TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 360, Number 10, October 2008, Pages 5201–5227
S 0002-9947(08)04514-5
Article electronically published on May 2, 2008

NECESSARY AND SUFFICIENT CONDITIONS FOR OPTIMALITY OF NONCONVEX, NONCOERCIVE AUTONOMOUS VARIATIONAL PROBLEMS WITH CONSTRAINTS

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ABSTRACT. We consider the classical autonomous constrained variational problem of minimization of $\int_a^b f(v(t), v'(t))dt$ in the class $\Omega := \{v \in W^{1,1}(a,b) : v(a) = \alpha, v(b) = \beta, v'(t) \geq 0$ a.e. in $(a,b)\}$, where $f : [\alpha,\beta] \times [0,+\infty) \to \mathbb{R}$ is a lower semicontinuous, nonnegative integrand, which can be nonsmooth, nonconvex and noncoercive.

We prove a necessary and sufficient condition for the optimality of a trajectory $v_0 \in \Omega$ in the form of a DuBois-Reymond inclusion involving the subdifferential of Convex Analysis. Moreover, we also provide a relaxation result and necessary and sufficient conditions for the existence of the minimum expressed in terms of an upper limitation for the assigned mean slope $\xi_0 = (\beta-\alpha)/(b-a)$. Applications to various noncoercive variational problems are also included.

1. Introduction

Let us consider the classical autonomous one-dimensional Lagrange problem

$$\text{(P)} \qquad \text{minimize } F(v) := \int_a^b f(v(t), v'(t))\ dt\ , \quad v \in \Omega$$

where $\Omega := \{v \in W^{1,1}(a,b) : v(a) = \alpha, v(b) = \beta, v'(t) \geq 0$ a.e. in $(a,b)\}$ and $f : [\alpha,\beta] \times [0,+\infty) \to \mathbb{R}$ is a lower semicontinuous, nonnegative integrand. Hence, f can be nonsmooth, nonconvex and noncoercive.

As is well known, the lack of convexity and coercivity does not allow the use of the classical direct methods of the Calculus of Variations and this problem can have no solution.

Nonconvex problems have been widely studied in the literature (see the survey [18] for a rather exhaustive bibliography) and even recently some papers appeared in which existence results, related to the solvability of the relaxed problem, have been established for free autonomous nonconvex problems. In particular, in [11] and [20], Fusco, Marcellini and Ornelas proved the solvability of nonconvex but coercive autonomous integrals of sum type, under the assumption that the convex

Received by the editors November 25, 2004 and, in revised form, June 30, 2006.
2000 *Mathematics Subject Classification.* Primary 49J30, 49J52, 49K30.
Key words and phrases. Constrained variational problems, autonomous Lagrangians, nonsmooth analysis, noncoercive problems, nonconvex problems, necessary and sufficient conditions, DuBois-Reymond condition.

envelope coincides with the integrand at the origin. We quote also the recent result [5] for nonconvex autonomous multiple integrals.

In [4], Celada and Perrotta studied scalar nonconvex integrands satisfying a growth condition slightly weaker than superlinearity, which roughly speaking requires that the infimum of the values at the origin of the support affine functions to f^{**} is $-\infty$, uniformly with respect to the state variable in compact sets; i.e.,

$$\lim_{|z|\to+\infty} \sup_{|s|\le R} \sup(f^{**}(s,z) - z\partial f^{**}(s,z)) = -\infty \quad \text{for every } R \ge 0 \tag{1}$$

where $\partial f^{**}(s,z)$ denotes the subdifferential of $f^{**}(s,\cdot)$ at z. Under this assumption they proved some conditions guaranteeing the solvability of problem (P) related to the solvability of (P^{**}) (see also [7] and [10] for similar results).

In [6], Cellina considered a pointwise version of condition (1) (see assumption (GA)), by avoiding the requirement of uniformity, and proved relaxation results and the Lipschitz continuity of minimizers in the vectorial case.

As observed in these papers, condition (1) is weaker than superlinearity (for instance the map $(s,z) \mapsto |z| - \sqrt{|z|}$ satisfies it). However, it does not hold for integrands having the structure $f(s,z) = a(s)h(z)$, with h superlinear but $a(s)$ assuming the value zero somewhere, or having an asymptotic straight line as $|z| \to +\infty$, as in Brachistochrone and Fermat's principle problems, which nevertheless admit the optimal trajectory.

The case of linear growth at infinity can instead be handled by means of the existence result proved for convex integrands in more general contexts by Clarke in [9], but nevertheless even this result is not applicable to integrands of the type $f(s,z) = a(s)h(z)$ with h superlinear but $\min a(s) = 0$.

In some classical examples of noncoercive problems, as the problem of the surface of revolution of minimal area, it is well known that the existence of the minimum is related to the relative position of the prescribed endpoints. This suggests that also in general a sufficient condition for the existence of the minimum could be obtained in terms of the boundary data (a,α), (b,β). Indeed, in [1], Botteron and Dacorogna achieved existence and nonexistence results for constrained functionals with nonautonomous integrand having the sum-type structure $f(t,s,z) = \phi(t,z) + \psi(t,s)$ just relatively to the prescribed mean slope $\xi_0 = \frac{\beta-\alpha}{b-a}$. They showed that under some technical assumptions on functions ϕ and ψ, if ξ_0 is sufficiently small the minimum exists, while if it is too large the minimum does not exist. A similar investigation was carried out in [14]–[16] for integrands not depending on the state variable, obtaining a necessary and sufficient condition for the existence of the minimum expressed in term of a limitation on the assigned slope ξ_0. The relevance of the boundary conditions and other specific parameters of the problem was also discussed by B. Mordukhovich in [19], where *individual existence theorems* were presented in the framework of optimal control problems.

Herein we investigate autonomous constrained problems, without a specific structure, under very mild assumptions on the integrand (lower semicontinuity and boundedness from below). The first aim of this paper is to analyze necessary conditions for the optimality of a given admissible trajectory in Ω for problem (P). In particular, we prove a nonsmooth version of the DuBois-Reymond necessary condition, expressed in terms of an inclusion involving the subdifferential of Convex Analysis, which turns out to be both necessary and sufficient for the optimality of a trajectory in Ω. Such a result is obtained by introducing a suitable Bolza problem

($\tilde{\mathrm{P}}$), whose integrand does not depend on the state variable, which is equivalent to problem (P) (see Theorem 4). The Euler-Lagrange inclusion for ($\tilde{\mathrm{P}}$), which is both necessary and sufficient, corresponds to the following DuBois-Reymond inclusion for problem (P):

$$f(v_0(t), v_0'(t)) - c \in v_0'(t)\partial f(v_0(t), v_0'(t)) \quad \text{a.e. } t$$

for some constant $c \leq \min_{s\in[\alpha,\beta]} f(s,0)$, which is necessary and sufficient for the optimality of v_0 (see Theorem 7 and Remark 3). We underline that in this condition $\partial f(s_0, z_0)$ denotes the subdifferential of $f(s_0, \cdot)$ at z_0, in the sense of Convex Analysis, even if the integrand f is nonconvex in general.

This result allows us to obtain a relaxation theorem for problem (P), stating that the minimum exists if and only if the relaxed problem (P^{**}) admits the minimum, attained at a trajectory whose gradient lies in the convex envelope of the contact set $C_{v_0(t)} = \{z > 0 : f(v_0(t), z) = f^{**}(v_0(t), z)\}$ (see Theorem 8).

The second part of the paper is devoted to the study of the existence of the minimum for (P^{**}), without any growth assumption. We prove a necessary and sufficient condition for the existence of the minimum expressed in terms of an upper limitation for the assigned slope $\xi_0 = \frac{\beta-\alpha}{b-a}$ (see Theorems 12 and 13). Results pertaining to the Lipschitz continuity or strict monotonicity of minimizers are also included.

We observe that when condition (GA) in [6] holds, then no limitation occurs for the slope ξ_0, but in some cases the existence of the minimum is guaranteed for every value of ξ_0 also when condition (GA) does not hold. For instance, in the case of product type integrands $f(s,z) = a(s)h(z)$, with $a \in C[\alpha,\beta]$ positive almost everywhere, $h \in C^1[0,+\infty)$ convex but not affine, denoted by $\ell := \lim_{z\to+\infty}(h(z) - zh'(z))$, the infimum of the value at the origin of the affine support functions of h, we obtain (see Corollary 16):

- If $\ell = -\infty$, then the minimum exists for every positive slope ξ_0. Moreover, if $m := \min_{s\in[\alpha,\beta]} a(s) > 0$, the minimizers are Lipschitz continuous.
- If $\ell = 0$ and $m > 0$, then the minimum exists for every positive slope ξ_0 and the minimizers are Lipschitz continuous.
- If $\ell = 0$ and $m = 0$, then the minimum does not exist for any positive slope ξ_0.
- If $\ell > -\infty, \ell \neq 0$, then there exists a positive threshold value ξ^* (explicitly defined), such that the minimum exists if and only if $\xi_0 \leq \xi^*$.

As an application, in the case of integrands $f(v(t), v'(t)) = a(v(t))\sqrt{1 + (v'(t))^2}$, we have $\ell = 0$. So, if $m = \min_{s\in[\alpha,\beta]} a(s) > 0$ the minimum exists for every slope $\xi_0 > 0$ with Lipschitz continuous minimizer. Otherwise, if $m = 0$ the minimum does not exist for any slope ξ_0. Note that in this case the growth condition (GA) considered in [6] is not satisfied. For a more detailed discussion, see Examples 1, 2.

Moreover, in the case of integrands $f(v(t), v'(t)) = a(v(t))h(v'(t))$ with h having superlinear growth, but $\min a(s) = 0$, the minimum exists for every slope $\xi_0 > 0$, even if with non-Lipschitz minimizer. We wish to point out that in this situation none of the quoted existence results seems applicable, since neither condition (GA) nor the condition considered in [9] are satisfied (see Example 5 and Remark 8 for a detailed discussion).

The paper is organized as follows: in Section 2 we introduce the equivalent Bolza problem and discuss the relation with (P), and in Section 3 we prove the necessary and sufficient condition for optimality expressed as a DuBois-Reymond inclusion. In Section 4 we prove a relaxation result and in Section 5 we analyze the existence of the minimum for (P^{**}), in relation to the value of the slope ξ_0, and discuss some examples. Section 6 contains the proofs of all the lemmas. Finally, in the Appendix we prove an extension of the classical Lyapunov theorem to possible nonsummable functions.

2. An equivalent Bolza problem

In this section we associate an equivalent Bolza problem, whose integrand does not depend on the state variable, to problem (P). To this aim, put

$$\tilde{\Omega} := \{u \in W^{1,1}(\alpha,\beta) : \ u(\alpha) = a, \ u(\beta) \le b, \ u'(\tau) > 0 \text{ a.e. in } (\alpha,\beta)\}.$$

Let us consider the map $\chi : \Omega \to \tilde{\Omega}$, defined by $v \mapsto \chi_v$ where

$$\chi_v(\tau) := a + \int_\alpha^\tau w_v'(\sigma)\, d\sigma \quad \text{and} \quad w_v(\tau) := \min\{t \in [a,b] : v(t) = \tau\} \tag{2}$$

and the map $\Psi : \tilde{\Omega} \to \Omega$, defined by $u \mapsto \Psi_u$ where

$$\Psi_u(t) := \begin{cases} u^{-1}(t) & \text{if } a \le t \le u(s^*), \\ s^* & \text{if } u(s^*) \le t \le u(s^*) + b - u(\beta), \\ u^{-1}(t - b + u(\beta)) & \text{if } u(s^*) + b - u(\beta) \le t \le b, \end{cases} \tag{3}$$

and $s^* := \min\{\tau \in [\alpha,\beta] : f(\tau,0) = \min_{s\in[\alpha,\beta]} f(s,0)\}$.

The next preliminary results concern the properties of the function w_v and the maps χ, Ψ.

Lemma 1. *Let $v \in \Omega$. Then*

i) *for every $\tau_0 \in (\alpha,\beta]$ there exists an increasing sequence $(t_n)_n$ converging to $w_v(\tau_0)$, such that $t_n = w_v(v(t_n))$, $n \in \mathbb{N}$;*
ii) *w_v is continuous at τ_0 if and only if $v^{-1}(\tau_0)$ is a singleton;*
iii) *if there exists $v'(t_0) > 0$, then there exists also $w_v'(v(t_0)) = \frac{1}{v'(t_0)}$.*

Lemma 2. *The maps χ and Ψ are well defined. Moreover, for every $u \in \tilde{\Omega}$ we have $\chi(\Psi(u)) = u$, and $\chi_v(\beta) = b$ if and only if $v'(t) > 0$ for a.e. $t \in [a,b]$.*

Remark 1. The previous lemma asserts that the map Ψ is a partial inverse of the map χ, which is not injective, since in general $v \ne \Psi(\chi(v))$. However, since $u = \chi(\Psi(u))$ for every $u \in \tilde{\Omega}$, the map χ is surjective.

Observe that the functions $\chi_v(\tau)$ and $\Psi_u(t)$ can be viewed as time parameterizations. A similar argument has been used in a different context, to deduce the Pontryagin Maximum Principle for strong local optimality from the analogous necessary condition for weak local optimality. This method is known as the Dubovitskii-Milyutin time transformation (see [12] or [13]). In the present context, the transformations $\tau = \Psi_u(t)$ and $t = w_v(\tau)$ have the same characteristics of the Dubovitskii-Milyutin ones, in particular w_v can be discontinuous. Nevertheless, the novelty here consists in the introduction of the regularized parametrization $t = \chi_v(\tau)$, which will allow us to obtain equivalent minimization problems (see Lemma 3 and Theorem 4 below). Indeed, in the derivation of the Pontryagin Maximum Principle, which is only a necessary condition, the regularity of one transformation was sufficient. Instead, our goal is the statement of necessary and sufficient conditions, and so we also need the regularity of an inverse (in some sense) transformation.

Let us now consider the functional $\tilde{F} : \tilde{\Omega} \to [0, +\infty]$ defined by

$$\tilde{F}(u) := \int_\alpha^\beta f\left(\tau, \frac{1}{u'(\tau)}\right) u'(\tau)\ d\tau \ + \ \mu(b - u(\beta))$$

where $\mu := f(s^*, 0) = \min_{s \in [\alpha,\beta]} f(s, 0)$. The following result concerns the behavior of the functionals F and $\tilde{F}$ with respect to the maps χ and Ψ.

Lemma 3. *We have*

$$F(\Psi(\chi(v))) = \tilde{F}(\chi(v)) \le F(v) \qquad \text{for every } v \in \Omega. \tag{4}$$

Moreover, $F(v) = \tilde{F}(\chi(v))$ *if and only if* $f(v(t), 0) = \mu$ *for a.e.* $t \in (a, b)$ *such that* $v'(t) = 0$.

Let us now consider the Bolza problem

$$(\tilde{\mathrm{P}}) \qquad \text{minimize} \left\{ \int_\alpha^\beta \tilde{f}(\tau, u'(\tau))\ d\tau \ + \mu(b - u(\beta)) \right\}, \quad u \in \tilde{\Omega}$$

where $\tilde{f} : [\alpha, \beta] \times (0, +\infty) \to \mathbb{R}$ is defined by

$$\tilde{f}(\tau, z) := f\left(\tau, \frac{1}{z}\right) z.$$

The above problem turns to be equivalent to (P) in the sense stated by the following theorem.

Theorem 4. *If* $v \in \Omega$ *is a minimizer for problem* (P)*, then* $\chi(v) \in \tilde{\Omega}$ *is a minimizer for problem* $(\tilde{\mathrm{P}})$*. Vice versa, if* $u \in \tilde{\Omega}$ *is a minimizer for problem* $(\tilde{\mathrm{P}})$*, then* $\Psi(u) \in \Omega$ *is a minimizer for problem* (P).

Proof. Assume that $v_0 \in \Omega$ is a minimizer for problem (P). By (4), $\Psi(\chi(v_0))$ is then a minimizer for (P). Hence, for every $u \in \tilde{\Omega}$, again by (4) we get

$$\tilde{F}(u) = F(\Psi(u)) \ge F(\Psi(\chi(v_0)) = \tilde{F}(\chi(v_0));$$

that is, $\chi(v_0)$ is a minimizer for problem $(\tilde{\mathrm{P}})$.

Vice versa, if u_0 is a minimizer for problem $(\tilde{\mathrm{P}})$, then for every $v \in \Omega$ we get

$$F(v) \ge \tilde{F}(\chi(v)) \ge \tilde{F}(u_0) = F(\Psi(u_0));$$

that is, $\Psi(u_0)$ is a minimizer for problem (P). □

3. Necessary and sufficient conditions for optimality

By using the equivalent Bolza problem, in this section we provide a necessary and sufficient condition for the optimality of a trajectory $v \in \Omega$, expressed in terms of a DuBois-Reymond inclusion, involving the subdifferential of Convex Analysis. This result will be achieved as a consequence of an analogous necessary and sufficient condition for optimality regarding the equivalent Bolza problem $(\tilde{\mathrm{P}})$, expressed in terms of an Euler-Lagrange condition.

Given a function $h:(0,+\infty)\to\mathbb{R}$, in what follows we will adopt the following notation:

(5)
$$h^-(z) := \sup_{0<\zeta<z} \frac{h(\zeta)-h(z)}{\zeta - z}\ ,\ h^+(z) := \inf_{\zeta>z} \frac{h(\zeta)-h(z)}{\zeta - z}\ ,\ \partial h(z) := [h^-(z), h^+(z)]$$

with the convention $\partial h(z) = \emptyset$ if $h^-(z) > h^+(z)$.

It is immediate to verify that $\partial h(z)$ is the subdifferential of h, restricted to $(0,+\infty)$, in the sense of Convex Analysis; that is,

$$\partial h(z) = \{c \in \mathbb{R} : h(\zeta) - h(z) \geq c(\zeta - z) \text{ for every } \zeta > 0\}.$$

Finally, let h^{**} denote the convex envelope of h (with respect to the family of convex functions defined in $(0,+\infty)$ which are less than or equal to h).

The following lemma analyzes the main properties of the transformation $\tilde{}$ which associates to the function h the function $\tilde{h}:(0,+\infty)\to\mathbb{R}$ defined by $\tilde{h}(z) := h(1/z)z$.

Lemma 5. *For every $h:(0,+\infty)\to\mathbb{R}$ we have*

i) $\tilde{\tilde{h}} = h$;
ii) $\tilde{h}$ *is convex in* (c,d) *if and only if* h *is convex in* $(\frac{1}{d},\frac{1}{c})$;
iii) $\tilde{h}$ *is affine in* (c,d) *if and only if* h *is affine in* $(\frac{1}{d},\frac{1}{c})$;
iv) $(\tilde{h})^{**} = (\widetilde{h^{**}})$;
v) $\tilde{h}^-(z) = h(1/z) - \frac{1}{z}\ h^+(1/z)$; $\quad \tilde{h}^+(z) = h(1/z) - \frac{1}{z}\ h^-(1/z)$;
vi) $\partial\tilde{h}(z) = h(1/z) - \frac{1}{z}\ \partial h(1/z)$.

Theorem 6. *A function $u_0 \in \tilde{\Omega}$ is a minimizer for problem $(\tilde{P})$ if and only if the following Euler-Lagrange condition (EL) (expressed according to the value of $u_0(\beta)$) holds:*

$(\mathrm{EL})_1$ *(when $u_0(\beta) = b$):* $\operatorname*{ess\,sup}_{\tau\in[\alpha,\beta]} \tilde{f}^-(\tau, u_0'(\tau)) \leq \min\{\mu, \operatorname*{ess\,inf}_{\tau\in[\alpha,\beta]} \tilde{f}^+(\tau, u_0'(\tau))\}$,

$(\mathrm{EL})_2$ *(when $u_0(\beta) < b$):* $\operatorname*{ess\,sup}_{\tau\in[\alpha,\beta]} \tilde{f}^-(\tau, u_0'(\tau)) \leq \mu \leq \operatorname*{ess\,inf}_{\tau\in[\alpha,\beta]} \tilde{f}^+(\tau, u_0'(\tau))$

where $\tilde{f}^-(\tau,\cdot)$, $\tilde{f}^+(\tau,\cdot)$ have the same meaning as in (5).

Proof. (Necessity). Let $u_0 \in \tilde{\Omega}$ be a minimizer for $(\tilde{\mathrm{P}})$. First consider the case when $u_0(\beta) = b$. Of course, u_0 is a minimizer for the functional $\tilde{F}$ also in the subclass of $\tilde{\Omega}$ consisting of those trajectories satisfying $u(\beta) = b$. Hence, by applying [16, Theorem 3.2] concerning the Euler-Lagrange inclusion for variational problems with open constraints (in our case $u_0'(\tau) \in (0,+\infty)$ for a.e. $\tau \in (\alpha,\beta)$), we have that there exists a constant $\bar{c} \in \mathbb{R}$ such that $\bar{c} \in \partial\tilde{f}(\tau, u_0'(\tau))$ for a.e. $\tau \in (\alpha,\beta)$. So, put

$$C := [\operatorname*{ess\,sup}_{\tau\in[\alpha,\beta]} \tilde{f}^-(\tau, u_0'(\tau)), \operatorname*{ess\,inf}_{\tau\in[\alpha,\beta]} \tilde{f}^+(\tau, u_0'(\tau))].$$

We have that $\bar{c} \in C$, so C is a compact, nonempty interval. Our goal is to show that $\mu \geq \operatorname*{ess\,sup}_{\tau\in[\alpha,\beta]} \tilde{f}^-(\tau, u_0'(\tau))$.

Assume the contrary. Then, $\mu < \tilde{f}^-(\tau, u_0'(\tau))$ in a subset $U \subset (\alpha, \beta)$ having positive measure. Thus, the multifunction defined by

$$\Xi(\tau) := \{\xi < u_0'(\tau) \ : \ \tilde{f}(\tau, \xi) - \tilde{f}(\tau, u_0'(\tau)) < \mu(\xi - u_0'(\tau))\}$$

takes measurable, nonempty values. Let $\xi(\tau)$ be a measurable selection of $\Xi(\tau)$ and put

$$\eta(\tau) := \begin{cases} \xi(\tau) & \text{for } \tau \in U \\ u_0'(\tau) & \text{for } \tau \in (\alpha, \beta) \setminus U \end{cases} \qquad \text{and} \qquad u(\tau) := a + \int_\alpha^\tau \eta(\sigma)\, d\sigma.$$

Of course, $u \in W^{1,1}(\alpha, \beta)$ with $u(\alpha) = a$ and

$$u(\beta) = a + \int_\alpha^\beta \eta(\tau) d\tau = a + \int_\alpha^\beta u_0'(\tau) d\tau + \int_U [\xi(\tau) - u_0'(\tau)] d\tau < u_0(\beta) = b;$$

that is, $u \in \tilde{\Omega}$. Moreover,

$$\begin{aligned} \tilde{F}(u) &= \int_U \tilde{f}(\tau, \xi'(\tau)) d\tau + \int_{(\alpha,\beta)\setminus U} \tilde{f}(\tau, u_0'(\tau)) d\tau + \mu(b - u(\beta)) \\ &< \int_\alpha^\beta \tilde{f}(\tau, u_0'(\tau)) d\tau + \mu \int_U [\xi(\tau) - u_0'(\tau)] d\tau + \mu(b - u(\beta)) \\ &= \int_\alpha^\beta \tilde{f}(\tau, u_0'(\tau))\, d\tau + \mu \int_\alpha^\beta [\eta(\tau) - u_0'(\tau)] d\tau + \mu(b - u(\beta)) = \tilde{F}(u_0), \end{aligned}$$

a contradiction and condition $(\mathrm{EL})_1$ is proved.

Let us consider now the case $u_0(\beta) < b$. Assume by contradiction that the set $S := \{\tau : \mu \notin \partial \tilde{f}(\tau, u_0'(\tau))\}$ has positive measure. For every $\tau \in S$ define

$$Q(\tau) := \{q \in \mathbb{R}^+ : \tilde{f}(\tau, q) - \tilde{f}(\tau, u_0'(\tau)) < \mu(q - u_0'(\tau))\}$$

and let $q(\tau)$ be a measurable selection of the multifunction Q. Let $E \subset S$ be a set having positive measure such that

$$\int_E [q(\tau) - u_0'(\tau)]\, d\tau < b - u_0(\beta) \tag{6}$$

and put

$$\gamma(\tau) := \begin{cases} q(\tau) & \text{for } \tau \in E \\ u_0'(\tau) & \text{for } \tau \in (\alpha, \beta) \setminus E \end{cases}, \qquad u(\tau) := a + \int_\alpha^\tau \gamma(\sigma)\, d\sigma.$$

By (6) we have $u(\beta) < b$; hence $u \in \tilde{\Omega}$ and

$$\begin{aligned} \tilde{F}(u) &= \int_E \tilde{f}(\tau, q(\tau)) d\tau + \int_{(\alpha,\beta)\setminus E} \tilde{f}(\tau, u_0'(\tau)) d\tau + \mu(b - u(\beta)) \\ &< \int_\alpha^\beta \tilde{f}(\tau, u_0'(\tau)) d\tau + \mu \int_E [q(\tau) - u_0'(\tau)] d\tau + \mu(b - u(\beta)) \\ &= \int_\alpha^\beta \tilde{f}(\tau, u_0'(\tau))\, d\tau + \mu \int_\alpha^\beta [\gamma(\tau) - u_0'(\tau)] d\tau + \mu(b - u(\beta)) = \tilde{F}(u_0), \end{aligned}$$

a contradiction.

(*Sufficiency*). Fix $u_0 \in \tilde{\Omega}$, with $u_0(\beta) = b$. By condition $(\mathrm{EL})_1$, putting $c := \operatorname*{ess\,sup}_{\tau\in[\alpha,\beta]} \tilde{f}^-(\tau, u_0'(\tau))$, we have $c \le \mu$ and for every $u \in \tilde{\Omega}$ we get

$$\begin{aligned}\tilde{F}(u) - \tilde{F}(u_0) &= \int_\alpha^\beta [\tilde{f}(\tau, u'(\tau)) - \tilde{f}(\tau, u_0'(\tau))]\, d\tau + \mu(b - u(\beta)) \\ &\ge \int_\alpha^\beta c[u'(\tau) - u_0'(\tau)]d\tau + \mu(b - u(\beta)) = (\mu - c)(b - u(\beta)) \ge 0;\end{aligned}$$

i.e., u_0 is a minimizer for problem $(\tilde{\mathrm{P}})$.

Instead, if condition $(\mathrm{EL})_2$ holds, then $\mu \in \partial f(\tau, u_0'(\tau))$ a.e. in $\tau \in (\alpha, \beta)$ and for every $u \in \tilde{\Omega}$ we get

$$\begin{aligned}\tilde{F}(u) - \tilde{F}(u_0) &= \int_\alpha^\beta [\tilde{f}(\tau, u'(\tau)) - \tilde{f}(\tau, u_0'(\tau))]\, d\tau + \mu(u_0(\beta) - u(\beta)) \\ &\ge \mu \int_\alpha^\beta (u'(\tau) - u_0'(\tau))d\tau + \mu(u_0(\beta) - u(\beta)) = 0;\end{aligned}$$

i.e., u_0 is a minimizer for problem $(\tilde{P})$. □

Remark 2. Note that condition (EL) could be written in the following equivalent more usual form:

$$\begin{aligned}&(\mathrm{EL})_1 \ : \ \text{there exists} \ \ c \le \mu \ : \ c \in \partial\tilde{f}(\tau, u_0'(\tau)) \text{ a.e. in } (\alpha, \beta), \\ &(\mathrm{EL})_2 \ : \ \mu \in \partial\tilde{f}(\tau, u_0'(\tau)) \quad \text{a.e. in } (\alpha, \beta).\end{aligned}$$

So the main novelty here is a limitation from above of the constant c in the case $u_0(\beta) = b$ and the exact determination of $c = \mu$ in the case $u_0(\beta) < b$.

From now on, we will adopt the following notation for every given trajectory $v \in \Omega$:

$$A_v := \{t : \text{ there exists } v'(t) > 0\}, \ B_v := \{t : \text{ there exists } v'(t) = 0\}. \tag{7}$$

Observe that by the absolute continuity of v, we have $|v(A_v)| = \beta - \alpha$ and

$$|E| = 0 \ \Leftrightarrow |v^{-1}(E) \cap A_v| = 0 \quad \text{ for every set } E \subset [\alpha, \beta]. \tag{8}$$

The main result of this section is the following.

Theorem 7. *A function* $v_0 \in \Omega$ *is a minimizer for problem* (P) *if and only if* $\partial f(v_0(t), v_0'(t)) \neq \emptyset$ *for a.e.* $t \in A_{v_0}$, *and the following DuBois-Reymond condition* (DBR) (*expressed according to the measure* $|B_{v_0}|$) *holds:*

$(\mathrm{DBR})_1$ (*when* $|B_{v_0}| = 0$) :

$$\begin{aligned}&\operatorname*{ess\,sup}_{t\in[a,b]} \{f(v_0(t), v_0'(t)) - v_0'(t) f^+(v_0(t), v_0'(t))\} \\ &\quad \le \ \min\{\mu, \operatorname*{ess\,inf}_{t\in[a,b]} \{f(v_0(t), v_0'(t)) - v_0'(t) f^-(v_0(t), v_0'(t))\}\},\end{aligned}$$

$(\mathrm{DBR})_2$ (*when* $|B_{v_0}| > 0$) : $f(v_0(t), 0) = \mu$ *for a.e.* $t \in B_{v_0}$ *and*

$$\begin{aligned}&\operatorname*{ess\,sup}_{t\in A_{v_0}} \{f(v_0(t), v_0'(t)) - v_0'(t) f^+(v_0(t), v_0'(t))\} \\ &\quad \le \mu \le \operatorname*{ess\,inf}_{t\in A_{v_0}} \{f(v_0(t), v_0'(t)) - v_0'(t) f^-(v_0(t), v_0'(t))\}\end{aligned}$$

where $f^-(v_0(t), \cdot)$, $f^+(v_0(t), \cdot)$ *have the same meaning as in* (5).

Proof. (Necessity). Let v_0 be a minimizer for problem (P). By applying Theorems 4 and 6 we get

$$\operatorname*{ess\,sup}_{\tau\in[\alpha,\beta]} \tilde f^-(\tau,\chi'_{v_0}(\tau)) \;\le\; \min\{\mu,\ \operatorname*{ess\,inf}_{\tau\in[\alpha,\beta]} \tilde f^+(\tau,\chi'_{v_0}(\tau))\};$$

that is, by (2) and property *v)* of Lemma 5,

$$\begin{aligned}&\operatorname*{ess\,sup}_{\tau\in[\alpha,\beta]} \{f(\tau,\tfrac{1}{w'_{v_0}(\tau)})-\tfrac{1}{w'_{v_0}(\tau)}f^+(\tau,\tfrac{1}{w'_{v_0}(\tau)})\}\\ &\le \min\{\ \mu, \operatorname*{ess\,inf}_{\tau\in[\alpha,\beta]} \{f(\tau,\tfrac{1}{w'_{v_0}(\tau)})-\tfrac{1}{w'_{v_0}(\tau)}f^-(\tau,\tfrac{1}{w'_{v_0}(\tau)})\}\}.\end{aligned}\tag{9}$$

Let $E:=\{\tau\in v_0(A_{v_0}):\ (9)\text{ holds}\}$ and put $A':=v_0^{-1}(E)$. By (8) we have $|A_{v_0}\setminus A'|=0$ and then for a.e. $t\in A_{v_0}$ we have $v_0(t)\in E$ and $v_0'(t)=\frac{1}{w'_{v_0}(v_0(t))}$ by property *iii)* of Lemma 1. So, we deduce

$$\begin{aligned}&\operatorname*{ess\,sup}_{t\in A_{v_0}} \{f(v_0(t),v_0'(t))-v_0'(t)f^+(v_0(t),v_0'(t))\}\\ &\quad\le\ \min\{\mu,\operatorname*{ess\,inf}_{t\in A_{v_0}} \{f(v_0(t),v_0'(t))-v_0'(t)f^-(v_0(t),v_0'(t))\}\}.\end{aligned}$$

Therefore, if $|B_{v_0}|=0$, the assertion is proved. Instead, if $|B_{v_0}|>0$, by Lemma 2 we get $\chi_{v_0}(\beta)<b$, so by applying the Euler-Lagrange condition $(\mathrm{EL})_2$ we obtain

$$\begin{aligned}&\operatorname*{ess\,sup}_{t\in A_{v_0}} \{f(v_0(t),v_0'(t))-v_0'(t)f^+(v_0(t),v_0'(t))\}\\ &\quad\le\mu\le \operatorname*{ess\,inf}_{t\in A_{v_0}} \{f(v_0(t),v_0'(t))-v_0'(t)f^-(v_0(t),v_0'(t))\}.\end{aligned}$$

Moreover, by Lemma 3 we have $f(v_0(t),v_0'(t))=\mu$ for a.e. $t\in B_{v_0}$, implying the assertion.

(*Sufficiency*). Let us fix a function $v_0\in\Omega$ satisfying condition $(\mathrm{DBR})_1$ (with $|B_{v_0}|=0$). Hence, $|v_0(A_{v_0})|=\beta-\alpha$ and by (2) and property *iii)* of Lemma 1 we achieve

$$\begin{aligned}&\operatorname*{ess\,sup}_{\tau\in[\alpha,\beta]} \{f(\tau,\frac{1}{\chi'_{v_0}(\tau)})-\frac{1}{\chi'_{v_0}(\tau)}f^+(\tau,\frac{1}{\chi'_{v_0}(\tau)})\}\\ &\quad\le\ \min\{\mu,\ \operatorname*{ess\,inf}_{\tau\in[\alpha,\beta]} \{f(\tau,\frac{1}{\chi'_{v_0}(\tau)})-\frac{1}{\chi'_{v_0}(\tau)}f^-(\tau,\frac{1}{\chi'_{v_0}(\tau)})\}\};\end{aligned}$$

hence by property v) of Lemma 5 we deduce

$$\operatorname*{ess\,sup}_{\tau\in[\alpha,\beta]} \tilde f^-(\tau,\chi'_{v_0}(\tau)) \;\le\; \min\{\mu,\operatorname*{ess\,inf}_{\tau\in[\alpha,\beta]} \tilde f^+(\tau,\chi'_{v_0}(\tau))\}.$$

Then the function $\chi(v_0)$ satisfies condition $(\mathrm{EL})_1$ and by virtue of Theorem 6 we deduce that $\chi(v_0)$ is a minimizer for problem $(\tilde{\mathrm{P}})$. Now, Theorem 4 guarantees that $\Psi(\chi(v_0))$ is a minimizer for problem (P). Since $|B_{v_0}|=0$, by applying Lemma 3 we get $F(v_0)=\tilde F(\chi(v_0))=F(\Psi(\chi(v_0)))$ and this implies that v_0 is a minimizer for problem (P).

Assume now that $v_0\in\Omega$ satisfies condition $(\mathrm{DBR})_2$ (with $|B_{v_0}|>0$). By Lemma 2 we get $\chi_{v_0}(\beta)<b$ and since $|v_0(A_{v_0})|=\beta-\alpha$, similarly to what done above we have

$$\operatorname*{ess\,sup}_{\tau\in[\alpha,\beta]} \tilde f^-(\tau,\chi'_{v_0}(\tau)) \;\le\; \mu\ \le \operatorname*{ess\,inf}_{\tau\in[\alpha,\beta]} \tilde f^+(\tau,\chi'_{v_0}(\tau)).$$

Therefore, the function $\chi(v_0)$ satisfies condition $(\mathrm{EL})_2$ and by virtue of Theorem 6 we deduce that $\chi(v_0)$ is a minimizer for problem $(\tilde{\mathrm{P}})$. Finally, since $f(v_0(t),0)=\mu$ for a.e. $t\in B_{v_0}$, by Lemma 3 we deduce that $F(v_0)=\tilde{F}(\chi(v_0))=F(\Psi(\chi(v_0)))$, implying that v_0 is a minimizer for problem (P). □

Remark 3. Similarly to what was observed in Remark 2, note that condition (DBR) in the previous result can be written in the following more usual form:

$(\mathrm{DBR})_1:$ there exists $c\le\mu$: $f(v_0(t),v_0'(t))-c\in v_0'(t)\ \partial f(v_0(t),v_0'(t))$ a.e. in (a,b),

$(\mathrm{DBR})_2:$ $f(v_0(t),v_0'(t))-\mu\in v_0'(t)\ \partial f(v_0(t),v_0'(t))$ a.e. in A_{v_0}.

We note the introduction of the limitation from above of the constant c in the first condition and its exact determination in the second one. Moreover, we underline that the subdifferential appearing in conditions (DBR) is in the sense of Convex Analysis, even if $f(s,\cdot)$ is nonconvex in general.

Remark 4. Notice that all the results stated in this section still hold if f is only assumed to be Borel-measurable, provided that $f(\cdot,0)$ is lower semicontinuous.

4. Relaxation

This section is devoted to discussing relaxation results for problem (P). To this purpose, for every $s\in[\alpha,\beta]$ let $f^{**}(s,\cdot)$ be the convex envelope of $f(s,\cdot)$, in $[0,+\infty)$. From now on let F^{**} and (P^{**}) respectively denote the functional F with the integrand f^{**} and the minimization problem (P) related to the functional F^{**}.

In what follows let C_s denote the contact set in the open half-line $(0,+\infty)$,

$$C_s:=\{z>0\ :\ f(s,z)=f^{**}(s,z)\},$$

and let $Bd(C_s)$ denote the boundary of C_s.

Theorem 8. *Assume that $f(s,\cdot)$ is continuous at the origin, for every $s\in[\alpha,\beta]$. Then, problem* (P) *admits the minimum if and only if problem* (P^{**}) *admits a minimizer $v_0\in\Omega$ such that*

$$v_0'(t)\in co(C_{v_0(t)})\qquad \text{for a.e. } t\in A_{v_0} \tag{10}$$

where A_{v_0} was defined by (7) *and $co(C_{v_0(t)})$ denotes the convex envelope of the set $C_{v_0(t)}$.*

Proof. First note that by the continuity of $f(s,\cdot)$ at the origin, $f^{**}(s,\cdot)$ (the convex envelope in the closed half-line $[0,+\infty)$) coincides with the convex envelope in the open half-line $(0,+\infty)$, is continuous on $[0,+\infty)$ and $f^{**}(s,0)=f(s,0)$ for every $s\in[\alpha,\beta]$.

(Necessity). Let v_0 be a minimizer for problem (P). First assume that $|B_{v_0}|=0$; then by Lemma 2 we get $\chi_{v_0}(\beta)=b$. Moreover, by Theorem 4, $\chi(v_0)$ is a minimizer for problem $(\tilde{\mathrm{P}})$ and satisfies condition $(\mathrm{EL})_1$ by virtue of Theorem 6. Hence, $\partial\tilde{f}(\tau,\chi_{v_0}'(\tau))$ is nonempty and, recalling property *iv)* of Lemma 5, this implies that

$$\tilde{f}(\tau,\chi_{v_0}'(\tau))=(\tilde{f})^{**}(\tau,\chi_{v_0}'(\tau))=(\widetilde{f^{**}})(\tau,\chi_{v_0}'(\tau))\quad \text{for a.e. } \tau\in(\alpha,\beta).$$

Therefore, since $|B_v|=0$, by (8) we get

$$f(v_0(t),v_0'(t))=f^{**}(v_0(t),v_0'(t))\quad \text{for a.e. } t\in(a,b), \tag{11}$$

which implies that $\partial f(v_0(t), v_0'(t)) = \partial f^{**}(v_0(t), v_0'(t))$ a.e. in (a,b). Consequently, by virtue of condition $(\mathrm{DBR})_1$ of Theorem 7, from (11) we deduce that

$$f^{**}(v_0(t), v_0'(t)) - c \in v_0'(t)\partial f^{**}(v_0(t), v_0'(t)) \quad \text{a.e. in } (a,b)$$

for some constant $c \le \mu$, implying that v_0 is a minimizer for problem (P^{**}). Moreover, by (11) it follows that $v_0'(t) \in C_{v_0(t)} \subset co(C_{v_0(t)})$ for a.e. $t \in (a,b)$.

Instead, when $|B_v| > 0$, all the above discussion holds for $c = \mu$, a.e. in A_{v_0}. Moreover, note that $f^{**}(v_0(t), 0) = f(v_0(t), 0) = \mu$ for a.e. $t \in B_{v_0}$; hence v_0 satisfies condition $(\mathrm{DBR})_2$ relative to function f^{**}, implying that v_0 is a minimizer for problem (P^{**}).

(*Sufficiency*). If v_0 is a minimizer for problem (P^{**}), then χ_{v_0} is a minimizer for problem $(\widetilde{\mathrm{P}^{**}})$. Moreover, by (10) we infer the existence of positive measurable functions $\xi_1, \xi_2 : A_{v_0} \to \mathbb{R}$, with $\xi_i(t) \in C_{v_0(t)}$ and $\xi_1(t) \le v_0'(t) \le \xi_2(t)$ a.e. in A_{v_0}, such that $\xi_i(t) = v_0'(t)$ whenever $v_0'(t) \in C_{v_0(t)}$ and $\xi_i(t) \in Bd(C_{v_0(t)})$ otherwise. Moreover, $f^{**}(v_0(t), \cdot)$ is affine in $[\xi_1(t), \xi_2(t)]$. Hence, by property *iv)* of Lemma 5 we get

$$(\tilde{f})^{**}(v_0(t), \frac{1}{\xi_i(t)}) = (\widetilde{f^{**}})(v_0(t), \frac{1}{\xi_i(t)}) = \tilde{f}(v_0(t), \frac{1}{\xi_i(t)}), \quad \text{a.e. in } A_{v_0}, i = 1,2.$$

Note that for every $\tau \in v_0(A_{v_0})$ there exists a unique $t := v_0^{-1}(\tau) \in A_{v_0}$ such that $v_0(t) = \tau$. So, putting $\phi_i(\tau) := \frac{1}{\xi_i(v_0^{-1}(\tau))}$ for every $\tau \in v_0(A_{v_0})$, $i = 1,2$, since $|v_0(A_{v_0})| = \beta - \alpha$, we obtain

$$(\tilde{f})^{**}(\tau, \phi_i(\tau)) = \tilde{f}(\tau, \phi_i(\tau)), \quad \text{a.e. in } (\alpha, \beta), i = 1,2. \tag{12}$$

Moreover, $\phi_2(\tau) \le \chi_{v_0}'(\tau) \le \phi_1(\tau)$. Then there exist measurable weight functions $\lambda_1(\tau)$, $\lambda_2(\tau) : [\alpha, \beta] \to [0,1]$ such that $\lambda_1(\tau) + \lambda_2(\tau) = 1$ and

$$\lambda_1(\tau)\ \phi_1(\tau) + \lambda_2(\tau)\ \phi_2(\tau) = \chi_{v_0}'(\tau), \quad \text{a.e. in } (\alpha, \beta). \tag{13}$$

From property *iii)* of Lemma 5, it follows that $(\tilde{f})^{**}(\tau, \cdot)$ is affine in $[\phi_2(\tau), \phi_1(\tau)]$.

Thus, by applying the extended Lyapunov theorem (see the Appendix) to the functions $g_i(\tau) := (\phi_i(\tau), (\tilde{f})^{**}(\tau, \phi_i(\tau)))$, $i = 1,2$, we deduce the existence of a decomposition of $[\alpha, \beta]$ into disjoint measurable subsets E_1, E_2, such that, putting $\psi(\tau) := \phi_i(\tau)$ for $\tau \in E_i$, $i = 1,2$, by (13) we have $\int_\alpha^\beta \psi(\tau)\, d\tau = \int_\alpha^\beta \chi_{v_0}'(\tau)\, dt$ and by (12),

$$\begin{aligned}\int_\alpha^\beta \tilde{f}(\tau, \psi(\tau))\, d\tau &= \int_{E_1} (\tilde{f})^{**}(\tau, \phi_1(\tau))\, d\tau + \int_{E_2} (\tilde{f})^{**}(\tau, \phi_2(\tau))\, d\tau \\ &= \int_\alpha^\beta [\lambda_1(\tau)(\tilde{f})^{**}(\tau, \phi_1(\tau)) + \lambda_2(\tau)(\tilde{f})^{**}(\tau, \phi_2(\tau))]\, d\tau = \int_\alpha^\beta (\tilde{f})^{**}(\tau, \chi_{v_0}'(\tau))\, d\tau.\end{aligned}$$

Therefore, putting $u_0(\tau) := a + \int_\alpha^\tau \psi(\sigma)\, d\sigma$, we have $u_0 \in \tilde{\Omega}$ with $u_0(\beta) = \chi_{v_0}(\beta)$. Moreover, for every $u \in \tilde{\Omega}$ we have

$$\begin{aligned}\tilde{F}(u_0) &= \int_\alpha^\beta \tilde{f}(\tau, \phi(\tau)) d\tau + \mu(b - u_0(\beta)) = \int_\alpha^\beta (\tilde{f})^{**}(\tau, \chi_{v_0}'(\tau))\, d\tau + \mu(b - \chi_{v_0}(\beta)) \\ &= (\tilde{F})^{**}(\chi_{v_0}) = (\widetilde{F^{**}})(\chi_{v_0}) \le (\widetilde{F^{**}})(u) = (\tilde{F})^{**}(u) \le \tilde{F}(u);\end{aligned}$$

i.e., u_0 is a minimizer for problem $(\tilde{\mathrm{P}})$ and consequently $\Psi(u_0) \in \Omega$ is a minimizer for problem (P). $\square$

As an immediate consequence of the previous theorem, the following result holds.

Corollary 9. *Under the same assumptions of Theorem 8, if $co(C_s) = (0,+\infty)$ for every $s \in [\alpha,\beta]$, then problem* (P) *is solvable if and only if problem* (P^{**}) *is solvable.*

Note that if for every $s \in [\alpha,\beta]$ the detachment set $D_s := \{z > 0 \ : \ f(s,z) \neq f^{**}(s,z)\}$ is bounded, with $\inf D_s > 0$, then all the assumptions of Corollary 9 are satisfied. This occurs, for instance, with integrands having one of the special structures

$$f(s,z) = a(s)h(z) \quad \text{or} \quad f(s,z) = a(s) + h(z)$$

when $h = h^{**}$ in $(0,\epsilon] \cup [M,+\infty)$, for some $\epsilon, M > 0$.

5. Necessary and sufficient condition for the existence of the minimum for (P^{**})

In view of the relaxation result stated in the previous section, it is crucial to investigate the existence of the minimum for problem (P^{**}), whose Lagrangian is not coercive in general. To this purpose, in this section we apply the Dubois-Reymond necessary and sufficient condition for optimality given by Theorem 7, in order to establish a necessary and sufficient condition for the existence of the minimum for (P^{**}), expressed as an upper limitation for the assigned mean slope $\xi_0 := (\beta-\alpha)/(b-a)$. Of course, these results are applicable directly to problem (P) if $f(s,\cdot)$ is convex or under the assumption of Corollary 9.

We need some preliminary results. Let $h : (0,+\infty) \to \mathbb{R}$ be a convex function. In this case the functions $h^-(z)$ and $h^+(z)$ defined in (5) respectively denote the left and right derivative of h at the point z. For every $z > 0$ put

$$g^+(z) := h(z) - zh^+(z), \quad g^-(z) := h(z) - zh^-(z),$$

and define

$$\gamma^-(y) := \max\{z > 0 : \ g^-(z) \geq y\}, \quad \gamma^+(y) := \min\{z > 0 : \ g^+(z) \leq y\},$$

for $\inf_{z>0} g^+(z) =: \lambda < y < \Lambda := \sup_{z>0} g^-(z)$.

Lemma 10. *If $\lambda < \Lambda$, then functions γ^- and γ^+ are well defined in (λ,Λ). Moreover, γ^- is left-continuous, γ^+ is right-continuous and they satisfy*

$$\gamma^+(y_2) \leq \gamma^-(y_2) \leq \gamma^+(y_1) \leq \gamma^-(y_1) \quad \textit{whenever } y_1 < y_2; \tag{14}$$

$$\gamma^+(g^+(z)) \leq z \leq \gamma^-(g^-(z)) \quad \textit{for every } z > 0; \tag{15}$$

$$g^+(\gamma^+(y)) \leq y \leq g^-(\gamma^-(y)) \quad \textit{for every } y \in (\lambda,\Lambda). \tag{16}$$

Remark 5. When $\Lambda \in \mathbb{R}$, we can extend the definition of the functions γ^+ and γ^- also at the point Λ, by setting

$$\gamma^+(\Lambda) := 0, \quad \gamma^-(\Lambda) := \lim_{y\to\Lambda^-} \gamma^-(y) \in [0,+\infty).$$

Similarly, when $\lambda \in \mathbb{R}$ we can extend the definition of the functions γ^+ and γ^- also at the point λ, by setting

$$\gamma^+(\lambda) := \lim_{y\to\lambda^+} \gamma^+(y) \in (0,+\infty], \quad \gamma^-(\lambda) := +\infty.$$

In this way, γ^+, γ^- are nonnegative extended-values functions, respectively right-continuous and left-continuous, satisfying (14) and $g^+(\gamma^+(\Lambda)) = g^+(0) = \Lambda$. Moreover, note that if $\gamma^-(\Lambda) > 0$, then $g^-(z) \equiv \Lambda$ in $(0, \gamma^-(\Lambda)]$ and $g^-(\gamma^-(\Lambda)) = \Lambda$. Analogously, if $\gamma^+(\lambda) < +\infty$, then $g^+(z) \equiv \lambda$ in $[\gamma^+(\lambda), +\infty)$ and $g^+(\gamma^+(\lambda)) = \lambda$.

Let us now consider the functions

$$g^-(s,z) := f^{**}(s,z) - z f^{**}_-(s,z), \quad g^+(s,z) := f^{**}(s,z) - z f^{**}_+(s,z)$$

for every $(s,z) \in [\alpha,\beta] \times (0,+\infty)$ where $f^{**}_-(s,z)$, $f^{**}_+(s,z)$ respectively denote the left and right derivative of $f^{**}(s,\cdot)$ at the point z. Moreover, put

$$\lambda(s) := \inf_{z>0} g^+(s,z) \in \mathbb{R} \cup \{-\infty\}, \quad \Lambda(s) := \sup_{z>0} g^-(s,z) \in \mathbb{R} \cup \{+\infty\},$$

and for every $\lambda(s) < y < \Lambda(s)$ put

$$\gamma^-(s,y) := \max\{z > 0 : \ g^-(s,z) \geq y\}; \quad \gamma^+(s,y) := \min\{z > 0 : \ g^+(s,z) \leq y\}.$$

Lemma 11. *If f^{**} is continuous on $[\alpha,\beta] \times [0,+\infty)$, then $g^+(\cdot,z)$ and $\Lambda(\cdot)$ are lower semicontinuous in $[\alpha,\beta]$; instead, $g^-(\cdot,z)$ and $\lambda(\cdot)$ are upper semicontinuous. Moreover, $\gamma^+(\cdot,y)$ [respectively $\gamma^-(\cdot,y)$] is lower [respectively upper] semicontinuous in the open set $H_y := \{s \in [\alpha,\beta] : \lambda(s) < y < \Lambda(s)\}$.*

Let us extend the definitions of γ^+ and γ^- also for $y = \lambda(s), \Lambda(s)$ by setting $\gamma^-(s,\lambda(s)) = +\infty$, $\gamma^+(s,\lambda(s)) = \lim_{y \to \lambda(s)^+} \gamma^+(s,y) \leq +\infty$, and $\gamma^+(s,\Lambda(s)) = 0$, $\gamma^-(s,\Lambda(s)) = \lim_{y \to \Lambda(s)^-} \gamma^-(s,y) \geq 0$ (see Remark 5). Finally, set

$$c_0 := \operatorname*{ess\,sup}_{s \in [\alpha,\beta]} \lambda(s),$$

the essential supremum of $\lambda(s)$, with $c_0 = -\infty$ if $\lambda(s) = -\infty$ for a.e. $s \in (\alpha,\beta)$.

So, $\gamma^+(s,y)$ and $\gamma^-(s,y)$ are defined for every $s \in [\alpha,\beta]$ and $y \in [\lambda(s), \Lambda(s)] \cap \mathbb{R}$, taking values in $[0,+\infty]$. We can define in the same set also the functions $1/\gamma^+(s,y)$ and $1/\gamma^-(s,y)$, with the conventions $\frac{1}{+\infty} = 0$ and $\frac{1}{0} = +\infty$. In this way, also such functions take values in $[0,+\infty]$ and respectively are right-continuous and left-continuous with respect to the variable $y \in [\lambda(s), \Lambda(s)] \cap \mathbb{R}$.

The necessary and sufficient condition for the existence of the minimum of (P**) is given by the following result, where we recall that $\mu = \min_{s \in [\alpha,\beta]} f(s,0)$.

Theorem 12. *Let f^{**} be continuous on $[\alpha,\beta] \times [0,+\infty)$ and assume that if $c_0 < \mu$ we have*

$$1/\gamma^-(s,c) \in L^1(\alpha,\beta) \quad \textit{for every } c \in (c_0,\mu). \tag{17}$$

If problem (P**) is solvable, then

$$\int_\alpha^\beta \frac{1}{\gamma^-(s,\hat{c})}\, ds \leq b - a \tag{18}$$

for some real constant $\hat{c} \in [c_0,\mu]$. Vice versa, if (18) holds for some constant $\hat{c} \in (c_0,\mu]$, then problem (P**) is solvable.

Moreover, if (18) holds for some constant $\hat{c} > \max_{s \in [\alpha,\beta]} \lambda(s)$, then the minimizers are Lipschitz continuous.

Finally, if v_0 *is a minimizer, we have*

$$v_0'(t) > 0 \quad a.e.\ in\ (\alpha,\beta) \qquad \Leftrightarrow \qquad \int_\alpha^\beta \frac{1}{\gamma^-(s,\mu)}\ ds \geq b-a.$$

Proof. First note that $\mu \leq \Lambda(s)$ for every $s \in [\alpha,\beta]$. Indeed, since $\lim_{z\to 0^+} f_+^{**}(s,z) \in \mathbb{R}\cup\{-\infty\}$, we have $\lim_{z\to 0^+} z f_+^{**}(s,z) \in [-\infty,0]$. Moreover, by the lower semicontinuity of $f(s,\cdot)$ we have $f^{**}(s,0) = f(s,0)$ for every $s \in [\alpha,\beta]$. Hence,

$$\mu \leq f(s,0) = f^{**}(s,0) \leq \lim_{z\to 0^+} g^+(s,z) = \Lambda(s).$$

Moreover, $\mu < \Lambda(s)$ whenever $f(s,0) > \mu$. Then, for every real constant $c \in [c_0,\mu]$ we have $\lambda(s) \leq c \leq \Lambda(s)$ a.e. in (α,β) and then the functions $1/\gamma^+(s,c)$, $1/\gamma^-(s,c)$ are well defined and measurable in $[\alpha,\beta]$, taking values in $[0,+\infty]$. Therefore there exist, finite or $+\infty$, the integrals $\displaystyle\int_\alpha^\beta \frac{1}{\gamma^+(s,c)}\ ds$ and $\displaystyle\int_\alpha^\beta \frac{1}{\gamma^-(s,c)}\ ds$.

(Necessity). If $v_0 \in \Omega$ is a minimizer for (P**), it satisfies condition (DBR) by virtue of Theorem 7. So, for some constant $c \leq \mu$ we have

$$f^{**}(v_0(t),v_0'(t)) - c \in v_0'(t)\ \partial f^{**}(v_0(t),v_0'(t)) \qquad \text{a.e. in } A_{v_0}.$$

Hence, we have $g^+(v_0(t),v_0'(t)) \leq c \leq g^-(v_0(t),v_0'(t))$ a.e. in A_{v_0}. Therefore, since $|v_0(A_{v_0})| = \beta-\alpha$, we infer

$$g^+(\tau,\frac{1}{\chi_{v_0}'(\tau)}) \leq c \leq g^-(\tau,\frac{1}{\chi_{v_0}'(\tau)}), \qquad \text{for a.e. } \tau \in (\alpha,\beta),$$

by property *iii)* of Lemma 1. This implies $\lambda(\tau) \leq c \leq \Lambda(\tau)$ a.e. in (α,β), and so $c_0 \leq c \leq \mu$.

As discussed at the beginning of the proof, the function $1/\gamma^-(\tau,c)$ is well defined, taking values in $[0,+\infty]$. Since $\gamma^-(\tau,c) \geq \gamma^-(\tau,g^-(\tau,\frac{1}{\chi_{v_0}'(\tau)})) \geq \frac{1}{\chi_{v_0}'(\tau)}$ for a.e. $\tau \in (\alpha,\beta)$, we get $0 \leq \frac{1}{\gamma^-(\tau,c)} \leq \chi_{v_0}'(\tau)$ a.e. in (α,β). So, $\int_\alpha^\beta \frac{1}{\gamma^-(\tau,c)}\ d\tau \leq b-a$ and (18) holds.

(*Sufficiency*). Let us distinguish two cases.

• *First case*: $\displaystyle\int_\alpha^\beta \frac{1}{\gamma^-(s,\mu)}\ ds \leq b-a.$

In this case, let $u_0(\tau) := a + \int_\alpha^\tau \frac{1}{\gamma^-(s,\mu)}\ ds$. Since $\mu > c_0$ we have $\gamma^-(\tau,\mu) < +\infty$; hence $u_0'(\tau) > 0$ for a.e. $\tau \in (\alpha,\beta)$, and then $u_0 \in \tilde{\Omega}$.

Put $v_0 := \Psi(u_0)$. Let us prove that v_0 is a minimizer for problem (P**). In order to do this, note that by (3) and (8) we have $v_0'(t) = \dfrac{1}{u_0'(v_0(t))} = \gamma^-(v_0(t),\mu)$ for a.e. $t \in A_{v_0}$, so by (16) we get $g^-(v_0(t),v_0'(t)) = g^-(v_0(t),\gamma^-(v_0(t),\mu)) \geq \mu$ a.e. in A_{v_0} (see also Remark 5 in the case $\mu = \Lambda(s)$). Similarly, since $\gamma^+(v_0(t),\mu) \leq v_0'(t)$, we get $\mu \geq g^+(v_0(t),\gamma^+(v_0(t),\mu)) \geq g^+(v_0(t),v_0'(t))$. Hence we have $g^+(v_0(t),v_0'(t)) \leq \mu \leq g^-(v_0(t),v_0'(t))$ for a.e. $t \in A_{v_0}$, and this means that $f^{**}(v_0(t),v_0'(t)) - \mu \in v_0'(t)\partial f^{**}(v_0(t),v_0'(t))$ a.e. in A_{v_0}. Moreover, by (3), we get $f^{**}(v_0(t),0) = f(v_0(t),0) = \mu$ for a.e. $t \in B_{v_0}$, so v_0 satisfies the sufficient condition (DBR)$_2$ of Theorem 7 and then is a minimizer for (P**).

- *Second case*: $\displaystyle\int_\alpha^\beta \frac{1}{\gamma^-(s,\mu)}\,ds > b-a.$

Put $\tilde{c} := \sup\{c > c_0 : \displaystyle\int_\alpha^\beta \frac{1}{\gamma^-(s,c)}\,ds \le b-a\}$. Since $\gamma^-(s,\cdot)$ is left-continuous and decreasing, by the monotone convergence theorem we get $\displaystyle\int_\alpha^\beta \frac{1}{\gamma^-(s,\tilde{c})}\,ds \le b-a$, and then $\tilde{c} < \mu$.

Let $(c_n)_n$ be a decreasing sequence converging to $\tilde{c}$, with $c_1 < \mu$. We have $\lambda(s) < c_1 < \Lambda(s)$ for a.e. $s \in [\alpha,\beta]$, so $\gamma^+(\cdot,c_1)$ is well defined and positive a.e. in $[\alpha,\beta]$. Moreover, since $\gamma^+(s,c_1) \ge \gamma^-(s,c)$ for every $c \in (c_1,\mu)$, by assumption (17) we infer $\dfrac{1}{\gamma^+(s,c_1)} \in L^1(\alpha,\beta)$. Therefore, since $\dfrac{1}{\gamma^+(s,c_n)} \le \dfrac{1}{\gamma^+(s,c_1)}$ a.e. in $s \in [\alpha,\beta]$, for every $n \in \mathbb{N}$, by applying the dominated convergence theorem and recalling the right-continuity of $\gamma^+(s,\cdot)$, we get $\displaystyle\lim_{n\to+\infty}\int_\alpha^\beta \frac{1}{\gamma^+(s,c_n)}\,ds = \int_\alpha^\beta \frac{1}{\gamma^+(s,\tilde{c})}\,ds$. But $\displaystyle\int_\alpha^\beta \frac{1}{\gamma^+(s,c_n)}\,ds \ge \int_\alpha^\beta \frac{1}{\gamma^-(s,c_n)}\,ds > b-a$, and we obtain

$$\int_\alpha^\beta \frac{1}{\gamma^-(s,\tilde{c})}\,ds \le b-a \le \int_\alpha^\beta \frac{1}{\gamma^+(s,\tilde{c})}\,ds < +\infty. \tag{19}$$

So, putting $\Gamma(\sigma) := \displaystyle\int_\alpha^\sigma \frac{1}{\gamma^-(s,\tilde{c})}\,ds + \int_\sigma^\beta \frac{1}{\gamma^+(s,\tilde{c})}\,ds$, we have $\Gamma(\alpha) \ge b-a \ge \Gamma(\beta)$. By (19) and the continuity of Γ, the existence of a value $\tilde{\sigma} \in [\alpha,\beta]$ such that $\Gamma(\tilde{\sigma}) = b-a$ follows. Hence, consider the function

$$u_0(\tau) := a + \int_\alpha^\tau \xi(s)\,ds, \qquad \text{where} \qquad \xi(s) := \begin{cases} 1/\gamma^-(s,\tilde{c}) & \text{for } s \in [\alpha,\tilde{\sigma}], \\ 1/\gamma^+(s,\tilde{c}) & \text{for } s \in (\tilde{\sigma},\beta]. \end{cases}$$

Since $\tilde{c} > c_0$, we have $u_0'(\tau) > 0$ for a.e. $\tau \in (\alpha,\beta)$. Then $u_0 \in \tilde{\Omega}$, with $u_0(\beta) = a + \int_\alpha^\beta \xi(s)\,ds = b$. Hence, putting $v_0 := \Psi(u_0)$, we have $v_0'(t) = \frac{1}{u_0'(v_0(t))}$ for a.e. $t \in (a,b)$. Therefore, $\gamma^+(v_0(t),\tilde{c}) \le v_0'(t) \le \gamma^-(v_0(t),\tilde{c})$ a.e. in (a,b) and by (16) we infer

$$\begin{aligned} g^+(v_0(t),v_0'(t)) &\le g^+(v_0(t),\gamma^+(v_0(t),\tilde{c})) \le \tilde{c} \\ &\le g^-(v_0(t),\gamma^-(v_0(t),\tilde{c})) \le g^-(v_0(t),v_0'(t)) \end{aligned}$$

a.e. in (a,b); i.e., v_0 satisfies condition $(\mathrm{DBR})_1$ of Theorem 7, implying that v_0 is a minimizer for problem (P^{**}).

As for the Lipschitz continuity of minimizers, first note that $\lambda(s)$ admits the maximum M in $[\alpha,\beta]$ owing to the upper semicontinuity (see Lemma 11). So, in the second case considered in the proof of the sufficient part, if $\hat{c} > M$, then $\tilde{c} > M$ and $\lambda(s) < \tilde{c} < \Lambda(s)$ for every $s \in [\alpha,\beta]$. Hence $\gamma^-(\cdot,\tilde{c})$ is defined and upper semicontinuous on the whole compact interval $[\alpha,\beta]$ (see Lemma 11). Putting $L := \max_{s\in[\alpha,\beta]} \gamma^-(s,\tilde{c})$, we deduce that $v_0'(t) \le \gamma^-(v_0(t),\tilde{c}) \le L$; i.e., v_0 is Lipschitz continuous.

Similarly, in the first case, for a chosen $c \in (M,\mu)$, we have $v_0'(t) \le \gamma^-(v_0(t),\mu) \le \gamma^-(v_0(t),c)$ for a.e. $t \in (a,b)$, and the proof proceeds as above.

Finally, the last assertion concerning the measure of A_{v_0} can be immediately verified in view of the proof of sufficiency. □

As can easily be verified in view of the proof of the sufficient part in the previous theorem, the requirement $\hat{c} > c_0$ really only serves to guarantee that $\hat{c} > \lambda(s)$ for a.e. $s \in (\alpha, \beta)$, in order to have $\gamma^-(s, \hat{c}) < +\infty$. Therefore, by the same proof of Theorem 12, one can prove also the following variant, useful when $c_0 \in \mathbb{R}$.

Theorem 13. *Let f^{**} be continuous in $[\alpha, \beta] \times [0, +\infty)$ and condition* (17) *be fulfilled. Moreover, assume that $\lambda(s) < c_0$ for a.e. $s \in (\alpha, \beta)$. Then, problem* (P^{**}) *admits a minimum if and only if*

$$c_0 \le \mu \quad \text{and} \quad \int_\alpha^\beta \frac{1}{\gamma^-(s, c_0)}\, ds \le b - a.$$

Furthermore, as an immediate consequence of Theorem 12 the following result holds.

Corollary 14. *Under the same assumptions of Theorem* 12, *if $c_0 < \mu$ and*

$$\lim_{c \to c_0^+} \int_\alpha^\beta \frac{1}{\gamma^-(s, c)}\, ds < b - a,$$

then problem (P^{**}) *is solvable.*

Instead, if $c_0 = -\infty$ and

$$\lim_{c \to -\infty} \int_\alpha^\beta \frac{1}{\gamma^-(s, c)}\, ds > b - a,$$

then (P^{**}) *is not solvable.*

Remark 6. Condition (18) can be reviewed as an upper limitation for the assigned mean slope $\xi_0 := (\beta - \alpha)/(b - a)$; indeed, it is equivalent to

$$\xi_0 \le 1 \Big/ \fint_\alpha^\beta \frac{1}{\gamma^-(s, \hat{c})}\, ds$$

where the last term denotes the mean value of the function $1/\gamma^-(s, \hat{c})$, which is positive whenever $c > c_0$. So, taking account of Corollary 14 too, we deduce that

- *if $c_0 > \mu$, then problem* (P^{**}) *does not admit the minimum for any positive slope ξ_0;*
- *if $c_0 < \mu$, then problem* (P^{**}) *admits the minimum when the slope ξ_0 is sufficiently small;*
- *if $c_0 < \mu$ and*

$$\lim_{c \to c_0^+} \int_\alpha^\beta \frac{1}{\gamma^-(s, c)}\, ds = 0, \tag{20}$$

  then the minimum exists for every positive slope ξ_0.

In view of what was just observed, it is interesting to establish conditions ensuring the validity of condition (20), in such a way that the minimum exists for every positive assigned slope $\xi_0 = \frac{\beta - \alpha}{b - a}$. The next result provides an answer to this question.

Theorem 15. *Under the same assumptions of Theorem* 12, *let $c_0 < \mu$ and put $H_0 := \{s \in [\alpha, \beta] : \ c_0 > \lambda(s)\}$.*

i) *If* $|H_0| > 0$, *then* $\int_\alpha^\beta \frac{1}{\gamma^-(s,c_0)}\, ds > 0$ *and condition* (18) *is an effective upper limitation for the range of slopes for which the minimum exists. In particular, if the minimum exists, then*

$$\xi_0 \le 1/\int_\alpha^\beta \frac{1}{\gamma^-(s,c_0)}\, ds. \tag{21}$$

Vice versa, if $|H_0| = \beta - \alpha$ *and* (21) *holds with* $c_0 \le \mu$, *then the minimum exists.*

ii) *If* $|H_0| = 0$ *and for a.e.* $s \in (\alpha, \beta)$ *the functions* $g^\pm(s,\cdot)$ *are not constant in any half-line, then the minimum exists for every* $\xi_0 > 0$.

Proof. For every $s \in H_0$ we have $\gamma^-(s,c_0) < +\infty$. So, if $|H_0| > 0$ we have $\int_\alpha^\beta \frac{1}{\gamma^-(s,c)}\, ds \ge \int_\alpha^\beta \frac{1}{\gamma^-(s,c_0)}\, ds > 0$ for every $c \ge c_0$. Hence the minimum does not exist if the slope ξ_0 does not satisfy (21). The last statement of *i)* is a consequence of Theorem 13.

Instead, if $|H_0| = 0$, then $c_0 = \lambda(s)$ a.e. in (α, β) and since $g^\pm(s,\cdot)$ are not constant in any half-line we have (see Remark 5)

$$\lim_{c \to c_0^+} \gamma^-(s,c) \ge \lim_{c \to c_0^+} \gamma^+(s,c) = \gamma^+(s,\lambda(s)) = +\infty \quad \text{a.e. in } (\alpha,\beta).$$

Therefore, by applying the dominated convergence theorem we achieve

$$\lim_{c \to c_0^+} \int_\alpha^\beta \frac{1}{\gamma^-(s,c)}\, ds = \lim_{c \to c_0^+} \int_\alpha^\beta \frac{1}{\gamma^+(s,c)}\, ds = 0$$

and the minimum exists for every $\xi_0 > 0$, as a consequence of Corollary 14. □

The previous results are very easily applicable to integrands having one of the following structures:

$$f(s,z) = a(s)h(z) \qquad \text{or} \qquad f(s,z) = a(s) + h(z)$$

with $a \in C[\alpha,\beta]$ and $h \in C[0,+\infty)$ convex (but not necessarily coercive). Indeed, put

$$M := \max_{s\in[\alpha,\beta]} a(s),\ m := \min_{s\in[\alpha,\beta]} a(s),$$
$$\ell := \inf_{z>0}(h(z) - zh^+(z)) = \inf_{z>0}(h(z) - zh^-(z)).$$

In other words, ℓ is the infimum of the values at the origin of the affine support functions of h. Finally, as defined at the beginning of this section, let

$$\gamma^-(y) = \max\{z > 0 : \ h(z) - zh^-(z) \ge y\}.$$

Corollary 16. *Let* $f(s,z) = a(s)h(z)$, *with* $a \in C[\alpha,\beta]$ *positive almost everywhere,* $h \in C[0,+\infty)$ *nonnegative, convex, but not affine. Then,*

- *If* $\ell = -\infty$ *the minimum exists for every slope* $\xi_0 > 0$; *moreover, if* $m > 0$ *the minimizers are Lipschitz continuous.*
- *If* $\ell = 0$, $m > 0$ *and* h *is not affine in any half-line, then the minimum exists for every slope* $\xi_0 > 0$ *and the minimizers are Lipschitz continuous. Instead, if* $\ell = 0$, $m = 0$ *and* h *is strictly convex, then the minimum does not exists for any slope* $\xi_0 > 0$.

- *If* $\ell > 0$ *and* $a(s) < M$ *a.e. in* (α,β)*, then the minimum exists if and only if* $M\ell \le mh(0)$ *and*

$$\int_\alpha^\beta \frac{1}{\gamma^-(M\ell/a(s))}\, ds \le b-a. \tag{22}$$

Moreover, if $M\ell < mh(0)$ *and*

$$\lim_{c\to(M\ell)^+}\int_\alpha^\beta \frac{1}{\gamma^-(c/a(s))}\, ds < b-a, \tag{23}$$

then the minimizers are Lipschitz continuous.

- *If* $-\infty < \ell < 0$ *and* $a(s) > m$ *a.e. in* (α,β)*, then the minimum exists if and only if*

$$\int_\alpha^\beta \frac{1}{\gamma^-(m\ell/a(s))}\, ds \le b-a. \tag{24}$$

Moreover, if $m > 0$ *and*

$$\lim_{c\to(m\ell)^+}\int_\alpha^\beta \frac{1}{\gamma^-(c/a(s))}\, ds < b-a, \tag{25}$$

then the minimizers are Lipschitz continuous.

Proof. First let us show that condition (17) is satisfied whenever $c_0 < \mu$. To this aim, note that $\mu = mh(0) \ge 0$. If $\mu > 0$, for every positive $c \in (c_0,\mu)$ we have $\frac{c}{a(s)} \le \frac{c}{m} < h(0)$, so $1/\gamma^-(\frac{c}{a(s)}) \le 1/\gamma^-(\frac{c}{m}) < +\infty$ and assumption (17) is satisfied. Instead, if $\mu = 0$, for every $c < 0$ we have $\frac{c}{a(s)} \le \frac{c}{M} < 0$ a.e. in (α,β); then $1/\gamma^-(\frac{c}{a(s)}) \le 1/\gamma^-(\frac{c}{M}) < +\infty$ and again assumption (17) is satisfied.

If $\ell = -\infty$, then $h^\pm(z)$ are not constant in any half-line; moreover, since $a(s) > 0$ a.e. in (α,β), $\lambda(s) = -\infty$ for a.e. $s \in (\alpha,\beta)$ and the existence of the minimum follows from *ii)* of Theorem 15. Moreover, if $m > 0$, then $\lambda(s) = -\infty$ for every $s \in [\alpha,\beta]$, i.e. $c_0 = -\infty = \max_{s\in[\alpha,\beta]} \lambda(s)$ and the minimizers are Lipschitz continuous as a consequence of Theorem 12.

Similarly, when $\ell = 0$, then $\lambda(s) = c_0 = 0$ for every $s \in [\alpha,\beta]$. Moreover, $h(0) > 0$ and $\mu = mh(0)$. So, if $m > 0$ we have $\mu > c_0 = \max_{s\in[\alpha,\beta]} \lambda(s)$, and the assertion again follows from *ii)* of Theorem 15, Corollary 14 and Theorem 12. Instead, if $m = 0$, then $c_0 = \mu = 0$ and if the minimum exists, by condition (DBR) the minimizer v_0 should satisfy

$$a(v_0(t))v_0'(t)h^-(v_0'(t)) \le a(v_0(t))h(v_0'(t)) \le a(v_0(t))v_0'(t)h^+(v_0'(t)) \quad \text{a.e. in } A_{v_0}.$$

By (8) we have $a(v_0(t)) > 0$ for a.e. $t \in A_{v_0}$. Hence

$$h^-(v_0'(t)) \le \frac{h(v_0'(t))}{v_0'(t)} \le h^+(v_0'(t)) \quad \text{a.e. in } A_{v_0};$$

that is, h should be affine on $[0, \operatorname{ess\,sup}_{t\in(a,b)} v_0'(t))$. Thus, since h is assumed to be strictly convex, the minimum does not exist for any slope $\xi_0 > 0$.

When $\ell > 0$ we have $c_0 = M\ell = \max_{s\in[\alpha,\beta]} \lambda(s)$ and $\mu = mh(0)$. Moreover, since $a(s) = M$ in a set of null measure, we have $c_0 > \lambda(s)$ a.e. in $[\alpha,\beta]$ and the existence of the minimum under assumption (22) follows from Theorem 13. Moreover, when

$M\ell < mh(0)$ and (23) holds, then the minimizers are Lipschitz continuous as a consequence of Theorem 12.

Similarly, when ℓ is real negative, then $c_0 = m\ell \le mh(0) = \mu$ and since $a(s) > m$ a.e. in (α, β), we get $c_0 > \lambda(s)$ a.e. in $[\alpha, \beta]$. So, the existence of the minimum again follows from Theorem 13. Finally, if $m > 0$, since h is not affine, then $m\ell < mh(0)$. Hence if (25) holds, then the minimizers are Lipschitz continuous as a consequence of Theorem 12, since $c_0 = \max_{s\in[\alpha,\beta]} \lambda(s)$. □

Remark 7. Note that the integrals in (22) and (24) are positive. So these conditions are an effective upper limitation on the admissible slopes and the minimum does not exist if ξ_0 is too large.

Moreover, observe that if h has a superlinear growth, then $\ell = -\infty$; instead, if h has a linear growth, then ℓ coincides with the value at the origin of the asymptotic straight line of h as $z \to +\infty$. So, if such a line passes through the origin and $m > 0$, then the minimum exists for every slope $\xi_0 > 0$; otherwise, when ξ_0 is too large, the minimum does not exist.

Corollary 17. *Let* $f(s, z) = a(s) + h(z)$, *with* $a \in C[\alpha, \beta]$ *nonnegative,* h *convex, nonnegative and continuous in* $[0, +\infty)$. *Then,*

- *if* $\ell = -\infty$, *then the minimum exists for every slope* $\xi_0 > 0$ *with Lipschitz continuous minimizers;*
- *if* $\ell > -\infty$ *and* $a(s) < M$ *a.e. in* (α, β), *then the minimum exists if and only if* $M + \ell \le m + h(0)$ *and*

$$\int_\alpha^\beta \frac{1}{\gamma^-(M + \ell - a(s))}\, ds \le b - a.$$

Moreover, if $M + \ell < m + h(0)$ *and*

$$\lim_{c\to(M+\ell)^+} \int_\alpha^\beta \frac{1}{\gamma^-(c - a(s))}\, ds < b - a,$$

then the minimizers are Lipschitz continuous.

Proof. Similarly to what was done in the proof of Corollary 16, observe that since $\mu = m + h(0)$, for every $c < \mu$ we have $c - a(s) \le c - m < h(0)$. So, $1/\gamma^-(c - a(s)) \le 1/\gamma^-(c - m) < +\infty$ and assumption (17) is satisfied. Hence, the proof proceeds as that of Corollary 16. □

Example 1. Let us consider the problem of the surface of revolution of minimal area with endpoints $(a, 0)$, (b, β). Then, the constraint $v(t) \ge 0$ can be replaced by the constraint $v'(t) \ge 0$ a.e. in (a, b). In this case $f(s, z) = s\sqrt{1 + z^2}$ and we have $\ell = 0$, but $\min_{s\in[0,\beta]} a(s) = 0$. So, by applying Corollary 16 we deduce that the minumum does not exist for any slope $\xi_0 > 0$. Indeed, the area of the degenerate surface consisting of the disk of radius β centered at the point $(b, 0)$, obtained by rotating the graph of the BV function $v(t) = 0$ for $t \in [a, b)$, $v(b) = \beta$, is smaller than the area of any surface generated by rotation of a monotone AC function connecting $(a, 0)$ and (b, β).

Instead, if we consider the endpoints (a, α) and (b, β), with $0 < \alpha < \beta$, and search for the surface of minimal area subjected to the constraint $v'(t) \ge 0$ a.e., then in this case $\min_{s\in[\alpha,\beta]} a(s) = \alpha > 0$ and the minimum exists for every slope

$\xi_0 > 0$. In other words, the presence of the constraint has a regularizing effect on the problem.

Example 2. Consider the more general case $f(s,z) = a(s)h(z)$ with $h(z) = \sqrt{1+z^2}+k$, $k \geq 0$. Then, $\ell = k$ and by virtue of Corollary 16 we get that if $k = 0$, then problem (P) with integrand $f(s,z) = a(s)h(z)$ admits a minimum for every $\xi_0 > 0$, provided that $a(s)$ is continuous and positive (as in the Brachistochrone or Fermat's principle problems).

Instead, when $k > 0$, easy computations lead to the conclusion that the minimum exists if and only if $Mk \leq m(k+1)$ and

$$b - a \geq \int_\alpha^\beta k \frac{M - a(s)}{\sqrt{a^2(s) - k^2(M - a(s))^2}}\ ds.$$

For instance, take $[\alpha, \beta] = [0, 1]$ and $a(s) = s + 1$. Then, for each slope $\xi_0 > 0$, the inequality $k \leq 1$ is a necessary condition for the existence of the minimum. Moreover, for $k = 1$ the minimum exists if and only if $b - a \geq 2/3$.

Observe that the existence of the minimum for this type of functional can also be derived by the results given in [2], in the wider class of increasing trajectories satisfying $v(a) \geq \alpha$ and $v(b) \leq \beta$, provided that the function $a(s)$ does not have a proper maximum point in the interior (α, β).

Example 3. Let us consider the classical problem of minimization of the nonconvex integral of Newton's problem

$$F(v) := \int_0^1 v(t) \frac{(v'(t))^3}{1 + (v'(t))^2}\ dt$$

in the class $\Omega := \{v \in W^{1,1}(a,b) : v(a) = 0, v(b) = \beta, v'(t) \geq 0 \text{ a.e.}\}$.

In this case, putting $h(z) = \frac{z^3}{1+z^2}$ we have that $h^{**}(z) = h(z)$ if $0 \leq z \leq 1$, while $h^{**}(z) = z - \frac{1}{2}$ for $z > 1$. Hence, $\ell = -\frac{1}{2}$, $m = \min a(s) = 0$, so $c_0 = \mu = 0$ and $\gamma^-(0) = 0$. Then, by virtue of Theorem 8 and Corollary 16, we deduce that the problem is not solvable for any $\beta > 0$. Indeed, it is well known that the functional

$$\int_0^1 v(t) \frac{(v'(t))^3}{1 + (v'(t))^2}\ dt + \frac{1}{2}(v(0))^2$$

admits the minimum in the class $\bar{\Omega} := \{v \in W^{1,1}(a,b) : v(a) \geq 0, v(b) = \beta, v'(t) \geq 0 \text{ a.e.}\}$ attained at a function satisfying $v(a) > 0$ (see, e.g., [3]).

Example 4. Consider $f(s,z) = a(s)e^{-z}$. Also in this case we have $\ell = 0$, so the minimum exists provided that $m := \min a(s) > 0$, even if the integrand does not satisfy any growth assumption. Instead, for $f(s,z) = a(s) + e^{-z}$, if $M - m > 1$, where $M := \max a(s)$, the minimum does not exist for any $\xi_0 > 0$; while, if $0 < M - m \leq 1$, then the minimum exists if and only if the slope ξ_0 is sufficiently small.

Example 5. Take $f(s,z) = (s^p + k)z^q$, $(s,z) \in [0,\beta] \times [0,+\infty)$, with $k \geq 0$, $p > 0$ and $q > 1$. In this case $\ell = -\infty$ and by virtue of Corollary 16 we deduce that the minimum exists for every slope $\xi_0 > 0$. Nevertheless, the minimizer is Lipschitz continuous only if $k > 0$. Indeed, when $k = 0$, by applying the procedure described in the proof of Theorem 12, after easy computations we obtain that the minimizer for $a = 0$ and $b = 1$ is $v_0(t) = \beta t^{q/(p+q)}$ which is not Lipschitz continuous, for any $\beta > 0$.

Remark 8. Observe that condition (1) in our notation becomes $\lambda(s) = -\infty$ for every $s \in [\alpha, \beta]$. In this case $c_0 = \lambda(s) = \max_{s\in[\alpha,\beta]} \lambda(s)$ for every $s \in [\alpha, \beta]$.

Note that the requirement $\lambda(s) = -\infty$ for every $s \in [\alpha, \beta]$ is stronger than $c_0 = -\infty$, since c_0 is the *essential* supremum of $\lambda(s)$. For instance, for Lagrangians of product-type $f(s, z) = a(s)h(z)$, with $\ell = \inf_{z>0}(h(z) - zh^-(z)) = -\infty$ (as in Example 5), these conditions coincide only when $m = \min_{s\in[\alpha,\beta]} a(s) > 0$. Instead, when $m = 0$ and $a(s) > 0$ a.e. in (α, β), condition (1) is not satisfied, but we have $c_0 = -\infty$ and our existence results can be applied. Even the existence result given in [9] is not applicable in this situation, since the assumption (H2) does not hold.

6. Proofs of lemmas

Proof of Lemma 1. First note that the function $w_v : [\alpha, \beta] \to [a, b]$ is well defined and strictly increasing since $v(w_v(\tau)) = \tau$, $\tau \in [\alpha, \beta]$.

Let us fix $\tau_0 \in (\alpha, \beta]$. We have $w_v(\tau_0) > a$ and $v(t) < \tau_0$ for every $t < w_v(\tau_0)$. Then, by the continuity of v, it is not constant in any left neighborhood of $w_v(\tau_0)$. Hence, for every $n \in \mathbb{N}$ there exist $t'_n, t''_n \in (w_v(\tau_0) - \frac{1}{n}, w_v(\tau_0))$ such that $v(t'_n) < v(t''_n)$. Putting $t_n := \min\{t : v(t) = v(t''_n)\}$, we get $w_v(\tau_0) - \frac{1}{n} < t'_n < t_n \le t''_n < w_v(\tau_0)$ and $t_n = w_v(v(t_n))$, for every $n \in \mathbb{N}$. Moreover, it is clear that the sequence $(t_n)_n$ can be taken to be increasing, and i) is proved.

In order to prove ii), first note that w_v is left continuous at every $\tau_0 \in (\alpha, \beta]$. In fact, by virtue of property i) we have that $\sup_{\tau<\tau_0} w_v(\tau) \ge \sup_{n\in\mathbb{N}} w_v(v(t_n)) = w_v(\tau_0)$ and by the monotonicity of w_v the left-continuity follows.

Let us now assume that $w_v(\tau)$ is also right-continuous at τ_0 and let there exist, by contradiction, $t_1 < t_2$ such that $v(t_1) = v(t_2) = \tau_0$. Then, $w_v(\tau_0) \le t_1 < t_2$, but for every $\tau > \tau_0$ we have $w_v(\tau) > t_2 > w_v(\tau_0)$, in contrast with the continuity of w_v at τ_0. Vice versa, if $v^{-1}(\tau_0)$ is a singleton, then for every $t > w_v(\tau_0)$ we have $v(t) > v(w_v(\tau_0)) = \tau_0$. Hence v is not constant in any right neighborhood of $w_v(\tau_0)$. Then, similarly to what was done in the proof of property *i)*, we can find a decreasing sequence $(t_n)_n$, converging to $w_v(\tau_0)$, such that $t_n = w_v(v(t_n))$, $n \in \mathbb{N}$. Therefore, $\inf_{\tau>\tau_0} w(\tau) \le \inf_{n\in\mathbb{N}} w_v(v(t_n)) = w_v(\tau_0)$, and this means that w_v is right-continuous at τ_0.

As for property iii), note that by the monotonicity of v, if there exists $v'(t_0) > 0$, then $v(t) \ne v(t_0)$ for every $t \ne t_0$; i.e., $v^{-1}(v(t_0))$ is a singleton. So, by property ii) we deduce that w_v is continuous at $\tau_0 := v(t_0)$. Therefore,

$$0 < \lim_{t\to t_0} \frac{v(t) - v(t_0)}{t - t_0} = \lim_{\tau\to\tau_0} \frac{v(w_v(\tau)) - v(w_v(\tau_0))}{w_v(\tau) - w_v(\tau_0)} = \lim_{\tau\to\tau_0} \frac{\tau - \tau_0}{w_v(\tau) - w_v(\tau_0)},$$

implying the existence of $w'_v(v(t_0)) = \frac{1}{v'(t_0)}$. □

Proof of Lemma 2. By the monotonicity of w_v, it follows that it is differentiable a.e. in (α, β), with summable derivative w'_v. Hence, we have $\chi_v \in W^{1,1}(\alpha, \beta)$ with $\chi_v(\alpha) = a$ and $\chi_v(\beta) \le a + w_v(\beta) - w_v(\alpha) = w_v(\beta) \le b$. It remains to prove that $\chi'_v(\tau) > 0$ a.e. in (α, β). By the absolute continuity of v, we have that $|v(A_v)| = \beta - \alpha$, where A_v was defined in (7). So, for a.e. $\tau \in (\alpha, \beta)$ there exists (a unique) $t \in A_v$ such that $\tau = v(t)$ and then $w_v(\tau) = t$. So, as a consequence of property iii) of Lemma 1, we get $w_v'(\tau) = \frac{1}{v'(t)} > 0$ for a.e. $t \in (\alpha, \beta)$. Thus, also $\chi_v'(\tau) > 0$ a.e. in (α, β). □

Let us now fix a function $u \in \tilde{\Omega}$. Of course it is invertible, with continuous and increasing inverse function. Moreover, since $u'(\tau) > 0$ a.e. in (α, β), we get

$$\int_a^t (u^{-1})'(s)\, ds = \int_\alpha^{u^{-1}(t)} (u^{-1})'(u(\sigma))u'(\sigma)d\sigma = \int_\alpha^{u^{-1}(t)} d\sigma = u^{-1}(t) - u^{-1}(a);$$

that is, $u^{-1} \in W^{1,1}(a,b)$. By construction, also $\Psi_u \in W^{1,1}(a,b)$, with $\Psi_u(a) = \alpha$, $\Psi_u(b) = \beta$ and $\Psi_u'(t) \geq 0$ a.e. in (a,b). Then, the map Ψ is well defined.

Observe now that $\Psi_u'(t) > 0$ for a.e. $t \in (a, u(s^*)) \cup (u(s^*) + b - u(\beta), b)$, so Ψ_u is invertible on the same set and $w_{\Psi_u}(\tau) = u(\tau)$ for every $\tau \in [\alpha, s^*]$, whereas $w_{\Psi_u}(\tau) = u(\tau) + b - u(\beta)$ for every $\tau \in (s^*, \beta]$. Hence, $w_{\Psi_u}'(\tau) = u'(\tau)$ for a.e. $\tau \in (\alpha, \beta)$ and by (2) we get $\chi(\Psi(u)) = u$ for every $u \in \tilde{\Omega}$.

By (8) for a.e. $t \in A_v$ we have $w_v'(v(t)) = \chi_v'(v(t))$. So, by property iii) of Lemma 1 we obtain

(26)
$$|A_v| = \int_{A_v} \chi_v'(v(t))v'(t)\, dt = \int_a^b \chi_v'(v(t))v'(t)\, dt = \int_\alpha^\beta \chi_v'(\tau)\, d\tau = \chi_v(\beta) - a\ ,$$

implying that $|B_v| = b - \chi_v(\beta)$. This concludes the proof. □

Proof of Lemma 3. By (8) and property iii) of Lemma 1, we have $v'(t) = \dfrac{1}{\chi_v'(v(t))}$ for a.e. $t \in A_v$. Hence,

$$\begin{aligned}\int_\alpha^\beta f\left(\tau, \frac{1}{\chi_v'(\tau)}\right) \chi_v'(\tau)\, d\tau &= \int_a^b f\left(v(t), \frac{1}{\chi_v'(v(t))}\right) \chi_v'(v(t))v'(t)\, dt \\ &= \int_{A_v} f(v(t), v'(t))\, dt.\end{aligned}$$

Moreover, by (26) we have $|(a,b) \setminus A_v| = b - \chi_v(\beta)$ and then

$$\begin{aligned}F(v) &= \int_{A_v} f(v(t), v'(t))\, dt + \int_{[a,b]\setminus A_v} f(v(t), v'(t))\, dt \\ &= \int_\alpha^\beta f\left(\tau, \frac{1}{\chi_v'(\tau)}\right) \chi_v'(\tau)\, d\tau + \int_{[a,b]\setminus A_v} f(v(t), 0)\, dt \\ &\geq \int_\alpha^\beta f\left(\tau, \frac{1}{\chi_v'(\tau)}\right) \chi_v'(\tau)\, d\tau + \mu(b - \chi_v(\beta)) = \tilde{F}(\chi(v)).\end{aligned}$$

Furthermore, note that $F(v) = \tilde{F}(\chi(v))$ if and only if $f(v(t), 0) = \mu$ for a.e. $t \notin A_v$. Then, since $\Psi_{\chi_v}(t) = s^*$ for a.e. t such that $\Psi_{\chi_v}'(t) = 0$, and recalling that $\chi(\Psi(u)) = u$ for every $u \in \tilde{\Omega}$, we deduce that

$$F(\Psi(\chi(v))) = \tilde{F}(\chi(\Psi(\chi(v)))) = \tilde{F}(\chi(v)).$$ □

Proof of Lemma 5. Property i) is immediate. As regards property ii), note that the convexity of h in (c,d) is equivalent to the inequality

$$\text{(27)} \qquad \begin{vmatrix} h(z_1) & h(z_2) & h(z_3) \\ 1 & 1 & 1 \\ z_1 & z_2 & z_3 \end{vmatrix} \geq 0 \quad \text{for every } \ c < z_1 < z_2 < z_3 < d.$$

Multiplying the i-th column by $w_i := 1/z_i$, $i = 1, 2, 3$, (27) is equivalent to

$$0 \leq \begin{vmatrix} \tilde{h}(w_1) & \tilde{h}(w_2) & \tilde{h}(w_3) \\ w_1 & w_2 & w_3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \tilde{h}(w_3) & \tilde{h}(w_2) & \tilde{h}(w_1) \\ 1 & 1 & 1 \\ w_3 & w_2 & w_1 \end{vmatrix}$$

for $\frac{1}{d} < w_3 < w_2 < w_1 < \frac{1}{c}$, which is equivalent to the convexity of $\tilde{h}$ in $(\frac{1}{d}, \frac{1}{c})$.

Observe now that if h is affine in $(\frac{1}{d}, \frac{1}{c})$, then both h and $-h$ are convex in the same interval. So, seeing as $-\tilde{h} = (\widetilde{-h})$, property ii) implies that $\tilde{h}$ is affine too. The vice versa follows from property i).

As for property iv), putting $H := \{g : \mathbb{R}^+ \to \mathbb{R} : g \text{ is convex and } g(z) \leq h(z)\}$ and $\tilde{H} := \{\gamma : \mathbb{R}^+ \to \mathbb{R} : \gamma \text{ is convex and } \gamma(z) \leq \tilde{h}(z)\}$, by property ii) we infer $\tilde{H} = \{\tilde{g} : g \in H\}$; hence

$$\begin{aligned} (\tilde{h})^{**}(z) &= \sup_{\gamma \in \tilde{H}} \gamma(z) = \sup_{g \in H} \tilde{g}(z) = \sup_{g \in H} z\ g(1/z) \\ &= z \sup_{g \in H} g(1/z) = z h^{**}(1/z) = (\widetilde{h^{**}})(z). \end{aligned}$$

Concerning property v), observe that

$$\begin{aligned} \tilde{h}^-(z) &= \sup_{0<\zeta<z} \frac{\tilde{h}(\zeta) - \tilde{h}(z)}{\zeta - z} = \sup_{0<\zeta<z} \frac{\zeta h(\frac{1}{\zeta}) - z h(\frac{1}{z})}{\zeta - z} = \sup_{\eta > 1/z} \frac{\frac{1}{z} h(\eta) - \eta h(\frac{1}{z})}{\frac{1}{z} - \eta} \\ &= \sup_{\eta>1/z} \frac{1}{z} \frac{h(\eta) - h(\frac{1}{z})}{\frac{1}{z} - \eta} + h(1/z) = h(1/z) - \frac{1}{z} \inf_{\eta > 1/z} \frac{h(\eta) - h(\frac{1}{z})}{\eta - \frac{1}{z}} \\ &= h(1/z) - \frac{1}{z}\ h^+(1/z). \end{aligned}$$

The relation for $\tilde{h}^+$ can be derived similarly.

Finally, property vi) is an immediate consequence of v). □

Proof of Lemma 10. Let $z_2 > z_1 > 0$ be fixed; put $R(z_1, z_2) := \frac{h(z_2) - h(z_1)}{z_2 - z_1}$. Since $h^+(z_1) \leq R(z_1, z_2) \leq h^-(z_2)$, we have $(z_2 - z_1)\ R(z_1, z_2) \leq z_2 h^-(z_2) - z_1 h^+(z_1)$, i.e. $g^-(z_2) \leq g^+(z_1)$. Hence, we have

$$g^+(z_2) \leq g^-(z_2) \leq g^+(z_1) \leq g^-(z_1) \quad \text{whenever } z_1 < z_2. \tag{28}$$

Therefore, we deduce

$$\lambda = \lim_{z \to +\infty} g^+(z) = \lim_{z \to +\infty} g^-(z), \qquad \Lambda = \lim_{z \to 0^+} g^-(z) = \lim_{z \to 0^+} g^+(z).$$

So, for every $y \in (\lambda, \Lambda)$ the sets $G_y^- := \{z > 0 :\ g^-(z) \geq y\}$ and $G_y^+ := \{z > 0 : g^+(z) \leq y\}$ are both nonempty, with $\sup G_y^- < +\infty$ and $\inf G_y^+ > 0$. Moreover, since h^+ is right-continuous and h^- is left-continuous, also g^+ and g^- satisfy the same properties and then there exist $\max G_y^-$ and $\min G_y^+$; that is, γ^- and γ^+ are well defined.

Observe that for every $y \in (\lambda, \Lambda)$ we have $\gamma^+(y) \leq \gamma^-(y)$. Indeed, if $\gamma^+(y) > \gamma^-(y)$, for a fixed value $\gamma^-(y) < \bar{z} < \gamma^+(y)$ we would have $g^-(\bar{z}) < y < g^+(\bar{z})$, in contradiction with (28). Moreover, for fixed $y_1 < y_2$, by (28) we have $z \leq \zeta$ for every $z \in G_{y_2}^-$ and $\zeta \in G_{y_1}^+$, so $\gamma^-(y_2) \leq \gamma^+(y_1)$ and (14) holds. Properties (15) and (16) are immediate consequences of the definitions of γ^+ and γ^-.

Finally, for a fixed $y_0 \in (\lambda, \Lambda)$, put $z^* := \inf_{\lambda<y<y_0} \gamma^-(y)$. Since $z^* \leq \gamma^-(y)$ for every $y < y_0$, by (28) and (16) we deduce that $g^-(z^*) \geq g^-(\gamma^-(y)) \geq y$ for every $y < y_0$, implying that $g^-(z^*) \geq y_0$. So, $z^* \leq \gamma^-(y_0)$ and this means that γ^- is left-continuous. The proof that γ^+ is right-continuous is analogous. □

Proof of Lemma 11. Since $f_+^{**}(\cdot, z)$ is increasing and right-continuous, it is upper semicontinuous, and the lower semicontinuity of $g^+(\cdot, z)$ immediately follows, for every $z > 0$.

Let us fix $s_0 \in [\alpha, \beta]$ and let $(s_n)_n$ be a sequence converging to s_0 such that $\lim_{n\to+\infty} \Lambda(s_n) = \liminf_{s\to s_0} \Lambda(s)$. Observe that by (28) we have $\Lambda(s) = \sup_{z>0} g^+(s, z)$. Hence, if $\lim_{n\to+\infty} \Lambda(s_n) < k$, then we have $g^+(s_n, z) < k$ for every $z > 0$ and n sufficiently large. So, by the lower semicontinuity of $g^+(\cdot, z)$, we deduce that $g^+(s_0, z) \leq k$ for every $z > 0$; hence $\Lambda(s_0) \leq k$. Thus, $\liminf_{s\to s_0} \Lambda(s) \geq \Lambda(s_0)$ and $\Lambda(\cdot)$ is lower semicontinuous. The proof of the upper semicontinuity of $g^-(\cdot, z)$ and $\lambda(\cdot)$ is analogous.

Let us now fix s_0, y with $s_0 \in H_y$. By virtue of what has been proved above, we deduce that H_y is an open set in $[\alpha, \beta]$; hence the functions $\gamma^+(\cdot, y)$ and $\gamma^-(\cdot, y)$ are well defined in a neighborhood of s_0. Let $(\xi_n)_n$ be converging to s_0, such that $\lim_{n\to+\infty} \gamma^+(\xi_n, y) = \liminf_{s\to s_0} \gamma^+(s, y)$. If $\lim_{n\to+\infty} \gamma^+(\xi_n, y) < k$, then $g^+(\xi_n, k) \leq g^+(\xi_n, \gamma^+(\xi_n, y)) \leq y$ for n sufficiently large. So, $g^+(s_0, k) \leq \liminf_{n\to+\infty} g^+(\xi_n, k) \leq y$ and then $\gamma^+(s_0, y) \leq \gamma^+(s_0, g^+(s_0, k)) \leq k$. Thus, $\liminf_{s\to s_0} \gamma^+(s, y) \geq \gamma^+(s_0, y)$ and $\gamma^+(\cdot, y)$ is lower semicontinuous. The proof for $\gamma^-(\cdot, y)$ is analogous. □

7. Appendix: The Lyapunov Theorem for non-summable functions

In this section we prove an extension of the classical Lyapunov Theorem (see [8, Theorem 16.1.v]) to possible non-summable vector-values functions.

For fixed vectors $v, w \in \mathbb{R}^n$ we put

$$v \vee w := (\max\{v^{(1)}, w^{(1)}\}, \dots, \max\{v^{(n)}, w^{(n)}\}),$$
$$v \wedge w := (\min\{v^{(1)}, w^{(1)}\}, \dots, \min\{v^{(n)}, w^{(n)}\}).$$

Moreover, for a given function $g : A \to \mathbb{R}^n$, $A \subset \mathbb{R}$, let $[g]^-, [g]^+$ respectively denote the negative and the positive part of g, i.e.

$$[g]^-(x) = -(g(x) \wedge 0), \qquad [g]^+(x) = g(x) \vee 0.$$

Theorem 7.1. *Let $g_j : A \to \mathbb{R}^n$, $j = 1, \dots, h$, be measurable functions on a set $A \subset \mathbb{R}$ with finite measure, and let $\lambda_j : A \to [0, 1]$, $j = 1, \dots, h$, be measurable weight functions with $\sum_{j=1}^h \lambda_j(x) \equiv 1$. Suppose that*

$$\sum_{j=1}^{h} \lambda_j g_j \in L^1(A) \tag{7.1}$$

and that at least one of the following conditions is satisfied:

$$[g_j]^- \in L^1(A), \quad for\ every\ j = 1, \dots, h, \tag{7.2′}$$

$$[g_j]^+ \in L^1(A), \quad for\ every\ j = 1, \dots, h. \tag{7.2″}$$

Then, there exists a decomposition $F_1, \dots, F_h$ of A into disjoint measurable subsets such that $g_j \in L^1(F_j)$ for all $j = 1, \dots, h$, and

$$\sum_{j=1}^{h} \int_{F_j} g_j(x)\,\mathrm{d}x = \int_A \sum_{j=1}^{h} \lambda_j(x) g_j(x)\,\mathrm{d}x.$$

Proof. Assume (7.2′) (the proof is analogous if (7.2″) holds). Set

$$L(x) := [g_1]^-(x) \ \vee \cdots \vee \ [g_h]^-(x) \quad \text{and} \quad \gamma_j(x) := g_j(x) + L(x).$$

Of course, $L \in L^1(A)$ and $\gamma_j^{(i)}(x) \geq 0$, for every $i = 1, \dots, n$, $j = 1, \dots, h$.
For all the integers $k \geq 1$, define

$$A_k = \{x \in A : \ \sum_{j=1}^{h} \|\gamma_j(x)\| \in [k-1, k)\}.$$

Of course, $A = \bigcup_{k=0}^{\infty} A_k$ and $A_{k_1} \cap A_{k_2} = \emptyset$ whenever $k_1 \neq k_2$. Moreover, each function γ_j is summable in A_k and we can apply the classical Lyapunov Theorem, deducing the existence of a decomposition $F_1^{(k)}, \dots, F_h^{(k)}$ of A_k into disjoint measurable subsets such that

$$\sum_{j=1}^{h} \int_{F_j^{(k)}} \gamma_j(x)\,\mathrm{d}x = \int_{A_k} \sum_{j=1}^{h} \lambda_j(x) \gamma_j(x)\,\mathrm{d}x. \tag{7.2}$$

Set $F_j := \bigcup_{k \in \mathbb{N}} F_j^{(k)}$, we have $A = \bigcup_{j=1}^{h} F_j$ and $F_{j_1} \cap F_{j_2} = \emptyset$ whenever $j_1 \neq j_2$. Moreover, since

$$0 \leq \int_{F_j^{(k)}} \gamma_j^{(i)}(x)\,\mathrm{d}x \leq \sum_{j=1}^{h} \int_{F_j^{(k)}} \gamma_j^{(i)}(x)\,\mathrm{d}x = \int_{A_k} \sum_{j=1}^{h} \lambda_j(x) \gamma_j^{(i)}(x)\,\mathrm{d}x,$$

for every $k \in \mathbb{N}$, $i = 1, \dots, n$, and $j = 1, \dots, h$, then we get

$$\sum_{k=1}^{\infty} \int_{F_j^{(k)}} \gamma_j^{(i)}(x)\,\mathrm{d}x \leq \sum_{k=1}^{\infty} \int_{A_k} \sum_{j=1}^{h} \lambda_j(x) \gamma_j^{(i)}\,\mathrm{d}x = \int_A \sum_{j=1}^{h} \lambda_j(x) \gamma_j^{(i)}(x)\,\mathrm{d}x < +\infty,$$

implying that $\gamma_j \in L^1(F_j)$ for every $j = 1, \dots, h$ and hence

$$\begin{aligned} \sum_{j=1}^{h} \int_{F_j} \gamma_j(x)\,\mathrm{d}x &= \sum_{j=1}^{h} \sum_{k=1}^{\infty} \int_{F_j^{(k)}} \gamma_j(x)\,\mathrm{d}x = \sum_{k=1}^{\infty} \sum_{j=1}^{h} \int_{F_j^{(k)}} \gamma_j(x)\,\mathrm{d}x \\ &= \sum_{k=1}^{\infty} \int_{A_k} \sum_{j=1}^{h} \lambda_j(x) \gamma_j(x)\,\mathrm{d}x = \int_A \sum_{j=1}^{h} \lambda_j(x) \gamma_j(x)\,\mathrm{d}x. \end{aligned}$$

Finally, by the definition of γ_j, recalling that $L \in L^1(A)$, we infer that $g_j \in L^1(F_j)$ for every $j = 1, \dots, n$ and

$$\begin{aligned} \sum_{j=1}^{h} \int_{F_j} g_j(x)\,\mathrm{d}x &= \sum_{j=1}^{h} \int_{F_j} \gamma_j(x)\,\mathrm{d}x - \int_A L(x)\,\mathrm{d}x \\ &= \int_A \sum_{j=1}^{h} \lambda_j(x) \gamma_j(x)\,\mathrm{d}x - \int_A \sum_{j=1}^{h} \lambda_j(x) L(x)\,\mathrm{d}x = \int_A \sum_{j=1}^{h} \lambda_j(x) g_j(x)\,\mathrm{d}x. \end{aligned}$$

□

Notice that the requirement of the validity of one of the assumptions (7.2′), (7.2″) cannot be replaced by the following weaker assumption

$$[g_j]^- \in L^1(A) \text{ or } [g_j]^+ \in L^1(A), \qquad \text{for every } j = 1, \dots, h,$$

as the following simple example shows.

Example. Let $g_1, g_2 : \ (0,1) \to \mathbb{R}$ be defined by $g_1(x) := \frac{1}{x}$, $g_2(x) := -\frac{1}{x}$, and take $\lambda_1(x) \equiv \lambda_2(x) \equiv \frac{1}{2}$. Then, obviously we have $\lambda_1(x)g_1(x) + \lambda_2(x)g_2(x) \equiv 0 \in L^1(0,1)$, and $[g_1]^-, [g_2]^+ \in L^1(0,1)$, but for any possible pair of disjoint measurable sets $F_1, F_2 \subset (0,1)$, with $F_1 \cup F_2 = (0,1)$, we have

$$\int_{F_1} g_1(x)\,\mathrm{d}x = +\infty \quad \text{ or } \quad \int_{F_2} g_2(x)\,\mathrm{d}x = -\infty.$$

In fact, if both the previous integrals are finite, then we get the contradictory conclusion

$$\int_0^1 \frac{1}{x}\mathrm{d}x = \int_{F_1} g_1(x)\,\mathrm{d}x - \int_{F_2} g_2(x)\,\mathrm{d}x < +\infty.$$

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