing. The interchange between $L_\pm$ is called the *switch* of a crossing. The *number of crossings* of a diagram D is written $c(D)$. The sum of signs of all crossings in D is called the *writhe* of D and denoted by $w(D)$.

Note that if $L_{\pm,0}$ are diagrams of closed braids $\beta_{\pm,0}$, then $\beta_\pm = \alpha\sigma_i^{\pm 1}\alpha'$ and $\beta_0 = \alpha\alpha'$ for some words α, α'. In that sense we can also understand the operation of crossing smoothing or crossing switch for braid(word)s.

A knot/link is called *positive/negative* if it has a diagram with all crossings positive/negative. Such a diagram is called also positive/negative. A diagram is called *n-almost positive*, if it has exactly n negative crossings. A knot/link is n-almost positive, if it has an n-almost positive, but no $n-1$-almost positive, diagram.

A link is called *braid positive*, if it is the closure of a positive braid, that is, if it has a positive braid representation, or alternatively a positive diagram as a closed braid. See, e.g., [Bu, Cr]. Beware that some authors, for example van Buskirk [Bu], confusingly call such links 'positive links' (which we use for a wider class here). Other authors call them 'positive braids', abusing the distinction between braids and their closures. Analogously one can define the property n-almost braid positive.

Smoothing out all crossings in D one obtains a collection of loops in the plane called *Seifert circles*. We write $s(D)$ for the number of Seifert circles of a diagram D. We call a Seifert circle A *opposite* to another Seifert circle B at a crossing p, if p joins A and B.

A Seifert circle is called *separating* if both regions of its complement in the plane contain other Seifert circles. Otherwise, it is called *non-separating*. A diagram is called *special* if all its Seifert circles are non-separating. (Contrarily the closed braid diagrams are those with only two non-separating Seifert circles.) Any link diagram decomposes as the *Murasugi sum* ($*$-product) of special diagrams (see [Cr2, §1]).

For a connected diagram two of the properties positive/negative, special and alternating, imply the third. A diagram with (all) these properties is called *special alternating*. A special alternating link is a link with a special alternating diagram. It is a link which is simultaneously positive and alternating (see [N]).

The *(canonical) Euler characteristic* $\chi(D)$ of a link diagram D is defined as $\chi(D) = s(D) - c(D)$, where $s(D)$ is, as before, the number of Seifert circles and $c(D)$ the number of crossings of D. If D is a diagram of a link with n components, the *(canonical) genus* $g(D)$ of D is given by

$$g(D) = \frac{2-n-\chi(D)}{2} = \frac{2-n+c(D)-s(D)}{2}.$$

These are the genus and Euler characteristic of the *canonical Seifert surface* of D, the one obtained by applying Seifert's algorithm on D. The *genus* $g(L)$ and *Euler characteristic* $\chi(L)$ of a link L are the minimal genus and maximal Euler characteristic of all Seifert surfaces of L, and the *canonical genus* $g_c(L)$ and *canonical Euler characteristic* $\chi_c(L)$ of L are the minimal genus and maximal Euler characteristic of all canonical Seifert surfaces of L, i.e. all Seifert surfaces obtained by applying Seifert's algorithm on some diagram D of L.

The importance of the canonical genus relies on the following classical fact:

Theorem 2.1. *An alternating/positive knot or link with an alternating/positive diagram of genus g has genus g. (In particular, for such knots or links canonical genus and ordinary genus coincide.)*

In the alternating case this was proved by [C, Mu3]. It can also be proved, in both cases, using [Ga]. For positive diagrams (and in particular positive braid representations) it follows from [Cr2], or from *Bennequin's inequality*, which we now state.

Theorem 2.2 ([Be, theorem 3])**.** *If β is a braid representation of a link L, then*

$$\chi(L) \leq n(\beta) - |[\beta]|.$$

This inequality admits several improvements. A first, and easy, observation is that by the braid algorithms of Yamada [Y] and Vogel [Vo] we obtain a version for a general link diagram D of L:

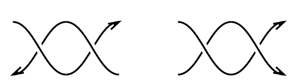
$$\chi(L) \leq s(D) - |w(D)|. \tag{2}$$

Later Rudolph [Ru2] showed that the r.h.s. in Bennequin's inequality is actually an estimate for the (smooth) slice Euler characteristic:

$$\chi_s(L) \leq s(D) - |w(D)|. \tag{3}$$

This inequality was further extended by showing that one can replace the l.h.s. with the invariants of Ozsváth, Szabó and Rasmussen on the one hand, and by slightly improving the r.h.s. on the other hand (adding a strongly negative Seifert circle term; see [K2]).

A *clasp* is a tangle made up of two crossings. According to the orientation of the strands we distinguish between reverse and parallel clasp.



reverse clasp parallel clasp

Definition 2.4. A *region* of a link diagram is a connected component of the complement of the (plane curve of) the diagram. An *edge* of D is the part of the plane curve of D between two crossings (clearly each edge bounds two regions). At each crossing p, exactly two of the four adjacent regions contain a part of the Seifert circles near p. We call these the *Seifert circle regions* of p. The other two regions are called the *non-Seifert circle regions* of p. We call two regions *opposite* at a crossing p, if p lies in the boundary of both regions, but they do not share any of the four edges bounded by p. If two regions share an edge, they are called *neighbored*. A diagram has a (canonical up to interchanging colors) black-white region coloring, given by the condition that neighbored regions have different colors. This is called the *checkerboard coloring*.

2.4. **The signature.** The *signature* σ is a $\mathbb{Z}$-valued invariant of knots and links. It has several definitions. The most common one is using Seifert surfaces and linking pairings. See, e.g., [Ro]. In the sequel, it will be more convenient to follow a rather different approach, using properties of the behaviour of σ under local transformations.

The signature is related to the *determinant* $\det(L) = |\Delta_L(-1)|$, where Δ_L is the *Alexander polynomial* of L. The Alexander polynomial $\Delta_L(t)$ can be specified by the (skein) relation

$$\Delta(L_+) \,-\, \Delta(L_-) \,=\, (t^{1/2} - t^{-1/2})\,\Delta(L_0)\,, \tag{4}$$

with $L_{\pm,0}$ as in (1), and the value 1 on the unknot. We have that $\sigma(L)$ has the opposite parity to the number of components of a link L, whenever $\Delta_L(-1) \neq 0$. This in particular always happens for L being a knot ($\Delta_L(-1)$ is always odd in this case), so that σ takes only even values on knots. Most of the early work on the signature was done by Murasugi [Mu], who showed several properties of this invariant.

Consider 3 links differing at just one crossing as in (1). Then

$$\sigma(L_+) - \sigma(L_-) \quad\in\quad \{0, 1, 2\}, \tag{5}$$

$$\sigma(L_\pm) - \sigma(L_0) \quad\in\quad \{-1, 0, 1\}. \tag{6}$$

(Note that when $L_\pm$ are knots, only 0 or 2 may occur on the right of (5), and keep in mind the footnote on page 1.) Further, Murasugi found the following important relation between $\sigma(K)$ and $\det(K)$ for a knot K:

$$\begin{aligned} \sigma(K) \equiv 0\,(4) &\quad\Longleftrightarrow\quad \det(K) \equiv 1\,(4), \\ \sigma(K) \equiv 2\,(4) &\quad\Longleftrightarrow\quad \det(K) \equiv 3\,(4). \end{aligned} \tag{7}$$

These conditions, together with the initial value $\sigma(\bigcirc) = 0$ for the unknot, and the additivity of σ under split union (denoted by '$\sqcup$') and connected sum (denoted by '#')

$$\sigma(L_1 \# L_2) \,=\, \sigma(L_1 \sqcup L_2) \,=\, \sigma(L_1) + \sigma(L_2)\,, \tag{8}$$

allow one to calculate σ for very many links. In particular, if we have a sequence of knots K_i

$$K_0 \to K_1 \to K_2 \cdots \to K_n$$

such that K_n is the unknot and K_i differs from K_{i-1} only by one crossing change, then (5) and (7) allow us to calculate inductively $\sigma(K_i)$ from $\sigma(K_{i+1})$, if $\det(K_i)$ is known.

From this the following property is evident for knots, which also holds for links: $\sigma(!L) = -\sigma(L)$, where $!L$ is the mirror image of L.

For general links of n components and maximal Euler characteristic χ of an (orientable compact spanning) Seifert surface, we have $\sigma \leq n - \chi$, but if no split component of the link bounds a disconnected Seifert surface, then $\sigma \leq n' - \chi$, where n' is the number of split components if L. (This condition is satisfied for very many links, including positive or alternating links, or links with non-zero Alexander polynomial.) Murasugi showed that this natural upper bound σ is exact for a special alternating link [Mu2], i.e. an alternating positive link. (It suffices to deal with the case $n' = 1$.) Simple examples show that this is not true for general positive, or positive braid links.

3. Special diagrams and Bennequin's inequality

3.1. **An algorithm for special diagrams.** In [BZ] it was proved that each link has a special diagram by a procedure on how to turn any given diagram of the link into a special one. However, the procedure in this proof alters drastically the initial diagram and offers no reasonable control on the complexity (canonical genus and crossing number) of the resulting special diagram. A much more economical procedure was given by Hirasawa [Hr] and rediscovered a little later independently in [St7, §7]. Hirasawa's move consists of laying a part of a separating Seifert circle s along itself in the opposite direction (we call this move *rerouting*, it is also called *wave move*), while changing the side of s depending on whether interior or exterior adjacent crossings to s are passed. See Figure 2.

The move of [St7] is similar, only that this type of rerouting is applied to the Seifert circle connected to s by a crossing c exterior to s. This move augments the canonical genus by one, but by properly choosing to reroute the strand above or below the rest of the diagram (that is, such that it passes all newly created crossings as over or undercrossings), one obtains a trivial parallel clasp involving c, whose resolution lowers the canonical genus back by one. Then we obtain an instance of Hirasawa's move.

Hereby, unlike in Hirasawa's original version of his algorithm, we take the freedom to alter the signs of the new crossings, as far as the isotopy type of the link, but *not* necessarily the isotopy type of the canonical Seifert surface is preserved. It is of importance to us only that the canonical genus of the diagram is preserved. We assume that this freedom is given throughout the rest of this section.

Hirasawa's algorithm is very economical – the number of new crossings added is linearly bounded in the crossing number, and even in the canonical Euler characteristic of the diagram started with. (Note, for example, that the braid algorithms of Yamada [Y] and Vogel [Vo] have quadratic growth.)

We start by an explicit estimate of the number of crossings and, what will be more important later, the number of negative crossings which need to be added.

The following notion will be of particular importance:

Definition 3.1. The *index* $\mathrm{ind}(s)$ of a separating Seifert circle s is defined as follows: denote for an inner crossing (i.e. attached from the inside) of s a letter 'i', and for an outer crossing a letter 'o' cyclically along s. Then $\mathrm{ind}(s)$ is by definition the minimal number of disjoint subwords of the form i^n or o^n $(n > 0)$ of this cyclic word. (For example, the Seifert circle s in Figure 2 has index 4.) A set of crossings corresponding to such a subword is called an *inner/outer group*. We call a group

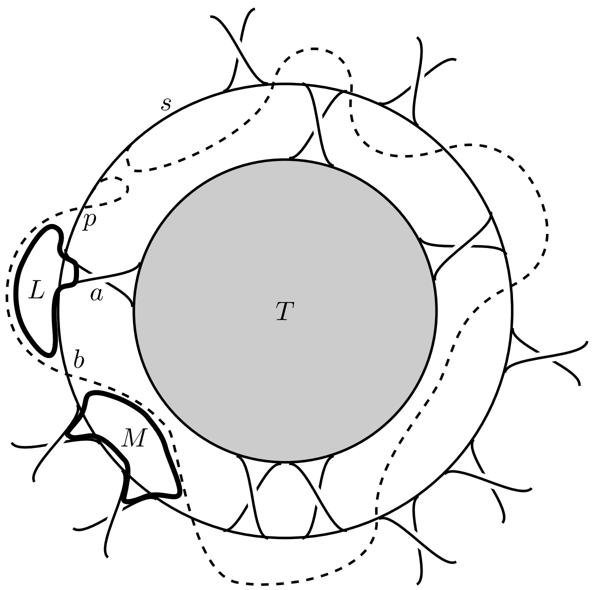


FIGURE 2

positive or negative depending on the sign of its crossings. (In special diagrams, it does not restrict us significantly to assume that no crossings of opposite sign occur within a group; these can always by cancelled in pairs.) We call a group *trivial* if it consists of a single crossing.

Put differently, $\mathrm{ind}(s)$ is one half of the number of sides of the polygon, along which the Murasugi sum corresponding to s is performed.

Denote by $c_i(s)$ the number of inner crossings adjacent (i.e. attached) to a Seifert circle s, by $c_o(s)$ the number of outer crossings, and by $c(s) = c_i(s)+c_o(s)$ the total number of crossings at s. (For example, for s in Figure 2, $c_i(s) = 5$ and $c_o(s) = 7$.)

Lemma 3.1. *A diagram can be made special by Hirasawa moves on separating Seifert circles s, under which at most*

a) $2\,\mathrm{ind}(s) - 1$ *crossings, or*
b) $2c_i(s) - 3$ *crossings, or*
c) $\mathrm{ind}(s) - 1$ *negative crossings*

are added.

Proof. Consider a separating Seifert circle s. W.l.o.g. assume by induction on the nesting of the Seifert circles that all interior Seifert circles of s, lying in T, are non-separating. Then

a) apply Hirasawa's move at a piece of s between an inner and outer crossing, and remove the nugatory crossing p arising at the side of the inner crossing,
b) follows from part a), if the number of inner crossings of s is strictly greater than its valence. Otherwise, each group of inner crossings is a single crossing. Then by a proper choice between under- and over-rerouting the dashed arc, we obtain a trivial clasp between a and b in Figure 2.

We assumed by induction that all Seifert circles inside s are disjoint. Then the resolution of the clasp of b and a joins the Seifert circle opposite to the Seifert circle in region L at a with the Seifert circle in region M, which also has an empty interior, so the new Seifert circle also has an empty interior.

c) By a proper choice between under- and over-rerouting the dashed arc, we can have the nugatory crossing arising in part a) to be negative, and the $2\,\mathrm{ind}(s)$ crossings created by the move alternate in sign.

As Hirasawa's move does not alter the Seifert picture (in particular the index and number of inner crossings of Seifert circles) outside s, we can continue by induction. □

Corollary 3.1. *Let a link L have a reduced diagram D with no trivial split components of $c = c(D)$ crossings, and canonical Euler characteristic $\chi = \chi(D)$. Then L has a special diagram D' with $\chi(D') = \chi(D)$ and*

$$c(D') \leq \min(3c-4, c-4\chi) - 3s_s\,, \tag{9}$$

where $s_s = s_s(D)$ is the number of separating Seifert circles of D.

Proof. By the terminal remark in the above proof we can estimate the number of new crossings for each separating Seifert circle separately, and directly from D, without the need to consider the intermediate diagrams of the moves. The result follows from that of part b) of the lemma. We have, writing in the sequel SC for 'Seifert circle' and SSC for 'separating Seifert circle',

$$c(D') - c(D) \,\leq \sum_{s \text{ SSC in } D} \big(2c_i(s) - 3\big). \tag{10}$$

Now the sum in (10) can be estimated more self-containedly in two different ways.

First, we have

$$\sum_{s \text{ SSC in } D} \big(2c_i(s) - 3\big) \,=\, 2 \sum_{s \text{ SSC in } D} c_i(s) - 3s_s \,\leq\, 2(c-2) - 3s_s.$$

To see the inequality on the right, note that any crossing is an inner crossing to at most one Seifert circle, and if every crossing is such, there is a Seifert circle s enclosing the entire diagram. Then this Seifert circle can be turned into one with empty interior, and the $c_i(s) \geq 2$ formerly inner crossings to s become outside any Seifert circle. (Here the reducedness and non-triviality of the split components of D are needed.) This gives the first alternative in the minimum in (9).

The second way to estimate the sum in (10) is to use $c_o(s) \geq 2$ (again $c_o = 1$ gives a non-reduced diagram). Then

$$\begin{aligned}
\sum_{s \text{ SSC in } D} \big(2c_i(s) - 3\big) \quad &\leq \quad \sum_{s \text{ SSC in } D} \big(2(c(s)-2) - 3\big) \\
&= \quad \sum_{s \text{ SSC in } D} (2c(s)-4) - 3s_s(D) \\
&\leq \quad \sum_{s \text{ SC in } D} (2c(s)-4) - 3s_s(D) \\
&\leq \quad 4c(D) - 4s(D) - 3s_s(D) \quad = \quad -4\chi(D) - 3s_s(D).
\end{aligned}$$

(Note that taking the non-separating Seifert circles into the sum with the second inequality, we use again that D is reduced.) This gives the second alternative in the minimum in (9). $\square$

3.2. **A Bennequin inequality for the signature.** Bennequin's inequality (2) gives a lower bound for the genus in terms of a braid representation of a knot or link, and was used in his discovery of non-standard contact structures on $\mathbb{R}^3$. For a positive knot/link, this inequality sharply estimates the genus. Hence its newer version (3), due to gauge-theoretic work of Rudolph [Ru2] and Kronheimer-Mrowka (see [KMr]), sharply estimates the 4-genus. Then one obtains explicit formulas for these invariants, and for braid positive knots/links from [BW] also for the unknotting/unlinking number. This in particular implies a famous conjecture of Milnor that for torus knots (or more generally knots of singularities) the smooth 4-ball genus is equal to the genus (or unknotting number). The (rather obvious) discussion can be found e.g. in [St5, K].

Recently, new signature-type concordance invariants, giving lower bounds for the 4-genus, were developed from Floer homology [OS] and Khovanov's homology [Ra] theory. Positive knots are again intrinsically linked to these invariants. In particular, Rasmussen's approach gives a new, combinatorial, proof of Bennequin's inequality and Milnor's conjecture.

One important difference between σ and its successors is that only the former is a concordance invariant in the topological category, while the latter apply only in the smooth category. This difference must be emphasized in view of the growing division in methods to treat both types of concordance. In that light the study of σ, a basic topological concordance invariant, gains new motivation; see for example Corollary 1.1.

One of Murasugi's original results about σ is that it, too, estimates (from below) the 4-genus of a knot. But the signature of torus knots (and links), found in [H, GLM], fails providing the sharp estimate desired for Milnor's conjecture. Many more examples illustrate that the signature does not conform to the lower bound in Bennequin's inequality. Such examples led to the question, encountered already in Bennequin's original work (see [Be, p. 121]), on how to modify his inequality to also be applicable to σ. We propose a solution to this problem, which is the main application of Lemma 3.1.

Theorem 3.1. *Let L be a positive non-split link of Euler characteristic $\chi = \chi(L)$, and D a positive connected diagram of L, with separating Seifert circles $s_1, \dots, s_{s_s(D)}$. Then*

$$\sigma(L) \geq 1 - \chi(L) - 2 \sum_{i=1}^{s_s(D)} \big(\mathrm{ind}(s_i) - 1\big). \tag{11}$$

More generally, for a connected diagram D of an arbitrary link L with separating Seifert circles $s_1, \dots, s_{s_s(D)}$, we have

$$\sigma(L) \geq w(D) - s(D) + 1 - 2 \sum_{i=1}^{s_s(D)} \big(\mathrm{ind}(s_i) - 1\big).$$

Proof. By part b) of Lemma 3.1, we can transform D into a special diagram of Euler characteristic $\chi = \chi(L)$ having at most

$$\sum_{i=1}^{s_s(D)} \bigl(\mathrm{ind}(s_i) - 1\bigr)$$

negative crossings. The claims then follow from Murasugi's formula [Mu2] $\sigma = 1-\chi$ for special alternating diagrams, and the fact that σ decreases by at most two under a crossing change. □

This inequality generalizes Murasugi's formula $\sigma = 1-\chi$ [Mu2] for special alternating links, and allows us to make a statement on the signature of a positive link having few Murasugi sum factors (or separating Seifert circles), and which are, in a sense, on the "opposite end" to positive braid links. (Note that braid diagrams are those with the maximal possible number of separating Seifert circles – all Seifert circles except two are separating.) Unfortunately, the estimates on both "ends" are still insufficient to give a non-trivial result on the general "intermediate" case. (Contrarily, the estimate for $1-\chi$ in Bennequin's inequality is sharp on positive diagrams.)

Remark 3.1. There are several ways to improve (11), in particular if s_i has a trivial (inner or outer) negative group, we can replace $\mathrm{ind}(s_i) - 1$ by $\mathrm{ind}(s_i) - 2$, by again making a and b in Figure 2 cancel.

3.3. **An application to torus knots.** Rather than as a signature estimate, the inequality (11) can be used on examples with small signature in the opposite direction to obtain restrictions on the Murasugi sum decomposition of the positive diagram. As an example, also applying Hirzebruch's formula, we have the following statement. (See Remark 4.1 for another example.)

Proposition 3.1. *Let the (p,q)-torus knot K with p,q odd, $(p,q) = 1$ and $p < q < \frac{4p}{3}+1$, be smoothly concordant to a positive prime knot K'. Then K' has no positive diagram D which can be obtained by connecting the endpoints (wiring) of tangles with ≥ 4 (positive) parallel half-twists. In particular, K' has no positive braid representation β, in which each generator occurs in power ≥ 4, i.e. in the word representation*

$$\beta = \prod_{i=1}^{m} \sigma_{k_i}^{r_i} \tag{12}$$

with $r_i \in \mathbb{N} \setminus \{0\}$ and $k_{i+1} \neq k_i$, all $r_i \geq 4$.

Proof. For K itself one can obtain the result, even for '4' replaced by '2', using some work of Murasugi [Mu4] and Thistlethwaite [Th] on the skein and Kauffman polynomial. However, we will just use the genus and signature, so that we obtain the result up to smooth concordance. (It follows from (2) and (3), that for a positive knot, the smooth 4-genus is equal to the genus.)

We apply Hirzebruch's formula [H, §2, (3)], for a (p,q)-torus knot, reading

$$\sigma = \frac{(p-1)(q-1)}{2} + 2\left(N_{p,q} + N_{q,p}\right), \tag{13}$$

where

$$N_{p,q} := \#\left\{ 1 \leq x \leq \frac{p-1}{2} \,:\, \exists\, y \in \mathbb{Z} \,:\, 0 > qx - py > -\frac{p}{2} \right\}.$$

It is easy to see that for $|p-q|=k$, we have

$$\left| N_{p,q} - \frac{p-1}{4} \right| \leq \frac{k}{4}. \tag{14}$$

To show that a diagram D of K' of the specified kind does not exist, we need to calculate for such a hypothetic D the estimate in (11). We have for the first term

$$1-\chi(K) \,=\, 1-\chi(K') \,=\, 1-\chi(D) \,=\, c(D)-s(D)+1 \,=\, (p-1)(q-1). \tag{15}$$

More work will be necessary to estimate the other term in (11). We claim the following inequality:

$$2\sum_{i=1}^{s_s(D)} \operatorname{ind}(s_i) \leq \frac{c(D)}{2} - 2s_n(D)\,, \tag{16}$$

with $s_n(D)=s(D)-s_s(D)$ the number of non-separating Seifert circles of D.

We must spend some effort in justifying the inequality (16). First, assume w.l.o.g. that each non-separating Seifert circle has empty interior (and not exterior).

We will then count

$$is(D) \,:=\, \sum_{i=1}^{s_s(D)} \operatorname{ind}(s_i)$$

with regard to the tangles attached from *inside* to the Seifert circles. To do so, mark all groups of crossings (tangles) T with the following property: T is attached from inside to a Seifert circle S, and the next group attached to S along its orientation is attached from the outside. This way, for any separating Seifert circle S exactly $\operatorname{ind}(S)$ groups attached from the inside to S are marked. Since a group is attached from the inside to at most one Seifert circle, the total number of groups marked is $is(D)$ (and not less).

Note that only two crossings of each marked group T are needed to account for $2\,is(D)$. Since any group has ≥ 4 crossings, we thus have

$$2\,is(D) \leq \frac{c_m(D)}{2}\,,$$

where $c_m(D)$ is the number of crossings of D in marked groups.

Let $n_n(D)$ be the number of non-marked groups in D. Then

$$c_m(D) \leq c(D) - 4n_n(D).$$

Thus for the inequality (16) it remains to prove

Sublemma 1. *$n_n(D) \geq s_n(D)$.*

Proof. We define a map

$$m \,:\, \{\,(S,p)\,\} \longrightarrow \{\,(N,s)\,\}\,,$$

where (S,p) is a pair consisting of a non-separating Seifert circle S in D and a group p attached (from the outside) to S, and s is a non-marked group in D and N a Seifert circle to which s is attached from the outside.

Define m by the following procedure. Let $s_0=p$ and $N_0=S$. Then for $i=0,1,2,\dots$ proceed like this: If s_i is non-marked, set $m(S,p)=(N_i,s_i)$. Otherwise s_i is attached to a Seifert circle N_{i+1} from the inside and, along the orientation of N_{i+1}, is followed by a group attached to N_{i+1} from the outside. Let s_{i+1} be this group; augment i by 1 and repeat the step. This iteration must terminate at some point because the number of Seifert circles separating s_i from s_0 grows.

We now claim that m is injective. Let s be a non-marked group and N a Seifert circle to which s is attached from the outside. Then there is at most one preimage (S,p) with $m(S,p)=(N,s)$. It is obtained as follows.

Let $i=0$, $S_0=N$ and $p_0=s$. Make the following step inductively over i. If S_i is non-separating, then $(S,p)=(S_i,p_i)$. If S_i is separating and p_i is preceded along the orientation of S_i by a group attached to S_i from the outside, there is no preimage (S,p). If it is preceded by a group attached to S_i from the inside, let p_{i+1} be this group and S_{i+1} the other Seifert circle to which p_{i+1} is attached (then from the outside). Then augment i by 1 and repeat the step.

Now, any non-separating Seifert circle S has at least two groups p attached to it (from the outside), unless D has a $(2,.)$-torus knot (diagram) factor under a connected sum, which can clearly be excluded assuming K' to be prime. Thus

$$\big|\{\,(S,p)\,\}\big| \geq 2s_n(D).$$

On the other hand, each non-marked group s is attached to at most two Seifert circles N from the outside. Thus

$$\big|\{\,(N,s)\,\}\big| \leq 2n_n(D).$$

The claim then follows from the injectivity of m. □

The sublemma shows (16). Using it, we now obtain from (11) and (15)

$$\begin{aligned}
\sigma &\geq 1-\chi-2\sum_{i=1}^{s_s(D)}\big(\operatorname{ind}(s_i)-1\big)\\
&= 1-\chi-2\sum_{i=1}^{s_s(D)}\operatorname{ind}(s_i)+2s_s(D)\\
&\geq (p-1)(q-1)-\frac{c(D)}{2}+2s_n(D)+2s_s(D)\\
&= (p-1)(q-1)+\frac{-(p-1)(q-1)-s(D)+1}{2}+2s(D)\\
&= \frac{(p-1)(q-1)}{2}+\frac{3s(D)}{2}+\frac{1}{2}\\
&\geq \frac{(p-1)(q-1)}{2}+\frac{3p}{2}+\frac{1}{2},
\end{aligned}$$

the last inequality coming from [Y] and the braid index p of K (see [Mu4]). Then we have a contradiction from (13) if

$$4\left(N_{p,q}+N_{q,p}\right) < 3p+1. \tag{17}$$

Let $k=q-p$. It follows from (14), that

$$N_{p,q}+N_{q,p} \leq \left(\frac{p-1}{4}+\frac{k}{4}\right)+\left(\frac{p+k-1}{4}+\frac{k}{4}\right) = \frac{p-1}{2}+\frac{3k}{4}.$$

Thus (17) holds in particular when $k<\frac{p}{3}+1$. □

Remark 3.2. The inequality (14) is often inexact, in particular because σ must be even (in fact, it should be divisible by 8). This way one can handle a few other cases, for example $(p,q)=(3,5)$.

We will use (11) to estimate the signature of closed positive 3- and 4-braids, and then also for general closed positive braids.

4. The Growth conjecture for braid positive links

In this section, we will be concerned with the proof of Theorem 1.1. For this proof some additional tools are necessary, most notably the braid scheme, introduced in [St]. Rudolph's original method of looking at subspaces of the homology group of the Seifert (fibre) surface, on which the Seifert form is positive definite, becomes algebraically difficult to push forward, and the subsequent improvements of his result used the easier-to-handle rules for the behaviour of the signature under local diagram moves, developed mainly by Murasugi [Mu2]. The proof of Theorem 1.1 will make use of this approach, too, and involves a combination of ideas in [Tr] and [St]. We thus begin with some, now more specific, preparations, recalling the method of [St]. Then we need to show first the case of closed positive 3-braids. This is obtained by applying (11). We then also discuss an estimate for 4-braids.

4.1. **Braid schemes.** We now recall the notion of a braid scheme, introduced in [St], since it will be the main object we will work with throughout the proof.

Definition 4.1. A *braid scheme* is a checkerboard diagram with integers putted on the black fields, e.g.

$$\begin{array}{ccccccc} & & -5 & & & & 6 \\ & -1 & & -2 & & 3 & \\ 1 & & 3 & & 2 & & 4 \\ & & & & & & \\ \sigma_1 & & \ldots & \ldots & \ldots & & \sigma_7 \end{array} \cdot \tag{18}$$

If integers are omitted, they are assumed to be 0.

We will denote the coordinates of the entries by the pair

$$(\,\text{row of the scheme, column} = \text{index of generator}\,).$$

Clearly, in each scheme entries with only even or only odd coordinate sum will occur. Beside the row coordinate we also leave the column coordinate unlimited, according to the inclusions of the braid groups $B_n \hookrightarrow B_{n+1}$. In subsequent drawings we assume the row coordinate to grow in the vertical direction from bottom to top, while the column coordinate to grow in the horizontal direction from left to right (so that pictures differing by a rotation by 90° are not equivalent).

We obtain a braid word from a braid scheme as follows. For each row, one writes the product of Artin generators, whose powers are given by the entries in this row of the scheme, and whose indices are given by the column coordinate of the entry. E.g., the braid word corresponding to the above scheme (18) is

$$(\sigma_1\sigma_3^3\sigma_5^2\sigma_7^4)\,(\sigma_2^{-1}\sigma_4^{-2}\sigma_6^3)\,(\sigma_3^{-5}\sigma_7^6).$$

Definition 4.2. A *reducing move* in a braid scheme is a local replacement of the type

$$\begin{array}{ccc} & x & \\ 0 & & 0 \\ & y & \end{array} \longrightarrow \begin{array}{ccc} & 0 & \\ 0 & & 0 \\ & x+y & \end{array}.$$

More formally, if x is a non-zero entry in the braid scheme in row ≥ 3, such that one row below its two neighbors (or its one neighbor, if it is the first or last generator) is/are zero, then replace x by 0 and the entry y situated two rows below x in the scheme by $x+y$. (This move just consists in applying commutativity relations to the braid word given by the scheme, so that the braid remains the same.)

Example 4.1. An instance of a sequence of reducing moves is the transformation

$$\begin{array}{ccccccc} & 3 & & -2 & & & \\ 0 & & 0 & & -4 & & \\ & 1 & & 0 & & & \\ 1 & & 2 & & 3 & & \end{array} \longrightarrow \begin{array}{ccccc} & 4 & & -2 & \\ 1 & & 2 & & -1 \end{array}.$$

A scheme is called *reduced* if it does not admit a reducing move. Clearly, any scheme can be reduced by finitely many reducing moves.

In the proof we will mainly consider schemes with all entries being 1. In this case a fragment of the scheme like

$$\begin{array}{ccc} & 1 & \\ 1 & & 0 \\ & 1 & \end{array} \quad \text{or} \quad \begin{array}{ccc} & 1 & \\ 0 & & 1 \\ & 1 & \end{array} \tag{19}$$

is the position where a YB relation can be applied.

4.2. **Positive 3-braids.** As an application of (11), we show

Theorem 4.1. *If $\beta \in B_3$ is positive non-split and $\beta \neq \sigma_1\sigma_2,\ \sigma_2\sigma_1$, then*

$$\sigma(\hat{\beta}) > \frac{1-\chi(\hat{\beta})}{2}.$$

Proof. Write

$$\beta = \sigma_1^{a_1}\sigma_2^{b_1}\dots\sigma_1^{a_i}\sigma_2^{b_i} \tag{20}$$

with i being the index of the middle Seifert circle. We regard β up to cyclic permutations of its letters, and thus in particular the sequence $(a_1, b_1, \dots, a_i, b_i)$ only in cyclic order. We will call this sequence the *cyclic sequence* of β. If i is minimal among all representations (20), then $\operatorname{ind}(\beta) = i$ per definition. By convention the index of $\sigma_{1,2}^k$ is 0. We assume, however, that $\operatorname{ind}(\beta) > 0$ since the other cases are trivial to check.

Write $\beta = \Delta^{2k}\alpha$ with α positive and $\Delta = \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ being the square root of the center generator of B_3. Since Δ^2 is normal, conjugacy of β passes through conjugacy of α. Assume k is maximal, i.e. α does not contain a Δ^2 as subword.

Let $r_\alpha = (a_{1,\alpha}, b_{1,\alpha}, \dots, a_{m,\alpha}, b_{m,\alpha})$ be a minimal length cyclic sequence of α. Then, because of minimality r_α contains no subsequence $(x, 1, y)$, with one of x or y being 1. Also it contains no such subsequence with $x = y = 2$, as $\sigma_1\sigma_2^2\sigma_1\sigma_2^2\sigma_1 = \sigma_1\Delta^2$. Thus $x + y \geq 5$.

Let the cyclic sequence of α contain l entries 1. Choose the minimal length cyclic sequence of α so that the number l is minimal (among all minimal length cyclic sequences). We now claim that, up to a few exceptional cases which we will handle ad hoc, the distance between two such entries '1' in r_α is at least three. (Distance here means 1+ number of intermediate entries, in cyclic order; so neighbored entries have distance one.) Let

$$\operatorname{est}(\alpha) := 1 - \chi(\alpha) - 2\left(\operatorname{ind}(\alpha) - 1\right),$$

with $\chi(\alpha) := -c(\alpha) + 3 = \chi(\hat{\alpha})$. So $\sigma(\hat{\alpha}) \geq \operatorname{est}(\alpha)$ when $\operatorname{ind}(\alpha) > 0$.

Case 1. Assume the distance between any two entries '1' in r_α is > 2. (This includes the case that r_α contains no '1's.) Let α' be the word obtained by deleting in α the l letters corresponding to all '1's in r_α. (In the braid diagram this means to smooth out the crossings.) Then the subsequence $(\dots, x, 1, y, \dots)$ in the cyclic sequence of α turns into $(\dots, x+y, \dots)$ for α'. We have

$$2\,\mathrm{ind}(\alpha') \le \frac{c(\alpha')}{2} - \frac{3l}{2}\,,$$

since for the braid with cyclic sequence $(2, 2, \dots, 2)$ we have $2\,\mathrm{ind} = c/2$, and we now have l different entries of the form $x+y \ge 5$, bringing at least 3 extra crossings each. (The condition of distance > 2 ensures that these extra crossings are not counted multiply for different entries '1'.) From this we find

$$\begin{aligned} \mathrm{est}(\alpha') &\ge 1 - \chi(\alpha') - \frac{c(\alpha')}{2} + \frac{3l}{2} + 2 \\ &= \frac{1-\chi(\hat\alpha')}{2} + 1 + \frac{3l}{2} \\ &= \frac{1-\chi(\beta)}{2} + 1 + l - 3k\,, \end{aligned} \tag{21}$$

since $1 - \chi(\hat\gamma) = c(\gamma) - 2$ for γ being either of β and α', and $c(\beta) = c(\alpha) + 6k = c(\alpha') + l + 6k$.

Now we use the fact that

$$\mathrm{ind}(\alpha'\Delta^2) \le \begin{cases} \mathrm{ind}(\alpha') + 1 & \text{if } \mathrm{ind}(\alpha') \ge 1, \\ \mathrm{ind}(\alpha') + 2 & \text{if } \mathrm{ind}(\alpha') = 0. \end{cases} \tag{22}$$

To see this w.l.o.g. cyclically conjugate a word of α' with minimal cyclic sequence so that it ends with σ_1, and if $\mathrm{ind}(\alpha') > 0$, it also starts with σ_2, and use the word representation $\Delta^2 = \sigma_1^2\sigma_2\sigma_1^2\sigma_2$. ($\alpha' = \sigma_2^k$ is handled as σ_1^k with indices interchanged.)

Therefore, $\mathrm{est}(\alpha'\Delta^2) \ge \mathrm{est}(\alpha') + 4$ for $\mathrm{ind}(\alpha') > 0$. Thus

$$\mathrm{est}(\alpha'\Delta^{2k}) \ge \mathrm{est}(\alpha') + 4k \tag{23}$$

for any positive α' of positive index. Then from (21) and (23),

$$\sigma(\widehat{\alpha'\Delta^{2k}}) \ge \mathrm{est}(\alpha'\Delta^{2k}) \ge \frac{1-\chi(\beta)}{2} + 1 + l + k.$$

Since $\widehat{\alpha'\Delta^{2k}}$ and $\hat\beta$ differ by l crossing smoothings, we have

$$\sigma(\hat\beta) \ge \sigma(\widehat{\alpha'\Delta^{2k}}) - l \ge \frac{1-\chi(\beta)}{2} + 1 + k. \tag{24}$$

Since $k \ge 0$, we are done under the assumption that no '1's in r_α have distance 2 and $\mathrm{ind}(\alpha') > 0$.

There remains the case when $\mathrm{ind}(\alpha') = 0$. Then $\mathrm{ind}(\alpha) \le 1$ because there were no entries '1' of distance 2 in r_α. Since $\mathrm{ind}(\alpha'\Delta^2) > 0$ for any positive α', we have from (22) and (23) that for any positive α' at least

$$\mathrm{est}(\alpha'\Delta^{2k}) \ge \mathrm{est}(\alpha') + 4k - 2. \tag{25}$$

Then (21) and (25) still provide, as above, the right estimate if $4k - 2 \ge 3k$, that is, $k > 1$.

If $k = 0$, then $\beta = \alpha$ and $\mathrm{ind}(\beta) = \mathrm{ind}(\alpha) \le 1$, which cases are easily checked (and lead to the exception of the unknot).

If $k = 1$, we must consider braids $\beta = \Delta^2\sigma_1^m\sigma_2^l$ with $m, l \geq 0$. Now we use the property that for any α with $\hat{\alpha}$ a knot we have $\sigma(\widehat{\sigma_1^{2i+2}\alpha}) = \sigma(\widehat{\sigma_1^{2i}\alpha}) + 2$ for any $i \geq 0$ except at most one, where $\sigma(\widehat{\sigma_1^{2i+2}\alpha}) = \sigma(\widehat{\sigma_1^{2i}\alpha})$. This relies on (5), (7) and the fact that the determinants of such knots form, up to sign, an arithmetic progression of the form $a + (4k+2)i$. (This progression can be obtained from the skein relation (4) for the Alexander polynomial.)

Thus if we show that $\sigma > \frac{1-\chi}{2} + 3$ for some odd m and l, we are through for all m', l' with $m' \geq m - 1$ and $l' \geq l - 1$ by applying twists to σ_1^m and σ_2^l and then possibly smoothing a crossing in each one of them. Thus we calculate σ for small m and l. We find

m	l	σ	$\frac{1-\chi}{2}$
0	0	4	2
1	0	5	5/2
1	1	6	3
3	1	8	4

This covers all cases up to symmetry in m and l.

Case 2. Assume that the distance between two '1's in r_α is two. (Distance one was excluded above by minimality of the cyclic sequence.) Then we have a subword in α of the form $\dots\sigma_2^p\sigma_1\sigma_2^q\sigma_1\sigma_2^r\dots$ with $p, q, r \geq 2$. We can assume that $q > 2$ or $p, r > 2$, as we excluded subsequences $(2, 1, 2)$ in r_α. If now some of p or r is > 2, then we can slide the σ_2^q to the left or right, becoming σ_1^q, obtaining a word with cyclic sequence of the same length but one '1' less. Thus $p = r = 2$. Then σ_1^q can be slided further to the left/right, and by repeating the argument we conclude that r_α consists only of '2's except for the subsequence $(1, q, 1)$, and $q > 2$.

It is easily seen that for the braids α with cyclic sequences of the form $r_\alpha = (1, q, 1, 2, 2, 2, 2, \dots)$ we have $4\,\mathrm{ind}(\alpha) = c + 4 - q$, and thus $\mathrm{est}(\alpha) = \frac{1-\chi(\alpha)}{2} + \frac{q-2}{2}$. The same argument with (22) for α instead of α' also shows

$$\sigma(\hat{\beta}) \geq \mathrm{est}(\beta) \geq \frac{1-\chi(\beta)}{2} + \frac{q-2}{2} + k \geq \frac{1-\chi(\beta)}{2} + \frac{q-2}{2}. \tag{26}$$

Since the cases $q \leq 2$ simplify, we are also done here. □

Remark 4.1. The result in Theorem 4.1 does not extend much further. The positive knot 14_{45657} of [St2] with $\sigma = \frac{1-\chi}{2} = 4$ is fibered and a single Murasugi sum (of two copies of the connected sum of a trefoil and two Hopf links). Also, the closures of the braids $(\sigma_1^2\sigma_2^2)^n$ have $\sigma \sim \frac{1-\chi}{2}$ as $n \to \infty$ by Murasugi's formulas [Mu].

Remark 4.2. Originally, I proved a special case of Theorem 4.1 from Murasugi's formulas [Mu], which was needed in the proof of Theorem 1.1. Now that Theorem 4.1 is available in full generality, the proof of Theorem 1.1 can be simplified, and the Murasugi formula arguments are redundant. Nonetheless one can likely also prove Theorem 4.1 completely from Murasugi's approach. Professor Murasugi informed me that Yoshiaki Uchida has an independent (and different) proof of Theorem 4.1, likely along such lines.

4.3. **Positive 4-braids.** One can use the 3-braid result for an estimate for 4-braids, whose proof is more involved, and which is less sharp, though.

Table 1. The minimal signatures of closures of positive non-split 4-braids for small values of χ'.

χ'	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\min\sigma$	0	1	2	3	4	4	4	5	6	5	6	7	8	8	8	9	10	9

Theorem 4.2. *If $\beta\in B_4$ is positive non-split, then* $\sigma(\hat\beta)\geq\dfrac{2}{11}\chi'(\hat\beta)-\dfrac{15}{11}$.

Thus we recover the result $\sigma>0$ in the cases $\chi'=1-\chi\geq 8$. The others are easy to check directly. Table 1 summarizes the minimal values of $\sigma(\hat\beta)$ for $1-\chi(\hat\beta)=[\beta]-3$ small. Although the values clearly stay above the bound of the theorem, the fact that the sequence is not monotonous should already hint to some caution. (The values in the table in fact also hold allowing for split braids, so that Theorem 4.2 extends by lifting the non-splitness condition on β.)

For the proof of Theorem 4.2 we use the method of Traczyk [Tr], as an analysis using Theorem 4.1 appears too complex.

Proof. Let $n=[\beta]$, $a=[\beta]_1$, $b=[\beta]_2$ and $c=[\beta]_3$. Then $a+b+c=n$. We assume $\min(a,b,c)\geq 2$; the other cases follow trivially or from Theorem 4.1. In the representation (12), let

$$r_a\;=\;\sum_{k_i=1}\lfloor r_i/2\rfloor\;,\quad r_b\;=\;\sum_{k_i=2}\lfloor r_i/2\rfloor\;,\text{ and }\quad r_c\;=\;\sum_{k_i=3}\lfloor r_i/2\rfloor\;.$$

Using Traczyk's argument, we have three options to estimate $\sigma(\hat\beta)$, by smoothing out one of the 3 generators. Then the contribution to the estimate of any of the $a/b/c$ crossings is negative, except for $r_a/r_b/r_c$, which are positive. (In a subword $\sigma_{k_i}^{r_i}$ only $r_i \bmod 2$ crossings need to be smoothed out, when $\lfloor r_i/2\rfloor$ are switched, and clasps resolved.)

Case 1. σ_1 is smoothed out. We obtain a braid word $\beta_{2,3}$ with $[\beta_{2,3}]=b+c$ and apply Theorem 4.1. Then

$$\sigma(\hat\beta)\geq\sigma(\hat\beta_{2,3})-a+2r_a\geq\frac{b+c-1}{2}-a+2r_a\;=\;\frac{n-3a}{2}+2r_a-\frac{1}{2}. \tag{27}$$

Case 2. σ_2 is smoothed out. We obtain a braid word $\beta_{1,3}$, the connected sum of two $(2,.)$-torus links. Thus $\sigma(\hat\beta_{1,3})=a+c-2$. Then

$$\sigma(\hat\beta)\geq a+c-b-2+2r_b\;=\;n-2b-2+2r_b.$$

Case 3. σ_3 is smoothed out. Analogously as for σ_1 we obtain

$$\sigma(\hat\beta)\geq\frac{n-3c}{2}+2r_c-\frac{1}{2}.$$

Then we claim that w.l.o.g. we can choose the word representation of β so that

$$r_a+r_b+r_c\;\geq\;\left\lfloor\frac{n}{6}\right\rfloor-1\;\geq\;\frac{n-11}{6}.$$

To see this, start with a word representation of β, for which $[\beta]_2$ is minimal, and subdivide it into subwords w of length 6. Then it suffices to consider such a subword w and show that one can obtain a σ_i^2 for each w with at most one exception. The only case when σ_i^2 is not already contained in w is up to symmetry $\langle 132132\rangle$, which transforms into $\langle 123121\rangle$, the half-twist braid, and $\langle 123123\rangle$, which transforms into $\left\langle 212^2 32\right\rangle$. Then pull all half-twist braids to the end of β (the interchange $\sigma_1 \leftrightarrow \sigma_3$ preserves σ_i^2), and for the l half-twist braids obtained, we have $l-1$ occurrences of σ_1^2 between two consecutive copies.

Putting the cases together, we find

$$\sigma(\hat{\beta}) \geq \min \left\{ \max\left(\frac{n-3a}{2}+2r_a-\frac{1}{2}, \frac{n-3c}{2}+2r_c-\frac{1}{2}, n-2b-2+2r_b\right) \;\middle|\; \begin{array}{c} 0 \leq a,\, b,\, c \\ 0 \leq r_a,\, r_b,\, r_c \\ a+b+c=n \\ r_a+r_b+r_c \geq \frac{n-11}{6} \\ 2r_a \leq a, 2r_b \leq b, 2r_c \leq c \end{array} \right\}.$$

To solve this optimization problem, we first simplify it. (We consider from now on the parameters continuous rather than discrete.) We note that clearly the minimum is attained when $r_a + r_b + r_c = \frac{n-11}{6}$. (To avoid degeneracies, assume that $n \geq 11$; the other cases must be checked directly.) Also we observe that the minimal point property is preserved under the operation

$$(a, r_a, b, r_b, c, r_c) \overset{x}{\longmapsto} \left(\frac{a+c}{2}, \frac{r_a+r_c}{2}, b, r_b, \frac{a+c}{2}, \frac{r_a+r_c}{2}\right),$$

since

$$\max\left(\frac{n-3a}{2}+2r_a, \frac{n-3c}{2}+2r_c\right)$$

does not increase under the operation x (it averages out the 2 alternatives in the maximum), and it preserves admissibility. Thus we assume w.l.o.g. $a = c$ and $r_a = r_c$, and have

$$\sigma(\hat{\beta}) \geq \min \left\{ \max\left(\frac{n-3a}{2}+2r_a-\frac{1}{2}, n-2b-2+2r_b\right) \;\middle|\; \begin{array}{c} 0 \leq a,\, b,\, r_a,\, r_b \\ 2a+b=n \\ 2r_a+r_b=\frac{n-11}{6} \\ 2r_a \leq a, 2r_b \leq b \end{array} \right\}. \tag{28}$$

Now assume that the minimum is attained at a point where one of the 2 alternatives in the maximum prevails. Then this is locally so, as both alternatives are continuous. But the first (resp. second) alternative can be decreased by perturbing a and r_a (resp. b and r_b) within range of admissibility. To see this, consider the first alternative (the r.h.s. of (27)). The replacement $a \to a + \varepsilon/2$, $b \to b - \varepsilon$, with $r_{a,b}$ kept constant, is admissible except if (i) $a = n/2$, in which case $r_b = b = 0$, and this alternative is not relevant for the maximum, or (ii) $2r_b = b$. In latter case (ii), together with $a \to a + \frac{\varepsilon}{2}$ (and $b \to b - \varepsilon$) we perturb $r_a \to r_a + \frac{\varepsilon}{4}$, $r_b \to r_b - \frac{\varepsilon}{2}$, which still decreases the first alternative, except if it is not admissible because of $r_b = 0$. Then, however, by assumption again $b = 2r_b = 0$, which we argued out in case (i). The case that the second alternative in the maximum in (28) prevails in a minimum point is dealt with in a similar way. Thus in a minimal point both alternatives in the maximum must be equal.

Then we consider

$$\frac{n-3a}{2} + 2r_a - \frac{1}{2} = n - 2b - 2 + 2r_b$$

with $2r_a + r_b = \frac{n-11}{6}$ and $2a + b = n$. We have by substitution

$$\frac{n - 3/2(n-b)}{2} + \frac{n-11}{6} - r_b - \frac{1}{2} = n - 2b - 2 + 2r_b\,, \tag{29}$$

and seek b and r_b so that this quantity is minimal for

$$0 \le 2r_b \le b \text{ and } 0 \le 2r_a = \frac{n-11}{6} - r_b \le \frac{n-b}{2} = a. \tag{30}$$

We get from (29)

$$\begin{aligned} r_b &= \frac{1}{3}\left(\frac{n - 3/2(n-b)}{2} + \frac{n-11}{6} - \frac{1}{2} - n + 2b + 2\right) \\ &= \frac{1}{3}\left(-\frac{5n}{4} + \frac{11}{4}b + \frac{3}{2} + \frac{n-11}{6}\right). \end{aligned} \tag{31}$$

The conditions (30) give

$$\frac{13n}{33} + \frac{4}{33} \le b \le \frac{19n}{33} - \frac{62}{33}. \tag{32}$$

Putting (31) into (29), we obtain

$$(29) = n - 2b - 2 + 2r_b = -\frac{b}{6} + \frac{5n}{18} - \frac{20}{9}.$$

Thus the minimal value is attained when the right inequality in (32) is sharp, and then it is $\frac{2}{11}n - \frac{21}{11}$, with $n = 4 - \chi$. Then the claim follows. □

4.4. **The proof for general braid positive links.** We now begin with the proof of Theorem 1.1.

Proof. Fix a sequence (K_i) of braid positive links $K_i = \hat{\alpha}_i$ with α_i positive and $\chi'(K_i) \to \infty$, and assume that $\sigma(K_i) \equiv \sigma_0$. We would like to produce a contradiction out of this assumption, by showing that $\sigma(K_i)$ has an unboundedly growing subsequence.

We can assume K_i to be connected, i.e. $n'(K_i) \equiv 1$, as we can replace split union by connected sum (both σ and χ' are additive under both operations), so that $[\alpha_i]_j = 0$ for some $j > 0$ implies $[\alpha_i]_{j'} = 0$ for any $j' \ge j$. Moreover, we can also assume that α_i are irreducible, that is, $[\alpha_i]_j \ne 1$ for any j.

First we put the α_i into the form of a scheme, that is, we write $\alpha_i = \alpha_{i,1} \cdot \cdots \cdot \alpha_{i,n_i}$ such that $\alpha_{i,j}$ and $\alpha_{i,j+1}$ consist just of (powers of) even and odd index generators (the parity for $\alpha_{i,1}$ we set to be the one of the first generator index appearing in α_i). Now we use Traczyk's method, considering the closed braid diagram as a positive diagram, and try to observe in which cases it can be applied.

If

$$\Big| \sum_{j \text{ even}} [\alpha_i]_j - \sum_{j \text{ odd}} [\alpha_i]_j \Big| \tag{33}$$

is unbounded in i (which more precisely should mean that it has an unboundedly growing subsequence), then Traczyk's method of smoothing out all crossings corresponding to even/odd index generators except one for each generator, and using the σ of the remaining connected sum of $(2, [\alpha_i]_j)$ torus links, shows $\sigma(K_i) \to \infty$. Therefore, assume that (33) is bounded by some constant b_0 independent on i.

Now consider generator squares in the braid words α_i. That is, when writing

$$\alpha_i \,=\, \prod_{j=1}^{[\alpha_i]} \sigma_{k_{i,j}}\,,$$

these generator squares are length-2-subwords starting at indices $j_l = j_{i,l}$, $1 \le l \le n_i$, such that $k_{i,j_l} = k_{i,j_l+1}$. When counting such generator squares, we assume that $j_{l+1} > j_l + 1$, that is, the pairs of letters are disjoint. So subwords of α_i of the form σ_j^n are counted as $\lfloor n/2 \rfloor$ squares, and not as $n-1$.

Assume the number n_i of generator squares in α_i is unbounded (on some subsequence). So in particular there are unboundedly many (at least $n_i/2$) even (or odd) index generator squares. Let $p \in \{0, 1\}$ be this index parity. Then choose in Traczyk's method the even (or odd) index generators to be smoothed out. Then for each generator square the smoothing out of both crossings does not augment σ, because it is realizable by a crossing switch. (If the pair of crossings is the last for the generator, smoothing out one of them even reduces σ by 1, as it corresponds to factoring out a positive Hopf link, as in case c) of [Tr, figure 2].)

Now when we smooth out generators (letters) with indices of the parity p, we know that σ does not increase at least $n_i/2$ times. Even if $\sum_{j \equiv p(2)} [\alpha_i]_j$ is larger for the chosen parity p, the defect on σ obtained by accounting for this difference is bounded by b_0, and we have

$$\sigma(K_i) \,\ge\, \frac{n_i}{2} - b_0 \underset{i \to \infty}{\longrightarrow} \infty.$$

Therefore, only finitely many of the powers of generators in α_i are higher than one. Modifying the braids α_i at some bounded number of positions, which changes σ just by a bounded quantity, we obtain braids α_i' with all generators appearing without squares. If we now show $\sigma(\hat{\alpha}_i') \to \infty$, then we would also have $\sigma(\hat{\alpha}_i) \to \infty$.

Therefore, we can w.l.o.g. assume that α_i in their scheme form consist only of entries '1'.

We now try to apply the case of a subdiagram admitting a Reidemeister III move (in our case a YB relation) in Traczyk's paper. These are the scheme fragments (19). If the number of such fragments is unbounded, then the number of fragments with sufficiently high distance to each other (so as the Reidemeister III moves to be applicable separately) is still unbounded, and Traczyk's argument in his case a) shows $\sigma(\hat{\alpha}_i) \to \infty$. Therefore, assume that the number of fragments (19) is bounded. This in particular implies that the strand number of α_i grows unboundedly (here in the meaning that every subsequence grows unboundedly), as it was observed in [St] that the number of fragments (19) is at least $O_n^{\asymp}\left(\dfrac{1}{n^2}\right)$ of the number of entries for an n-string braid (scheme).

We now apply a modification of Traczyk's idea and produce a contradiction by Theorem 4.1.

We take $k \in \{0, 1, 2\}$, where $\displaystyle\sum_{j \equiv k\,(3)} [\alpha_i]_j$ is minimal, and smooth out all crossings corresponding to σ_{3m+k} except one for each generator.

According to Traczyk's idea, we sacrifice at most ${}^1\!/_3\,\chi'$. Then from the remaining link L' with $\chi'(L') \geq {}^2\!/_3\,\chi'(L)$ we need to show that $\sigma(L') \geq {}^1\!/_2\,\chi'(L') + \epsilon_n$, with ϵ_n growing when the strand number n of L' (and L) grows. But now L' is a connected sum of (positive) closed 3-braid links L_i. Thus, it suffices to show that for these links

$$\sigma(L_i) \geq {}^1\!/_2\;\chi'(L_i) + \epsilon \tag{34}$$

for some $\epsilon > 0$ independent on i and L_i.

But we have shown this for $\epsilon = {}^1\!/_2$ in Theorem 4.1. Thus we are done. □

Remark 4.3. The examples given in Remark 4.1, beside the fact that the arguments for positive braids made heavy use of their group structure, suggest how difficult it is to show Conjecture 1.1 in general. At least, the method for braids can very unlikely be extended.

4.5. **Some corollaries.** It is useful to remark that we have in fact proved a qualitative estimate for σ in Theorem 1.1, although the bound was not stated explicitly. We include it here, since it is related to the type of results we prove.

Corollary 4.1. *There is a positive constant C such that for any non-split positive braid link L, $\sigma(L) \geq C\,\sqrt[3]{\chi'(L)}$.*

Proof. By the argument in the proof of Theorem 1.1, there is a constant C_1 such that $\sigma(\hat\beta) \geq C_1\lfloor[\beta]/n^2\rfloor$ for $\beta \in B_n$, since for at least $O(n^{-2})$ of β's crossings we can apply a clasp resolution or a Yang-Baxter relation, and the argument of Traczyk ensures that each such fragment of the braid contributes 1 to the signature.

On the other hand, for braids β of variable strand number n, we obtain from Theorem 4.1 that when smoothing out generators of indices congruent to each other mod 3, we have a positive contribution to the signature from any of the 3-braid links remaining in the connected sum after the compensation of the crossing smoothings. Thus we have $\sigma(\hat\beta) > O(n)$.

Combining both estimates gives the result. □

From Theorem 1.1, we immediately have

Corollary 4.2. *There exist only finitely many braid positive (or n-almost braid positive[2]) links without trivial split components of a given signature.*

Proof. This is a consequence of Theorem 1.1 and the fact that there are only finitely many braid positive links with given χ'. This fact follows from Bennequin's inequality or from Cromwell's work [Cr2] on homogeneous fibred links. (See also [St4].) □

Corollary 4.3. *Let D be a positive diagram of a given number n of Seifert circles. Then*

$$\sigma(D) \geq O_n^{\asymp}\left(\frac{1}{n^2}\right)\cdot\,\chi'(D) - O_n(n^2)\,,$$

[2]Closures of braids with exactly n negative crossings.

where $O_n^{\asymp}$ and O_n denote the asymptotical behaviour of the expression as $n \to \infty$. For braid diagrams the inequality holds with the O_n term being zero.

Proof. For braid diagrams this follows from the above mentioned observation of [St] that the number of fragments of the scheme as in (19) (but now with the entries '1' possibly replaced by some higher integers) is at least $O_n^{\asymp}\left(\frac{1}{n^2}\right)$ of the number of entries of the scheme. Each such fragment allows us to apply at least one of Traczyk's cases c) or a).

For arbitrary diagrams the statement can be recurred to braid diagrams by Vogel's algorithm [Vo]. Vogel gave an estimate for the number of moves needed to obtain a braid diagram out of a diagram of n Seifert circles. As each such move generates one negative crossing, we obtain the second correction term. $\square$

Corollary 4.4. *If (D_i) is a sequence of pairwise distinct n-almost positive diagrams of given σ, then the number of Seifert circles of D_i grows unboundedly, when $i \to \infty$.*

Proof. This is a straightforward consequence of the previous corollary. $\square$

If Conjecture 1.1 is true in full generality, we would have that in fact all D_i arise from finitely many diagrams by the so-called $\bar{t}_2'$ moves introduced in [St3] (replacement of a crossing by three antiparallel half-twists). These moves will also be used in some further related results, which we discuss below.

5. Extensions and problems

In the last section we spend some words concerning Conjecture 1.1 for positive knots and its possible extension to Tristram-Levine signatures. Let us recall that the Growth conjecture is equivalent to the finiteness of the sets

$$\Sigma_\sigma = \{\, g(K) : K \text{ positive knot}, \sigma(K) = \sigma \,\}.$$

Our aim in a separate paper [St10] will be to show how one can prove, at least in theory, that any initial number of sets Σ_σ is finite, provided this is true. (Note that if Σ_σ is infinite, then so is $\Sigma_{\sigma'}$ for any $\sigma' > \sigma$.) Namely, we show that there exists an algorithmically determinable collection of knots, such that if Σ_σ is finite, only finitely many of the determined knots need to be checked to establish this finiteness.

Theorem 5.1. *For all $n > 1$ there is a set C_n of positive knots with two properties:*

1) *C_n is finite and algorithmically constructible.*
2) *For all $\sigma \in 2\mathbb{N}$ we have:*

$$\exists \text{ positive knot } K \text{ of genus } g \geq n \text{ with } \sigma(K) \leq \sigma \iff \exists K \in C_n \text{ with } \sigma(K) \leq \sigma.$$

To verify, using this theorem, that Σ_σ is finite, one uses induction on σ and examines

$$C = C_{\max \bigcup_{\sigma' \leq \sigma} \Sigma_{\sigma'} + 1}.$$

We know $\Sigma_{\sigma'}$ for $\sigma' < \sigma$ by induction. If some $K \in C$ is found with $\sigma(K) = \sigma$ and $g(K) \notin \Sigma_\sigma$, we can join $g(K)$ to Σ_σ, and seek further such knots K (within the updated set C) until none are found.

Further evidence for the Growth conjecture is given by the following result on the average value of σ for given genus.

Theorem 5.2. *Let*

$$P_{g,n} := \{\, K \text{ positive knot}, g(K) = g, c(K) \leq n \,\},$$

where $c(K)$ denotes the crossing number of K. Then

$$\lim_{n\to\infty} \frac{1}{|P_{g,n}|} \left(\sum_{K \in P_{g,n}} \sigma(K) \right) = 2g.$$

(Note that $P_{g,n}$ is always finite, and becomes non-empty for fixed g when n is large enough. Note also that in general the crossing number $c(K)$ of a positive knot K may not be admitted by a positive diagram, as shown in [St8].)

This theorem means that generically the value of σ for fixed genus is the maximal possible. From this point of view, the philosophy behind the Growth conjecture is that 'when the generic value is the maximal possible, the minimal value should not be too small.' Theorem 5.2 is a consequence of a (largely unrelated to the subject of this paper) extension of the asymptotical denseness result for special alternating knots in [SV], which is proved in a separate paper [St6].

The referee asked the natural question on the status of Tristram-Levine signatures σ_ξ and nullities n_ξ in relation to the Growth conjecture. We finish the exposition with some related remarks.

These invariants were defined in [Ts, Le] for a complex number ξ of unit norm, and $\sigma = \sigma_{-1}$. The signatures σ_ξ satisfy (5), (6) and (8), and so one easily sees that if L is positive, then $\sigma_\xi(L) \geq 0$ for all ξ. However, apart from this trivial fact, there seems not much one can say. A central theme in our work is the result $\sigma = 1 - \chi$ of [Mu2] for special alternating links. A simple look at $(2, n)$-torus knots shows that Murasugi's equality fails for any σ_ξ with $\xi \neq -1$, even replacing σ_ξ by $\sigma_\xi + n_\xi$, as in the Murasugi-Tristram inequality. Then our Theorem 3.1, that builds on and extends Murasugi's result, fails too. By taking connected sums one sees that some σ_ξ can remain 0 even for arbitrarily large genus. So it is not even quite clear what weaker version of (11) to aim at. (Certainly for $\Re e\, \xi > 1/2$ no non-trivial bound can be expected.) Even with such a weaker inequality, our proof of Theorem 1.1 breaks down, because in Theorem 4.1 we obtained using (11) an estimate just above the critical lower limit. In any case a treatment of these difficulties leads to a long and painful path ...

With this said, it is also not surprising that the preceding results, summarized in the Introduction, do not address σ_ξ for $\xi \neq -1$. Their proofs meet similar difficulties and, as far as we can tell, mostly do not apply to $\xi \neq -1$. On the other hand, in the approach of [Ta], which is extended in Theorem 5.1, only crossing changes are used (and no smoothings), so it does yield information about some σ_ξ for $\xi \neq -1$. This will also be briefly discussed in the paper [St10] that proves Theorem 5.1.

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