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TOPOLOGICAL STRUCTURE OF (PARTIALLY) HYPERBOLIC SETS WITH POSITIVE VOLUME

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Abstract. We consider both hyperbolic sets and partially hyperbolic sets attracting a set of points with positive volume in a Riemannian manifold. We obtain several results on the topological structure of such sets for diffeomorphisms whose differentiability is larger than one. We show in particular that there are no partially hyperbolic horseshoes with positive volume for such diffeomorphisms. We also give a description of the limit set of almost every point belonging to a hyperbolic set or a partially hyperbolic set with positive volume.

1. Introduction

Since the 60s hyperbolic sets have played an important role in the development of the Theory of Dynamical Systems. These are invariant (by a smooth map) compact sets over which the tangent bundle splits into two invariant subbundles, one of them contracting and the other one expanding under the action of the derivative of the map. In this work we are concerned with discrete dynamical systems (smooth transformations of a manifold), but our techniques also proved useful in the continuous setting (vector fields in a manifold), especially for the study of singular-hyperbolic sets as done in [2]. In the last decades an increasing emphasis has been put on the dynamics of partially hyperbolic sets. These are compact invariant sets for which the tangent bundle splits into two invariant subbundles having contracting/expanding behavior in one direction and the other one being dominated by it. Precise definitions of all these objects will be given in the next section.

In this context a special role has been played by the horseshoes, which have been introduced by Smale and, as shown in [17], always exist near a transverse homoclinic point associated to some hyperbolic periodic point of saddle type, i.e. a point whose orbit asymptotically approaches that saddle point, both in the past and in the future. Horseshoes can be used to show that transverse homoclinic points are always accumulated by periodic points, but the dynamical richness of these objects goes far beyond the initial application by Smale, and since then many

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other results have been proved using horseshoes. These are Cantor sets which are, in dynamical terms, topologically conjugated to full shifts.

Special interest lies in the horseshoes that appear when one unfolds a homoclinic tangency. Knowing how *fat* these horseshoes are can have several implications in the dynamical behavior after the homoclinic bifurcation. In this setting we mention the *thickness*, which has been used by Newhouse [10] to prove the existence of infinitely many sinks, and the *Hausdorff dimension*, which has been used by Palis, Takens and Yoccoz to study the prevalence of hyperbolicity after the unfolding of a homoclinic tangency; see [11, 12, 13].

One interesting issue we will address is the *volume* of horseshoes. As shown by Bowen in [6], there are C^1 diffeomorphisms with hyperbolic horseshoes of positive volume. On the other hand, Bowen has proved in [7, Theorem 4.11] that a basic set (locally maximal hyperbolic set with a dense orbit) of a C^2 diffeomorphism which attracts a set with positive volume, necessarily attracts a neighborhood of itself. In particular, the unstable manifolds through points of this set must be contained in it, and consequently C^2 diffeomorphisms have no horseshoes with positive volume.

For diffeomorphisms whose differentiability is greater than one, we prove the nonexistence of horseshoes with positive volume in a more general context of sets with some partially hyperbolic structure. Using our framework in the context of hyperbolic sets, we are able to show that Bowen's result still holds without the local maximality assumption, i.e. a transitive hyperbolic set which attracts a set with positive volume necessarily attracts a neighborhood of itself. Furthermore, we are able to prove that there are no proper transitive hyperbolic sets with positive volume for diffeomorphisms whose differentiability is greater than one. Similar results for sets with nonempty interior have already been obtained in [1, Theorem 1] and in [8, Theorem 1.1]. On the other hand, as described in [1, Remark 2.1] or in [8, Example 2], there exist (nontransitive) hyperbolic sets with positive volume which do not attract neighborhoods of themselves.

Let us mention two more important results in this direction. It follows from [18, Theorem 2] that proper uniformly partially hyperbolic sets supporting a unique equilibrium state and attracting open neighborhoods of themselves necessarily have zero volume. In the conservative setting, [5, Theorem 15] gives that a hyperbolic set for a volume preserving C^2 diffeomorphism either has zero volume or coincides with the whole manifold.

In this work we also give a good description of the limit set of almost every point in a hyperbolic set with positive volume: there is a finite number of basic sets for which the ω-limit set of Lebesgue almost every point is contained in one of these basic sets. We are also able to prove in a partially hyperbolic setting that these ω-limit sets are contained in the closure of finitely many hyperbolic periodic points.

2. Statement of results

Let $f : M \to M$ be a diffeomorphism of a compact connected Riemannian manifold M. We say that f is C^{1+} if f is C^1 and Df is Hölder continuous. We use Leb to denote a normalized volume form on the Borel sets of M that we call *Lebesgue measure.* Given a submanifold $\gamma \subset M$ we use Leb_γ to denote the measure on γ induced by the restriction of the Riemannian structure to γ. A set $\Lambda \subset M$ is said to be *invariant* if $f(\Lambda) = \Lambda$, and *positively invariant* if $f(\Lambda) \subset \Lambda$.

2.1. **Partially hyperbolic sets.** Let K be a positively invariant compact set, and define

$$\Lambda = \bigcap_{n\geq 0} f^n(K).$$

Suppose that there is a continuous splitting $T_K M = E^{cs} \oplus E^{cu}$ of the tangent bundle restricted to K, and assume that this splitting is Df-invariant over Λ. We say that this is a *dominated splitting* (over Λ) if there is a constant $0 < \lambda < 1$ such that for some choice of a Riemannian metric on M

$$\|Df \mid E^{cs}_x\| \cdot \|Df^{-1} \mid E^{cu}_{f(x)}\| \leq \lambda, \quad \text{for every } x \in \Lambda.$$

We call E^{cs} the *centre-stable bundle* and E^{cu} the *centre-unstable bundle*. Λ is said to be *partially hyperbolic* if additionally E^{cs} is *uniformly contracting* or E^{cu} is *uniformly expanding*, meaning that there exists $0 < \lambda < 1$ such that

$$\|Df \mid E^{cs}_x\| \leq \lambda, \quad \text{for every } x \in \Lambda,$$

or

$$\|Df^{-1} \mid E^{cu}_{f(x)}\| \leq \lambda, \quad \text{for every } x \in \Lambda.$$

We say that f is *nonuniformly expanding along the centre-unstable direction* in K if there is $c > 0$ such that for Lebesgue almost every $x \in K$

$$\text{(NUE)} \qquad \liminf_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^{-1} \mid E^{cu}_{f^j(x)}\| < -c.$$

Condition NUE means that the derivative has *expanding behavior in the centre-unstable direction in average* over the orbit of x for an infinite number of times. If condition NUE holds for every point in a compact invariant set Λ, then E^{cu} is *uniformly expanding in the centre-unstable direction in* Λ. This is not necessarily the case if NUE occurs only Lebesgue almost everywhere. A class of diffeomorphisms with a dominated splitting $TM = E^{cs} \oplus E^{cu}$ for which NUE holds Lebesgue almost everywhere in M and E^{cu} is not uniformly expanding can be found in [4, Appendix A].

We say that an embedded disk $\gamma \subset M$ is an *unstable manifold*, or an *unstable disk*, if $\operatorname{dist}(f^{-n}(x), f^{-n}(y)) \to 0$ exponentially fast as $n \to \infty$, for every $x, y \in \gamma$. Similarly, γ is called a *stable manifold*, or a *stable disk*, if $\operatorname{dist}(f^n(x), f^n(y)) \to 0$ exponentially fast as $n \to \infty$, for every $x, y \in \gamma$. It is well known that every point in a hyperbolic set possesses a local stable manifold $W^s_{\mathrm{loc}}(x)$ and a local unstable manifold $W^u_{\mathrm{loc}}(x)$ which are disks tangent to E^s_x and E^u_x at x respectively. A compact invariant set Λ is said to be *horseshoe-like* if there are local stable and local unstable manifolds through all its points which intersect Λ in a Cantor set.

Theorem A. *Let $f : M \to M$ be a C^{1+} diffeomorphism and let $K \subset M$ be a forward invariant compact set with a continuous splitting $T_K M = E^{cs} \oplus E^{cu}$ dominated over $\Lambda = \bigcap_{n\geq 0} f^n(K)$. If* NUE *holds for a positive Lebesgue set of points $x \in K$, then Λ contains some local unstable disk.*

The next result is a direct consequence of Theorem A whenever E^{cu} is uniformly expanding. If, on the other hand, E^{cs} is uniformly contracting, then we just have to apply Theorem A to f^{-1}.

Corollary B. *Let $f : M \to M$ be a C^{1+} diffeomorphism and let $K \subset M$ be a compact invariant set with* $\mathrm{Leb}(K) > 0$ *having a continuous splitting $T_K M = E^{cs} \oplus E^{cu}$ for which $\Lambda = \bigcap_{n\geq 0} f^n(K)$ is partially hyperbolic.*

(1) *If E^{cs} is uniformly contracting, then Λ contains a local stable disk.*
(2) *If E^{cu} is uniformly expanding, then Λ contains a local unstable disk.*

In particular, C^{1+} diffeomorphisms have no partially hyperbolic horseshoe-like sets with positive volume. The same conclusion holds for partially hyperbolic sets intersecting a local stable disk or a local unstable disk in a positive Lebesgue measure subset, as Corollary D below shows.

Theorem C. *Let $f : M \to M$ be a C^{1+} diffeomorphism and let $K \subset M$ be a forward invariant compact set with a continuous splitting $T_K M = E^{cs} \oplus E^{cu}$ dominated over $\Lambda = \bigcap_{n\geq 0} f^n(K)$. Assume that there is a local unstable disk γ such that* NUE *holds for every x in a positive* Leb_γ *subset of $\gamma \cap K$. Then Λ contains some local unstable disk.*

The next result is an immediate consequence of Theorem C, in the case that E^{cu} is uniformly expanding, and a consequence of the same theorem applied to f^{-1} when E^{cs} is uniformly contracting. Actually, we shall prove a stronger version of this result in Theorem 5.1.

Corollary D. *Let $f : M \to M$ be a C^{1+} diffeomorphism and let $K \subset M$ be a forward invariant compact set having a continuous splitting $T_K M = E^{cs} \oplus E^{cu}$ dominated over $\Lambda = \bigcap_{n\geq 0} f^n(K)$.*

(1) *If E^{cs} is uniformly contracting and there is a local stable disk γ such that* $\mathrm{Leb}_\gamma(\gamma \cap K) > 0$*, then Λ contains a local stable disk.*
(2) *If E^{cu} is uniformly expanding and there is a local unstable disk γ such that* $\mathrm{Leb}_\gamma(\gamma \cap K) > 0$*, then Λ contains a local unstable disk.*

Using the previous results we are able to give a description of the ω-limit of Lebesgue almost every point in a partially hyperbolic whose centre-unstable direction displays nonuniform expansion in a subset with positive volume. Recall that the *ω-limit* of $x \in M$ is the set of accumulation points of its orbit.

Theorem E. *Let $f : M \to M$ be a C^{1+} diffeomorphism and let $K \subset M$ be a forward invariant compact set with* $\mathrm{Leb}(K) > 0$ *having a continuous splitting $T_K M = E^{cs} \oplus E^{cu}$ for which $\Lambda = \bigcap_{n\geq 0} f^n(K)$ is partially hyperbolic. Assume that E^{cs} is uniformly contracting and* NUE *holds for Lebesgue almost every $x \in K$. Then there are hyperbolic periodic points $p_1, \ldots, p_k \in \Lambda$ such that:*

(1) *$\overline{W^u(p_i)} \subset \Lambda$ for each $1 \leq i \leq k$;*
(2) *for* Leb *almost every $x \in K$ there is $1 \leq i \leq k$ with $\omega(x) \subset \overline{W^u(p_i)}$.*

Moreover, if E^{cu} has dimension one, then for each $1 \leq i \leq k$

(3) *$\overline{W^u(p_i)}$ attracts an open neighborhood of itself.*

This last conclusion also holds whenever E^{cs} is uniformly contracting. Actually, more can be said in the case of uniformly hyperbolic sets with positive volume as we shall see in the next subsection.

2.2. **Hyperbolic sets.** We say that a compact invariant set Λ is *hyperbolic* if there is a Df-invariant splitting $T_\Lambda M = E^s \oplus E^u$ of the tangent bundle restricted to Λ and a constant $\lambda < 1$ such that (for some choice of a Riemannian metric on M) for every $x \in \Lambda$

$$\|Df \mid E^s_x\| < \lambda \quad \text{and} \quad \|Df^{-1} \mid E^u_x\| < \lambda.$$

We are able to prove that transitive hyperbolic sets with positive volume necessarily coincide with the whole manifold, i.e. the diffeomorphism is Anosov.

Theorem F. *Let $f : M \to M$ be a C^{1+} diffeomorphism and let $\Lambda \subset M$ be a transitive hyperbolic set.*

1. *If Λ has positive volume, then $\Lambda = M$.*
2. *If Λ attracts a set with positive volume, then Λ attracts a neighborhood of itself.*

The main reason why we cannot generalize the results in this subsection to the context of partially hyperbolic sets is that the length of local stable/unstable manifolds may shrink to zero when iterated back/forth, respectively. The next result gives a description of the ω-limit of Lebesgue almost every point in a hyperbolic set with positive volume. Taking f^{-1}, a similar decomposition holds for α-limits.

Theorem G (Spectral decomposition)**.** *Let $f : M \to M$ be a C^{1+} diffeomorphism and let $\Lambda \subset M$ be a hyperbolic set with positive volume. There are hyperbolic sets $\Omega_1, \dots, \Omega_q \subset \Lambda$ such that:*

1. *for* Leb *almost every $x \in \Lambda$ there is $1 \le i \le q$ such that $\omega(x) \subset \Omega_i$;*
2. *Ω_j attracts a neighborhood of itself in M, for each $1 \le j \le q$;*
3. *$f|\Omega_k$ is transitive;*
4. $\mathrm{Per}(f)$ *is dense in Ω_j, for each $1 \le j \le q$.*

Moreover, for each $1 \le k \le q$ there is a decomposition of Ω_k into disjoint hyperbolic sets $\Omega_k = \Omega_{k,1} \cup \dots \cup \Omega_{k,n_k}$ such that:

5. *$f(\Omega_{k,i}) = \Omega_{k,i+1}$, for $1 \le i < n_k$, and $f(\Omega_{k,n_k}) = \Omega_{k,1}$;*
6. *$f^{n_k} : \Omega_{k,i} \to \Omega_{k,i}$ is topologically mixing for every $1 \le i \le n_k$.*

2.3. **Overview.** This paper is organized in the following way. In Section 3 we present some results from [4] on the Hölder control of the tangent direction of certain submanifolds, and in Section 4 we derive some bounded distortion results. Theorem A and Theorem C are actually corollaries of a slightly more general result that we present at the beginning of Section 5. Let us mention that the results in Section 5 (especially Lemma 5.4) are not a consequence of the results in [4], since we are using a weaker form of nonuniform expansion in NUE. Theorem E is proved in Section 6. Finally, in Section 7 we prove Theorem F and Theorem G.

3. Hölder control of tangent direction

In this section we present some results from [4, Section 2] concerning the Hölder control of the tangent direction of submanifolds. Though the results in [4] are stated for C^2 diffeomorphims, they are valid for diffeomorphisms of class C^{1+}, as observed in [4, Remark 2.3].

Let K be a positively invariant compact set for which there is a continuous splitting $T_K M = E^{cs} \oplus E^{cu}$ of the tangent bundle restricted to K which is Df-invariant over

$$\Lambda = \bigcap_{n\geq 0} f^n(K).$$

We fix continuous extensions of the two bundles E^{cs} and E^{cu} to some compact neighborhood U of Λ, that we still denote by E^{cs} and E^{cu}. Replacing K by a forward iterate of it, if necessary, we may assume that $K \subset U$.

Given $0 < a < 1$, we define the *centre-unstable cone field* $\left(C_a^{cu}(x)\right)_{x\in U}$ *of width* a by

$$C_a^{cu}(x) = \left\{v_1 + v_2 \in E_x^{cs} \oplus E_x^{cu} \text{ such that } \|v_1\| \leq a\|v_2\|\right\}. \tag{1}$$

We define the *centre-stable cone field* $\left(C_a^{cs}(x)\right)_{x\in U}$ *of width* a in a similar way, just reversing the roles of the subbundles in (1).

We fix $a > 0$ and U small enough so that, up to slightly increasing $\lambda < 1$, the domination condition remains valid for any pair of vectors in the two cone fields, i.e.

$$\|Df(x)v^{cs}\| \cdot \|Df^{-1}(f(x))v^{cu}\| \leq \lambda\|v^{cs}\|\,\|v^{cu}\|,$$

for every $v^{cs} \in C_a^{cs}(x)$, $v^{cu} \in C_a^{cu}(f(x))$, and any $x \in U \cap f^{-1}(U)$. Note that the centre-unstable cone field is positively invariant:

$$Df(x)C_a^{cu}(x) \subset C_a^{cu}(f(x)), \quad \text{whenever } x, f(x) \in U.$$

Indeed, the domination property together with the invariance of E^{cu} over Λ implies that

$$Df(x)C_a^{cu}(x) \subset C_{\lambda a}^{cu}(f(x)) \tag{2}$$

for every $x \in \Lambda$. This extends to any $x \in U \cap f^{-1}(U)$ just by continuity, slightly increasing $\lambda < 1$, if necessary.

Remark 3.1. The invariance of the splitting $T_K M = E^{cs} \oplus E^{cu}$ is used in [4] to derive conclusions for the points in the small neighborhood U of K. Though we are taking here the invariance of the splitting restricted to Λ, since we are assuming $K \subset U$, where U is a small neighborhood of Λ, the results of [4, Section 2.1] are still valid in our situation. See also [4, Remark 2.1].

We say that an embedded C^1 submanifold $N \subset U$ is *tangent to the centre-unstable cone field* if the tangent subspace to N at each point $x \in N$ is contained in the corresponding cone $C_a^{cu}(x)$. Then $f(N)$ is also tangent to the centre-unstable cone field, if it is contained in U, by the domination property.

We choose $\delta_0 > 0$ small enough so that the inverse of the exponential map $\exp_x$ is defined on the δ_0 neighborhood of every point x in U. From now on we identify this neighborhood of x with the corresponding neighborhood U_x of the origin in $T_x N$, through the local chart defined by $\exp_x^{-1}$.

Reducing δ_0, if necessary, we may suppose that E_x^{cs} is contained in the centre-stable cone $C_a^{cs}(y)$ of every $y \in U_x$. In particular, the intersection of $C_a^{cu}(y)$ with E_x^{cs} reduces to the zero vector. Then, the tangent space to N at y is parallel to the graph of a unique linear map $A_x(y) : T_x N \to E_x^{cs}$. Given constants $C > 0$ and $0 < \zeta \leq 1$, we say that *the tangent bundle to N is (C,ζ)-Hölder* if for every $y \in N \cap U_x$ and $x \in V_0$

$$\|A_x(y)\| \leq C d_x(y)^\zeta, \tag{3}$$

where $d_x(y)$ denotes the distance from x to y along $N \cap U_x$, defined as the length of the shortest curve connecting x to y inside $N \cap U_x$.

Recall that we have chosen the neighborhood U and the cone width a sufficiently small so that the domination property remains valid for vectors in the cones $C_a^{cs}(z)$, $C_a^{cu}(z)$, and for any point z in U. Then, there exist $\lambda_1 \in (\lambda, 1)$ and $\zeta \in (0, 1]$ such that

$$\|Df(z)v^{cs}\| \cdot \|Df^{-1}(f(z))v^{cu}\|^{1+\zeta} \le \lambda_1 < 1 \tag{4}$$

for every norm 1 vectors $v^{cs} \in C_a^{cs}(z)$ and $v^{cu} \in C_a^{cu}(z)$, at any $z \in U$. Then, up to reducing $\delta_0 > 0$ and slightly increasing $\lambda_1 < 1$, condition (4) remains true if we replace z by any $y \in U_x$, with $x \in U$ (taking $\|\cdot\|$ to mean the Riemannian metric in the corresponding local chart).

We fix ζ and λ_1 as above. Given a C^1 submanifold $N \subset U$, we define

$$\kappa(N) = \inf\{C > 0 : \text{the tangent bundle of } N \text{ is } (C, \zeta)\text{-Hölder}\}. \tag{5}$$

The next result appears in [4, Corollary 2.4].

Proposition 3.2. *There exists $C_1 > 0$ such that, given any C^1 submanifold $N \subset U$ tangent to the centre-unstable cone field,*

1. *there exists $n_0 \ge 1$ such that $\kappa(f^n(N)) \le C_1$ for every $n \ge n_0$ such that $f^k(N) \subset U$ for all $0 \le k \le n$;*
2. *if $\kappa(N) \le C_1$, then the same is true for every iterate $f^n(N)$ such that $f^k(N) \subset U$ for all $0 \le k \le n$;*
3. *in particular, if N and n are as in (2), then the functions*

$$J_k : f^k(N) \ni x \longmapsto \log|\det\big(Df \mid T_x f^k(N)\big)|, \quad 0 \le k \le n,$$

are (L, ζ)-Hölder continuous with $L > 0$ depending only on C_1 and f.

4. Hyperbolic times and bounded distortion

Let $K \subset M$ be a forward invariant compact set and let $\Lambda \subset K \subset U$ be as in Section 3. The following notion will allow us to derive *uniform behaviour* (expansion, distortion) from the nonuniform expansion.

Definition 4.1. Given $\sigma < 1$, we say that n is a *σ-hyperbolic time* for $x \in K$ if

$$\prod_{j=n-k+1}^{n} \|Df^{-1} \mid E^{cu}_{f^j(x)}\| \le \sigma^k, \quad \text{for all } 1 \le k \le n.$$

If $a > 0$ is taken sufficiently small in the definition of our cone fields, and we choose $\delta_1 > 0$ also small so that the δ_1-neighborhood of K should be contained in U, then by continuity

$$\|Df^{-1}(f(y))v\| \le \frac{1}{\sqrt{\sigma}}\|Df^{-1}|E^{cu}_{f(x)}\|\,\|v\|, \tag{6}$$

whenever $x \in K$, $\text{dist}(x, y) \le \delta_1$ and $v \in C_a^{cu}(f(y))$.

Given any disk $\Delta \subset M$, we use $\text{dist}_\Delta(x, y)$ to denote the distance between $x, y \in \Delta$ measured along Δ. The distance from a point $x \in \Delta$ to the boundary of Δ is $\text{dist}_\Delta(x, \partial\Delta) = \inf_{y \in \partial\Delta} \text{dist}_\Delta(x, y)$.

Lemma 4.2. *Take any C^1 disk $\Delta \subset U$ of radius δ, with $0 < \delta < \delta_1$, tangent to the centre-unstable cone field. There is $n_0 \geq 1$ such that for $x \in \Delta \cap K$ with $\mathrm{dist}_\Delta(x, \partial\Delta) \geq \delta/2$ and $n \geq n_0$ a σ-hyperbolic time for x, then there is a neighborhood V_n of x in Δ such that:*

(1) *f^n maps V_n diffeomorphically onto a disk of radius δ_1 around $f^n(x)$ tangent to the centre-unstable cone field;*
(2) *for every $1 \leq k \leq n$ and $y, z \in V_n$,*
$$\mathrm{dist}_{f^{n-k}(V_n)}(f^{n-k}(y), f^{n-k}(z)) \leq \sigma^{k/2}\,\mathrm{dist}_{f^n(V_n)}(f^n(y), f^n(z));$$
(3) *for every $1 \leq k \leq n$ and $y \in V_n$,*
$$\prod_{j=n-k+1}^{n} \|Df^{-1} \mid E^{cu}_{f^j(y)}\| \leq \sigma^{k/2}.$$

Proof. First we show that $f^n(\Delta)$ contains some disk of radius δ_1 around $f^n(x)$, as long as

$$n > 2\frac{\log(\delta/(2\delta_1))}{\log(\sigma)}. \tag{7}$$

Define Δ_1 as the connected component of $f(\Delta) \cap U$ containing $f(x)$. For $k \geq 1$, we inductively define $\Delta_{k+1} \subset f^{k+1}(\Delta)$ as the connected component of $f(\Delta_k) \cap U$ containing $f^{k+1}(x)$. We shall prove that Δ_n contains some disk of radius δ_1 around $f^n(x)$, for n as in (7). Observe that since $\Delta_j \subset U$, the invariance (2) gives that for every $j \geq 1$

$$T_w\Delta_j \subset C^{cu}_{\lambda^j a}(w), \quad \text{for every } w \in \Delta_j. \tag{8}$$

Let η_0 be a curve of minimal length in Δ_n connecting $f^n(x)$ to $f^n(y) \in \Delta_n$ for which $\mathrm{dist}_{\Delta_n}(f^n(x), f^n(y)) < \delta_1$. For $0 \leq k \leq n$, writing $\eta_k = f^{-k}(\eta_0)$ we have $\eta_k \subset \Delta_{n-k}$. We prove by induction that $\mathrm{length}(\eta_k) < \sigma^{k/2}\delta_1$, for $0 \leq k \leq n$. Let $1 \leq k \leq n$ and assume that

$$\mathrm{length}(\eta_j) < \sigma^{j/2}\delta_1, \quad \text{for } 0 \leq j \leq k-1.$$

Denote by $\dot\eta_0(w)$ the tangent vector to the curve η_0 at the point w. Using the fact that $\eta_k \subset \Delta_{n-k}$ and (8) we have

$$Df^{-j}(w)\dot\eta_0(w) \in C^{cu}_{\lambda^{n-j}a}(f^{-j}(w)) \subset C^{cu}_a(f^{-j}(w)).$$

Then, by the choice of δ_1 in (6) and the definition of σ-hyperbolic time,

$$\|Df^{-k}(w)\dot\eta_0(w)\| \leq \sigma^{-k/2}\,\|\dot\eta_0(w)\| \prod_{j=n-k+1}^{n} \|Df^{-1}|E^{cu}_{f^j(x)}\| \leq \sigma^{k/2}\|\dot\eta_0(w)\|.$$

Hence,

$$\mathrm{length}(\eta_k) \leq \sigma^{k/2}\,\mathrm{length}(\eta_0) < \sigma^{k/2}\delta_1\,.$$

This completes our induction.

In particular we have $\mathrm{length}(\eta_n) < \sigma^{n/2}\delta_1$. Moreover, the k preimage of the ball of radius δ_1 in Δ_n centered at $f^n(x)$ is contained in U for each $1 \leq k \leq n$. If η_n is a curve in Δ connecting x to $y \in \partial\Delta$, then we must have

$$n < 2\frac{\log(\delta/(2\delta_1))}{\log(\sigma)}.$$

Hence $f^n(\Delta)$ contains some disk of radius δ_1 around $f^n(x)$ for n as in (7).

Now let D_1 be the disk of radius δ_1 around $f^n(x)$ in $f^n(\Delta)$ and let $V_n = f^{-n}(D_1)$, for n as in (7). Take any $y, z \in V_n$ and let η_0 be a curve of minimal length in D_1 connecting $f^n(y)$ to $f^n(z)$. Defining $\eta_k = f^{-n+k}(\eta_0)$, for $1 \le k \le n$, and arguing as before, we inductively prove that for $1 \le k \le n$

$$\operatorname{length}(\eta_k) \le \sigma^{k/2} \operatorname{length}(\eta_0) = \sigma^{k/2} \operatorname{dist}_{f^n(V_n)}(f^n(y), f^n(z)),$$

which implies that for $1 \le k \le n$

$$\operatorname{dist}_{f^{n-k}(V_n)}(f^{n-k}(y), f^{n-k}(z)) \le \sigma^{k/2} \operatorname{dist}_{f^n(V_n)}(f^n(y), f^n(z)).$$

This completes the proof of the first two items of the lemma.

Given $y \in V_n$ we have $\operatorname{dist}(f^j(x), f^j(y)) \le \delta_1$ for every $1 \le j \le n$, which together with (6) gives

$$\prod_{j=n-k+1}^{n} \|Df^{-1} \mid E^{cu}_{f^j(y)}\| \le \sigma^{-k/2} \prod_{j=n-k+1}^{n} \|Df^{-1} \mid E^{cu}_{f^j(x)}\| \le \sigma^{k/2}.$$

Recall that $f^j(x) \in K$ for every j, and n is a σ-hyperbolic time for x. $\square$

We shall sometimes refer to the sets V_n as *hyperbolic pre-balls* and to their images $f^n(V_n)$ as *hyperbolic balls*. Notice that the latter are indeed balls of radius δ_1.

Corollary 4.3 (Bounded distortion)**.** *There exists $C_2 > 1$ such that given Δ as in Lemma 4.2 with $\kappa(\Delta) \le C_1$, and given any hyperbolic pre-ball $V_n \subset \Delta$ with $n \ge n_0$, then for all $y, z \in V_n$*

$$\frac{1}{C_2} \le \frac{|\det Df^n \mid T_y\Delta|}{|\det Df^n \mid T_z\Delta|} \le C_2.$$

Proof. For $0 \le i < n$ and $y \in \Delta$, we denote $J_i(y) = |\det Df \mid T_{f^i(y)} f^i(\Delta)|$. Then,

$$\log \frac{|\det Df^n \mid T_y\Delta|}{|\det Df^n \mid T_z\Delta|} = \sum_{i=0}^{n-1} \big(\log J_i(y) - \log J_i(z)\big).$$

By Proposition 3.2, $\log J_i$ is (L, ζ)-Hölder continuous, for some uniform constant $L > 0$. Moreover, by Lemma 4.2, the sum of all $\operatorname{dist}_{f^j(\Delta)}(f^j(y), f^j(z))^\zeta$ over $0 \le j \le n$ is bounded by $2\delta_1/(1 - \sigma^{\zeta/2})$. Then it suffices to take

$$C_2 = \exp(2\delta_1 L/(1 - \sigma^{\zeta/2})). \qquad \square$$

5. A local unstable disk inside Λ

Now we are able to prove Theorems A and C. These will be obtained as corollaries of the next slightly more general result, as we shall see. Take $K \subset M$ as a forward invariant compact set and let $\Lambda \subset K \subset U$ be as before.

Theorem 5.1. *Let $f : M \to M$ be a C^{1+} diffeomorphism and let $K \subset M$ be a forward invariant compact set with a continuous splitting $T_K M = E^{cs} \oplus E^{cu}$ dominated over $\Lambda = \bigcap_{n \ge 0} f^n(K)$. Assume that there is a disk Δ tangent to the centre-unstable cone field intersecting K in a positive* Leb_Δ *set of points where* NUE *holds. Then Λ contains some local unstable disk.*

Let us show that Theorem 5.1 implies Theorem A. Assume that NUE holds for Lebesgue almost every $x \in K$ with $\mathrm{Leb}(K) > 0$. Choosing a Leb density point of K, we laminate a neighborhood of that point into disks tangent to the centre-unstable cone field contained in U. Since the relative Lebesgue measure of the intersections of these disks with K cannot be all equal to zero, we obtain some disk Δ as in the assumption of Theorem 5.1.

For showing that Theorem 5.1 implies Theorem C, we just have to observe that local unstable manifolds are tangent to the centre-unstable subspaces and these vary continuously with the points in K, thus being tangent to the centre-unstable cone field.

In the remainder of this section we shall prove Theorem 5.1. Let $H \subset K$ be the set of points where NUE holds. Let Δ be a disk tangent to the centre-unstable cone field intersecting H in a positive Leb_Δ. Since NUE remains valid under positive iteration, by Proposition 3.2 we may assume that $\kappa(\Delta) < C_1$. It is no restriction to assume that H intersects the sub-disk of Δ of radius $\delta/2$, for some $0 < \delta < \delta_1$, in a positive Leb_Δ subset, and we do so.

The following lemma is due to Pliss [15], and a proof of it in this precise form can be found in [4, Lemma 3.1].

Lemma 5.2. *Given $A \geq c_2 > c_1 > 0$ there exists $\theta > 0$ such that for any real numbers $a_1, \ldots, a_N$ with $a_j \leq A$ and*

$$\sum_{j=1}^{N} a_j \geq c_2 N, \quad \text{for every } 1 \leq j \leq N,$$

there are $l > \theta N$ and $1 < n_1 < \cdots < n_l \leq N$ so that

$$\sum_{j=n+1}^{n_i} a_j \geq c_1(n_i - n), \quad \text{for every } 0 \leq n < n_i \text{ and } 1 \leq i \leq l.$$

Corollary 5.3. *There is $\sigma > 0$ such that every $x \in H$ has infinitely many σ-hyperbolic times.*

Proof. Given $x \in H$, by NUE we have infinitely many positive integers N for which

$$\sum_{j=1}^{N} \log \|Df^{-1}|E^{cu}_{f^j(x)}\| \leq -cN.$$

Then it suffices to take $c_1 = c/2$, $c_2 = c$, $A = \sup \big| \log \|Df^{-1}|E^{cu}\| \big|$, and $a_j = -\log \|Df^{-1}|E^{cu}_{f^j(x)}\|$ in the previous lemma. □

Note that under assumption NUE we are unable to prove the existence of a positive frequency of hyperbolic times at infinity, as in [4, Corollary 3.2]. This would be possible if we had taken lim sup instead of lim inf in the definition of NUE. The existence of infinitely many hyperbolic times is enough for what follows.

Lemma 5.4. *Let O be an open set in Δ such that* $\mathrm{Leb}_\Delta(O \cap H) > 0$. *Given any small $\rho > 0$ there is a hyperbolic time n, a hyperbolic pre-ball $V \subset O$ and $W \subset V$ such that $\Delta_n = f^n(W)$ is a disk of radius $\delta_1/4$ tangent to the centre-unstable cone field, and* $\mathrm{Leb}_{\Delta_n}(f^n(H)) \geq (1-\rho)\mathrm{Leb}_{\Delta_n}(\Delta_n)$.

Proof. Take a small number $\epsilon > 0$. Let C be a compact subset of $O \cap H$ and let A be an open neighborhood of $O \cap H$ in Δ such that

$$\mathrm{Leb}_\Delta(A \setminus C) < \epsilon \mathrm{Leb}_\Delta(C).$$

It follows from Corollary 5.3 and Lemma 4.2 that we can choose for each $x \in C$ a σ-hyperbolic time $n(x)$ and a hyperbolic pre-ball V_x such that $V_x \subset A$. Recall that V_x is the neighborhood of x which is mapped diffeomorphically by $f^{n(x)}$ onto a ball $B_{\delta_1}(f^{n(x)}(x))$ of radius δ_1 around $f^{n(x)}(x)$, tangent to the centre-unstable cone field. Let $W_x \subset V_x$ be the pre-image of the ball $B_{\delta_1/4}(f^{n(x)}(x))$ of radius $\delta_1/4$ under this diffeomorphism. By compactness there are $x_1, ..., x_m \in C$ such that $C \subset W_{x_1} \cup ... \cup W_{x_s}$. Writing

$$\{n_1, ..., n_s\} = \{n(x_1), ..., n(x_m)\}, \quad \text{with } n_1 < n_2 < ... < n_s, \tag{9}$$

let $I_1 \subset \mathbb{N}$ be a maximal set of $\{1, ..., m\}$ such that if $i \in I_1$ then $n(x_i) = n_1$ and $W_{x_i} \cap W_{x_j} = \emptyset$ for all $j \in I_1$ with $j \neq i$. Inductively we define I_k for $2 \leq k \leq s$ as follows: Supposing that I_{k-1} has already been defined, let $I_k \subset \mathbb{N}$ be a maximal set of $\{1, \ldots, m\}$ such that if $i \in I_k$, then $n(x_i) = n_k$ and $W_{x_i} \cap W_{x_j} = \emptyset$ for all $j \in I_k$ with $j \neq i$, and also $W_{x_i} \cap W_{x_j} = \emptyset$ for all $j \in I_1 \cup ... \cup I_{k-1}$.

Let $I = I_1 \cup \ldots \cup I_s$. By maximality, each W_{x_j}, for $1 \leq j \leq m$, intersects some W_{x_i} with $i \in I$ and $n(x_j) \geq n(x_i)$. Thus, given any $1 \leq j \leq m$ and taking $i \in I$ such that $W_{x_j} \cap W_{x_i} \neq \emptyset$ and $n(x_j) \geq n(x_i)$, we get

$$f^{n(x_i)}(W_{x_j}) \cap B_{\delta_1/4}(f^{n(x_i)}(x_i)) \neq \emptyset.$$

Lemma 4.2 assures us that

$$\operatorname{diam}(f^{n(x_i)}(W_{x_j})) \leq \frac{\delta_1}{2} \sigma^{(n(x_j)-n(x_i))/2} \leq \frac{\delta_1}{2},$$

and so

$$f^{n(x_i)}(W_{x_j}) \subset B_{\delta_1}(f^{n(x_i)}(x_i)).$$

This implies that $W_{x_j} \subset V_{x_i}$. Hence $\{V_{x_i}\}_{i \in I}$ is a covering of C. It follows from Corollary 4.3 that there is a uniform constant $\gamma > 0$ such that

$$\frac{\operatorname{Leb}_\Delta(W_{x_i})}{\operatorname{Leb}_\Delta(V_{x_i})} \geq \gamma, \quad \text{for every } i \in I.$$

Hence

$$\begin{aligned}
\operatorname{Leb}_\Delta \Big(\bigcup_{i \in I} W_{x_i} \Big) &= \sum_{i \in I} \operatorname{Leb}_\Delta(W_{x_i}) \\
&\geq \sum_{i \in I} \gamma \operatorname{Leb}_\Delta(V_{x_i}) \\
&\geq \gamma \operatorname{Leb}_\Delta \Big(\bigcup_{i \in I} V_{x_i} \Big) \\
&\geq \gamma \operatorname{Leb}_\Delta(C).
\end{aligned}$$

Setting

$$\rho = \min \left\{ \frac{\operatorname{Leb}_\Delta(W_{x_i} \setminus C)}{\operatorname{Leb}_\Delta(W_{x_i})} : i \in I \right\},$$

we have

$$\begin{aligned}
\varepsilon\,\mathrm{Leb}_\Delta(C) &\geq \mathrm{Leb}_\Delta(A\setminus C)\\
&\geq \mathrm{Leb}_\Delta\big(\bigcup_{i\in I} W_{x_i}\setminus C\big)\\
&\geq \sum_{i\in I}\mathrm{Leb}_\Delta(W_{x_i}\setminus C)\\
&\geq \rho\,\mathrm{Leb}_\Delta\big(\bigcup_{i\in I} W_{x_i}\big)\\
&\geq \rho\gamma\,\mathrm{Leb}_\Delta(C).
\end{aligned}$$

This implies that $\rho<\varepsilon/\gamma$. Since $\varepsilon>0$ can be taken arbitrarily small, we may choose W_{x_i} with the relative Lebesgue measure of C in W_{x_i} arbitrarily close to 1. Then, by bounded distortion, the relative Lebesgue measure of $f^{n(x_i)}(H)\supset f^{n(x_i)}(C)$ in $f^{n(x_i)}(W_{x_i})$, which is a disk of radius $\delta_1/4$ around $f^{n(x_i)}(x_i)$ tangent to the centre-unstable cone field, is also arbitrarily close to 1. Observe that since points in H have infinitely many σ-hyperbolic times, we may take the integer $n(x_i)$ arbitrarily large, as long as n_1 in (9) is also taken large enough. $\square$

Proposition 5.5. *There are a sequence of sets $W_1\supset W_2\supset\cdots$ and a sequence of positive integers $n_1\leq n_2\leq\cdots$ such that:*

1. *W_k is contained in some hyperbolic pre-ball with hyperbolic time n_k;*
2. *$\Delta_k=f^{n_k}(W_k)$ is a disk of radius $\delta_1/4$, centered at some point x_k, tangent to the centre-unstable cone field;*
3. *$f^{n_k}(W_{k+1})$ is contained in the disk of radius $\delta_1/8$ centered at x_k;*
4. $\displaystyle\lim_{k\to\infty}\frac{\mathrm{Leb}_{\Delta_k}(f^{n_k}(H))}{\mathrm{Leb}_{\Delta_k}(\Delta_k)}=1.$

Proof. Take a constant $0<\rho<1$ such that for any disk D of radius $\delta_1/4$ centered at some point x tangent to the centre-unstable cone field the following holds: *if* $\mathrm{Leb}_D(A)\geq(1-\rho)\mathrm{Leb}_D(D)$ *for some $A\subset D$, then we must have* $\mathrm{Leb}_{D^*}(A)>0$, *where $D^*\subset D$ is the disk of radius $\delta_1/8$ centered at the same point x.* Note that it is possible to make a choice of ρ in these conditions only depending on the radius of the disk and the dimension of the disk. Surely, once we have chosen some ρ satisfying the required property, then any smaller number still has that property.

We shall use Lemma 5.4 successively in order to define the sequence of sets $(W_k)_k$ and hyperbolic times $(n_k)_k$ inductively. Let us start with $O=\Delta$ and $0<\rho<1$ with the property above. By Lemma 5.4 there are $n_1\geq 1$ and $W_1\subset V_1\subset O$, where V_1 is a hyperbolic pre-ball with hyperbolic time n_1, such that $\Delta_1=f^{n_1}(W_1)$ is a disk of radius $\delta_1/4$ centered at some point x_1, tangent to the centre-unstable cone field, with

$$\frac{\mathrm{Leb}_{\Delta_1}(f^{n_1}(H))}{\mathrm{Leb}_{\Delta_1}(\Delta_1)}\geq 1-\rho.$$

Considering $\Delta_1^*\subset\Delta_1$ the disk of radius $\delta_1/8$ centered at x_1, then by the choice of ρ we have $\mathrm{Leb}_{\Delta_1^*}(f^{n_1}(H))>0$. Let $O_1\subset W_1$ be the part of W_1 which is sent by f^{n_1} diffeomorphically onto Δ_1^*. We have $\mathrm{Leb}_\Delta(O_1\cap H)>0$.

Next we apply Lemma 5.4 to $O = O_1$ and $\rho/2$ in the place of ρ. Then we find a hyperbolic time n_2 and $W_2 \subset O_1$ such that $\Delta_2 = f^{n_2}(W_2)$ satisfies

$$\frac{\mathrm{Leb}_{\Delta_2}(f^{n_2}(H))}{\mathrm{Leb}_{\Delta_2}(\Delta_2)} \geq 1 - \frac{\rho}{2}.$$

Observe that $W_2 \subset O_1 \subset W_1$. Then we take $O_2 \subset W_2$ as that part of W_2 which is sent by f^{n_2} diffeomorphically onto the disk Δ_2^* of radius $\delta_1/8$ and proceed inductively. □

The next proposition gives the conclusion of Theorem 5.1.

Proposition 5.6. *The sequence $(\Delta_k)_k$ has a subsequence converging to a local unstable disk Δ_∞ of radius $\delta_1/4$ inside Λ.*

Proof. Let $(\Delta_k)_k$ be the sequence of disks given by Proposition 5.5 and let $(x_k)_k$ be the sequence of points at which these disks are centered. Up to taking subsequences, we may assume that the centers of the disks converge to some point x. Using Ascoli-Arzela, a subsequence of the disks converges to some disk Δ_∞ centered at x. We necessarily have $\Delta_\infty \subset \Lambda$.

Note that each Δ_k is contained in the n_k-iterate of Δ, which is a disk tangent to the centre-unstable cone field. The domination property implies that the angle between Δ_k and E^{cu} goes uniformly to 0 as $n \to \infty$. In particular, Δ_∞ is tangent to E^{cu} at every point in $\Delta_\infty \subset \Lambda$. By Lemma 4.2, given any $n \geq 1$, then f^{-n} is a $\sigma^{n/2}$-contraction on Δ_k for every large k. Passing to the limit, we get that f^{-n} is a $\sigma^{n/2}$-contraction in the E^{cu} direction over Δ_∞ for every $n \geq 1$. The fact that the Df-invariant splitting $T_\Lambda M = E^{cs} \oplus E^{cu}$ is dominated implies that any expansion Df may exhibit along the complementary direction E^{cs} is weaker than the expansion in the E^{cu} direction. Then there exists a unique unstable manifold $W^u_{\mathrm{loc}}(x)$ tangent to E^{cu} and which is contracted by the negative iterates of f; see [14]. Since Δ_∞ is contracted by every f^{-n}, and all its negative iterates are tangent to the centre-unstable cone field, then Δ_∞ is contained in $W^u_{\mathrm{loc}}(x)$. □

6. Existence of hyperbolic periodic points

Here we prove Theorem E. By Proposition 5.5 there exist a sequence of sets $W_1 \supset W_2 \supset \cdots$ contained in Δ and a sequence of positive integers $n_1 \leq n_2 \leq \cdots$ such that:

(1) W_k is contained in some hyperbolic pre-ball with hyperbolic time n_k;
(2) $\Delta_k = f^{n_k}(W_k)$ is a disk of radius $\delta_1/4$, centered at some point x_k, tangent to the centre-unstable cone field;
(3) $f^{n_k}(W_{k+1})$ is contained in the disk Δ_k^* of radius $\delta_1/8$ centered at x_k.

Taking a subsequence, if necessary, we have by Proposition 5.6 that the sequence of disks $(\Delta_k)_k$ accumulates on a local unstable disk Δ_∞ of radius $\delta_1/4$ which is contained in Λ. Our aim now is to prove that Λ contains the unstable manifold of some periodic point.

Similarly to (6), we choose $\delta > 0$ small so that $W^s_\delta(z)$ is defined for every $z \in \Lambda$, the 2δ-neighborhood of Λ is contained in U, and

$$\|Df^{-1}(f(y))v\| \leq \sigma^{-1/4} \|Df^{-1}|E^{cu}_{f(x)}\| \, \|v\|, \tag{10}$$

whenever $x \in U$, $\mathrm{dist}(x, y) \leq 2\delta$, and $v \in C^{cu}_a(y)$.

Proposition 6.1. *Given $\Lambda_1 \subset \Lambda$ with $\mathrm{Leb}(\Lambda_1) > 0$, there exist a hyperbolic periodic point $p \in \Lambda$ and $\delta_2 > 0$ (not depending on p) such that:*

1. $\overline{W^u(p)} \subset \Lambda$;
2. *the size of $W^u_{loc}(p)$ is at least δ_2;*
3. $\mathrm{Leb}_{W^u_{loc}(p)}$ *almost every point in $W^u_{loc}(p)$ belongs to H;*
4. *there is $x \in \Lambda_1$ with $\omega(x) \subset \overline{W^u(p)}$.*

Proof. Let x denote the center of the accumulation disk Δ_∞. Let us consider the cylinder

$$\mathcal{C}_\delta = \bigcup_{y \in \Delta_\infty} W^s_\delta(y),$$

and the projection along local stable manifolds

$$\pi\colon \mathcal{C}_\delta \longrightarrow \Delta_\infty.$$

Slightly diminishing the radius of the disk Δ_∞, if necessary, we may assume that there is a positive integer k_0 such that for every $k \geq k_0$

$$\pi(\Delta_k \cap \mathcal{C}_\delta) = \Delta_\infty \quad \text{and} \quad \Delta^*_k \subset \mathcal{C}_\delta. \tag{11}$$

For each $k \geq k_0$ let

$$\pi_k\colon \Delta_\infty \longrightarrow \Delta_k$$

be the projection along the local stable manifolds. Notice that these projections are continuous and $\pi \circ \pi_k = \mathrm{id}_{\Delta_\infty}$. Take a positive integer $k_1 > k_0$ sufficiently large so that

$$\pi(\Delta_{k_1} \cap \mathcal{C}_{\delta/2}) = \Delta_\infty \quad \text{and} \quad \lambda^{n_{k_1} - n_{k_0}} \leq \frac{1}{4}. \tag{12}$$

We have

$$\Delta_{k_1} = f^{n_{k_1}}(W_{k_1}) \subset f^{n_{k_1}-n_{k_0}}\left(f^{n_{k_0}}(W_{k_0+1})\right) \subset f^{n_{k_1}-n_{k_0}}(\Delta^*_{k_0}),$$

which together with (11) and (12) implies that there is some disk $\Delta_0 \subset \Delta_\infty$ such that

$$\pi \circ f^{n_{k_1}-n_{k_0}} \circ \pi_{k_0}(\Delta_0) = \Delta_\infty.$$

Thus there must be some $z \in \Delta_0 \subset \Delta_\infty$ which is a fixed point for the continuous map $\pi \circ f^{n_{k_1}-n_{k_0}} \circ \pi_{k_0}$. This means that there are $z_{k_0}, z_{k_1} \in W^s_\delta(z)$ with $z_{k_0} \in \Delta_{k_0}$ and $z_{k_1} \in \Delta_{k_1}$ such that $f^{n_{k_1}-n_{k_0}}(z_{k_0}) = z_{k_1}$. Letting $\gamma = W^s_\delta(z)$, we have $\mathrm{dist}_\gamma(w, z_{k_1}) \leq 2\delta$ for every $w \in \gamma$. This implies that

$$\mathrm{dist}_\gamma(f^{n_{k_1}-n_{k_0}}(w), z_{k_1}) = \mathrm{dist}_\gamma(f^{n_{k_1}-n_{k_0}}(w), f^{n_{k_1}-n_{k_0}}(z_{k_0})) \leq 2\delta\lambda^{n_{k_1}-n_{k_0}},$$

which together with (12) gives

$$\mathrm{dist}_\gamma(f^{n_{k_1}-n_{k_0}}(w), z) \leq \mathrm{dist}_\gamma(f^{n_{k_1}-n_{k_0}}(w), z_{k_1}) + \mathrm{dist}_\gamma(z_{k_1}, z) \leq \delta.$$

We conclude that $f^{n_{k_1}-n_{k_0}}(W^s_\delta(z)) \subset W^s_\delta(z)$. Since $W^s_\delta(z)$ is a topological disk, this implies that $W^s_\delta(z)$ must necessarily contain some periodic point p of period $m = n_{k_1} - n_{k_0}$. As $z \in \Delta_\infty$ and $p \in W^s_\delta(z)$, it follows that $p \in \Lambda$, by closeness of Λ.

Let us now prove that p is a hyperbolic point. As $p \in W^s_\delta(z)$, it is enough to show that $\|Df^{-m} \mid E^{cu}_{f^m(p)}\| < 1$. Let $q = W^s_\delta(z) \cap f^{n_{k_0}}(W_{k_1})$. Observe that since $p \in \Lambda \cap W^s_\delta(z)$, then q belongs to the 2δ-neighborhood of Λ, which is contained in U. Since W_{k_1} is contained in some hyperbolic pre-ball with hyperbolic time n_{k_1}, it follows from Lemma 4.2 that for every $1 \leq j \leq n_{k_1}$ and $y \in W_{k_1}$,

$$\|Df^{-j} \mid E^{cu}_{f^{n_{k_1}}(y)}\| \leq \sigma^{j/2}.$$

In particular, taking $j = m = n_{k_1} - n_{k_0}$ and $y = f^{-n_{k_0}}(q)$, we have

$$\|Df^{-m} \mid E^{cu}_{f^m(q)}\| \le \sigma^{m/2}.$$

The choice of δ in (10) together with the fact that $p, q \in W^s_\delta(z)$ implies that

$$\|Df^{-m} \mid E^{cu}_{f^m(p)}\| \le \prod_{j=1}^{m} \|Df^{-1} \mid E^{cu}_{f^j(p)}\| \tag{13}$$

$$\le \sigma^{-m/4} \prod_{j=1}^{m} \|Df^{-1} \mid E^{cu}_{f^j(q)}\|$$

$$\le \sigma^{m/4}. \tag{14}$$

Thus we have proved the hyperbolicity of p.

Now, since p is a hyperbolic periodic point, there is $W^u_{\mathrm{loc}}(p)$ a local unstable manifold through p tangent to the centre-unstable bundle. As Δ_∞ transversely cuts the local stable manifold through p, then using the λ-lemma we deduce that the positive iterates of Δ_∞ accumulate on the unstable manifold through p. Since these iterates are all contained in Λ and Λ is a closed set, we must have $W^u(p) \subset \Lambda$, which then implies that $\overline{W^u(p)} \subset \Lambda$. Thus we have proved the first part of the result.

By (13) and (14) we deduce that every multiple of m is a $\sigma^{1/4}$-hyperbolic time for p. Then we choose $\delta_2 > 0$ such that an inequality as in (6) holds with δ_2 in the place of δ_1 and $\sigma^{1/8}$ in the place of $\sigma^{1/2}$. Using Lemma 4.2 with $W^u_{\mathrm{loc}}(p)$ in the place of Δ and taking a sufficiently large $\sigma^{1/4}$-hyperbolic time for p we deduce that there is a hyperbolic pre-ball inside $W^u_{\mathrm{loc}}(p)$. This imples that its image by the hyperbolic time, which is a disk of radius δ_2 around p, is contained in the local unstable manifold of p. This gives the second part of the result.

Observe that as long as we take the local unstable manifold through p small enough, then every point in $W^u_{\mathrm{loc}}(p)$ belongs to the local stable manifold of some point in Δ_∞. By construction, Δ_∞ is accumulated by the disks $\Delta_k = f^{n_k}(W_k)$ which, by Proposition 5.5, satisfy

$$\lim_{k\to\infty} \frac{\mathrm{Leb}_{\Delta_k}(f^{n_k}(H))}{\mathrm{Leb}_{\Delta_k}(\Delta_k)} = 1. \tag{15}$$

Since H is positively invariant, we have

$$\lim_{k\to\infty} \frac{\mathrm{Leb}_{\Delta_k}(H)}{\mathrm{Leb}_{\Delta_k}(\Delta_k)} = 1.$$

Now let $\varphi\colon \Lambda \to \mathbb{R}$ be the continuous function given by

$$\varphi(x) = \log \|Df^{-1} \mid E^{cu}_x\|.$$

Since Birkhoff's time averages are constant for points in a same local stable manifold and the local stable foliation is absolutely continuous, we deduce that

$$\frac{\mathrm{Leb}_{\Delta_\infty}(H)}{\mathrm{Leb}_{\Delta_\infty}(\Delta_\infty)} = 1.$$

The same conclusion holds for the local unstable manifold of p in the place of Δ_∞ by the same reason.

Now let us prove the last item. Since H has full Lebesgue measure in Λ and $\Lambda_1 \subset \Lambda$ has positive Lebesgue measure, we may start our construction with the set $H_1 = H \cap \Lambda_1$ in the place of H intersecting the disk Δ in a positive Leb_Δ measure set of points. Although we do not have invariance of H_1, by (15) we still have the property that the iterates of $H_1 \subset \Lambda_1$ accumulate on the whole Δ_∞. Since the stable manifolds through points in $W^u_{\text{loc}}(p)$ intersect Δ_∞, there must be points in Λ_1 accumulating on $W^u_{\text{loc}}(p)$. $\square$

Let p_1 be a hyperbolic periodic point as in Proposition 6.1. Let B_1 be the basin of $\overline{W^u(p_1)}$, i.e. the set of points x whose ω-limit is contained in $\overline{W^u(p_1)}$. If $\text{Leb}(\Lambda \setminus B_1) = 0$, then we have proved the theorem. Otherwise, let $\Lambda_1 = \Lambda \setminus B_1$. Using again Proposition 6.1 we obtain a point $p_2 \in \Lambda$ such that the basin B_2 of $\overline{W^u(p_2)}$ attracts some point of Λ_1. By definition of Λ_1 we must have $\overline{W^u(p_1)} \neq \overline{W^u(p_2)}$.

We proceed inductively, thus obtaining periodic points $p_1, \dots, p_n \in \Lambda$ with $\overline{W^u(p_i)} \neq \overline{W^u(p_j)}$ for every $i \neq j$. This process must stop after a finite number of steps. Actually, if there were an infinite sequence of points as above, by compactness, choosing p_{i_1}, p_{i_2} sufficiently close, using the inclination lemma we would get $\overline{W^u(p_{i_1})} = \overline{W^u(p_{i_2})}$.

So far we have proved the first two items of Theorem E. Assume now that E^{cu} has dimension one. We want to show that each $\overline{W^u(p_i)}$ attracts an open set containing $\overline{W^u(p_i)}$. Given $1 \leq i \leq k$, by Proposition 6.1 we can find at least one point on each connected component of $W^u(p_i) \setminus \{p_i\}$ belonging to H. Since these points have infinitely many hyperbolic times, then each connected component of $W^u(p_i) \setminus \{p_i\}$ must necessarily have infinite arc length; recall Lemma 4.2. This implies that each point $x \in W^u(p_i)$ has an unstable arc $\gamma^u(x) \subset W^u(p_i)$ of a fixed length passing through it. Let

$$B(x) = \bigcup_{y \in \gamma^u(x)} W^s_\delta(y).$$

By domination, the angles of $\gamma^u(x)$ and the local stable manifolds $W^s_\delta(y)$ with $y \in \gamma^u(x)$ are uniformly bounded away from zero. Thus, $B(x)$ must contain some ball of uniform radius (not depending on x), and so the set $\bigcup_{x \in W^u(p_i)} B(x)$ is a neighborhood of $\overline{W^u(p_i)}$. Since, for each $x \in W^u(p_i)$, the points in $B(x)$ have their ω-limit set contained in $\overline{W^u(p_i)}$, we are done.

7. Hyperbolic sets with positive volume

In this section we prove Theorem F and Theorem G. Since in the present situation $Df|E^u_x$ is uniformly expanding, then we have NUE for every $x \in \Lambda$.

7.1. **Transitive case.** Assume first that Λ has positive volume. It follows from Corollary B that Λ must contain some local unstable disk. The first item of Theorem F is a consequence of the following folklore lemma whose proof we give here for the sake of completeness.

Lemma 7.1. *If Λ is a transitive hyperbolic set containing the local unstable manifold of some point, then Λ contains the local unstable manifolds of all its points.*

Proof. Take $\delta > 0$ small such that $W^s_\delta(x)$ and $W^u_\delta(y)$ intersect at most in one point, for every $x, y \in \Lambda$, and assume that $W^u_\delta(x_0) \subset \Lambda$ for some $x_0 \in \Lambda$. Let $z \in \Lambda$ be a point with dense orbit in Λ. It is no restriction to assume that $W^s_\delta(z)$ intersects $W^u_\delta(x_0)$, and we do so. Let $x_1 = W^s_\delta(z) \cap W^u_\delta(x_0)$. We also have $W^u_\delta(x_1) \subset \Lambda$. Given any point $y \in \Lambda$, we take a sequence of integers $0 = n_1 < n_2 < \cdots$ such that $f^{n_k}(z) \to y$, when $k \to \infty$. Since $x_1 \in W^s(z)$, we also have $x_k := f^{n_k}(x_1) \to y$, when $k \to \infty$. The local unstable manifolds through the points $x_1, x_2, \ldots$ are necessarily contained in Λ and accumulate on a disk $D(y)$ contained in Λ and containing y. Since the local unstable disks are tangent to the unstable spaces, the continuity of these spaces implies that $T_w D(y) = E^u_w$ for every $w \in D(y)$. By uniqueness of the unstable foliation, we must have $D(y)$ coinciding with the local unstable manifold through y. □

Using the previous lemma applied to f^{-1}, we have that Λ must also contain the stable manifolds through its points. Then we easily deduce that every point in Λ belongs in the interior of Λ, thus showing that Λ is an open set. Since Λ is assumed to be closed, we conclude that $\Lambda = M$, thus having proved the first part of Theorem F.

Lemma 7.2. *Let Λ be a hyperbolic set attracting a set with positive volume. Then there is a point in Λ whose local unstable manifold is contained in Λ.*

Proof. We fix continuous extensions (not necessarily continuous) of the two bundles E^{cs} and E^{cu} to some neighborhood U of Λ. Let A be the set of points which are attracted to Λ under positive iteration. Since A has positive volume, there must be some compact set $C \subset A$ with positive volume, and some $N \in \mathbb{N}$ such that $f^n(C) \subset U$ for every $n \geq N$. Letting

$$K = \bigcup_{n \geq N} f^n(C) \cup \Lambda,$$

we have that K is a compact forward invariant set with positive volume for which

$$\Lambda = \bigcap_{n \geq 1} f^n(K).$$

The conclusion of the lemma then follows from Theorem E. □

The second part of Theorem F can now be easily deduced from Lemma 7.1 and Lemma 7.2. Actually, it follows from the lemmas that

$$\bigcup_{x \in \Lambda} W^s_\delta(x)$$

is a neighborhood of Λ whose points are attracted to Λ under positive iteration.

7.2. **Nontransitive case.** Here we consider the case of hyperbolic sets with positive volume not necessarily transitive and prove Theorem G.

Let $\Sigma = \overline{W^u(p)} \subset \Lambda$, where p is a hyperbolic periodic point given by Proposition 6.1. We claim that Σ contains the local unstable manifolds of all its points. Indeed, if $x \in \Sigma$, then there is a sequence $(x_n)_n$ of points in $W^u(p)$ converging to x. The continuous variation of the local unstable manifolds gives that the local unstable manifolds of the points x_n, which are contained in Σ, accumulate on the local unstable manifold of x. By closeness, the local unstable manifold of x must be contained in Σ. Thus, defining

$$A = \bigcup_{x \in \Sigma} W^s_\delta(x),$$

we have that A is a neighborhood of Σ whose points have their ω-limit set contained in Σ. Since Σ is a hyperbolic set with a local product structure attracting an open neighborhood of itself, then by [9, Theorem 18.3.1] there are hyperbolic invariant sets $\Omega_1, \dots, \Omega_s \subset \Sigma \subset \Lambda$ verifying (3)-(6) of Theorem G. Moreover, their union is the set of nonwandering points of f in Σ,

$$NW(f|\Sigma) = \Omega_1 \cup \dots \cup \Omega_s.$$

Since $L(f|\Sigma) \subset NW(f|\Sigma)$, this implies that $\omega(x) \subset \Omega_1 \cup \dots \cup \Omega_s$ for every $x \in A$. Recall that every point in A belongs to the stable manifold of some point in Σ. Now since $\Omega_1, \dots, \Omega_s$ are disjoint compact invariant sets, given $x \in A$, we must even have $\omega(x) \subset \Omega_i$ for some $1 \le i \le s$. Reordering these sets if necessary, let $\Omega_1, \dots, \Omega_q$, for some $q \le s$, be those which attract a set with positive Lebesgue measure. By Theorem F and transitivity, each $\Omega_1, \dots, \Omega_q$ attracts a neighborhood of itself.

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