MIXED NORM ESTIMATES
FOR CERTAIN GENERALIZED RADON TRANSFORMS

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Abstract. We obtain essentially optimal local mixed norm inequalities for certain generalized Radon transforms defined by integration over curves.

1. Introduction

In this paper we investigate the mapping properties in Lebesgue-type spaces of certain generalized Radon transforms defined by integration over curves.

Let $X$ and $Y$ be open subsets of $\mathbb{R}^d$, $d \geq 2$, and let $Z$ be a smooth submanifold of $X \times Y \subset \mathbb{R}^{2d}$ of dimension $d + 1$. Assume that the projections $\pi_1 : Z \to X$ and $\pi_2 : Z \to Y$ are submersions at each point of $Z$. For each $y \in Y$, let

$$\gamma_y = \{ x \in X : (x, y) \in Z \} = \pi_1 \pi_2^{-1}(y).$$

In this case, $\gamma_y$ are smooth curves in $X$ which vary smoothly with $y \in Y$. For every $y \in Y$, choose a smooth, nonnegative measure $\sigma_y$ on $\gamma_y$ which varies smoothly with $y$ in the natural sense. A generalized Radon transform $T$ (see, e.g., [2, 10, 15]) is defined as an operator taking functions on $X$ to functions on $Y$ via

$$Tf(y) = \int_{\gamma_y} f d\sigma_y,$$

The adjoint of this operator has a similar form:

$$T^* g(x) = \int_{\gamma^*_x} g d\sigma^*_x,$$

where

$$\gamma^*_x = \{ y : (x, y) \in Z \} = \pi_2 \pi_1^{-1}(\{x\}) \subset Y$$

and $\sigma^*_x$ is a nonnegative measure on $\gamma^*_x$ with a smooth density which varies smoothly with $x$.

Tao and Wright [16] have formulated and proved a nearly optimal characterization of the local $(L^p, L^q)$ mapping properties of these operators. We extend their result to the mixed-norm setting and obtain essentially optimal local mixed-norm inequalities for these operators, under one additional dimensional restriction. Previously this result was obtained for a model operator in [17, 6, 4]. See [11, 6, 12, 13, 14, 16, 17] for various examples and prior work, and [2] for a partially alternative development of the unmixed norm theory.
Mixed norms on $Y$. Throughout the entire discussion, $X, Y$ will denote sufficiently small neighborhoods of $x_0, y_0$ for some fixed point $(x_0, y_0) \in Z$. Let $\Pi : Y \to \mathbb{R}$ be a submersion such that the fibers $\Pi^{-1}(t)$ are transverse to the curves $\gamma_t^x$. This means that the restriction of $\Pi$ to $\gamma_t^x$ is a diffeomorphism for each $x \in X$. Choose coordinates so that $\Pi(y_0) = 0$. Let $\lambda_t$ be Lebesgue measure on the $(d - 1)$-dimensional surface $\Pi^{-1}(t)$. To a function $f : Y \to \mathbb{R}$ we associate the mixed norms

$$
\|f\|_{L^s L^r(Y)} = \|f\|_{q,r} := \left( \int_{\mathbb{R}} \left[ \int_{\Pi^{-1}(t)} |f(s)|^r d\lambda_t(s) \right]^{q/r} \ dt \right)^{1/q}.
$$

The integral with respect to $t$ is taken over a small neighborhood of the origin in $\mathbb{R}$; we may assume that this neighborhood is contained in $[-1, 1]$. Here, and throughout this paper, $t$ is restricted to lie in a one-dimensional manifold. This is not a natural restriction, but our analysis yields reasonably satisfactory results only in this special case.

This situation is quite general, and is satisfied for generic choices of the jets, up to a certain order depending only on $d$, of $X, Y, \Pi$ at $(x_0, y_0)$. Begin with any manifold $Z$ of dimension $d + 1$, and a point $z_0 \in Z$. In a neighborhood of $z_0$ construct two real vector fields $V_X, V_Y$ which are linearly independent at $z_0$, and which generate a Lie algebra that spans the tangent space to $Z$ at $z_0$. Such vector fields exist in any dimension, and hence for any $d$ there exists $N$ such that any pair $(V_X, V_Y)$ with generic jet up to order $N$ at $z_0$ will satisfy this condition. Choose two codimension-one manifolds $X, Y \subset Z$ in a neighborhood of $z_0$, such that the integral curves of $V_X, V_Y$ passing through $z_0$ are transverse to $X, Y$, respectively, at $z_0$. There are natural submersions $\pi_X, \pi_Y$ from $Z$ to $X, Y$, defined by flowing from $z \in Z$ to $X, Y$ along the integral curves of $V_X, V_Y$, respectively. The only requirement on $\Pi : Y \to \mathbb{R}$ is then that $V_X(\pi_Y \circ \Pi)(z_0) \neq 0$. Equivalently, $D\Pi(z_0)$ is required not to annihilate the push-forward under $\pi_Y$ of $V_X(z_0)$. This is a single linear condition on $D\Pi(z_0)$, so any generic map $\Pi$ satisfies the hypothesis. In particular, one can consider such mixed norms for each of the concrete examples discussed in [15].

A fundamental example which motivates our investigation has $X = \mathbb{R}^d, Y$ equal to the manifold of all lines in $\mathbb{R}^d$, and $Z = \{(x, L) \in X \times Y : x \in L\}$. To any line $L$ is associated $\Pi(L)$, the unique line parallel to $L$ containing the origin. $\Pi$ maps $Y$ to the manifold $G$ of all one-dimensional subspaces of $\mathbb{R}^d$. The x-ray transform maps $f : \mathbb{R}^d \to \mathbb{C}$ to $Tf : Y \to \mathbb{C}$, with $Tf(L)$ equal to the integral of $f$ over $L$. $Y$ can naturally be parametrized as the set of all $(\theta, x) \in G \times \mathbb{R}^{d-1}$ such that $x \perp \theta$; $L = x + \theta$ is a line parallel to $\theta$. Thus one has mixed norms $L^q_g L^r_x$ for $Y$. Let $B$ be a ball of finite radius in $\mathbb{R}^d$. A strong form of conjectured bounds for the Kakeya maximal function would say that $T$ is bounded from $L^q(B)$ to $L^q_g L^r_x(Y)$, for all $r < \infty$ and $q > d$. The inequality holds for $d = 2$ and is unknown for $d \geq 3$. This example fails to satisfy our hypotheses in two respects for $d \geq 3$. Firstly, $Y$ has dimension $2d - 2 > d$. This means that the operator formally adjoint to $T$ involves integration over submanifolds of dimension $d - 1 > 1$, rather than over curves. Secondly, $G$ has dimension strictly greater than one. In our view, this example amply motivates a general investigation of mixed-norm inequalities for generalized Radon transforms, even though such a general framework neglects special features of the motivating problem.
We say that $T$ is of strong mixed type $(p, q, r)$ if $T$ maps $L^p(X)$ to $L^q L^r(Y)$ boundedly. We are mainly interested in local estimates. We assume throughout the discussion that $T$ is $L^p$-improving, which means that for each $p \in (1, \infty)$, there exists $q > p$ such that $T$ maps $L^p(X)$ to $L^q(Y)$. See [5] and [11] for characterizations of this property.

Our theorem is a characterization of the exponents $(p, q, r)$ for which $T$ is bounded. Before proceeding, let us record some simple facts about these exponents.

(i) Because of the transversality hypothesis described above, $T$ is of strong mixed type $(p, \infty, p)$ for all $p \in [1, \infty]$.

(ii) Since we are working in a bounded region, whenever $T$ is of strong mixed type $(p, q, r)$, it is also of strong mixed type $(p_1, q_1, r_1)$ whenever $p_1 \geq p$, $q_1 \leq q$, and $r_1 \leq r$.

(iii) Because of (i) and (ii), $T$ is of strong mixed type $(p, q, r)$ whenever $p \geq r$.

Two-parameter Carnot-Carathéodory balls. Tao and Wright [16] related the set of all exponents $(p, q)$ for which the operator $T$ maps $L^p(X)$ to $L^q(Y)$ to the geometry of $Z$. To describe this relation, choose smooth nowhere-vanishing linearly independent real vector fields $V_1$, $V_2$ on $Z$ whose integral curves are the fibers of $\pi_1$, $\pi_2$ respectively. Equivalently, at each point $z \in Z$, $V_j$ spans the nullspace of $D\pi_j$, for $j = 1, 2$. The $L^p$-improving property is equivalent to $V_1$, $V_2$ satisfying the bracket condition, i.e., $V_1$ and $V_2$ together with their iterated commutators span the tangent space to $Z$ at each point in $Z$; see [5].

**Definition 1.** Let $z_0 \in Z$ and $0 < \delta_1, \delta_2 \ll 1$. The two-parameter Carnot-Carathéodory ball $B(z_0, \delta_1, \delta_2)$ consists of all the points $z \in Z$ such that there exists an absolutely continuous function $\varphi : [0, 1] \to Z$ satisfying

(i) $\varphi(0) = z_0$, $\varphi(1) = z$

(ii) for almost every $t \in [0, 1]$,

$$\varphi'(t) = a_1(t)V_1(\varphi(t)) + a_2(t)V_2(\varphi(t))$$

with $|a_1(t)| < \delta_1$, $|a_2(t)| < \delta_2$.

The metric properties of Carnot-Carathéodory balls were studied extensively in [11]. The discussion there is phrased in terms of the one-parameter family of balls naturally associated to a family of vector fields satisfying the bracket condition. These balls depend on a center point, a radius $r$, and a family \{W_j\} of vector fields. They can equivalently be viewed as depending on a center point and the family \{rW_j\} of vector fields, with the radius redefined to be identically one. In these terms, the proofs in [11] go through more generally, for balls of radius one assigned to families of vector fields \{U_j^\alpha\} satisfying the bracket condition for each parameter $\alpha$, with appropriate uniformity as $\alpha$ varies. In particular, for the vector fields \{\delta_1V_1, \delta_2V_2\}, provided that $0 < \delta_1, \delta_2 \leq c_0$ for sufficiently small $c_0$, the conclusions of [11] hold uniformly in $\delta_1, \delta_2$ under a supplementary hypothesis of weak comparability, which is discussed below. See [16, 2].

It will be convenient in our proof to parametrize the curves $\gamma_x^s$ by $t$ so that $\Pi(\gamma_x^s(t)) \equiv t$. With this parametrization, the measure $\sigma_x$ on $\gamma_x^s$ is equivalent to $dt$, uniformly in $x$. We rescale $V_1$ if necessary so that for each $z \in Z$ and sufficiently small $s \in \mathbb{R}$,

$$\Pi\sigma_2(e^{sV_1}z) = \Pi\sigma_2(z) + s.$$
Definition 2. Let $0 < \theta \leq 1$, and let $A$ be a positive constant. We say that $0 < \delta_1, \delta_2 \leq 1$ are $(\theta, A)$-weakly comparable, and write $\delta_1 \sim_{\theta, A} \delta_2$, if $\delta_1 \leq A\delta_2^\theta$ and $\delta_2 \leq A\delta_1^\theta$.

The following lemma collects basic facts about the balls $B(z, \delta_1, \delta_2)$.

Lemma 1. Let $K$ be a compact subset of $Z$. Assume that $\delta_1 \sim_{(\theta, A)} \delta_2$ are sufficiently small, and let $z \in K$. Then $B = B(z, \delta_1, \delta_2)$ satisfies

(i) $|B| \sim |B(z, 2\delta_1, 2\delta_2)|$,
(ii) $|B| \sim |\pi_1(B)|\delta_1 \sim |\pi_2(B)|\delta_2$,
(iii) $|\Pi(\pi_2(B))| \sim \delta_1$,
(iv) $\|\chi_{\pi_2(B)}\|_{q', r'} \sim |B|^{1 - \frac{1}{q'} \frac{r}{r'} - \frac{1}{\delta_1^\theta}}\delta_2^{\frac{1}{\delta_2^\theta} - \frac{1}{r'}}$,
(v) $\frac{|B|}{|\pi_1(B)|^{\frac{r}{p}}\|\chi_{\pi_2(B)}\|_{q', r'}} \sim |B|^{\frac{1}{p} + \frac{1}{q'} - \frac{1}{r'}}\delta_1^{\frac{1}{\delta_1^\theta} - \frac{1}{r'}}\delta_2^{1 - \frac{1}{r'}}$.

Here $1 \leq p, q, r \leq \infty$, and $q', r'$ are the exponents conjugate to $q, r$, respectively.

The notation $A \sim C$ means that the ratio $A/C$ is bounded above and below by quantities depending on $Z, \theta, A$, and the compact set $K$, but not on $\delta_1, \delta_2$. In the absence of weak comparability, the doubling property (i) fails in general for two-parameter Carnot-Carathéodory balls associated to $C^\infty$ vector fields satisfying the bracket condition; see [2].

For a sketch of the proof of Lemma 1 see [3] below.

Statement of results. Recall that $T$ is said to be of restricted weak type $(p, q)$ if for all Lebesgue measurable sets $E \subset X$ and $F \subset Y$,

(2) $\langle T\chi_E, \chi_F \rangle \lesssim |E|^{1/p}|F|^{1/q'}$,

where $q'$ denotes the exponent conjugate to $q$. In our setup,

(3) $\langle T\chi_E, \chi_F \rangle \approx |\pi_1^{-1}(E) \cap \pi_2^{-1}(F)|$,

where $|\cdot|$ denotes Lebesgue measure on $Z$. We test the inequality (2) on the Carnot-Carathéodory balls $B(z, \delta_1, \delta_2)$ under the restriction that $\delta_1 \sim_{(\theta, A)} \delta_2$. Let $E = \pi_1(B(z, \delta_1, \delta_2)), F = \pi_2(B(z, \delta_1, \delta_2))$. Using (3), Lemma 1 and restricting attention to the nontrivial case where $q > p$, the inequality (2) reads

(4) $|B(z, \delta_1, \delta_2)| \gtrsim \delta_1^{c_1}\delta_2^{c_2}$,

where

(5) $c_1 = \frac{p^{-1}}{p^{-1} - q^{-1}}, \quad c_2 = \frac{1 - q^{-1}}{p^{-1} - q^{-1}}$.

Define

$C_{\theta, A}(T) := \{ (c_1, c_2) : \inf \left( \frac{|B(z, \delta_1, \delta_2)|}{\delta_1^{c_1}\delta_2^{c_2}} \right) > 0 \}$,

where the infimum is taken over all $z \in Z$ and over all pairs $\delta_1, \delta_2$ that satisfy $\delta_1 \sim_{(\theta, A)} \delta_2$. Define

$\mathcal{C}(T) := \bigcap_{0 < \theta \leq 1} \bigcap_{A \geq 1} C_{\theta, A}(T)$.
According to (1), (2) cannot hold for \((p,q)\) if the corresponding \((c_1,c_2)\) does not belong to \(\mathcal{C}(T)\). Tao and Wright \[16\] proved that for all \((c_1,c_2)\) in the interior of \(\mathcal{C}(T)\), (2) holds for the exponents \((p,q)\) defined by (3).

In this note we extend this result to mixed norms. We say that \(T\) is of restricted weak mixed type \((p,q,r)\) if for all \(E \subset X\) and \(F \subset Y\),
\[
\langle T \chi_E, \chi_F \rangle \lesssim |E|^{1/p} \| \chi_F \|_{q',r'}.
\]
By interpolation, the strong mixed type estimates can be obtained from these inequalities, except for exponents corresponding to boundary points of \(\mathcal{C}(T)\).

The two-parameter Carnot-Carathéodory balls defined above also dictate the allowed exponent triples \((p,q,r)\) for mixed norm inequalities, under certain additional restrictions on the exponents \(p,q,r\):

**Definition 3.** Let \(P_{\theta,A}(T)\) be the set of all exponents \((p,q,r)\) satisfying
(i) \(1 \leq p \leq q \leq r \leq \infty\),
(ii) \[
\sup_{z,\delta_1,\delta_2} \frac{|B(z,\delta_1,\delta_2)|}{\pi_1(B(z,\delta_1,\delta_2))^{1/p} \| \chi_{\pi_2(B(z,\delta_1,\delta_2))} \|_{q',r'}} < \infty,
\]
where \(q', r'\) are the conjugates of \(q, r\) respectively, and the supremum is taken over all \(z \in Z\) and \(\delta_1 \sim_{\theta,A} \delta_2\).

Using Lemma 1 we can rewrite the second condition in the definition of \(P_{\theta,A}(T)\) as in (4) with
\[
c_1 = \frac{p^{-1} + q^{-1} - r^{-1}}{p^{-1} - r^{-1}}, \quad c_2 = \frac{1 - r^{-1}}{p^{-1} - r^{-1}}.
\]
Define
\[
P(T) := \bigcap_{0 < \theta \leq 1} \bigcap_{A \geq 1} P_{\theta,A}(T) = \{ (p,q,r) : r \geq q \geq p, (c_1, c_2) \in \mathcal{C}(T) \}.
\]
We assume always that \(r > p\), since otherwise the desired inequality holds automatically so long as \(\pi_1, \pi_2\) are submersions, as discussed above.

It is natural to conjecture that if \(1 \leq p \leq q \leq r \leq \infty\), then \(T\) is of restricted weak mixed type \((p,q,r)\) if and only if \((p,q,r) \in P(T)\); we prove this conjecture except at the endpoints. In an appendix we explain the presence of the additional restriction \(p \leq q \leq r\).

**Theorem 1.** Let \(T\) be a generalized Radon transform of the class described above, and let \(1 \leq p \leq q \leq r \leq \infty\). If \((p,q,r)\) is in the interior of \(P(T)\), then \(T\) maps \(L^p(X)\) to \(L^q L^r(Y)\) boundedly. Moreover, if \((p,q,r) \notin P(T)\), then \(T\) does not map \(L^p(X)\) to \(L^q L^r(Y)\) boundedly.

2. A BRIEF ACCOUNT OF \[16\]

Given \(E \subset X\) and \(F \subset Y\), let
\[
\alpha_1 = \frac{\langle T \chi_E, \chi_F \rangle}{|E|}, \quad \alpha_2 = \frac{\langle T \chi_E, \chi_F \rangle}{|F|}.
\]

\[1\] Tao and Wright defined the set \(\mathcal{C}(T)\) differently. An analysis of the two-parameter balls along the lines of \[11\] establishes the equivalence of these two definitions.
follows from an estimate of the form
\[ |\Omega| := |\pi_1^{-1}(E) \cap \pi_2^{-1}(F)| \approx \langle T \chi_E, \chi_F \rangle \gtrsim \alpha_1 c_1 \alpha_2 c_2, \]
where \( c_1, c_2 \) are as in \((5)\). Moreover, a loss of an arbitrarily small power of \( \alpha_1 \alpha_2 \) is of no consequence since \((c_1, c_2)\) belongs to the interior of \( C(T) \). Throughout the remainder of the discussion we will denote
\begin{equation}
\Omega := \pi_1^{-1}(E) \cap \pi_2^{-1}(F).
\end{equation}

Let \( \varepsilon > 0 \) be a small exponent to be specified near the end of the proof, and let \( C_\varepsilon \) be a large constant. All constants in the ensuing discussion depend on \( X, Y, Z, c_1, c_2 \) and the quantities given as subscripts.

**Definition 4.** Let \( S \subset [-1, 1] \) be a measurable set of positive Lebesgue measure. We say that \( S \) is central with width \( w > 0 \) if
(i) \( S \subset [-C_\varepsilon w, C_\varepsilon w] \) and 
(ii) \( |I \cap S| \leq C_\varepsilon (|I|/w)^\varepsilon |S| \) for all intervals \( I \).

Let \( \Omega \) be a Lebesgue measurable subset of \( Z \), having positive measure. We will assume throughout the discussion that \( \pi_j(\Omega) \) has positive measure, for both \( j = 1, 2 \). Define
\begin{equation}
\alpha_j = \frac{|\Omega|}{|\pi_j(\Omega)|} \quad \text{for} \quad j = 1, 2,
\end{equation}
\begin{equation}
\alpha := \min(\alpha_1, \alpha_2).
\end{equation}

Fix any \( \rho > 0 \); \( \rho \) will eventually be chosen to be arbitrarily small at the conclusion of the proof.

**Definition 5.** Let \( \Omega \subset Z \) and let \( \alpha_1, \alpha_2 \) be as defined in \((10)\). Let \( j \) be an integer. A \( j \)-subsheaf \( \Omega' \) of \( \Omega \), of width \( w_j \), is a subset of \( \Omega \) of measure \( \geq C^{-1}_\rho \alpha^\rho |\Omega| \) such that for all \( x \in \Omega' \), the set
\[ \{|t| \ll 1 : e^{IV_j}(x) \in \Omega'\} \]
is a central set of width \( w_j \) and measure \( \geq C_N \varepsilon \alpha^{CN\varepsilon+C/N} \alpha_j \). Here \( V_j = V_1 \) if \( j \) is odd, and \( V_j = V_2 \) if \( j \) is even.

In \((16)\) it is proved (see Corollary 8.3) that for any \( \rho \in (0, 1] \) there exists \( C_\rho < \infty \) such that for any measurable set \( \Omega \subset Z \) there exists a nested sequence of subsets of \( \Omega \)
\[ \Omega_0 \subset \Omega_1 \subset \ldots \subset \Omega_{d+1} \subset \Omega \]
such that: For each \( j \), \( \Omega_j \) is a \( j \)-sheaf of \( \Omega \) with width \( w_j \),
\[ C_\rho^{-1} \alpha^\rho \delta_j \leq w_j \leq \delta_j, \]
where \( \delta_j \) is a 2-periodic sequence (that is, \( \delta_{j+2} = \delta_j \) for all \( 0 \leq j \leq d - 1 \)) with the properties
\[ C_\rho^{-1} \alpha^\rho \alpha_j \leq \delta_j \leq 1, \]
and
\[ \delta_1 \leq C_\rho \delta_2, \quad \delta_2 \leq C_\rho \delta_1^\rho. \]
In particular, \( \delta_1, \delta_2 \) are weakly comparable, even though \( \alpha_1, \alpha_2 \) need not be; this ultimately explains why only balls with weakly comparable radii \( \delta_1, \delta_2 \) need be taken into account in the hypothesis of our theorem.
Using this construction, it is proved [16] that there exists some ball \( B(z, \delta_1, \delta_2) \) such that
\[
|B(z, \delta_1, \delta_2) \cap \Omega| \geq c \alpha^\rho \left( \frac{\alpha_1}{\delta_1} \right)^{(d+2)/2} \left( \frac{\alpha_2}{\delta_2} \right)^{(d+1)/2} |B(z, \delta_1, \delta_2)|,
\]
where \( \rho > 0 \) can be made arbitrarily small by choosing \( \rho \) sufficiently small, and where \( c > 0 \) depends on \( \rho \) but not on \( \Omega, \alpha_j, \delta_j \). This implies that for arbitrarily small \( \rho > 0 \),
\[
|\Omega| \gtrsim C \alpha^\rho \delta_1 \delta_2 \left( \frac{\alpha_1}{\delta_1} \right)^{(d+2)/2} \left( \frac{\alpha_2}{\delta_2} \right)^{(d+1)/2},
\]
for all \((c_1, c_2)\) in the interior of \( C(T) \). This finishes the proof in the nonmixed-norm case since\(^{2}\) \( c_1, c_2 \geq d \geq [(d+2)/2], [(d+1)/2] \). The roles of \( X \) and \( Y \) are interchangeable. Therefore there is the alternative bound
\[
|\Omega| \gtrsim C \alpha^\rho \delta \left( \frac{\alpha_1}{\delta_1} \right)^{(d+2)/2},
\]
for all \((c_1, c_2)\) in the interior of \( C(T) \).

3. Proof of Theorem 1

The second conclusion of the theorem is immediate. If \((p,q,r) \notin P(T)\), then \((p,q,r) \notin P_{\theta,A}(T)\) for some \( \theta, A \). Therefore \( T \) cannot be of mixed type \((p,q,r)\).

For the first conclusion, it suffices to prove that for all \((p,q,r)\) in the interior of \( P(T) \) and for all \( \eta > 0 \), \( \beta > 0 \), and \( F \subset Y \),
\[
\langle T \chi_E; \chi_F \rangle \leq C_{p,q,r,\eta} \beta^{-\eta} |E|^{1/p} \| \chi_F \|_{q',r'},
\]
where \( E := \{x \in X : \beta < T^* \chi_F(x) \leq 2\beta\} \).

Let \( \Omega := \pi_1^{-1}(E) \cap \pi_2^{-1}(F) \). Note that
\[
|\Omega| \approx \langle T \chi_E; \chi_F \rangle = \langle \chi_E; T^* \chi_F \rangle \approx \beta |E|.
\]
For each \( x \in E \), let \( \mathcal{F}(x) := \{t \in \mathbb{R} : \gamma_x^* (t) \in F\} \subset \Pi(F) \subset [-1,1] \). Note that \( |\mathcal{F}(x)| \approx \beta \).

Let \( \eta > 0 \) be a small constant to be specified at the end of the proof. Fix a small constant \( c_\eta > 0 \). Let \( I(x) \subset [-1,1] \) be a dyadic interval of minimal length so that
\[
|I(x) \cap \mathcal{F}(x)| \geq c_\eta |I(x)|^{\eta} |\mathcal{F}(x)|;
\]
we choose \( c_\eta \) to guarantee that either \( I = [-1,0] \) or \( I = [0,1] \) satisfies \((17)\). No interval of length \( < (c_\eta |\mathcal{F}(x)|)^{1/(1-\eta)} \) can satisfy \((17)\), so there must exist at least one dyadic interval of minimal length among all those satisfying the inequality.

This implies that for any dyadic subinterval \( J \subset I(x) \),
\[
|J \cap \mathcal{F}(x)| \leq \frac{|J|^{\eta}}{|I(x)|^{\eta}} |I(x) \cap \mathcal{F}(x)|.
\]
Let
\[
E_{m,k} = \{x \in E : |I(x)| \approx 2^m \beta, |I(x) \cap \mathcal{F}(x)| \approx 2^k \}.
\]
Note that \( \beta^{\eta/(1-\eta)} \lesssim 2^m \approx \beta^{-1} \) and \( \beta^{1/(1-\eta)} \lesssim 2^k \lesssim \beta.\)

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\(^{2}\) The bounds \( c_1, c_2 \geq d \) are a simple consequence of an equivalent definition of \( C(T) \) given in [16].
By the pigeonhole principle there exists a pair $m, k$ such that $\tilde{E} = E_{m,k}$ satisfies
\begin{equation}
\langle T\chi_{E}, \chi_{F} \rangle \leq C_{n}\beta^{-q}(T\chi_{E}, \chi_{F}) .
\end{equation}
Choose one such pair, with which we work exclusively henceforth.
Partition $\mathbf{R}$ into intervals $I_{n}$, each of length $C2^{m}\beta$ where $C$ is a sufficiently large fixed constant. Set
\begin{equation}
E_{n} = \{ x \in \tilde{E} : I(x) \cap I_{n} \neq \emptyset \},
\end{equation}
\begin{equation}
F_{n} = F \cap \Pi^{-1}(I_{n-1} \cup I_{n} \cup I_{n+1}).
\end{equation}
Note that
\begin{equation}
\langle T\chi_{E}, \chi_{F} \rangle \lesssim \sum_{n} \langle T\chi_{E_{n}}, \chi_{F_{n}} \rangle .
\end{equation}
Let $\Omega^{n} := \pi_{1}^{-1}(E_{n}) \cap \pi_{2}^{-1}(F_{n})$. Then
\begin{equation}
2^{k}|E_{n}| \lesssim |\Omega^{n}| \leq \beta|E_{n}|
\end{equation}
by Fubini’s theorem. On the one hand, $|\pi_{1}^{-1}(x)| \leq \beta$ for all $x \in E$. On the other hand, $2^{k} \sim |I(x) \cap F(x)|$ and for $x \in E_{n}$, $I(x) \cap F(x) \subset (I_{n-1} \cup I_{n} \cup I_{n+1}) \cap F(x) \subset \pi_{2}^{-1}(F_{n})$. Using (18) with $I = I_{n-1} \cup I_{n} \cup I_{n+1}$, for each $x \in E_{n}$ and any dyadic interval $J$, we obtain
\begin{equation}
|J \cap F(x)| \lesssim |J|^{\eta}(2^{m}\beta)^{-\eta}2^{k} .
\end{equation}
Let $\alpha_{n,i} = |\Omega^{n}|/|\pi_{i}(\Omega^{n})|$, $i = 1, 2$. Then
\begin{equation}
2^{k} \lesssim \alpha_{n,1} \approx \beta ,
\end{equation}
by (21). Let $\alpha_{n} = \min(\alpha_{n,1}, \alpha_{n,2})$.

The following lemma follows from a simple application of Hölder’s inequality; see [4].

**Lemma 2.** Let $F \subset Y$. For $r \geq q$,
\begin{equation}
\|\chi_{F}\|_{q',r'} \geq |F|^{1/r'}|\Pi(F)|^{1/q'-1/r'} .
\end{equation}

We aim to prove that $\langle T\chi_{E_{n}}, \chi_{F_{n}} \rangle \lesssim |E_{n}|^{1/p}\|\chi_{F_{n}}\|_{q',r'}$. Via the preceding lemma, this would follow from
\begin{equation}
|\Omega^{n}| \approx \langle T\chi_{E_{n}}, \chi_{F_{n}} \rangle \gtrsim \alpha_{n,1}^{\gamma_{1}}\alpha_{n,2}^{\gamma_{2}}|\Pi(F_{n})|^{\gamma_{3}} ,
\end{equation}
where
\begin{equation}
\gamma_{1} = \frac{p^{-1}}{p^{-1} - r^{-1}} , \quad \gamma_{2} = \frac{1 - r^{-1}}{p^{-1} - r^{-1}} , \quad \gamma_{3} = \frac{q^{-1} - r^{-1}}{p^{-1} - r^{-1}} .
\end{equation}

Whereas the exponent $\frac{1}{q} - \frac{1}{p}$ in Lemma 2 was nonpositive, here all the exponents $\gamma_{j}$ are nonnegative.

The construction of Tao and Wright [16] produces a nested sequence of subsets $\Omega^{n}_{0} \subset \Omega^{n}_{1} \subset \ldots \subset \Omega^{n}_{d+1} \subset \Omega^{n}$ satisfying: For each $j$, $\Omega^{n}_{j}$ is a $j$-sheaf of $\Omega^{n}$ with width $w_{j}$,
\begin{equation}
C_{p}^{-1}\alpha_{n,j}^{\delta_{j}} \leq w_{j} \leq C_{p}\delta_{j} ,
\end{equation}
where $\delta_{j}$ is a 2-periodic sequence with the property
\begin{equation}
C_{p}^{-1}\alpha_{n,j}^{\delta_{j}} \leq \delta_{j} \leq 1 ,
\end{equation}
and $|\Omega^{n}_{j}| \geq c_{p}\alpha_{n,j}^{\delta_{j}}|\Omega^{n}|$. Moreover, $\delta_{1}, \delta_{2}$ are weakly comparable.
Now, we prove that the construction above guarantees a lower bound for $\delta_1$. Indeed, let $j$ be odd. Using (22), we have
\[
|\Omega^n_j| \leq |E_n| \sup_{J,x} |J \cap F_n(x)| \leq |E_n| w_j^n (2^m \beta)^{-n} 2^k,
\]
where the supremum is taken over all intervals $J$ of length $w_j$ and all points $x \in E_n$. Using (27), (21) and the fact that $\Omega^n_j$ is a $j$-sheaf of $\Omega^n$, we get
\[
c_p \alpha_n^2 |E_n| \lesssim C_p \alpha_n^2 |\Omega^n| \leq |\Omega^n_j| \leq |E_n| w_j^n (2^m \beta)^{-n} 2^k.
\]
This implies that for odd $j$,
\[
\delta_1 \gtrsim w_j \gtrsim 2^m \beta C_p^{1/\eta} \alpha_n^2/\eta \gtrsim |\Omega(F_n)| C_p^{1/\eta} \alpha_n^2/\eta.
\]
Using the fundamental information (14), we obtain
\[
|\Omega^n| \gtrsim \alpha_n^2 \alpha_n^{C_p/\eta} \delta_1 \delta_2 \left( \frac{\alpha_n}{\delta_1} \right)^{(d+1)/2} \left( \frac{\alpha_n}{\delta_2} \right)^{(d+2)/2},
\]
for any $(c_1, c_2)$ belonging to the interior of $C(T)$. Note that for $d \geq 2$ and for such $(c_1, c_2)$, we have $0 \leq \gamma_3 < 1 \leq c_1 - [(d + 1)/2]$ and $c_2 > [(d + 2)/2]$. Using this, (26) and (29), we obtain
\[
|\Omega^n| \gtrsim \alpha_n^2 \alpha_n^{C_p/\eta} \delta_1 \alpha_n^{c_1-\gamma_3} \alpha_n^{c_2-\gamma_3} \gtrsim \alpha_n^2 \alpha_n^{C_p/\eta} |\Omega(F_n)| \gamma_3 \alpha_n^{c_1-\gamma_3} \alpha_n^{c_2-\gamma_3},
\]
for all $(c_1, c_2)$ in the interior of $C(T)$.

Given exponents $(p, q, r) \in P(T)$, define $\gamma_1, \gamma_2, \gamma_3$ by (23). Choose $\rho > 0$, depending on $\eta$, and $(c_1, c_2)$ in the interior of $C(T)$, so that
\[
\frac{C \rho}{\eta} + c_1 - \gamma_3 + \theta \leq \gamma_1, \quad \frac{C \rho}{\eta} + c_2 + \theta \leq \gamma_2.
\]
Therefore, since $\gamma_3 \geq 0$ (which is a consequence of the assumptions that $r > p$ and $r \geq q$),
\[
|\Omega^n| \gtrsim \alpha_n^2 \alpha_n^{\gamma_3} |\Omega(F_n)| \gamma_3,
\]
and hence
\[
\langle T \chi_{E_n}, \chi_F \rangle \lesssim |E_n|^{1/p} \|\chi_F\|_{q', r'}.
\]
Using this and then Hölder’s inequality in (20) (here we use the assumption $q \geq p$), we have
\[
\langle T \chi_{E}, \chi_F \rangle \lesssim \beta^-\eta \sum_n \langle T \chi_{E_n}, \chi_F \rangle
\]
\[
\lesssim \beta^-\eta \sum_n |E_n|^{1/p} \|\chi_F\|_{q', r'}
\]
\[
\lesssim \beta^-\eta \left[ \sum_n |E_n| \right]^{1/p} \left[ \sum_n \|\chi_F\|_{q', r'}^q \right]^{1/q'}
\]
\[
\leq \beta^-\eta |E|^{1/p} \|\chi_F\|_{q', r'}.
\]
In the last inequality we have used the bounded overlap property of the collections $\{E_n\}$ and $\{F_n\}$. Since $\eta > 0$ can be chosen to be arbitrarily small, this finishes the proof of Theorem 1.
4. Proof of Lemma [1]

We refer the reader to [2] for the proofs of (i) and (ii).

Let $B$ be the ball $B(z_0, \delta_1, \delta_2)$. Note that $\{e^{sV_1}z_0 : |s| < \delta_1\} \subset B$ and by [1], $|\Pi\pi_2(\{e^{sV_1}z_0 : |s| < \delta_1\})| \approx \delta_1$. Now, we prove that $|\Pi\pi_2(B)| \lesssim \delta_1$. Let $z \in B$ and let $\varphi$ be a curve connecting $z_0$ to $z$ as in Definition [1]. Let $\psi = \Pi \circ \pi_2 \circ \varphi$. Since $V_2$ is contained in the kernel of $D\pi_2$, $|\psi(t)| < \delta_1$ and $|V_1| \lesssim 1$, we have

$$|\psi'(t)| \lesssim \delta_1.$$ 

Therefore $|\Pi\pi_2(B)| \lesssim \delta_1$. This proves (iii).

We have actually shown that the 1-dimensional measure of $B(z_0, 2\delta_1, 2\delta_2) \cap \pi_1^{-1}(x)$ is $\approx \delta_1$ for every $x \in \pi_1(B)$. This also proves (ii) for $j = 1$. In fact it implies that if $A$ is a subset of a $k$-dimensional smooth submanifold of $X$, then $\delta_1$ times the $k$-dimensional measure of $A$ is comparable to the $(k+1)$-dimensional measure of $\pi_1^{-1}(A) \cap B$. The corresponding statement holds for $j = 2$, as well.

To prove (iv), define $f(t, \delta_1, \delta_2) = |\pi_2^{-1}\Pi^{-1}(t) \cap B(z_0, \delta_1, \delta_2)|$, where $| \cdot |$ signifies the $d$-dimensional measure of a subset of $Z$. To simplify notation we suppose that $\Pi(\pi_2(z_0)) = 0$; this can be achieved by a change of coordinates. Note that if $A \subset B(z_0, \delta_1, \delta_2)$, then for $s < \delta_1$, $e^{sV_1}A \subset B(z_0, 2\delta_1, 2\delta_2)$. Choose $t_0 \in [-\delta_1, \delta_1]$ so that $f(t_0, \delta_1, \delta_2) = \max_{|t| \leq \delta_1} f(t, \delta_1, \delta_2)$. Then

$$|B(z_0, 2\delta_1, 2\delta_2)| \gtrsim \int_{|s| \leq \delta_1} |e^{sV_1} f(t_0)|ds \gtrsim \delta_1 f(t_0).$$

Now $|B(z_0, \delta_1, \delta_2)| \gtrsim |B(z_0, 2\delta_1, 2\delta_2)|$; the proof given by Nagel, Stein, and Wainger [1] of the volume doubling property carries over to two-parameter balls with $(\theta, \delta)$–weakly comparable radii, with a bound depending on $\theta$ and on $\delta$ but not on $\delta, z_0$. We conclude that $f(t) \lesssim |B(z_0, \delta_1, \delta_2)|/\delta_1$ whenever $|t| \leq \delta_1$.

Similar reasoning shows that if $|t|, |t'| \leq \delta_1$, then $f(t', \delta_1, \delta_2) \leq Cf(t, 3\delta_1, 3\delta_2)$. Since $\Pi(\pi_2(B(z_0, \delta_1, \delta_2))) \subset [-\delta_1, \delta_1]$, there exists some $t' \in [-\delta_1, \delta_1]$ for which $f(t', \delta_1, \delta_2) \gtrsim |B(z_0, \delta_1, \delta_2)|/\delta_1$. By combining these two observations, we conclude the reverse inequality $f(t, 3\delta_1, 3\delta_2) \gtrsim |B(z_0, \delta_1, \delta_2)|/\delta_1$ whenever $|t| \leq \delta_1$.

The lower bound for $\|\chi_{\pi_2(t)}\|_{q', r'}$ in (iv) follows from this together with the observation that $f(t) \approx |\pi_2(f(t))|\delta_2$. The upper bound follows in the same way from the upper bound $f(t) \lesssim |B(z_0, \delta_1, \delta_2)|/\delta_1$.

Finally, conclusion (v) follows from (ii) and (iv). 

5. On the hypothesis $p \leq q \leq r$

One reason why the restriction $r \geq q \geq p$ in Theorem [1] is natural is as follows.

**Proposition 3.** Suppose that $|B(z, \delta_1, \delta_2)|$ is comparable to $|B(z', \delta_1, \delta_2)|$, uniformly for all $z, z'$ and all weakly comparable $\delta_1, \delta_2$. Then all valid mixed norm inequalities for $T$ are implied by the conjectured inequalities. That is:

(i) If $T$ is of restricted weak mixed type $(p, q, r)$ with $r > p > q$, then $(p, p, r) \in P(T)$.

(ii) If $T$ is of restricted weak mixed type $(p, q, r)$ with $q > r > p$, then $(p, q, r)$ is an interpolant between $(1, \infty, 1)$ and some $(p_1, q_1, r_1) \in P(T)$.

In case (i), the restricted weak mixed type bound for $(p, p, r)$ implies that for $(p, q, r)$ by Hölder’s inequality, since we are working in a bounded region and $q < p$. In case (ii), the conclusion is that $(p^{-1}, q^{-1}, r^{-1})$ belongs to the line segment with
endpoints \((1, 0, 1)\) and \((p^{-1}, q^{-1}, r^{-1})\). Since any generalized Radon transform \(T\) is of strong type \((1, \infty, 1)\), the restricted weak mixed type \((p, q, r)\) inequality follows from the \((p_1, q_1, r_1)\) inequality by interpolation.

**Proof.** (i) If \((p, p, r) \notin P(T)\), then there exist \(\theta, A\) and a sequence of balls \(B_n(z_n, \delta_{n,1}, \delta_{n,2})\) with \(\delta_{n,1} \sim_{(\theta, A)} \delta_{n,2}\) satisfying

\[
|B_n|^{\frac{1}{p} - \frac{1}{r}} \delta_{n,1}^{\frac{1}{q} + \frac{1}{r}} \delta_{n,2}^{1 - \frac{1}{r}} \to \infty,
\]
as \(n \to \infty\). Choose \(N_n \approx \delta_{n,1}^{-1}\) balls of size comparable to \(B_n\) with disjoint projections under \(\Pi \circ \pi_2\); this is possible by Lemma 1. Let \(U_n\) be the union of these balls. Note that \(|\pi_1(U_n)| \lesssim N_n|\pi_1(B_n)|\).

\[
\frac{|U_n|}{|\pi_1(U_n)|^{1/p}\|\pi_2(U_n)\|^{q/r}} \gtrsim \frac{N_n|B_n|}{\left[\frac{N_n|B_n|}{|\delta_{n,1}^{1/q} \delta_{n,2}^{1/q'}}\right]^{1/q'}} \delta_{n,1}^{1/q} \delta_{n,2}^{1/q'}
\]

\[
\approx \left(B_n^{\frac{1}{p} - \frac{1}{r}} \delta_{n,1}^{\frac{1}{q} + \frac{1}{r}} \delta_{n,2}^{1 - \frac{1}{r}}\right) (N_n \delta_{n,1})^{1/q - 1/p}
\]

\[
\approx |B_n|^{\frac{1}{p} - \frac{1}{r}} \delta_{n,1}^{\frac{1}{q} + \frac{1}{r}} \delta_{n,2}^{1 - \frac{1}{r}} \to \infty,
\]
as \(n \to \infty\). Thus, \(T\) cannot be of restricted weak mixed type \((p, q, r)\).

(ii) For \(s \in [0, 1]\) define \(p_1, q_1, r_1\) by

\[
\frac{1}{p} = 1 - s + \frac{s}{p_1}, \quad \frac{1}{q} = \frac{s}{q_1}, \quad \frac{1}{r} = 1 - s + \frac{s}{r_1}.
\]

A bit of algebra shows that \(r_1 \geq q_1 \geq p_1\) if and only if \(\frac{1}{q} + \frac{1}{p} \leq \frac{1}{q_1} + \frac{1}{p_1}\), and moreover that \(0 < \frac{1}{q} + \frac{1}{p} < 1\) under our assumption that \(q > r > p\). Thus it is possible to choose \(s \in (0, 1)\) so that \(r_1 \geq q_1 \geq p_1\). Fix such a parameter \(s\).

If \((p_1, q_1, r_1) \in P(T)\), then we have the conclusion of case (ii). Otherwise there exist \(\theta, A\) and a sequence of balls \(B_n = B_n(z_n, \delta_{n,1}, \delta_{n,2})\) with \(\delta_{n,1} \sim_{(\theta, A)} \delta_{n,2}\) such that

\[
|B_n|^{\frac{1}{p} - \frac{1}{q_1}} \delta_{n,1}^{\frac{1}{q_1} + \frac{1}{r_1}} \delta_{n,2}^{1 - \frac{1}{r_1}} \to \infty.
\]

Note that for these balls,

\[
\frac{|B_n|}{|\pi_1(B_n)|^{1/p}\|\pi_2(B_n)\|^{q/r'}} \approx |B_n|^{\frac{1}{p} - \frac{1}{r_1}} \delta_{n,1}^{\frac{1}{q_1} + \frac{1}{r_1}} \delta_{n,2}^{1 - \frac{1}{r_1}}
\]

\[
= \left(|B_n|^{\frac{1}{p} - \frac{1}{q_1}} \delta_{n,1}^{\frac{1}{q_1} + \frac{1}{r_1}} \delta_{n,2}^{1 - \frac{1}{r_1}}\right)^s \to \infty.
\]

Thus \(T\) cannot be of restricted weak mixed type \((p, q, r)\).

Case (ii) is valid for all operators \(T\), without the hypothesis that balls of equal bi-radii have uniformly comparable measures.

**References**


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