

## THE WHITNEY EXTENSION PROBLEM AND LIPSCHITZ SELECTIONS OF SET-VALUED MAPPINGS IN JET-SPACES

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ABSTRACT. We study a variant of the Whitney extension problem (1934) for the space  $C^{k,\omega}(\mathbf{R}^n)$ . We identify  $C^{k,\omega}(\mathbf{R}^n)$  with a space of *Lipschitz* mappings from  $\mathbf{R}^n$  into the space  $\mathcal{P}_k \times \mathbf{R}^n$  of polynomial fields on  $\mathbf{R}^n$  equipped with a certain metric. This identification allows us to reformulate the Whitney problem for  $C^{k,\omega}(\mathbf{R}^n)$  as a Lipschitz selection problem for set-valued mappings into a certain family of subsets of  $\mathcal{P}_k \times \mathbf{R}^n$ . We prove a Helly-type criterion for the existence of Lipschitz selections for such set-valued mappings defined on finite sets. With the help of this criterion, we improve estimates for finiteness numbers in finiteness theorems for  $C^{k,\omega}(\mathbf{R}^n)$  due to C. Fefferman.

### 1. THE MAIN PROBLEM AND MAIN RESULTS

Let  $\omega : \mathbf{R}_+ \rightarrow \mathbf{R}_+$  be a continuous concave function satisfying  $\omega(0) = 0$ . We let  $\dot{C}^{k,\omega}(\mathbf{R}^n)$  denote the space of all functions  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  with continuous derivatives of all orders up to  $k$ , for which the seminorm

$$\|f\|_{\dot{C}^{k,\omega}(\mathbf{R}^n)} := \sum_{|\alpha|=k} \sup_{x,y \in \mathbf{R}^n, x \neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{\omega(\|x - y\|)}$$

is finite. By  $C^{k,\omega}(\mathbf{R}^n)$  we denote the Banach subspace of  $\dot{C}^{k,\omega}(\mathbf{R}^n)$  defined by the norm

$$\|f\|_{C^{k,\omega}(\mathbf{R}^n)} := \sum_{|\alpha| \leq k} \sup_{x \in \mathbf{R}^n} |D^\alpha f(x)| + \|f\|_{\dot{C}^{k,\omega}(\mathbf{R}^n)}.$$

Throughout the paper we let  $S$  denote an arbitrary closed subset of  $\mathbf{R}^n$ .

In this paper we study the following extension problem.

**Problem.** Given a positive integer  $k$  and an arbitrary function  $f : S \rightarrow \mathbf{R}$ , what is a necessary and sufficient condition for  $f$  to be the restriction to  $S$  of a function  $F \in C^{k,\omega}(\mathbf{R}^n)$ ?

This is a variant of a classical problem which is known in the literature as the Whitney Extension Problem [22, 23]. It has attracted a lot of attention in recent years. We refer the reader to [4]–[7], [9]–[15], [2, 3] and [24, 25] and references therein for numerous results in this direction, and for a variety of techniques for obtaining them.

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This note is devoted to the *phenomenon of “finiteness”* in the Whitney problem. It turns out that, in many cases, Whitney-type problems for different spaces of smooth functions can be reduced to the same kinds of problems, but for finite sets with *prescribed numbers of points*.

For the space  $C^{1,\omega}(\mathbf{R}^n)$  (with  $\omega(t) = t^p$ ,  $0 < p \leq 1$ ) and for the Zygmund space, this phenomenon has been studied in the author’s papers [17, 18]. The case of an arbitrary  $\omega$  was treated in joint papers with Yu. Brudnyi [4, 7]. It was shown that *a function  $f$  defined on  $S$  can be extended to a function  $F \in C^{1,\omega}(\mathbf{R}^n)$  with  $\|F\|_{C^{1,\omega}(\mathbf{R}^n)} \leq \gamma = \gamma(n)$  provided its restriction  $f|_{S'}$  to every subset  $S' \subset S$  consisting of at most  $N(n) = 3 \cdot 2^{n-1}$  points can be extended to a function  $F_{S'} \in C^{1,\omega}(\mathbf{R}^n)$  with  $\|F_{S'}\|_{C^{1,\omega}(\mathbf{R}^n)} \leq 1$ . (Moreover, the value  $3 \cdot 2^{n-1}$  is sharp [18, 7].)*

This result is an example of *“the finiteness property”* of the space  $C^{1,\omega}(\mathbf{R}^n)$ . We call the number  $N$  appearing in formulations of finiteness properties *“the finiteness number”*.

In his pioneering work [23], H. Whitney characterized the restriction of the space  $C^k(\mathbf{R})$ ,  $k \geq 1$ , to an arbitrary subset  $S \subset \mathbf{R}$  in terms of divided differences of functions. An application of Whitney’s method to the space  $C^{k,\omega}(\mathbf{R})$  implies the finiteness property for this space with the finiteness number  $N = k + 2$ .

An impressive breakthrough in the solution of the Whitney problem for  $C^{k,\omega}$ -spaces has recently been made by C. Fefferman [9]-[15]. In this paper we will consider two of his remarkable results related to the finiteness property and its generalizations for the space  $C^{k,\omega}(\mathbf{R}^n)$ . Here is the first of them:

**Theorem 1.1** (C. Fefferman [9, 11]). *There is a positive integer  $N = N(k, n)$  such that the following is true: Suppose we are given a function  $\omega$ , a set  $S \subset \mathbf{R}^n$ , and functions  $f : S \rightarrow \mathbf{R}$  and  $\xi : S \rightarrow \mathbf{R}_+$ . Assume that, for any  $S' \subset S$  with at most  $N$  points, there exists a function  $F_{S'} \in C^{k,\omega}(\mathbf{R}^n)$  with  $\|F_{S'}\|_{C^{k,\omega}(\mathbf{R}^n)} \leq 1$ , and*

$$|F_{S'}(x) - f(x)| \leq \xi(x) \text{ for all } x \in S'.$$

*Then there exists  $F \in C^{k,\omega}(\mathbf{R}^n)$ , with  $\|F\|_{C^{k,\omega}(\mathbf{R}^n)} \leq \gamma$  and*

$$|F(x) - f(x)| \leq \gamma \cdot \xi(x), \quad x \in S.$$

*Here  $\gamma = \gamma(k, n)$  is a constant depending only on  $k$  and  $n$ .*

In particular, if the function  $\xi$  is chosen to be identically zero, Theorem 1.1 shows that the space  $C^{k,\omega}(\mathbf{R}^n)$  possesses the finiteness property *for all  $k, n \geq 1$ .*

An upper bound for the finiteness number  $N(k, n)$  given in [9, 11] is

$$(1.1) \quad N(k, n) \leq (\dim \mathcal{P}_k + 1)^{3 \cdot 2^{\dim \mathcal{P}_k}}.$$

Here  $\mathcal{P}_k$  stands for the space of polynomials of degree at most  $k$  defined on  $\mathbf{R}^n$ . (Recall that  $\dim \mathcal{P}_k = \binom{n+k}{k}$ .)

Our first result, Theorem 1.2, states that the expression bounding  $N(k, n)$  in (1.1) can be replaced by a considerably smaller expression which depends on  $\dim \mathcal{P}_k$  exponentially.

**Theorem 1.2.** *Theorem 1.1 holds with the finiteness number  $N(k, n) = 2^{\dim \mathcal{P}_k}$ .*

*Remark 1.3.* In the spring of 2005 I learned that E. Bierstone and P. Milman obtained an improvement of estimate (1.1) to a bound that is exponential. Recently

P. Milman kindly drew my attention to the fact that their result gives precisely the estimate  $2^{\dim \mathcal{P}_k}$  for the spaces  $C^k(\mathbf{R}^n)$  and  $C^{k,\omega}(\mathbf{R}^n)$ ; see [1].

In fact there are many different versions of the Whitney extension problem. These versions arise when one considers a possibly different space of smooth functions on  $\mathbf{R}^n$  and a possibly different collection of given information about the function on the set  $S$ . In his classical paper [22], Whitney solved a version for the space  $C^k(\mathbf{R}^n)$  in the case where the given information about the function includes its values and the values of *all* of its partial derivatives of all orders up to  $k$  on the set  $S$ . Using Whitney’s extension method G. Glaeser [16] proved a similar result for the space  $C^{k,\omega}(\mathbf{R}^n)$ . Let us recall its formulation.

Given a  $k$ -times differentiable function  $f$  and  $x \in \mathbf{R}^n$ , we let  $T_x^k(f)$  denote the Taylor polynomial of  $f$  at  $x$  of degree at most  $k$ :

$$T_x^k(f)(y) := \sum_{|\alpha| \leq k} \frac{1}{\alpha!} (D^\alpha f)(x)(y-x)^\alpha, \quad y \in \mathbf{R}^n.$$

**Theorem 1.4** (Whitney-Glaeser). *Given a family of polynomials  $\{P_x \in \mathcal{P}_k : x \in S\}$  there is a function  $F \in C^{k,\omega}(\mathbf{R}^n)$  such that  $T_x^k(F) = P_x$  for every  $x \in S$  if and only if there is a constant  $\lambda > 0$  such that for every  $\alpha, |\alpha| \leq k$  we have*

$$(1.2) \quad |D^\alpha P_x(x)| \leq \lambda \text{ for all } x \in S,$$

and

$$(1.3) \quad \max\{|D^\alpha(P_x - P_y)(x)|, |D^\alpha(P_x - P_y)(y)|\} \leq \lambda \|x - y\|^{k-|\alpha|} \omega(\|x - y\|),$$

for all  $x, y \in S$ . Moreover,

$$\inf\{\|F\|_{C^{k,\omega}(\mathbf{R}^n)} : T_x^k(F) = P_x, x \in S\} \approx \inf \lambda$$

with constants of equivalence depending only on  $k$  and  $n$ .

Observe that Theorem 1.4 can be interpreted as a finiteness theorem with the finiteness number  $N = 2$ . In fact, the inequalities (1.2) and (1.3) depend on at most 2 (arbitrary) points of  $S$  so that the sufficiency part of this result can be reformulated as follows: There is a function  $F \in C^{k,\omega}(\mathbf{R}^n)$  with  $\|F\|_{C^{k,\omega}(\mathbf{R}^n)} \leq \gamma(k, n)$  satisfying  $T_x^k(F) = P_x, x \in S$ , provided that for every two-point set  $S' \subset S$  there exists a function  $F_{S'} \in C^{k,\omega}(\mathbf{R}^n)$  with  $\|F_{S'}\|_{C^{k,\omega}(\mathbf{R}^n)} \leq 1$  such that  $T_x^k(F_{S'}) = P_x, x \in S'$ .

In [12] C. Fefferman considered a version of the Whitney problem in which the family of polynomials  $\{P_x \in \mathcal{P}_k : x \in S\}$  is replaced by a family  $\{G(x) : x \in S\}$  of convex centrally-symmetric subsets of  $\mathcal{P}_k$ . He raised the following question: How can we decide whether there exist  $F \in C^{k,\omega}(\mathbf{R}^n)$  and a constant  $A > 0$  such that

$$T_x^k(F) \in A \odot G(x) \text{ for all } x \in S ?$$

Here  $A \odot G(x)$  denotes the dilation of  $G(x)$  with respect to its center by a factor of  $A$ .

Let  $P_x \in \mathcal{P}_k$  be the center of the set  $G(x)$ . This means that  $G(x)$  can be represented in the form  $G(x) = P_x + \sigma(x)$  where  $\sigma(x) \subset \mathcal{P}_k$  is a convex family of polynomials which is centrally symmetric with respect to 0. It is shown in [12] that, under certain conditions on the sets  $\sigma(x)$ , the finiteness property holds. We say

that a set  $\sigma(x) \subset \mathcal{P}_k$  is “Whitney  $\omega$ -convex” (with Whitney constant  $A$ ) at  $x \in \mathbf{R}^n$  if the following two conditions are satisfied:

(i)  $\sigma(x)$  is closed, convex and symmetric with respect to 0.

(ii) Suppose  $P \in \sigma(x)$ ,  $Q \in \mathcal{P}_k$  and  $\delta \in (0, 1]$ . Assume that  $P$  and  $Q$  satisfy the estimates

$$|\partial^\beta P(x)| \leq \omega(\delta)\delta^{k-|\beta|} \text{ and } |\partial^\beta Q(x)| \leq \delta^{-|\beta|}$$

for all  $|\beta| \leq k$ . Then  $T_x^k(P \cdot Q) \in A\sigma(x)$ . (See [12], p. 579.)

**Theorem 1.5** ([12]). *Given integers  $k, n \geq 1$  there is a constant  $N = N(k, n)$  for which the following holds: For each  $x \in S$ , suppose we are given a polynomial  $P_x \in \mathcal{P}_k$ , and a Whitney  $\omega$ -convex set  $\sigma(x)$  with Whitney constant  $A$ . Suppose that for every subset  $S'$  of  $S$  with cardinality at most  $N$  there exists a function  $F_{S'} \in C^{k,\omega}(\mathbf{R}^n)$  such that  $\|F_{S'}\|_{C^{k,\omega}(\mathbf{R}^n)} \leq 1$  and*

$$T_x^k(F_{S'}) \in P_x + \sigma(x) \text{ for all } x \in S'.$$

*Then there exists a function  $F \in C^{k,\omega}(\mathbf{R}^n)$ , satisfying  $\|F\|_{C^{k,\omega}(\mathbf{R}^n)} \leq \gamma$  and*

$$(1.4) \quad T_x^k(F) \in P_x + \gamma \cdot \sigma(x), \quad x \in S.$$

*Here  $\gamma$  depends only on  $k, n$  and the Whitney constant  $A$ .*

A particular case of this result for  $\sigma(x) = \{P \in \mathcal{P}_k : D^\alpha P(x) = 0, |\alpha| \leq k - 1\}$  and  $\omega(t) = t^p, 0 < p \leq 1$ , with the finiteness number  $N = 3 \cdot 2^{\binom{n+k-2}{k}}$  has been proved in [6].

Analogously to Theorem 1.2, our second result in this paper gives an explicit upper bound for a finiteness number.

**Theorem 1.6.** *Theorem 1.5 holds with the finiteness number*

$$N(k, n) = 2^{\min\{\ell+1, \dim \mathcal{P}_k\}},$$

*where  $\ell = \max_{x \in S} \dim \sigma(x)$ .*

*Remark 1.7.* Of course  $\ell$  necessarily satisfies  $\ell \leq \dim \mathcal{P}_k = \binom{n+k}{k}$ .

In fact both of our new estimates for finiteness numbers are corollaries of the following theorem which is the main result of this paper.

**Theorem 1.8.** *Let  $G$  be a mapping defined on a finite set  $S \subset \mathbf{R}^n$  which assigns a convex set of polynomials  $G(x) \subset \mathcal{P}_k$  of dimension at most  $\ell$  to every point  $x$  of  $S$ . Suppose that, for every subset  $S'$  of  $S$  consisting of at most  $2^{\min\{\ell+1, \dim \mathcal{P}_k\}}$  points, there exists a function  $F_{S'} \in C^{k,\omega}(\mathbf{R}^n)$  such that  $\|F_{S'}\|_{C^{k,\omega}(\mathbf{R}^n)} \leq 1$  and  $T_x^k(F_{S'}) \in G(x)$  for all  $x \in S'$ . Then there is a function  $F \in C^{k,\omega}(\mathbf{R}^n)$  satisfying  $\|F\|_{C^{k,\omega}(\mathbf{R}^n)} \leq \gamma$  and*

$$T_x^k(F) \in G(x) \text{ for all } x \in S.$$

*Here  $\gamma$  depends only on  $k, n$  and  $\text{card } S$ .*

Comparing this result with Theorem 1.5, let us note that here there are no restrictions on  $G$ . Moreover, here  $T_x^k(F)$  belongs to  $G(x)$  itself and not merely to its dilation as in (1.4). However the price that we have to pay to obtain such a general result is that we have to permit the constant  $\gamma$  (controlling the  $C^{k,\omega}$ -norm of the function  $F$ ) to depend on the number of points of  $S$ .

We can use the rather informal and imprecise terminology “ $C^{k,\omega}(\mathbf{R}^n)$  has the weak finiteness property” to express the kind of result obtained in Theorem 1.8

where  $\gamma$  depends on the number of points of  $S$ . The fact that such a weak finiteness property holds, strongly suggests that we can reasonably hope to establish an analogous “strong finiteness property”, by which we mean a result with  $\gamma$  depending only on  $k$  and  $n$ . Such a result may possibly require some additional very mild conditions to be imposed on the mapping  $G$ .

The weak finiteness property also provides an upper bound for the finiteness constant whenever the strong finiteness property holds. For instance, Fefferman’s Theorems 1.1 and 1.5 reduce the problem to a set of cardinality at most  $N(k, n)$  while the weak finiteness property decreases this number to  $2^{\dim \mathcal{P}_k}$  (as in Theorem 1.2) or to  $2^{\min\{l+1, \dim \mathcal{P}_k\}}$  (Theorem 1.6).

We prove Theorem 1.8 in Section 4. The proof is based on an approach presented in Sections 2 and 3.

The crucial ingredient in this approach is an isomorphism between the space  $C^{k, \omega}(\mathbf{R}^n)|_S$  and a certain space of Lipschitz mappings from  $S$  into the product  $\mathcal{P}_k \times \mathbf{R}^n$  equipped with a certain metric  $d_\omega$ . We define  $d_\omega$  and study its properties in Section 2. One of these properties, which is obtained in Proposition 2.5, is a useful formula for calculating  $d_\omega$ , namely

$$d_\omega(T, T') \approx \max_{|\alpha| \leq k} \{ \omega(\|x - x'\|), \varphi_\alpha(|D^\alpha(P - P')(x)|), \varphi_\alpha(|D^\alpha(P - P')(x')|) \}$$

where  $T = (P, x)$  and  $T' = (P', x')$  are any two elements of  $\mathcal{P}_k \times \mathbf{R}^n$ , and  $\varphi_\alpha := \omega((s^{k-|\alpha|}\omega(s))^{-1})$  (here  $(s^{k-|\alpha|}\omega(s))^{-1}$  means the *inverse function*). For instance, for  $\omega(t) = t, t > 0$ ,

$$d_\omega(T, T') \approx \max_{|\alpha| \leq k} \left\{ \|x - x'\|, |D^\alpha(P - P')(x)|^{\frac{1}{k-|\alpha|+1}}, |D^\alpha(P - P')(x')|^{\frac{1}{k-|\alpha|+1}} \right\}.$$

Recall that in this case the space  $C^{k, \omega}(\mathbf{R}^n)$  coincides with the Sobolev space of functions  $f \in C^k(\mathbf{R}^n)$  whose distributional partial derivatives of order  $k + 1$  belong to  $L_\infty(\mathbf{R}^n)$ .

We refer to the set

$$\mathcal{P}_k \times \mathbf{R}^n := \{(P, x) : P \in \mathcal{P}_k, x \in \mathbf{R}^n\}$$

as *the space of (potential) k-jets*. This name and also the definition of  $d_\omega$  are motivated by the Whitney-Glaeser extension theorem, Theorem 1.4.

Given  $T = (P, x) \in \mathcal{P}_k \times \mathbf{R}^n$  and  $\lambda \in \mathbf{R}$  we define  $\lambda \circ T := (\lambda P, x)$ . Then inequality (1.3) of the Whitney-Glaeser extension theorem can be reformulated as follows: for every  $T_x := (P_x, x), T_y := (P_y, y) \in \mathcal{P}_k \times S$

$$(1.5) \quad d_\omega(\lambda^{-1} \circ T_x, \lambda^{-1} \circ T_y) \leq \omega(\|x - y\|).$$

We define a metric on  $S$  by setting  $r_\omega(x, y) := \omega(\|x - y\|)$  for all  $x, y \in S$  and we let  $S_\omega$  be the metric space  $S_\omega := (S, r_\omega)$ . We also consider  $\mathcal{P}_k \times \mathbf{R}^n$  as a metric space with respect to  $d_\omega$ , i.e., we set  $\mathcal{T}_{k, n} := (\mathcal{P}_k \times \mathbf{R}^n, d_\omega)$ . Let  $\text{Lip}(S_\omega, \mathcal{T}_{k, n})$  denote the space of Lipschitz mappings from  $S$  (equipped with the metric  $r_\omega$ ) into  $\mathcal{P}_k \times \mathbf{R}^n$  (with the metric  $d_\omega$ ). Inequality (1.5) motivates us to equip this space with a “norm” by setting

$$(1.6) \quad \|T\|_{LO(S)} := \inf\{ \lambda : \|\lambda^{-1} \circ T\|_{\text{Lip}(S_\omega, \mathcal{T}_{k, n})} \leq 1 \}.$$

We call  $\|\cdot\|_{LO(S)}$  the Lipschitz-Orlicz norm. We use it to define a second “norm” by setting

$$(1.7) \quad \|T\|_{LO(S)}^* := \max_{|\alpha| \leq k} \sup_{x \in S} |D^\alpha P_x(x)| + \|T\|_{LO(S)}$$

and we introduce the subspace  $\mathbf{Lip}(S_\omega, \mathcal{T}_{k,n})$  of  $\text{Lip}(S_\omega, \mathcal{T}_{k,n})$  of “bounded” Lipschitz mappings  $T(x) = (P_x, z_x)$ ,  $x \in S$ , defined by the finiteness of the “norm” (1.7).

Now the Whitney-Glaeser extension theorem implies the following

**Proposition 1.9.** *Given a family of polynomials  $\{P_x \in \mathcal{P}_k : x \in S\}$ , there is a function  $F \in C^{k,\omega}(\mathbf{R}^n)$  such that  $T_x^k(F) = P_x$  for every  $x \in S$  if and only if the mapping  $T(x) := (P_x, x)$ ,  $x \in S$ , belongs to  $\mathbf{Lip}(S_\omega, \mathcal{T}_{k,n})$ . Moreover,*

$$\inf\{\|F\|_{C^{k,\omega}(\mathbf{R}^n)} : T_x^k(F) = P_x, x \in S\} \approx \|T\|_{LO(S)}^*$$

with constants of equivalence depending only on  $k$  and  $n$ .

Applying this proposition to  $S = \mathbf{R}^n$  we obtain an interesting isomorphism between  $C^{k,\omega}(\mathbf{R}^n)$  and a certain subfamily of  $\text{Lip}(\mathbf{R}_\omega^n, \mathcal{T}_{k,n})$ . Namely, every function  $F \in C^{k,\omega}(\mathbf{R}^n)$  gives rise to a Lipschitz mapping from  $\mathbf{R}_\omega^n := (\mathbf{R}^n, \omega(\|\cdot\|))$  into  $\mathcal{P}_k \times \mathbf{R}^n$  defined by the formula  $T(x) := (T_x^k(F), x)$ ,  $x \in \mathbf{R}^n$ . On the other hand, every Lipschitz mapping from  $\mathbf{R}_\omega^n$  to  $\mathcal{P}_k \times \mathbf{R}^n$  of the form  $T(x) := (P_x, x)$ ,  $x \in \mathbf{R}^n$ , generates a function  $F(x) := P_x(x)$ ,  $x \in \mathbf{R}^n$ , such that  $F \in C^{k,\omega}(\mathbf{R}^n)$  and  $T_x^k(F) = P_x$ ,  $x \in \mathbf{R}^n$ .

Let us restate this more concisely: The mapping

$$C^{k,\omega}(\mathbf{R}^n) \ni F \mapsto T(x) := (T_x^k(F), x) \in \text{Lip}(\mathbf{R}_\omega^n, \mathcal{T}_{k,n})$$

and its inverse mapping

$$\text{Lip}(\mathbf{R}_\omega^n, \mathcal{T}_{k,n}) \ni T(x) := (P_x, x) \mapsto F(x) := P_x(x) \in C^{k,\omega}(\mathbf{R}^n)$$

provide an isomorphism between  $C^{k,\omega}(\mathbf{R}^n)$  and the subfamily of  $\text{Lip}(\mathbf{R}_\omega^n, \mathcal{T}_{k,n})$  consisting of all elements of the form  $T(x) := (P_x, x)$ ,  $x \in \mathbf{R}^n$ . Moreover, Proposition 1.9 states that this isomorphism in some sense “preserves restrictions”.

The above ideas and results are presented in Section 2. They show that even though Whitney’s problem deals with restrictions of  $k$ -times differentiable functions, it is also a problem about Lipschitz mappings defined on subsets of  $\mathbf{R}^n$  and taking values in a very non-linear metric space  $\mathcal{T}_{k,n} = (\mathcal{P}_k \times \mathbf{R}^n, d_\omega)$ . More specifically, the Whitney problem can be reformulated as a problem about *Lipschitz selections of set-valued mappings* from  $S$  into  $2^{\mathcal{T}_{k,n}}$ . We study this problem in Section 3. We remark that the Lipschitz selection method has already been used to obtain a solution to the Whitney problem for the space  $C^{1,\omega}(\mathbf{R}^n)$ ; see [18, 20, 7].

We recall some relevant definitions: Let  $X = (\mathcal{M}, \rho)$  and  $Y = (\mathcal{T}, d)$  be metric spaces and let  $\mathcal{G} : \mathcal{M} \rightarrow 2^{\mathcal{T}}$  be a set-valued mapping, i.e., a mapping which assigns a subset  $\mathcal{G}(x) \subset \mathcal{T}$  to each  $x \in \mathcal{M}$ . A function  $g : \mathcal{M} \rightarrow \mathcal{T}$  is said to be a *selection* of  $\mathcal{G}$  if  $g(x) \in \mathcal{G}(x)$  for all  $x \in \mathcal{M}$ . If a selection  $g$  is an element of  $\text{Lip}(X, Y)$ , then it is said to be a *Lipschitz selection* of the mapping  $\mathcal{G}$ . (For various results and techniques related to the problem of the existence of Lipschitz selections in the case where  $Y = (\mathcal{T}, d)$  is a Banach space, we refer the reader to [19, 20, 21] and the references therein.)

It turns out that Theorem 1.8, the “weak finiteness” theorem, is equivalent to the following Helly-type criterion for the existence of a Lipschitz selection.

**Theorem 1.10.** *Let  $S \subset \mathbf{R}^n$  be a finite set and let  $\mathcal{G}(x) = (G(x), x), x \in S$ , be a set-valued mapping such that for each  $x \in S$  the set  $G(x) \subset \mathcal{P}_k$  is a convex set of polynomials of dimension at most  $\ell$ . Suppose that there exists a constant  $K > 0$  such that, for every subset  $S' \subset S$  consisting of at most  $2^{\min\{\ell+1, \dim \mathcal{P}_k\}}$  points, the restriction  $\mathcal{G}|_{S'}$  has a Lipschitz selection  $g_{S'} \in \text{Lip}(S', \mathcal{T}_{k,n})$  with  $\|g_{S'}\|_{LO(S')} \leq K$ . Then  $\mathcal{G}$ , considered as a map on all of  $S$ , has a Lipschitz selection  $g \in \text{Lip}(S, \mathcal{T}_{k,n})$  with  $\|g\|_{LO(S)} \leq \gamma K$ , where the constant  $\gamma$  depends only on  $k, n$  and  $\text{card } S$ .*

The proof of this result relies on some methods and ideas developed for the case of set-valued mappings which take their values in Banach spaces; see, e.g. Shvartsman [19, 20, 21]. In particular, an analog of Theorem 1.10 for Banach spaces has been proved in [20]. Our strategy will be to adapt that proof to the case of the metric space  $\mathcal{T}_{k,n} = (\mathcal{P}_k \times \mathbf{R}^n, d_\omega)$ . As in the case of Banach spaces our adapted proof will be based on Helly’s intersection theorem [8] and a combinatorial result about a structure of finite metric graphs (Proposition 3.1).

2.  $C^{k,\omega}(\mathbf{R}^n)$  AS A SPACE OF LIPSCHITZ MAPPINGS

The point of departure for our approach is inequality (1.3) of the Whitney extension theorem. This inequality motivates the definition of a certain special metric on the set  $\mathcal{P}_k \times \mathbf{R}^n$  which allows us to identify the restriction  $C^{k,\omega}(\mathbf{R}^n)|_S$  with a space of Lipschitz mappings from  $S$  into  $\mathcal{P}_k \times \mathbf{R}^n$ .

Recall that  $\omega$  is a non-negative concave function on  $\mathbf{R}_+$ , and therefore  $\omega$  is non-decreasing. Without loss of generality we may assume that  $\omega$  is a strictly increasing concave function on  $\mathbf{R}_+$ . (In fact, for every positive concave  $\omega : \mathbf{R}_+ \rightarrow \mathbf{R}_+$  there is a concave strictly increasing function  $\omega^*$  such that  $\omega^* \leq \omega$ . Therefore  $\tilde{\omega} := \omega + \omega^*$  is a concave strictly increasing function satisfying  $\omega \leq \tilde{\omega} \leq 2\omega$ .)

Now let us define a metric on  $\mathcal{P}_k \times \mathbf{R}^n$ . Since for every multi-index  $\alpha, |\alpha| < k$ , the function  $s^{k-|\alpha|}\omega(s)$  is strictly increasing, the inverse function  $\psi_\alpha := (s^{k-|\alpha|}\omega(s))^{-1}$  is well-defined. Therefore the function

$$(2.1) \quad \varphi_\alpha := \omega(\psi_\alpha) = \omega((s^{k-|\alpha|}\omega(s))^{-1})$$

is also well-defined.

For  $|\alpha| = k$  we put  $\varphi_\alpha(t) := t$ . It can be readily seen that for all  $\alpha, |\alpha| < k$ ,

$$\varphi'_\alpha(t) = \frac{1}{\psi_\alpha^{k-|\alpha|}(t) + (k - |\alpha|)\psi_\alpha^{k-|\alpha|-1}(t)\omega(\psi_\alpha(t))/\omega'(\psi_\alpha(t))}.$$

Since  $\psi_\alpha$  and  $\omega$  are non-decreasing and  $\omega'$  is non-increasing,  $\varphi'_\alpha$  is non-increasing, so that  $\varphi_\alpha$  is a concave function.

Fix two  $k$ -jets  $T_0 = (P_0, x_0), T_1 = (P_1, x_1) \in \mathcal{P}_k \times \mathbf{R}^n$  and put

$$(2.2) \quad \begin{aligned} &\delta_\omega(T_0, T_1) \\ &:= \max_{|\alpha| \leq k} \{ \omega(\|x_0 - x_1\|), \varphi_\alpha(|D^\alpha(P_0 - P_1)(x_0)|), \varphi_\alpha(|D^\alpha(P_0 - P_1)(x_1)|) \}. \end{aligned}$$

Clearly,

$$(2.3) \quad \delta_\omega((P, x_0), (P, x_1)) = \omega(\|x_0 - x_1\|) \text{ for every } P \in \mathcal{P}_k, x, y \in \mathbf{R}^n.$$

Recall that  $\lambda \circ T := (\lambda P, x)$  where  $T = (P, x) \in \mathcal{P}_k \times \mathbf{R}^n$  and  $\lambda \in \mathbf{R}$ . In these settings inequality (1.3) of the Whitney-Glaeser extension theorem means the following:

$$\delta_\omega(\lambda^{-1} \circ T_x, \lambda^{-1} \circ T_y) \leq \omega(\|x - y\|), T_x := (P_x, x), T_y := (P_y, y).$$

Now we define a metric  $d_\omega$  on  $\mathcal{P}_k \times \mathbf{R}^n$  by letting

$$(2.4) \quad d_\omega(T, T') := \inf \sum_{i=0}^{m-1} \delta_\omega(T_i, T_{i+1})$$

where the infimum is taken over all finite families  $\{T_0, T_1, \dots, T_m\} \subset \mathcal{P}_k \times \mathbf{R}^n$  such that  $T_0 = T$  and  $T_m = T'$ .

In particular, since  $\omega$  is subadditive, by this definition for every  $x, y \in \mathbf{R}^n$

$$d_\omega((P, x), (Q, y)) \geq \omega(\|x - y\|), \quad P, Q \in \mathcal{P}_k,$$

and by (2.3)

$$(2.5) \quad d_\omega((P, x), (P, y)) = \omega(\|x - y\|) \text{ for every } P \in \mathcal{P}_k.$$

The main result of this section is the following

**Theorem 2.1.** *For every  $T, T' \in \mathcal{P}_k \times \mathbf{R}^n$  we have*

$$d_\omega(T, T') \leq \delta_\omega(T, T') \leq d_\omega(e^n \circ T, e^n \circ T').$$

A proof of the theorem is based on a series of auxiliary lemmas.

**Lemma 2.2.** *For every  $t_1, t_2 > 0$  and every multi-index  $\alpha, \beta$  such that  $|\alpha| + |\beta| \leq k$  we have*

$$\varphi_\alpha(t_1^{|\beta|} t_2) \leq \max\{\omega(t_1), \varphi_{\alpha+\beta}(t_2)\}.$$

*Proof.* If  $|\alpha| = k$ , then  $\beta = 0$  so that there is nothing to prove. Therefore we will assume that  $|\alpha| < k$ . In this case by (2.1)  $\varphi_\alpha(t_1^{|\beta|} t_2) = \omega(u)$  where

$$u := (s^{k-|\alpha|} \omega(s))^{-1}(t_1^{|\beta|} t_2).$$

Hence  $t_1^{|\beta|} t_2 = u^{k-|\alpha|} \omega(u)$ .

Suppose that  $|\alpha| + |\beta| < k$ . Then  $\varphi_{\alpha+\beta}(t_2) = \omega(v)$  where

$$v := (s^{k-|\alpha|-|\beta|} \omega(s))^{-1}(t_2)$$

so that  $t_2 = v^{k-|\alpha|-|\beta|} \omega(v)$ .

Put  $w := \max\{t_1, v\}$ . Then

$$(2.6) \quad \max\{\omega(t_1), \varphi_{\alpha+\beta}(t_2)\} = \max\{\omega(t_1), \omega(v)\} = \omega(\max\{t_1, v\}) = \omega(w).$$

Hence

$$t_1^{|\beta|} t_2 = t_1^{|\beta|} v^{k-|\alpha|-|\beta|} \omega(v) \leq w^{|\beta|} w^{k-|\alpha|-|\beta|} \omega(w) = w^{k-|\alpha|} \omega(w).$$

But  $t_1^{|\beta|} t_2 = u^{k-|\alpha|} \omega(u)$  so that

$$u^{k-|\alpha|} \omega(u) = t_1^{|\beta|} t_2 \leq w^{k-|\alpha|} \omega(w).$$

Since  $t^{k-|\alpha|} \omega(t)$  is a strictly increasing function, this implies  $u \leq w$ . Recall also that  $\varphi_\alpha(t_1^{|\beta|} t_2) = \omega(u)$ . Then by (2.6)

$$\varphi_\alpha(t_1^{|\beta|} t_2) = \omega(u) \leq \omega(w) = \max\{\omega(t_1), \varphi_{\alpha+\beta}(t_2)\}.$$

It remains to consider the case  $|\alpha| + |\beta| = k$ . In this case by the definition of  $\varphi_\alpha$  we have  $\varphi_{\alpha+\beta}(t_2) = t_2$ . If  $\sup_{t>0} \omega(t) \leq t_2$ , then

$$\varphi_\alpha(t_1^{|\beta|} t_2) = \omega(u) \leq t_2 = \varphi_{\alpha+\beta}(t_2)$$



and the lemma follows. If  $t_2 < \sup_{t>0} \omega(t)$ , then there is  $v > 0$  such that  $t_2 = \omega(v)$ . This shows that equality  $t_2 = v^{k-|\alpha|-|\beta|} \omega(v)$  holds for the case  $|\alpha| + |\beta| = k$  as well. The lemma is proved.  $\square$

**Lemma 2.3.** *Let  $Q \in \mathcal{P}_k$  and let  $a, b \in \mathbf{R}^n$ . Then for every multi-index  $\alpha, |\alpha| \leq k$ , we have*

$$|D^\alpha Q(b)| \leq \sum_{|\beta| \leq k-|\alpha|} \frac{1}{|\beta|!} |D^{\alpha+\beta} Q(a)| \cdot \|b - a\|^{|\beta|}.$$

*Proof.* Since  $D^\alpha Q$  is a polynomial of degree at most  $k - |\alpha|$ , Taylor's formula for  $D^\alpha Q$  at  $a$  gives

$$D^\alpha Q(b) = \sum_{|\beta| \leq k-|\alpha|} \frac{1}{|\beta|!} D^{\alpha+\beta} Q(a) (b - a)^\beta$$

which immediately implies the required inequality of the lemma.  $\square$

**Lemma 2.4.** *Let  $T = (P, x), T' = (P', x') \in \mathcal{P}_k \times \mathbf{R}^n$  and let*

$$\{T_i = (P_i, x_i) : i = 0, 1, \dots, m\}$$

*be a finite family of elements of  $\mathcal{P}_k \times \mathbf{R}^n$  such that  $T_0 = T, T_m = T'$ . Then for every  $\alpha, |\alpha| \leq k$ ,*

$$|D^\alpha(P - P')(x)| \leq e^n \max_{|\beta| \leq k-|\alpha|} \sum_{i=0}^{m-1} |D^{\alpha+\beta}(P_i - P_{i+1})(x_i)| \cdot \|x - x_i\|^{|\beta|}.$$

*Proof.* By Lemma 2.3

$$|D^\alpha(P_i - P_{i+1})(x)| \leq \sum_{|\beta| \leq k-|\alpha|} \frac{1}{|\beta|!} |D^{\alpha+\beta}(P_i - P_{i+1})(x_i)| \cdot \|x - x_i\|^{|\beta|}$$

so that

$$\begin{aligned} |D^\alpha(P - P')(x)| &\leq \sum_{i=0}^{m-1} |D^\alpha(P_i - P_{i+1})(x)| \\ &\leq \sum_{i=0}^{m-1} \sum_{|\beta| \leq k-|\alpha|} \frac{1}{|\beta|!} |D^{\alpha+\beta}(P_i - P_{i+1})(x_i)| \cdot \|x - x_i\|^{|\beta|} \\ &= \sum_{|\beta| \leq k-|\alpha|} \frac{1}{|\beta|!} \sum_{i=0}^{m-1} |D^{\alpha+\beta}(P_i - P_{i+1})(x_i)| \cdot \|x - x_i\|^{|\beta|}. \end{aligned}$$

Hence

$$\begin{aligned} &|D^\alpha(P - P')(x)| \\ &\leq \left( \sum_{|\beta| \leq k-|\alpha|} \frac{1}{|\beta|!} \right) \max_{|\beta| \leq k-|\alpha|} \sum_{i=0}^{m-1} |D^{\alpha+\beta}(P_i - P_{i+1})(x_i)| \cdot \|x - x_i\|^{|\beta|} \\ &\leq e^n \max_{|\beta| \leq k-|\alpha|} \sum_{i=0}^{m-1} |D^{\alpha+\beta}(P_i - P_{i+1})(x_i)| \cdot \|x - x_i\|^{|\beta|}, \end{aligned}$$

proving the lemma.  $\square$

We are now in a position to prove Theorem 2.1.

*Proof of Theorem 2.1.* The first inequality follows from (2.4). Let us prove the second inequality. Consider a family  $\{T_i = (P_i, x_i) : i = 0, 1, \dots, m\} \subset \mathcal{P}_k \times \mathbf{R}^n$  such that  $T_0 = T := (P, x), T_m = T' := (P', x')$ . Thus  $P_i \in \mathcal{P}_k, x_i \in \mathbf{R}^n$  for every  $i = 0, \dots, m$  and  $P_0 = P, P_m = P', x_0 = x, x_m = x'$ . Let us prove that

$$(2.7) \quad \delta_\omega(T, T') \leq \sum_{i=0}^{m-1} \delta_\omega(e^n \circ T_i, e^n \circ T_{i+1}).$$

Let us fix a multi-index  $\alpha, |\alpha| \leq k$  and estimate  $\varphi_\alpha(|D^\alpha(P - P')(x)|)$ . By Lemma 2.4

$$|D^\alpha(P - P')(x)| \leq e^n \max_{|\beta| \leq k - |\alpha|} \sum_{i=0}^{m-1} |D^{\alpha+\beta}(P_i - P_{i+1})(x_i)| \cdot \|x - x_i\|^{|\beta|}.$$

Put  $\tilde{P}_i := e^n P_i$ . Then the latter inequality implies

$$|D^\alpha(P - P')(x)| \leq \max_{|\beta| \leq k - |\alpha|} \sum_{i=0}^{m-1} |D^{\alpha+\beta}(\tilde{P}_i - \tilde{P}_{i+1})(x_i)| \cdot \|x - x_i\|^{|\beta|}.$$

Since  $\|x - x_i\| \leq \sum_{i=0}^{m-1} \|x_i - x_{i+1}\|$ , we obtain

$$|D^\alpha(P - P')(x)| \leq \max_{|\beta| \leq k - |\alpha|} \left( \sum_{i=0}^{m-1} \|x_i - x_{i+1}\| \right)^{|\beta|} \cdot \left( \sum_{i=0}^{m-1} |D^{\alpha+\beta}(\tilde{P}_i - \tilde{P}_{i+1})(x_i)| \right).$$

Recall that the function  $\varphi_\alpha$  defined by (2.1) is non-decreasing. Hence

$$\begin{aligned} & \varphi_\alpha(|D^\alpha(P - P')(x)|) \\ & \leq \max_{|\beta| \leq k - |\alpha|} \varphi_\alpha \left( \left( \sum_{i=0}^{m-1} \|x_i - x_{i+1}\| \right)^{|\beta|} \cdot \left( \sum_{i=0}^{m-1} |D^{\alpha+\beta}(\tilde{P}_i - \tilde{P}_{i+1})(x_i)| \right) \right). \end{aligned}$$

By Lemma 2.2

$$\begin{aligned} & \varphi_\alpha \left( \left( \sum_{i=0}^{m-1} \|x_i - x_{i+1}\| \right)^{|\beta|} \cdot \left( \sum_{i=0}^{m-1} |D^{\alpha+\beta}(\tilde{P}_i - \tilde{P}_{i+1})(x_i)| \right) \right) \\ & \leq \max \left\{ \omega \left( \sum_{i=0}^{m-1} \|x_i - x_{i+1}\| \right), \varphi_{\alpha+\beta} \left( \sum_{i=0}^{m-1} |D^{\alpha+\beta}(\tilde{P}_i - \tilde{P}_{i+1})(x_i)| \right) \right\}. \end{aligned}$$

Since  $\omega$  and  $\varphi_{\alpha+\beta}$  are concave functions on  $\mathbf{R}_+$ , they are subadditive so that

$$(2.8) \quad \omega \left( \sum_{i=0}^{m-1} \|x_i - x_{i+1}\| \right) \leq \sum_{i=0}^{m-1} \omega(\|x_i - x_{i+1}\|)$$

and

$$\varphi_{\alpha+\beta} \left( \sum_{i=0}^{m-1} |D^{\alpha+\beta}(\tilde{P}_i - \tilde{P}_{i+1})(x_i)| \right) \leq \sum_{i=0}^{m-1} \varphi_{\alpha+\beta}(|D^{\alpha+\beta}(\tilde{P}_i - \tilde{P}_{i+1})(x_i)|).$$

Hence by (2.2) we have

$$\begin{aligned}
 & \varphi_\alpha(|D^\alpha(P - P')(x)|) \\
 & \leq \max_{|\beta| \leq k - |\alpha|} \left\{ \sum_{i=0}^{m-1} \omega(\|x_i - x_{i+1}\|), \sum_{i=0}^{m-1} \varphi_{\alpha+\beta}(|D^{\alpha+\beta}(\tilde{P}_i - \tilde{P}_{i+1})(x_i)|) \right\} \\
 & \leq \max_{|\beta| \leq k - |\alpha|} \left\{ \sum_{i=0}^{m-1} \max\{\omega(\|x_i - x_{i+1}\|), \varphi_{\alpha+\beta}(|D^{\alpha+\beta}(\tilde{P}_i - \tilde{P}_{i+1})(x_i)|)\} \right\} \\
 & \leq \sum_{i=0}^{m-1} \max_{|\beta| \leq k - |\alpha|} \{\omega(\|x_i - x_{i+1}\|), \varphi_{\alpha+\beta}(|D^{\alpha+\beta}(\tilde{P}_i - \tilde{P}_{i+1})(x_i)|)\} \\
 & \leq \sum_{i=0}^{m-1} \delta_\omega((\tilde{P}_i, x_i), (\tilde{P}_{i+1}, x_{i+1})) = \sum_{i=0}^{m-1} \delta_\omega(e^n \circ T_i, e^n \circ T_{i+1}).
 \end{aligned}$$

(Recall that  $\tilde{P}_i := e^n P_i$ .) In a similar way we prove that

$$\varphi_\alpha(|D^\alpha(P - P')(x')|) \leq \sum_{i=0}^{m-1} \delta_\omega(e^n \circ T_i, e^n \circ T_{i+1}).$$

Combining this with (2.8) and the definition of (2.2) we obtain the required inequality (2.7). By this inequality

$$\delta_\omega(T, T') \leq \inf \sum_{i=0}^{m-1} \delta_\omega(e^n \circ T_i, e^n \circ T_{i+1})$$

where the infimum is taken over all families  $\{T_0, T_1, \dots, T_m\} \subset \mathcal{P}_k \times \mathbf{R}^n$  such that  $T_0 = T$  and  $T_m = T'$ . By (2.4) this infimum is equal to  $d_\omega(e^n \circ T, e^n \circ T')$ , and the proof is finished.  $\square$

The following proposition presents two formulae for the calculation of the metric  $d_\omega$ .

**Proposition 2.5.** *For every  $T = (P, x), T' = (P', x') \in \mathcal{P}_k \times \mathbf{R}^n$  we have*

(i)

$$e^{-n} \delta_\omega(T, T') \leq d_\omega(T, T') \leq \delta_\omega(T, T');$$

(ii)

$$e^{-n} d_\omega(T, T') \leq \max_{|\alpha| \leq k} \{\omega(\|x - x'\|), \varphi_\alpha(|D^\alpha(P - P')(x)|)\} \leq e^n d_\omega(T, T').$$

*Proof.* (i) By Theorem 2.1 for every  $T, T' \in \mathcal{P}_k \times \mathbf{R}^n$  we have

$$d_\omega(T, T') \leq \delta_\omega(T, T') \leq d_\omega(e^n \circ T, e^n \circ T')$$

so that  $\delta_\omega(e^{-n} \circ T, e^{-n} \circ T') \leq d_\omega(T, T')$ . On the other hand, since  $\varphi_\alpha$  is a concave function, for every  $\lambda \geq 1$  we obtain

$$\begin{aligned}
 & \delta_\omega(\lambda \circ T, \lambda \circ T') \\
 & := \max_{|\alpha| \leq k} \{\omega(\|x - x'\|), \varphi_\alpha(|D^\alpha(\lambda P - \lambda P')(x)|), \varphi_\alpha(|D^\alpha(\lambda P - \lambda P')(x')|)\} \\
 & \leq \lambda \max_{|\alpha| \leq k} \{\omega(\|x - x'\|), \varphi_\alpha(|D^\alpha(P - P')(x)|), \varphi_\alpha(|D^\alpha(P - P')(x')|)\} \\
 & = \lambda \delta_\omega(T, T')
 \end{aligned}$$

so that

$$\delta_\omega(T, T') = \delta_\omega(e^n \circ (e^{-n} \circ T), e^n \circ (e^{-n} \circ T')) \leq e^n \delta_\omega(e^{-n} \circ T, e^{-n} \circ T') \leq e^n d_\omega(T, T'),$$

proving (i).

(ii) By (i) and (2.2) we have to prove that  $\delta_\omega(T, T') \leq e^n I$  where

$$I := \max_{|\alpha| \leq k} \{ \omega(\|x - x'\|), \varphi_\alpha(|D^\alpha(P - P')(x)|) \}.$$

This is equivalent to the inequality

$$(2.9) \quad \varphi_\alpha(|D^\alpha(P - P')(x')|) \leq e^n I, \quad |\alpha| \leq k.$$

By Lemma 2.3

$$\begin{aligned} |D^\alpha(P - P')(x')| &\leq \left( \sum_{|\beta| \leq k - |\alpha|} \frac{1}{\beta!} \right) \max_{|\beta| \leq k - |\alpha|} |D^{\alpha+\beta}(P - P')(x)| \cdot \|x - x'\|^{|\beta|} \\ &\leq e^n \max_{|\beta| \leq k - |\alpha|} |D^{\alpha+\beta}(P - P')(x)| \cdot \|x - x'\|^{|\beta|}. \end{aligned}$$

Since  $\varphi_\alpha(\lambda t) \leq \lambda \varphi_\alpha(t)$ ,  $\lambda \geq 1$ , this implies

$$\varphi_\alpha(|D^\alpha(P - P')(x')|) \leq e^n \max_{|\beta| \leq k - |\alpha|} \varphi_\alpha(|D^{\alpha+\beta}(P - P')(x)| \cdot \|x - x'\|^{|\beta|}).$$

But by Lemma 2.2 for every  $\beta$ ,  $|\beta| \leq k - |\alpha|$ , we have

$$\begin{aligned} &\varphi_\alpha(|D^{\alpha+\beta}(P - P')(x)| \|x - x'\|^{|\beta|}) \\ &\leq \max\{ \omega(\|x - x'\|), \varphi_{\alpha+\beta}(|D^{\alpha+\beta}(P - P')(x)|) \} \\ &\leq I, \end{aligned}$$

proving (2.9) and the lemma. □

In the next section we will need the following variant of the triangle inequality for  $d_\omega$ .

**Lemma 2.6.** *Let  $\{T_i = (P_i, x_i) : i = 0, 1, \dots, m\}$  be a family of elements of  $\mathcal{P}_k \times \mathbf{R}^n$  such that*

$$(2.10) \quad d_\omega(T_i, T_{i+1}) \leq \omega(\|x_i - x_{i+1}\|), \quad i = 0, \dots, m - 1.$$

*Suppose that for some  $\lambda \geq 1$  we have*

$$\sum_{i=0}^{m-1} \|x_i - x_{i+1}\| \leq \lambda \|x_0 - x_m\|, \quad \text{and} \quad \sum_{i=0}^{m-1} \omega(\|x_i - x_{i+1}\|) \leq \lambda \omega(\|x_0 - x_m\|).$$

*Then*

$$(2.11) \quad d_\omega(\tau^{-1} \circ T_0, \tau^{-1} \circ T_m) \leq \omega(\|x_0 - x_m\|)$$

*where  $\tau := e^{2n} \lambda^{k+1}$ .*

*Proof.* By Lemma 2.4

$$|D^\alpha(P_0 - P_m)(x_0)| \leq e^n \max_{|\beta| \leq k - |\alpha|} \sum_{i=0}^{m-1} |D^{\alpha+\beta}(P_i - P_{i+1})(x_i)| \cdot \|x_0 - x_i\|^{|\beta|}.$$

By Theorem 2.1 and (2.10)

$$\delta_\omega(e^{-n} \circ T_i, e^{-n} \circ T_{i+1}) \leq d_\omega(T_i, T_{i+1}) \leq \omega(\|x_i - x_{i+1}\|), \quad i = 0, \dots, m - 1,$$

so that by (2.2)

$$\varphi_{\alpha+\beta}(e^{-n}|D^{\alpha+\beta}(P_i - P_{i+1})(x_i)|) \leq \omega(\|x_i - x_{i+1}\|).$$

Since  $\omega$  is strictly increasing, by the definition of (2.1) for  $\varphi_\alpha$  we have

$$|D^{\alpha+\beta}(P_i - P_{i+1})(x_i)| \leq e^n \|x_i - x_{i+1}\|^{k-|\alpha|-|\beta|} \omega(\|x_i - x_{i+1}\|).$$

Hence

$$|D^\alpha(P_0 - P_m)(x_0)| \leq e^{2n} \max_{|\beta| \leq k-|\alpha|} \sum_{i=0}^{m-1} \|x_i - x_{i+1}\|^{k-|\alpha|-|\beta|} \omega(\|x_i - x_{i+1}\|) \|x_0 - x_i\|^{|\beta|}.$$

Clearly,  $\|x_l - x_j\| \leq \sum_{i=0}^{m-1} \|x_i - x_{i+1}\|$  for every non-negative integer  $l$  and  $j$  so that

$$\begin{aligned} |D^\alpha(P_0 - P_m)(x_0)| &\leq e^{2n} \max_{|\beta| \leq k-|\alpha|} \sum_{i=0}^{m-1} \left( \sum_{i=0}^{m-1} \|x_i - x_{i+1}\| \right)^{k-|\alpha|-|\beta|} \\ &\quad \cdot \left( \sum_{i=0}^{m-1} \|x_i - x_{i+1}\| \right)^{|\beta|} \omega(\|x_i - x_{i+1}\|) \\ &= e^{2n} \left( \sum_{i=0}^{m-1} \|x_i - x_{i+1}\| \right)^{k-|\alpha|} \left( \sum_{i=0}^{m-1} \omega(\|x_i - x_{i+1}\|) \right). \end{aligned}$$

Hence

$$\begin{aligned} |D^\alpha(P_0 - P_m)(x_0)| &\leq e^{2n} \lambda^{k+1-|\alpha|} \|x_0 - x_m\|^{k-|\alpha|} \omega(\|x_0 - x_m\|) \\ &\leq e^{2n} \lambda^{k+1} \|x_0 - x_m\|^{k-|\alpha|} \omega(\|x_0 - x_m\|). \end{aligned}$$

In a similar way we obtain

$$|D^\alpha(P_0 - P_m)(x_m)| \leq e^{2n} \lambda^{k+1} \|x_0 - x_m\|^{k-|\alpha|} \omega(\|x_0 - x_m\|), \quad |\alpha| \leq k.$$

In view of (2.2) this implies

$$\delta_\omega((e^{2n} \lambda^{k+1})^{-1} \circ T_0, (e^{2n} \lambda^{k+1})^{-1} \circ T_m) \leq \omega(\|x_0 - x_m\|).$$

It remains to note that by Theorem 2.1  $d_\omega \leq \delta_\omega$ , and the lemma follows. □

We turn to the proof of Proposition 1.9. As usual given metric spaces  $X = (\mathcal{M}, \rho)$  and  $Y = (\mathcal{T}, d)$  we let  $\text{Lip}(X, Y)$  denote the space of Lipschitz mappings from  $\mathcal{M}$  into  $\mathcal{T}$ . This (in general non-linear) space of mappings  $F : \mathcal{M} \rightarrow \mathcal{T}$  is equipped with the standard “seminorm”

$$\|F\|_{\text{Lip}(X, Y)} := \inf\{\lambda : d(F(x), F(y)) \leq \lambda \rho(x, y), \quad x, y \in \mathcal{M}\}.$$

Recall that  $\mathcal{T}_{k,n} := (\mathcal{P}_k \times \mathbf{R}^n, d_\omega)$  and  $S_\omega := (S, r_\omega)$ , where  $r_\omega(x, y) := \omega(\|x - y\|)$ ,  $x, y \in S$ . Recall also that the space  $\text{Lip}(S_\omega, \mathcal{T}_{k,n})$  is normalized by the Lipschitz-Orlicz norm defined by formula (1.6). In more detail, for every mapping  $T(x) = (P_x, z_x)$ ,  $x \in S$ ,

$$\begin{aligned} \|T\|_{LO(S)} &:= \inf\{\lambda : d_\omega(\lambda^{-1} \circ T(x), \lambda^{-1} \circ T(y)) \leq \omega(\|x - y\|) \text{ for all } x, y \in S\} \\ &:= \inf\{\lambda : d_\omega((\lambda^{-1} P_x, z_x), (\lambda^{-1} P_y, z_y)) \leq \omega(\|x - y\|) \text{ for all } x, y \in S\}. \end{aligned}$$

In Section 1 we have also defined the space  $\mathbf{Lip}(S_\omega, \mathcal{T}_{k,n})$  of all Lipschitz mappings  $T(x) = (P_x, z_x)$ ,  $x \in S$ , from  $\text{Lip}(S_\omega, \mathcal{T}_{k,n})$  such that  $\sup_{x \in S} |D^\alpha P_x(x)| < \infty$

for every  $\alpha, |\alpha| \leq k$ . This space is equipped with the “norm”  $\|\cdot\|_{LO(S)}^*$  defined by (1.7).

*Proof of Proposition 1.9. (Necessity).* Let  $F \in C^{k,\omega}(\mathbf{R}^n)$ . We have to prove that the mapping

$$T = T(x) = (P_x, x), \quad x \in S,$$

where  $P_x := T_x^k(F)$ , belongs to  $\mathbf{Lip}(S_\omega, \mathcal{T}_{k,n})$ . By the Whitney-Glaeser extension theorem (the necessity part) inequalities (1.2) and (1.3) are satisfied with  $\lambda := c(k, n)\|F\|_{C^{k,\omega}(\mathbf{R}^n)}$ .

Put  $T_x := (P_x, x)$ . Then inequality (1.3) is equivalent to

$$\max\{\varphi_\alpha(\lambda^{-1}|D^\alpha(P_x - P_y)(x)|), \varphi_\alpha(\lambda^{-1}|D^\alpha(P_x - P_y)(y)|)\} \leq \omega(\|x - y\|)$$

so that

$$\begin{aligned} & \delta_\omega(\lambda^{-1} \circ T_x, \lambda^{-1} \circ T_y) \\ & := \max_{|\alpha| \leq k} \{\omega(\|x - y\|), \varphi_\alpha(|D^\alpha(\tilde{P}_x - \tilde{P}_y)(x)|), \varphi_\alpha(|D^\alpha(\tilde{P}_x - \tilde{P}_y)(y)|)\} \\ & \leq \omega(\|x - y\|) \end{aligned}$$

where  $\tilde{P}_x := \lambda^{-1}P_x, \tilde{P}_y := \lambda^{-1}P_y$ .

Since  $d_\omega \leq \delta_\omega$  (see Theorem 2.1) this implies

$$(2.12) \quad d_\omega(\lambda^{-1} \circ T_x, \lambda^{-1} \circ T_y) \leq \omega(\|x - y\|), \quad x, y \in S,$$

which by definition (1.6) is equivalent to the inequality  $\|T\|_{LO(S)} \leq \lambda$ . From this and (1.2) we obtain that  $T \in \mathbf{Lip}(S_\omega, \mathcal{T}_{k,n})$  and

$$\|T\|_{LO(S)}^* \leq 2\lambda = 2c(k, n)\|F\|_{C^{k,\omega}(\mathbf{R}^n)}.$$

*(Sufficiency).* Assume that the mapping  $T = T(x) = (P_x, x), x \in S$ , belongs to  $\mathbf{Lip}(S_\omega, \mathcal{T}_{k,n})$ . Put  $\lambda := 2\|T\|_{LO(S)}^*$ . Then by (1.7) inequality (1.2) of the Whitney-Glaeser extension theorem is satisfied. We prove that inequality (1.3) is true as well.

By (1.6)  $\|\lambda^{-1} \circ T\|_{\mathbf{Lip}(S_\omega, \mathcal{T}_{k,n})} \leq 1$  so that  $T$  satisfies inequality (2.12). By Theorem 2.1

$$\delta_\omega((e^n \lambda)^{-1} \circ T_x, (e^n \lambda)^{-1} \circ T_y) \leq d_\omega(\lambda^{-1} \circ T_x, \lambda^{-1} \circ T_y)$$

so that

$$\delta_\omega((e^n \lambda)^{-1} \circ T_x, (e^n \lambda)^{-1} \circ T_y) \leq \omega(\|x - y\|), \quad x, y \in S.$$

This inequality and definition (2.2) of  $\delta_\omega$  imply that for every  $\alpha, |\alpha| \leq k$ , and every  $x, y \in S$  we have

$$(2.13) \quad \varphi_\alpha(|D^\alpha((e^n \lambda)^{-1}P_x - (e^n \lambda)^{-1}P_y)(x)|) \leq \omega(\|x - y\|)$$

and

$$(2.14) \quad \varphi_\alpha(|D^\alpha((e^n \lambda)^{-1}P_x - (e^n \lambda)^{-1}P_y)(y)|) \leq \omega(\|x - y\|).$$

Recall that  $\varphi_\alpha := \omega((s^{k-|\alpha|}\omega(s))^{-1})$  and by our assumption  $\omega$  is a strictly increasing function. This shows that (2.13) and (2.14) are equivalent to the required inequality (1.3) (with  $e^n \lambda$  instead of  $\lambda$ ).

Thus conditions (1.2) and (1.3) of the Whitney-Glaeser extension theorem are satisfied which implies the existence of a function  $F \in C^{k,\omega}(\mathbf{R}^n)$  such that  $T_x^k(F) = P_x, x \in S$ , and  $\|F\|_{C^{k,\omega}(\mathbf{R}^n)} \leq c(k, n)e^n \lambda$ . The proposition is proved.  $\square$

*Remark 2.7.* Proposition 1.9 allows us to reformulate the Whitney-Glaeser theorem, Theorem 1.4, as an extension theorem for Lipschitz mappings from  $\mathbf{Lip}(\mathbf{R}_\omega^n, \mathcal{T}_{k,n})$ .

**Proposition 2.8.** *Suppose we are given a family of polynomials  $P_x \in \mathcal{P}_k, x \in S$ , such that the mapping  $T(x) := (P_x, x), x \in S$ , belongs to  $\mathbf{Lip}(S_\omega, \mathcal{T}_{k,n})$ . Then  $T$  can be extended to a Lipschitz mapping  $\tilde{T}(x) = (\tilde{P}_x, x) \in \mathbf{Lip}(\mathbf{R}_\omega^n, \mathcal{T}_{k,n})$  satisfying  $\|\tilde{T}\|_{LO(\mathbf{R}^n)}^* \leq c(k, n)\|T\|_{LO(S)}^*$ .*

*Proof.* Since  $T \in \mathbf{Lip}(S_\omega, \mathcal{T}_{k,n})$  by Proposition 1.9 there is a function  $F \in C^{k,\omega}(\mathbf{R}^n)$  such that  $T_x^k(F) = P_x, x \in S$ , and  $\|F\|_{C^{k,\omega}(\mathbf{R}^n)} \leq c(k, n)\|T\|_{LO(S)}^*$ . Again applying Proposition 1.9 (necessity) to the function  $F$  (on  $\mathbf{R}^n$ ) we conclude that the mapping  $\tilde{T}(x) := (T_x^k(F), x), x \in \mathbf{R}^n$ , provides the required extension of  $T$  from  $S$  on all of  $\mathbf{R}^n$ . Its norm in  $\mathbf{Lip}(\mathbf{R}_\omega^n, \mathcal{T}_{k,n})$  satisfies the inequality

$$\|\tilde{T}\|_{LO(\mathbf{R}^n)}^* \leq c_1(k, n)\|F\|_{C^{k,\omega}(\mathbf{R}^n)} \leq c_1(k, n)c(k, n)\|T\|_{LO(S)}^*.$$

The proposition is proved. □

### 3. LIPSCHITZ SELECTIONS OF POLYNOMIAL SET-VALUED MAPPINGS

In this section we deal with the Lipschitz selection problem for the pair of metric spaces  $S_\omega := (S, \omega(\|\cdot\|))$  and  $\mathcal{T}_{k,n} := (\mathcal{P}_k \times \mathbf{R}^n, d_\omega)$ . Our goal is to prove Theorem 1.10. A proof of this result is based on the classical Helly theorem and a combinatorial lemma on a structure of finite metric graphs. For its formulation we let  $(\mathcal{M}, \rho)$  denote a metric space. Let  $T$  be a (graph-theoretic) tree whose set of vertices coincides with  $\mathcal{M}$ . If vertices  $z, z'$  are joined by an edge, we write  $z \leftrightarrow z'$ . This tree generates a new metric

$$\rho_T(x, y) := \sum_{i=0}^{n-1} \rho(z_i, z_{i+1})$$

where  $\{z_0, z_1, \dots, z_n\}$  is the *unique path* in  $T$  joining  $x$  and  $y$ , i.e.,  $z_0 = x, z_n = y, z_i \neq z_j$  for  $i \neq j$  and  $z_j$  joined to  $z_{j+1}$  by an edge ( $z_j \leftrightarrow z_{j+1}$ ).

Clearly,  $\rho \leq \rho_T$ . As usual, we let  $\text{deg}_T x$  denote the degree of a vertex  $x$  in  $T$ , i.e., the number of edges incident to  $x$ . Given  $a \in \mathbf{R}$ , we let  $]a[$  denote an integer  $m$  such that  $m - 1 < a \leq m$ .

**Proposition 3.1** ([20]). *For every finite metric space  $(\mathcal{M}, \rho)$  there is a tree  $T$  such that*

$$\rho(x, y) \leq \rho_T(x, y) \leq \eta\rho(x, y), \quad x, y \in \mathcal{M},$$

and

$$\max_{x \in \mathcal{M}} \text{deg}_T x \geq ]\log_2(\text{card } \mathcal{M})[.$$

Here  $\eta = \eta(\text{card } \mathcal{M})$  is a constant depending only on the cardinality of  $\mathcal{M}$ .

We turn to

*Proof of Theorem 1.10.* Recall that  $\mathcal{G}(x) = (G(x), x), x \in S$ , where  $G(x)$  is a convex subset of  $\mathcal{P}_k$ . Observe also that the theorem's statement can be readily reduced to the case  $K = 1$ . To this end it suffices to consider a set-valued mapping  $\tilde{\mathcal{G}}(x) = (K^{-1}G(x), x)$  and make use of the fact that given a mapping  $g(x) = (P_x, x), x \in S$ , its norm  $\|g\|_{LO(S)} \leq K$  iff  $\|\tilde{g}\|_{LO(S)} \leq 1$  where  $\tilde{g}(x) = (K^{-1}P_x, x), x \in S$ .

We prove the theorem by induction on  $m := \text{card } S$ . Put

$$(3.1) \quad \ell_{\mathcal{G}} := \min\{\ell + 1, \dim \mathcal{P}_k\}.$$

If  $m = 2^{\ell_{\mathcal{G}}}$ , there is nothing to prove. Suppose that the theorem is true for every set  $S$  with  $\text{card } S \leq m$  where  $m \geq 2^{\ell_{\mathcal{G}}}$ , and prove the result for a set  $S$  consisting of  $m + 1$  points.

Thus  $\text{card } S = m + 1$  and we may assume that the restriction  $\mathcal{G}|_{S'}$  to every subset  $S' \subset S$  consisting of at most  $m$  points has a Lipschitz selection  $g_{S'} \in \text{Lip}(S'_\omega, \mathcal{T}_{k,n})$  such that  $\|g_{S'}\|_{LO(S')} \leq 1$ . We have to prove that the set-valued mapping  $\mathcal{G}$  on all of  $S$  has a Lipschitz selection  $g \in \text{Lip}(S_\omega, \mathcal{T}_{k,n})$  with  $\|g\|_{LO(S)} \leq \gamma(k, n, m)$ .

Let us apply Proposition 3.1 to the metric space  $(S, \rho)$  with  $\rho(x, y) := \|x - y\|$ . By this proposition there is a tree  $T$  with vertices in  $S$  and a vertex  $x_0 \in S$  such that  $\rho \leq \eta(m)\rho_T$  and

$$\deg_T x_0 \geq \lceil \log_2(\text{card } S) \rceil \geq \lceil \log_2(m + 1) \rceil \geq \lceil \log_2(2^{\ell_{\mathcal{G}}} + 1) \rceil = \ell_{\mathcal{G}} + 1.$$

We let  $I(x_0)$  denote the family of vertices  $\{y_1, y_2, \dots, y_p\}$  incident to  $x_0$ . Thus the number of these vertices

$$(3.2) \quad p = \deg_T x_0 \geq \ell_{\mathcal{G}} + 1.$$

For every vertex  $y \in I(x_0)$  we define a subtree  $T_y$  of the tree  $T$  whose set of vertices  $S_y$  consists of all  $z \in S$  for which the (unique) path connecting  $z$  and  $y$  in  $T$  does not contain the vertex  $x_0$ . (We supply  $T_y$  with the tree structure induced by  $T$ .) Clearly, the trees  $T_y$  and  $T_{y'}$  have no common vertices for different  $y, y' \in I(x_0)$ .

For each vertex  $y \in I(x_0)$  (i.e.,  $y \leftrightarrow x_0$ ) we let  $\text{Or}(y)$  denote a family of polynomials  $P \in \mathcal{P}_k$  such that the following holds: For each vertex  $z \in S_y$  of  $T_y$  there is a polynomial  $P_z \in G(z)$  such that  $P_y = P$  and for every  $z, z' \in S_y$ ,  $z \leftrightarrow z'$ , we have

$$(3.3) \quad \delta_\omega(e^{-n} \circ T_z, e^{-n} \circ T_{z'}) \leq \omega(\|z - z'\|),$$

where  $T_z := (P_z, z), T_{z'} := (P_{z'}, z')$ .

Since  $\text{card } S_y < \text{card } S$ , by the assumption the restriction  $\mathcal{G}|_{S_y}$  has a Lipschitz selection  $g_{S_y} : S_y \rightarrow \mathcal{P}_k \times \mathbf{R}^n$  with  $\|g_{S_y}\|_{LO(S_y)} \leq 1$ . In other words, for each  $z \in S_y$  there is a polynomial  $P_z \in G(z)$  such that

$$d_\omega((P_z, z), (P_{z'}, z')) \leq \omega(\|z - z'\|).$$

Then

$$(3.4) \quad P_y \in \text{Or}(y).$$

In fact, by Theorem 2.1

$$\delta_\omega((e^{-n}P_z, z), (e^{-n}P_{z'}, z')) \leq d_\omega((P_z, z), (P_{z'}, z')) \leq \omega(\|z - z'\|),$$

proving (3.4).

We have also proved that  $\text{Or}(y) \neq \emptyset$  for every  $y \in I(x_0)$ . Recall that inequality (3.3) is equivalent to inequalities (1.3) of the Whitney-Glaeser extension theorem. The left-hand sides of these inequalities are subadditive and positively homogeneous functions of polynomials  $P_x, P_y$ . This and the definition of  $\text{Or}(y)$  show that for every  $y \in I(x_0)$  the set  $\text{Or}(y)$  is *convex*.

Given  $y \in I(x_0)$  we put

$$(3.5) \quad U(y) := \{P \in \mathcal{P}_k : \text{there is } \tilde{P} \in \text{Or}(y) \text{ such that } \delta_\omega((\theta^{-1}P, x_0), (\theta^{-1}\tilde{P}, y)) \leq \omega(\|x_0 - y\|)\}$$



where  $\theta := 3^{k+1}e^{3n}$ . Prove that

$$(3.6) \quad G(x_0) \bigcap \left\{ \bigcap_{y \in I(x_0)} U(y) \right\} \neq \emptyset.$$

But before we do this let us show how the proof of the theorem can be completed.

Property (3.6) implies the existence of polynomials  $P_{x_0} \in G(x_0)$ ,  $P_y \in \text{Or}(y) \subset G(y)$ ,  $y \in I(x_0)$ , such that

$$(3.7) \quad \delta_\omega((\theta^{-1}P_{x_0}, x_0), (\theta^{-1}P_y, y)) \leq \omega(\|x_0 - y\|).$$

In turn, since  $P_y \in \text{Or}(y)$  for  $y \in I(x_0)$ , by (3.3) there exist polynomials  $P_z \in G(z)$ ,  $z \in S_y$ , such that

$$(3.8) \quad \delta_\omega((e^{-n}P_z, z), (e^{-n}P_{z'}, z')) \leq \omega(\|z - z'\|), \quad z \leftrightarrow z', \quad z, z' \in S_y.$$

Now polynomials  $P_x$  are defined for all  $x \in S$ . Put

$$g(x) := (P_x, x), \quad x \in S.$$

Then  $g : S \rightarrow \mathcal{T}_{k,n}$  is a selection of  $\mathcal{G}$ . Let us show that  $g \in \text{Lip}(S_\omega, \mathcal{T}_{k,n})$  and  $\|g\|_{LO(S)}$  is bounded by a constant depending only on  $k, n$  and  $m$ . In fact, by (3.7) and (3.8) for every two vertices  $z$  and  $z'$  of the tree  $\mathbf{T}$  joined by an edge ( $z \leftrightarrow z'$ ) we have

$$\delta_\omega(\theta^{-1} \circ g(z), \theta^{-1} \circ g(z')) \leq \omega(\|z - z'\|).$$

By Theorem 2.1  $d_\omega \leq \delta_\omega$  so that

$$(3.9) \quad d_\omega(\theta^{-1} \circ g(z), \theta^{-1} \circ g(z')) \leq \omega(\|z - z'\|), \quad z \leftrightarrow z', \quad z, z' \in S.$$

To estimate  $d_\omega(g(x), g(y))$  for arbitrary  $x, y \in S$  we will make use of Lemma 2.6. Since  $x, y$  are vertices of the tree  $\mathbf{T}$ , there is the unique path  $\{z_0, z_1, \dots, z_q\}$  in  $\mathbf{T}$  joining  $x$  and  $y$  (i.e.,  $z_0 = x$ ,  $z_q = y$  and  $z_i \leftrightarrow z_{i+1}$ ,  $i = 0, 1, \dots, q - 1$ ). Clearly,  $q \leq m$  (recall that  $\text{card } S = m + 1$ ).

We put  $T_i := \theta^{-1} \circ g(z_i) = (\theta^{-1}P_{z_i}, z_i)$  so that by (3.9)

$$d_\omega(T_i, T_{i+1}) \leq \omega(\|z_i - z_{i+1}\|), \quad i = 0, 1, \dots, q - 1.$$

Recall that  $\rho(x, y) := \|x - y\| \leq \eta \rho_{\mathbf{T}}(x, y)$  where  $\eta = \eta(m)$  is the constant from Proposition 3.1. Hence

$$\sum_{i=0}^{q-1} \|z_i - z_{i+1}\| \leq \eta \|z_0 - z_q\| (= \eta \|x - y\|).$$

On the other hand,

$$\begin{aligned} \sum_{i=0}^{q-1} \omega(\|z_i - z_{i+1}\|) &\leq q \max_{i=0, \dots, q-1} \omega(\|z_i - z_{i+1}\|) \leq q\omega \left( \sum_{i=0}^{q-1} \|z_i - z_{i+1}\| \right) \\ &\leq m \omega(\eta \|z_0 - z_q\|) \leq m\eta \omega(\|z_0 - z_q\|) \end{aligned}$$

(recall that  $\omega$  is a concave non-negative function on  $\mathbf{R}_+$  so that  $\omega(\lambda t) \leq \lambda \omega(t)$ ,  $\lambda \geq 1$ ).

Let us apply Lemma 2.6 to the family  $\{T_i, i = 0, \dots, q - 1\}$ , points  $\{z_i, i = 0, \dots, q - 1\}$  and a parameter  $\lambda := m\eta$ . By this lemma

$$d_\omega(\tau^{-1} \circ T_0, \tau^{-1} \circ T_q) \leq \omega(\|z_0 - z_q\|)$$

where  $\tau := e^{2n\lambda^{k+1}}$ ; see (2.11). Since  $z_0 = x, z_q = y$  and  $T_0 = (P_{z_0}, z_0) = (P_x, x) = g(x), T_q = (P_{z_q}, z_q) = (P_y, y) = g(y)$ , we obtain

$$d_\omega(\tau^{-1} \circ g(x), \tau^{-1} \circ g(y)) \leq \omega(\|x - y\|), \quad x, y \in S.$$

Hence  $\|g\|_{LO(S)} \leq \tau = \tau(k, n, m)$ , proving that  $g$  is a Lipschitz selection of  $\mathcal{G}$ .

Thus it remains to prove (3.6). This property readily follows from Helly's theorem and the induction assumption. We put

$$(3.10) \quad F(x_0) := G(x_0), \quad F(y) := U(y), \quad y \in I(x_0),$$

and  $\tilde{I} := \{x_0, y_1, y_2, \dots, y_p\} (= x_0 \cup I(x_0))$ . Then property (3.6) is equivalent to

$$\bigcap \{F(y) : y \in \tilde{I}\} \neq \emptyset.$$

By (3.2)

$$\text{card } \tilde{I} = 1 + \text{card } I(x_0) \geq p + 1 \geq \ell_G + 2.$$

Moreover, all the sets  $F(y), y \in \tilde{I}$ , are convex subsets of the finite-dimensional space  $\mathcal{P}_k$ , and the dimension of one of them, of the set  $F(x_0) := G(x_0)$ , is at most  $\ell$ . Therefore by Helly's theorem it suffices to prove that

$$\bigcap \{F(y) : y \in I'\} \neq \emptyset$$

for every subfamily  $I' \subset \tilde{I}$  consisting of at most

$$\min\{\ell + 2, \dim \mathcal{P}_k + 1\} = \ell_G + 1$$

elements. (Recall that  $\ell_G$  is defined by (3.1).)

Since  $\text{card } \tilde{I} \geq \ell_G + 2$  and  $\text{card } I' \leq \ell_G + 1$ , there is a point  $\tilde{y} \in \tilde{I}$  such that  $\tilde{y} \notin I'$ . Then by the assumption for the set  $S' := S \setminus \{\tilde{y}\}$  the restriction  $\mathcal{G}|_{S'}$  has a Lipschitz selection  $g_{S'} : S' \rightarrow \mathcal{T}_{k,n}$  with  $\|g_{S'}\|_{LO(S')} \leq 1$ . Thus  $g_{S'}(y) = (P_y, y), y \in S'$ , where

$$(3.11) \quad P_y \in G(y), \quad y \in S',$$

and

$$(3.12) \quad d_\omega(g_{S'}(y), g_{S'}(y')) \leq \omega(\|y - y'\|), \quad y, y' \in S'.$$

We let  $\bar{y}$  denote the point nearest to  $x_0$  (in the metric  $\|\cdot\|$ ) from the family  $I'$ . (Clearly,  $\bar{y} = x_0$  whenever  $x_0 \in I'$ .) Prove that

$$(3.13) \quad P_{\bar{y}} \in F(y) \text{ for every } y \in I'.$$

In fact, if  $y = x_0$ , then  $x_0 \in I'$  so that  $\bar{y} = x_0$ . Therefore by (3.10)  $F(y) = G(x_0)$  so that (3.13) follows from (3.11). Thus later on we may assume that  $y \neq x_0$ .

As we have proved above (see (3.4))  $P_y \in \text{Or}(y), y \in I'$ . Moreover, by (3.12)

$$d_\omega(g_{S'}(y), g_{S'}(\bar{y})) \leq \omega(\|y - \bar{y}\|).$$

On the other hand, by (2.5)

$$d_\omega((P_{\bar{y}}, \bar{y}), (P_{\bar{y}}, x_0)) = \omega(\|\bar{y} - x_0\|).$$

But by definition of  $\bar{y}$

$$\|x_0 - \bar{y}\| + \|\bar{y} - y\| \leq 2\|x_0 - \bar{y}\| + \|x_0 - y\| \leq 3\|x_0 - y\|$$

and

$$\omega(\|x_0 - \bar{y}\|) + \omega(\|\bar{y} - y\|) \leq 2\omega(\|x_0 - \bar{y}\|) + \omega(\|x_0 - y\|) \leq 3\omega(\|x_0 - y\|).$$

Now let us apply Lemma 2.6 to  $T_0 := g_{S'}(y) = (P_y, y)$ ,  $T_1 := g_{S'}(\bar{y}) = (P_{\bar{y}}, \bar{y})$  and  $T_2 := (P_{\bar{y}}, x_0)$  with  $\lambda = 3$ . Then by the lemma

$$d_\omega((3^{k+1}e^{2n})^{-1} \circ T_0, (3^{k+1}e^{2n})^{-1} \circ T_2) \leq \omega(\|x_0 - y\|)$$

so that by Theorem 2.1

$$\begin{aligned} & \delta_\omega((3^{k+1}e^{3n})^{-1} \circ T_0, (3^{k+1}e^{3n})^{-1} \circ T_2) \\ & \leq d_\omega((3^{k+1}e^{2n})^{-1} \circ T_0, (3^{k+1}e^{2n})^{-1} \circ T_2) \\ & \leq \omega(\|x_0 - y\|). \end{aligned}$$

Recall that  $\theta := 3^{k+1}e^{3n}$ . Hence

$$\delta_\omega(\theta^{-1} \circ T_0, \theta^{-1} \circ T_2) = \delta_\omega((\theta^{-1}P_y, y), (\theta^{-1}P_{\bar{y}}, x_0)) \leq \omega(\|x_0 - y\|).$$

But  $P_y \in \text{Or}(y)$  so that by definition (3.5)  $P_{\bar{y}} \in U(y) = F(y)$  (recall that  $y \neq x_0$ ). Theorem 1.10 is completely proved.  $\square$

This theorem implies a similar result for the space  $\mathbf{Lip}(S_\omega, \mathcal{T}_{k,n})$  of “bounded” Lipschitz mappings.

**Theorem 3.2.** *Let  $\mathcal{G}(x) = (G(x), x)$ ,  $x \in S$ , be a set-valued mapping from a finite set  $S \subset \mathbf{R}^n$  into  $2^{\mathcal{P}_k \times \mathbf{R}^n}$  such that for each  $x \in S$  the set  $G(x) \subset \mathcal{P}_k$  is a convex set of polynomials of dimension at most  $\ell$ . Suppose that for every subset  $S' \subset S$  consisting of at most  $2^{\min\{\ell+1, \dim \mathcal{P}_k\}}$  points the restriction  $\mathcal{G}|_{S'}$  has a Lipschitz selection  $g_{S'} \in \mathbf{Lip}(S', \mathcal{T}_{k,n})$  with  $\|g_{S'}\|_{LO(S')}^* \leq K$ . Then  $\mathcal{G}$  on all of  $S$  has a Lipschitz selection  $g \in \mathbf{Lip}(S, \mathcal{T}_{k,n})$  with  $\|g\|_{LO(S)}^* \leq \gamma(k, n, \text{card } S)K$ .*

*Proof.* As in the proof of Theorem 1.10 it suffices to prove the result for  $K = 1$ . Given  $x \in S$  we put

$$H(x) := \{P \in \mathcal{P}_k : \max_{|\alpha| \leq k} |D^\alpha P_x(x)| \leq 1\}.$$

We define a set-valued mapping  $\tilde{\mathcal{G}}$  by letting

$$(3.14) \quad \tilde{\mathcal{G}}(x) := (G(x) \cap H(x), x), \quad x \in S.$$

Put  $\ell_{\mathcal{G}} := \min\{\ell + 1, \dim \mathcal{P}_k\}$  and prove that for every subset  $S' \subset S$  of cardinality  $\text{card } S' \leq 2^{\ell_{\mathcal{G}}}$  the restriction  $\tilde{\mathcal{G}}|_{S'}$  has a Lipschitz selection  $\tilde{g}_{S'} \in \text{Lip}(S'_\omega, \mathcal{T}_{k,n})$  with  $\|\tilde{g}_{S'}\|_{LO(S')} \leq 1$ . In fact, by the theorem’s hypothesis  $\mathcal{G}|_{S'}$  has a selection  $g_{S'} \in \text{Lip}(S'_\omega, \mathcal{T}_{k,n})$  such that  $\|g_{S'}\|_{LO(S')}^* \leq 1$ . Thus  $g_{S'}(x) = (P_{(S',x)}, x)$  where the polynomial  $P_{(S',x)}$ ,  $x \in S'$ , satisfies the following conditions: (i)  $P_{(S',x)} \in G(x)$ ,  $x \in S'$ ; (ii)  $|D^\alpha P_{(S',x)}(x)| \leq 1$  for all  $|\alpha| \leq k$  and  $x \in S'$ , and (iii)

$$(3.15) \quad d_\omega((P_{(S',x)}, x), (P_{(S',y)}, y)) \leq \omega(\|x - y\|), \quad x, y \in S'.$$

Hence  $P_{(S',x)} \in H(x) \cap G(x)$ ,  $x \in S'$ , so that the mapping  $\tilde{g}_{S'}(x) := (P_{(S',x)}, x)$ ,  $x \in S'$ , provides the required selection of  $\tilde{\mathcal{G}}|_{S'}$ . By (3.15) its Lipschitz-Orlicz norm in  $\text{Lip}(S'_\omega, \mathcal{T}_{k,n})$  does not exceed 1.

By Theorem 1.10  $\tilde{\mathcal{G}}$  on all of  $S$  has a Lipschitz selection  $g(x) := (P_x, x), x \in S$ , with  $\|g\|_{LO(S)} \leq \gamma_1(k, n, \text{card } S)$ . Since  $g$  is a selection of  $\tilde{\mathcal{G}}$ , by (3.14) it is a selection of  $\mathcal{G}$  as well. Moreover, by (3.14)  $P_x \in H(x), x \in S$ , so that  $\max_{|\alpha| \leq k} |D^\alpha P_x(x)| \leq 1$  for all  $x \in S$ . Hence

$$\|g\|_{LO(S)}^* = \max_{|\alpha| \leq k} \sup_{x \in S} |D^\alpha P_x(x)| + \|g\|_{LO(S)} \leq 1 + \gamma_1(k, n, \text{card } S).$$

The theorem is proved. □

#### 4. THE WEAK FINITENESS PROPERTY OF THE SPACE $C^{k,\omega}(\mathbf{R}^n)$

*Proof of Theorem 1.8.* The result easily follows from Proposition 1.9 and Theorem 3.2. In fact, we let  $\mathcal{G}$  denote a set-valued mapping  $\mathcal{G}(x) := (G(x), x), x \in S$ .

Fix a set  $S' \subset S$  of cardinality at most  $2^{\ell_{\mathcal{G}}}$  where  $\ell_{\mathcal{G}} := \min\{\ell + 1, \dim \mathcal{P}_k\}$ . By the theorem’s hypothesis there is a function  $F_{S'} \in C^{k,\omega}(\mathbf{R}^n)$  with  $\|F_{S'}\|_{C^{k,\omega}(\mathbf{R}^n)} \leq 1$  satisfying

$$(4.1) \quad T_x^k(F_{S'}) \in G(x), \quad x \in S'.$$

We put  $T_{S'}(x) := (T_x^k(F_{S'}), x), x \in S'$ . By (4.1)  $T_{S'}$  is a selection of the restriction  $\mathcal{G}|_{S'}$ . Moreover, by Proposition 1.9 (the “only if” part) the mapping  $T_{S'} : S' \rightarrow \mathcal{T}_{k,n}$  belongs to  $\mathbf{Lip}(S_\omega, \mathcal{T}_{k,n})$ , and its Lipschitz-Orlicz norm satisfies the inequality

$$\|T_{S'}\|_{LO(S')}^* \leq c_1(k, n) \|F_{S'}\|_{C^{k,\omega}(\mathbf{R}^n)} \leq c_1(k, n).$$

Thus  $T_{S'}$  is a Lipschitz selection of  $\mathcal{G}|_{S'}$ . Since  $S'$  is an arbitrary subset of  $S$  of cardinality at most  $2^{\ell_{\mathcal{G}}}$ , by Theorem 3.2 there is a selection  $T(x) = (P_x, x)$  of  $\mathcal{G}$  defined on all of  $S$  and satisfying  $\|T\|_{LO(S')}^* \leq c_1(k, n)\gamma(k, n, \text{card } S)$ .

In particular,  $P_x \in G(x), x \in S$ . Now by Proposition 1.9 (the “if” part) there is a function  $F \in C^{k,\omega}(\mathbf{R}^n)$  with

$$\|F\|_{C^{k,\omega}(\mathbf{R}^n)} \leq c_2(k, n) \|T\|_{LO(S')}^* \leq c_2(k, n)c_1(k, n)\gamma(k, n, \text{card } S)$$

such that  $T_x^k(F) = P_x, x \in S$ . Hence  $T_x^k(F) \in G(x), x \in S$ , and the theorem follows. □

*Remark 4.1.* Theorem 1.2 for  $\xi \equiv 0$  implies the following finiteness property of the space  $C^{k,\omega}(\mathbf{R}^n)$ : *A function  $f$  defined on a subset  $S \subset \mathbf{R}^n$  can be extended to a function  $F \in C^{k,\omega}(\mathbf{R}^n)$  with  $\|F\|_{C^{k,\omega}(\mathbf{R}^n)} \leq \gamma(k, n)$ , provided its restriction  $f|_{S'}$  to every subset  $S' \subset S$  consisting of at most  $N(k, n) = 2^{\dim \mathcal{P}_k}$  points can be extended to a function  $F_{S'} \in C^{k,\omega}(\mathbf{R}^n)$  with  $\|F_{S'}\|_{C^{k,\omega}(\mathbf{R}^n)} \leq 1$ .*

In particular,  $\dim \mathcal{P}_1 = n + 1$  so that  $N(1, n) \leq 2^{n+1}$ . Recall that the sharp value of the finiteness number for  $k = 1$  equals  $3 \cdot 2^{n-1} = \frac{3}{4} \cdot 2^{n+1}$ . This shows that the estimate  $2^{\dim \mathcal{P}_k}$  is rather far from the optimal one and can apparently be decreased considerably.

In the next paper we will prove that the finiteness number  $N(k, n)$  does not exceed  $(k + 1) \cdot 2^{\dim \mathcal{P}_k - k}$ . We conjecture that the sharp value of the finiteness number in the above finiteness property for  $C^{k,\omega}(\mathbf{R}^n)$  is

$$N(k, n) = \prod_{m=0}^k (k - m + 2)^{\binom{n+m-2}{m}}.$$

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