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# DOUBLE POISSON ALGEBRAS

MICHEL VAN DEN BERGH

Abstract. In this paper we develop Poisson geometry for non-commutative algebras. This generalizes the bi-symplectic geometry which was recently, and independently, introduced by Crawley-Boevey, Etingof and Ginzburg.

Our (quasi-)Poisson brackets induce classical (quasi-)Poisson brackets on representation spaces. As an application we show that the moduli spaces of representations associated to the deformed multiplicative preprojective algebras recently introduced by Crawley-Boevey and Shaw carry a natural Poisson structure.

Contents



## 1. Introduction

In this introduction we assume that $k$ is an algebraically closed field of characteristic zero. We start with our original motivating example (taken from [8]). Let $Q = (Q, I, h, t)$ be a finite quiver with vertex set $I = \{1, \dots, n\}$ and edge set $Q$. The maps $t, h : Q \to I$ associate with every edge its starting and ending vertex. We let $\bar{Q}$ be the double of $Q$. $\bar{Q}$ is obtained from $Q$ by adjoining for every arrow $a$ an opposite arrow $a^*$. We define $\epsilon : \bar{Q} \to \{\pm 1\}$ as the function which is 1 on $Q$ and $-1$ on $\bar{Q} - Q$.

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The author is a senior researcher at the FWO.

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ be a dimension vector and fix scalars $q = (q_1, \dots, q_n) \in (k^*)^n$. Put $\mathcal{R}_\alpha = \prod_{a \in \bar{Q}} M_{\alpha_{t(a)} \times \alpha_{h(a)}}$. The group $\mathrm{Gl}_\alpha = \prod_i \mathrm{Gl}_{\alpha_i}$ acts on $\mathcal{R}_\alpha$ by conjugation.

Let $\mathcal{S}_{\alpha,q}$ be the $\mathrm{Gl}_\alpha$ invariant subscheme of $\mathcal{R}_\alpha$ consisting of matrices $(X_a)_{a \in \bar{Q}}$ such that $1 + X_a X_{a^*}$ is invertible for all $a \in \bar{Q}$ and such that the following equations are satisfied for all $i \in I$:

$$\prod_{a \in \bar{Q}, h(a) = i} (1 + X_a X_{a^*})^{\epsilon(a)} = q_i$$

(it is shown in [8] that $\mathcal{S}_{\alpha,q}$ is independent of the ordering on these products). We prove the following result in this paper.

**Theorem 1.1.** *The GIT quotient $\mathcal{S}_{\alpha,q} /\!\!/ \mathrm{Gl}_\alpha$ is in a natural way a Poisson variety.*

This result is not unexpected since if $Q$ is a "star", then it is shown in [8] that the points in $\mathcal{S}_{\alpha,q} /\!\!/ \mathrm{Gl}_\alpha$ correspond to local systems on $\mathbb{P}^1$ whose monodromy lies in the closure of specific conjugacy classes. The result then follows from the work of Atiyah and Bott [3].

Different proofs of the Atiyah-Bott result were given in [1, 2] using quasi-Hamiltonian reduction and fusion. It is possible to give a proof of Theorem 1.1 in the same spirit. First one considers the small quiver consisting of two vertices and two arrows $a, a^*$ and one shows that in that case $\mathcal{R}_\alpha$ is quasi-Hamiltonian. This is then extended to general quivers using a process called "fusion". Finally we obtain a Poisson structure on $\mathcal{S}_{\alpha,q}$ by quasi-Hamiltonian reduction.

While working out this proof I noticed that all computations could be done directly in the path algebra $k\bar{Q}$ of $\bar{Q}$ (suitably localized). If computations are organized this way explicit matrices occur, somewhat as an afterthought, only in the very last step. Trying to understand why this is so then became the second motivation for writing this paper.

So we restart this introduction! Throughout $A$ is a $k$-algebra which for simplicity we assume to be finitely generated. For $N \in \mathbb{N}$ the associated representation space of $A$ is defined as

$$\mathrm{Rep}(A, N) = \mathrm{Hom}(A, M_N(k)).$$

The group $\mathrm{Gl}_N$ acts on $\mathrm{Rep}(A, N)$ by conjugation on $M_N(k)$.

A well-known philosophy in non-commutative algebraic geometry (probably first formulated by Maxim Kontsevich) is that for a property of the non-commutative ring $A$ to have geometric meaning it should induce standard geometric properties on all $\mathrm{Rep}(A, N)$. The case of symplectic geometry was worked out in [4, 11, 12]. In this paper we discuss Poisson geometry. More precisely we work out what kind of structure we need on $A$ in order that all $\mathrm{Rep}(A, N)$ are Poisson varieties.

To motivate our definitions we have to look in more detail at the coordinate ring $\mathcal{O}(\mathrm{Rep}(A, N))$ of $\mathrm{Rep}(A, N)$. For every $a \in A$ we have a corresponding matrix valued function $(a_{ij})_{i,j=1,\dots,N}$ on $\mathrm{Rep}(A, N)$. It is easy to see that the ring $\mathcal{O}(\mathrm{Rep}(A, N))$ is generated by the functions $a_{ij}$, subject to the relations

$$(ab)_{ij} = a_{il} b_{lj}$$

(where here and below we sum over repeated indices). Hence to define a Poisson bracket $\{-,-\}$ on $\mathrm{Rep}(A, N)$ we have to fix the values of $\{a_{ij}, b_{uv}\}$ for all $a, b \in A$. Now $\{a_{ij}, b_{uv}\}$ depends on four indices, so it is natural to assume that it comes from an element of $A \otimes A$. This leads to the following definition. A *double bracket* on $A$ is a bilinear map

$$\{\!\{-,-\}\!\} : A \times A \to A \otimes A,$$

which is a derivation in its second argument (for the outer bimodule structure on $A$) and which satisfies

$$\{\!\{a,b\}\!\} = -\{\!\{b,a\}\!\}^\circ$$

where $(u \otimes v)^\circ = v \otimes u$. We say that $A$ is a double Poisson algebra if $\{\!\{-,-\}\!\}$ satisfies in addition a natural analog of the Jacobi identity (see §2.3). A special case of one of our results is the following (see §7.5).

**Proposition 1.2.** *If $A, \{\!\{-,-\}\!\}$ is a double Poisson algebra, then $\mathcal{O}(\mathrm{Rep}(A,n))$ is a Poisson algebra, with Poisson bracket given by*

$$\{a_{ij}, b_{uv}\} = \{\!\{a,b\}\!\}'_{uj}\{\!\{a,b\}\!\}''_{iv} \tag{1.1}$$

*where by convention we write an element $x$ of $A \otimes A$ as $x' \otimes x''$ (i.e. we drop the summation sign).*

**Example 1.3** (see Examples 2.3.3, 7.5.3)**.** If $A = k[t]/(t^n)$, then $\mathrm{Rep}(A,n)$ consists of nilpotent $n \times n$ matrices. $A$ has a double Poisson bracket which is uniquely defined by the property

$$\{\!\{t,t\}\!\} = t \otimes 1 - 1 \otimes t.$$

This double bracket induces the standard Poisson bracket on nilpotent matrices.

Let $\{\!\{-,-\}\!\}$ be a double Poisson bracket on $A$. We define the associated bracket as

$$\{-,-\} : A \times A \to A : (a,b) \mapsto \{\!\{a,b\}\!\}'\{\!\{a,b\}\!\}''.$$

**Proposition 1.4.** *Assume that $A, \{\!\{-,-\}\!\}$ is a double Poisson algebra. Then the following holds:*

1. *$\{-,-\}$ is a derivation in its second argument and vanishes on commutators in its first argument.*
2. *$\{-,-\}$ is anti-symmetric modulo commutators.*
3. *$\{-,-\}$ makes $A$ into a left Loday algebra* [13, 16]*. I.e. $\{-,-\}$ satisfies the following version of the Jacobi identity:*

$$\{a,\{b,c\}\} = \{\{a,b\},c\} + \{b,\{a,c\}\}.$$

4. *$\{-,-\}$ makes $A/[A,A]$ into a Lie algebra.*

In commutative geometry it is customary to describe a Poisson bracket on a smooth variety $X$ in terms of a bi-vector field, i.e. in terms of a section $P$ of $\bigwedge^2 T_X$ satisfying $\{P,P\} = 0$ where $\{-,-\}$ is the so-called *Schouten-Nijenhuis bracket* on $\Gamma(X, \bigwedge T_X)$. Our next aim is to give a non-commutative version of this.

In the rest of this introduction we assume for simplicity that $A$ is *smooth*, by which we mean that $A$ is finitely generated and $\Omega_A = \ker(A \otimes A \to A)$ is a projective $A$-bimodule. It is easy to see that this implies that all spaces $\mathrm{Rep}(A,N)$ are smooth over $k$.

We first have to find the correct non-commutative analogue of a vector field. There are in fact *two* good answers to this. If we insist that a vector field on $A$ induces vector fields on all $\mathrm{Rep}(A,N)$, then a vector field on $A$ should simply be a derivation $\Delta : A \to A$. The induced derivation $\delta$ on $\mathcal{O}(\mathrm{Rep}(A,N))$ is then given by

$$\delta(a_{ij}) = \Delta(a)_{ij}.$$

A second point of view is that a vector field $\Delta$ on $A$ should induce *matrix valued vector fields* $(\Delta_{ij})_{i,j=1,\dots,n}$ on all $\operatorname{Rep}(A,N)$. Since now $\Delta_{ij}(a_{uv})$ depends on four indices, $\Delta(a)$ should be an element of $A\otimes A$.

In this paper we accept the second point of view; i.e. vector fields on $A$ will be elements of $D_A \overset{\text{def}}{=} \operatorname{Der}(A, A\otimes A)$ where as usual we put the outer bimodule structure on $A\otimes A$. The corresponding matrix valued vector fields on $\operatorname{Rep}(A,N)$ are then given by

$$\Delta_{ij}(a_{uv}) = \Delta(a)'_{uj}\Delta(a)''_{iv}.$$

$D_A$ contains a remarkable element $E$ which acts as $E(a) = a\otimes 1 - 1\otimes a$. We will call this element the *gauge element* since we have

**Proposition 1.5.** *The matrix valued vector field $(E_{ji})_{ij}$ on $\operatorname{Rep}(A,n)$ is the derivative of the action of $\operatorname{Gl}_N$ by conjugation.*

The importance of $\operatorname{Der}(A, A\otimes A)$ was first emphasized in [6].

Starting with $D_A$ we define *the algebra of poly-vector fields $DA$* on $A$ as the tensor algebra $T_A D_A$ of $D_A$ where we make $D_A$ into an $A$-bimodule by using the inner bimodule structure on $A\otimes A$.

Another main result of this paper is (see §3.2)

**Proposition 1.6.** *The graded algebra $DA$ has the structure of a* double Gerstenhaber algebra*, i.e. a (super) double Poisson algebra with a double Poisson bracket $\{\!\{-,-\}\!\}$ of degree $-1$.*

We call $\{\!\{-,-\}\!\}$ the *Schouten-Nijenhuis bracket* on $DA$. It is somewhat hard to construct, but as we will see below, in the case of quivers it takes a very trivial form.

The elements of $DA$ define matrix valued poly-vector fields on $\operatorname{Rep}(A,N)$ by the rule

$$(\delta_1\cdots\delta_m)_{ij} = \delta_{1,il_1}\delta_{1,l_1l_2}\cdots\delta_{m,l_mj}.$$

The compatibility between the matrix valued poly-vector fields and the Schouten brackets on $DA$ and $\Gamma(\operatorname{Rep}(A,N), T_{\operatorname{Rep}(A,N)})$ is given by a formula which is entirely similar to (1.1):

$$\{P_{ij}, Q_{uv}\} = \{\!\{P,Q\}\!\}'_{uj}\{\!\{P,Q\}\!\}''_{iv}.$$

Let us write $\operatorname{tr}(P) = P_{ii}$. Then the previous formula yields a morphism of graded Lie algebras

$$\operatorname{tr}: DA/[DA,DA] \to \Gamma(\operatorname{Rep}(A,N), \textstyle\bigwedge T_{\operatorname{Rep}(A,N)})^{\operatorname{Gl}_N}.$$

To reconnect with double Poisson structures on $A$ we show that there is a bijection

$$\big(DA/[DA,DA]\big)_2 \leftrightarrow \{\text{double brackets on } A\}$$

which sends $\delta_1\delta_2$ for $\delta_1,\delta_2\in D_A$ to the double bracket

$$\{\!\{a,b\}\!\} = \delta_2(b)'\delta_1(a)''\otimes\delta_1(a)'\delta_2(b)'' - \delta_1(b)'\delta_2(a)''\otimes\delta_2(a)'\delta_1(b)''.$$

An element $P\in(DA)_2$ corresponds to a double Poisson bracket if and only if

$$\{P,P\} = 0 \qquad \text{modulo commutators.}$$

Having a rudimentary differential geometric formalism in place we can now define various related notions. For example we say that $\mu \in A$ is a *moment map* for a double Poisson bracket $P$ if the following identity holds in $D_A$:

$$\{P, \mu\} = -E.$$

The reason is of course that if $\mu \in A$, then the corresponding matrix valued function $(\mu_{ij})_{ij}$ defines a moment map $\operatorname{Rep}(A, N) \to M_N$ for the action of $\mathrm{Gl}_N$ on $\operatorname{Rep}(A, N)$ (where we identify, as is customary, $M_N$ with its dual through the trace map).

We can also define the corresponding multiplicative notions (see [1]). An element $P \in (DA)_2$ is said to be a *quasi-Poisson bracket* if the following identity holds:

$$\{P, P\} = \frac{1}{6}E^3 \qquad \text{modulo commutators}$$

and an element $\Phi \in A$ is a multiplicative moment map for $P$ if it is a unit and

$$\{P, \Phi\} = -\frac{1}{2}(E\Phi + \Phi E)$$

in $D_A$. Again these notions induce the corresponding notions on representation spaces.

**Proposition 1.7.** (1) *Proposition* 1.4 *goes through unmodified for double quasi-Poisson algebras.*

(2) *Let $A$ be either a double Poisson algebra with moment map $\mu$ or a double quasi-Poisson algebra with moment map $\Phi$. Put $A^\lambda = A/(\mu - \lambda)$ in the first case with $\lambda \in k$ and $A^q = A/(\Phi - q)$ with $q \in k^*$ in the second case. Then the associated (quasi-)Poisson brackets on $\mathcal{O}(\operatorname{Rep}(A, N))$ induce Poisson brackets on $\mathcal{O}(\operatorname{Rep}(A, N))^{\mathrm{Gl}_N}$ and $\mathcal{O}(\operatorname{Rep}(A^{\lambda \text{ or } q}, N))^{\mathrm{Gl}_N}$.*

The second part of this theorem is an application of (quasi) Hamiltonian reduction [1].

Now we discuss quivers. Thus we return to the setting in the beginning of this introduction. In order for things to work nicely we must set things up in a relative setting. I.e. we let $B$ be a fixed commutative semi-simple algebra of the form $ke_1 \oplus \cdots \oplus ke_n$ with $e_i^2 = e_i$. A $B$-algebra is a $k$-algebra $A$ equipped with a morphism of $k$-algebras $B \to A$. For $B$-algebras we may define relative versions of the notions introduced above, e.g., $D_{A/B} = \operatorname{Der}_B(A, A \otimes A)$, $D_B A = T_A D_{A/B}$. Representation spaces are now indexed by $n$-tuples $(\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. By definition

$$\operatorname{Rep}(A, \alpha) = \operatorname{Hom}_B(A, M_N(k))$$

where $N = \alpha_1 + \cdots + \alpha_n$ and we view $B$ as being diagonally embedded in $M_N(k)$.

Now let $A = kQ$. In this case the idempotents $e_i$ are the paths of length zero corresponding to the vertices of $Q$. For $a \in Q$ we define the element $\frac{\partial}{\partial a} \in D_B A$ which on $b \in Q$ acts as

$$\frac{\partial b}{\partial a} = \begin{cases} e_{t(a)} \otimes e_{h(a)} & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $D_{A/B}$ is generated by $\left(\frac{\partial}{\partial a}\right)_{a \in Q}$ as an $A$-bimodule. Hence $D_B A$ is the tensor algebra over $A$ generated by $\left(\frac{\partial}{\partial a}\right)_a$. The matrix valued vector field corresponding to $\frac{\partial}{\partial a}$ is given by

$$\left(\frac{\partial}{\partial a}\right)_{ij} = \begin{cases} \dfrac{\partial}{\partial a_{ji}} & \text{if } \phi(i) = h(a),\ \phi(j) = t(a), \\ 0 & \text{otherwise,} \end{cases}$$

where for $i \in \{1, \ldots, N\}$ we put $\phi(i) = p \in \{1, \ldots, n\}$ if $i$ is in the $p$-th subinterval of $[1 \mathinner{..} N]$ when we decompose the latter into intervals of length $(\alpha_p)_p$.

The Schouten bracket on $A$ is as follows. Let $a, b \in Q$. Then

$$\begin{aligned} \{\!\{a, b\}\!\} &= 0, \\ \{\!\{\frac{\partial}{\partial a}, b\}\!\} &= \begin{cases} e_{t(a)} \otimes e_{h(a)} & \text{if } a = b, \\ 0 & \text{otherwise,} \end{cases} \\ \{\!\{\frac{\partial}{\partial a}, \frac{\partial}{\partial b}\}\!\} &= 0. \end{aligned}$$

We prove

**Theorem 1.8** (Theorem 6.3.1 in the body of the paper)**.** *$A = k\bar{Q}$ has a double Poisson bracket given by*

$$P = \sum_{a \in Q} \frac{\partial}{\partial a} \frac{\partial}{\partial a E^*}$$

*and a corresponding moment map*

$$\mu = \sum_{a \in Q} [a, a^*].$$

This theorem is more or less a reformulation of known results. The induced Lie algebra structure on $k\bar{Q}/[k\bar{Q}, k\bar{Q}]$ is the so-called *necklace Lie algebra* [4, 11, 12]. However it is noteworthy that this Lie algebra structure is induced from a Loday algebra structure on $k\bar{Q}$. In §6.4 we work out what it is.

The algebra $A^\lambda$ introduced in Proposition 1.7 is the so-called *deformed preprojective algebra* $\Pi^\lambda$ (see [9]). The Poisson bracket on $\operatorname{Rep}(\Pi^\lambda, \alpha)/\operatorname{Gl}(\alpha)$ is obtained from the standard Poisson bracket on $\mathcal{R}_\alpha = \operatorname{Rep}(k\bar{Q}, \alpha)$ given by

$$\sum_{a \in Q} \frac{\partial}{\partial (X_a)_{ij}} \frac{\partial}{\partial (X_{a^*})_{ji}}$$

in the notation of the first paragraph.

We then prove the main result of this paper.

**Theorem 1.9** (Theorem 6.7.1)**.** *Let $A$ be obtained from $k\bar{Q}$ by inverting all elements $(1 + aa^*)_{a \in \bar{Q}}$. Fix an arbitrary total ordering on $\bar{Q}$. Then $A$ has a quasi-Poisson bracket given by*

$$P = \frac{1}{2}\left(\sum_{a \in \bar{Q}} \left(\epsilon(a)(1 + a^* a)\frac{\partial}{\partial a}\frac{\partial}{\partial a^*}\right) - \sum_{a < b \in \bar{Q}} \left(\frac{\partial}{\partial a^*} a^* - a\frac{\partial}{\partial a}\right)\left(\frac{\partial}{\partial b^*} b^* - b\frac{\partial}{\partial b}\right)\right)$$

*and a corresponding moment map given by*

$$\Phi = \prod_{a \in \bar{Q}} (1 + aa^*)^{\epsilon(a)}.$$

*In the definition of $\Phi$ the product is taken with respect to the chosen ordering on $\bar{Q}$.*

The algebra $A_q$ introduced in Proposition 1.7 is now the *deformed multiplicative preprojective algebra* $\Lambda^q$ as introduced in [8]. Combining the previous theorem with Proposition 1.7 proves Theorem 1.1 since $\mathcal{S}_{\alpha,q} = \mathrm{Rep}(\Lambda^q, \alpha)$.

**Relation with bi-symplectic geometry.** The first version of this paper was written independently of the paper [7], which appeared around the same time on the ArXiv and which discusses a non-commutative analogue of symplectic geometry. In an appendix we outline the connection between the two papers. In particular we prove that an algebra with a bi-symplectic form is a double Poisson algebra. This allows us to strengthen some results of [7]. For example: if $A$ has a bi-symplectic form, then the associated Lie bracket on $A/[A, A]$ is obtained from a Loday bracket on $A$.

**Relation with Crawley-Boevey's Poisson structures.** In [5] Crawley-Boevey introduces non-commutative Poisson structures and shows that they induce classical Poisson structures on moduli spaces of representations. In the commutative case a Poisson structure is the same as a classical Poisson structure.

We show below that a double Poisson bracket or a double quasi-Poisson bracket induces a Poisson structure (Lemmas 2.6.2 and 5.1.3). By considering commutative algebras one easily sees that the converse is false.

The concept of a double Poisson structure is more adapted to algebras which are smooth in a non-commutative sense [10]. For example a semi-simple algebra has no Poisson structures [5, Rem. 1.2] but it has many double Poisson structures [17].

**A note on the organization of this paper.** The reader will find that this paper is rather peculiarly organized. We have seen that our motivating example is not a double Poisson algebra but only a double quasi-Poisson algebra. But it seemed difficult to treat double quasi-Poisson algebras without first introducing the algebra of poly-vector fields and its Schouten bracket. This Schouten bracket is a graded version of a double Poisson bracket. But again it seemed unreasonable to start this paper with graded double Poisson brackets since the many signs would have obscured the simplicity of the theory. Jean-Louis Loday pointed out to me that the sign problems can be mitigated by writing the definitions in terms of functions instead of elements. However since the Schouten bracket has degree $-1$ some signs would still remain.[1]

So we have chosen to treat double Poisson brackets first, and then to accept the (routine) generalizations of our statements to super Poisson brackets without further proof or discussion.

## 2. Double brackets and double Poisson algebras

2.1. **Generalities.** Throughout we work over a field $k$ of characteristic zero although this is not an essential condition. Unadorned tensor products are over $k$. If $V$, $W$ are $k$-vector spaces, then an element $a \in V \otimes W$ is written as $a' \otimes a''$. This is a shorthand for $\sum_i a'_i \otimes a''_i$. A similar convention is sometimes used for longer tensor products. We put $a^\circ = a'' \otimes a'$, i.e. $a^\circ = \sum_i a''_i \otimes a'_i$.

[1] Unless one is prepared to raise the level of abstraction by writing the formulas in terms of operators which take functions as arguments!

If $(V_i)_{i=1,\dots,n}$ are $k$-vector spaces and $s \in S_n$, then for $a = a_1 \otimes \cdots \otimes a_n \in V_1 \otimes \cdots \otimes V_n$ we put

$$\tau_s(a) = a_{s^{-1}(1)} \otimes \cdots \otimes a_{s^{-1}(n)}$$

so that $\tau_{st}(a) = \tau_s(\tau_t(a))$.

Below we fix a $k$-algebra $A$. Throughout we denote the multiplication map $A^{\otimes n} \to A$ by $m$. We will also view $A^{\otimes n}$ as an $A$-bimodule via the *outer* bimodule structure

$$b(a_1 \otimes \cdots \otimes a_n)c = ba_1 \otimes \cdots \otimes a_n c.$$

Of course $A^{\otimes n}$ has many other bimodule structures. For $n = 2$ we will frequently use the *inner* bimodule structure on $A^{\otimes 2}$ given by

$$b * (a_1 \otimes a_2) * c = a_1 c \otimes b a_2.$$

If $B$ is a (not necessarily commutative) $k$-algebra, then a $B$-algebra will be a $k$-algebra equipped with an (unnamed) $k$-algebra map $B \to A$.

An element of $\mathrm{Der}_B(A, A \otimes A)$ will be called a *double $B$-derivation*.

## 2.2. **Double brackets.**

**Definition 2.2.1.** An $n$-bracket is a linear map

$$\{\!\{-, \cdots, -\}\!\} : A^{\otimes n} \to A^{\otimes n},$$

which is a derivation $A \to A^{\otimes n}$ in its last argument for the outer bimodule structure on $A^{\otimes n}$, i.e.

(2.1)
$$\{\!\{a_1, a_2, \dots, a_{n-1}, a_n a_n'\}\!\} = a_n \{\!\{a_1, a_2, \dots, a_{n-1}, a_n'\}\!\} + \{\!\{a_1, a_2, \dots, a_{n-1}, a_n\}\!\} a_n',$$

and which is cyclically anti-symmetric in the sense

$$\tau_{(1\cdots n)} \circ \{\!\{-, \cdots, -\}\!\} \circ \tau_{(1\cdots n)}^{-1} = (-1)^{n+1} \{\!\{-, \cdots, -\}\!\}.$$

If $A$ is a $B$-algebra, then an $n$-bracket is *$B$-linear* if it vanishes when its last argument is in the image of $B$.

Clearly a 1-bracket is just a derivation $A \to A$. We will call a 2- and a 3-bracket respectively a *double* and a *triple* bracket. A double bracket satisfies

$$\{\!\{a, b\}\!\} = -\{\!\{b, a\}\!\}^\circ, \tag{2.2}$$

$$\{\!\{a, bc\}\!\} = b\{\!\{a, c\}\!\} + \{\!\{a, b\}\!\}c. \tag{2.3}$$

The formulas (2.2), (2.3) imply that $\{\!\{-, -\}\!\}$ is a derivation $A \to A \otimes A$ in its first argument for the inner bimodule structure on $A \otimes A$. I.e.

$$\{\!\{ab, c\}\!\} = a * \{\!\{b, c\}\!\} + \{\!\{a, c\}\!\} * b \tag{2.4}$$

where by "$*$" we mean the inner action. Combining (2.3) and (2.4) we obtain

$$\begin{aligned} &\{\!\{a_1 \cdots a_m, b_1 \cdots b_n\}\!\} \\ &\quad = \sum_{p,q} b_1 \cdots b_{q-1} \{\!\{a_p, b_q\}\!\}' a_{p+1} \cdots a_m \otimes a_1 \cdots a_{p-1} \{\!\{a_p, b_q\}\!\}'' b_{q+1} \cdots b_n. \end{aligned} \tag{2.5}$$

2.3. **The double Jacobi identity.** If $a \in A$, $b = b_1 \otimes \cdots \otimes b_n \in A^{\otimes n}$, then we define

$$\{\!\{a, b\}\!\}_L = \{\!\{a, b_1\}\!\} \otimes b_2 \otimes \cdots \otimes b_n,$$
$$\{\!\{a, b\}\!\}_R = b_1 \otimes \cdots \otimes b_{n-1} \otimes \{\!\{a, b_n\}\!\}.$$

Associated to a double bracket $\{\!\{-,-\}\!\}$ we define a trinary operation $\{\!\{-,-,-\}\!\}$ as follows:

$$\{\!\{a, b, c\}\!\} = \{\!\{a, \{\!\{b, c\}\!\}\}\!\}_L + \tau_{(123)}\{\!\{b, \{\!\{c, a\}\!\}\}\!\}_L + \tau_{(132)}\{\!\{c, \{\!\{a, b\}\!\}\}\!\}_L.$$

Or in more intrinsic notation,

$$\{\!\{-,-,-\}\!\} = \{\!\{-, \{\!\{-,-\}\!\}\}\!\}_L + \tau_{(123)}\{\!\{-, \{\!\{-,-\}\!\}\}\!\}_L \tau_{(123)}^{-1} + \tau_{(123)}^2\{\!\{-, \{\!\{-,-\}\!\}\}\!\}_L \tau_{(123)}^{-2}.$$

So $\{\!\{-,-,-\}\!\}$ is cyclically invariant, in the sense that

$$\{\!\{-,-,-\}\!\} = \tau_{(123)} \circ \{\!\{-,-,-\}\!\} \circ \tau_{(123)}^{-1}. \tag{2.6}$$

**Proposition 2.3.1.** $\{\!\{-,-,-\}\!\}$ *is a triple bracket.*

*Proof.* The cyclic invariance property has already been established. We now check the derivation property:

$$\begin{aligned} \{\!\{a, \{\!\{b, cd\}\!\}\}\!\}_L &= \{\!\{a, \{\!\{b, c\}\!\}d\}\!\}_L + \{\!\{a, c\{\!\{b, d\}\!\}\}\!\}_L \\ &= \{\!\{a, \{\!\{b, c\}\!\}\}\!\}_L d + \{\!\{a, c\}\!\}\{\!\{b, d\}\!\} + c\,\{\!\{a, \{\!\{b, d\}\!\}\}\!\}_L \end{aligned} \tag{2.7}$$

where in the second line we use the convention that $(x \otimes y)(s \otimes t) = x \otimes ys \otimes t$. We will often use the same convention below:

$$\begin{aligned} \{\!\{b, \{\!\{cd, a\}\!\}\}\!\}_L &= \{\!\{b, c * \{\!\{d, a\}\!\}\}\!\}_L + \{\!\{b, \{\!\{c, a\}\!\} * d\}\!\}_L \\ &= \{\!\{b, \{\!\{d, a\}\!\}' \otimes c\{\!\{d, a\}\!\}''\}\!\}_L + \{\!\{b, \{\!\{c, a\}\!\}'d \otimes \{\!\{c, a\}\!\}''\}\!\}_L \\ &= \{\!\{b, \{\!\{d, a\}\!\}'\}\!\}' \otimes \{\!\{b, \{\!\{d, a\}\!\}'\}\!\}'' \otimes c\{\!\{d, a\}\!\}'' \\ &\quad + \{\!\{b, \{\!\{c, a\}\!\}'\}\!\}' \otimes \{\!\{b, \{\!\{c, a\}\!\}'\}\!\}''d \otimes \{\!\{c, a\}\!\}'' \\ &\quad + \{\!\{c, a\}\!\}'\{\!\{b, d\}\!\}' \otimes \{\!\{b, d\}\!\}'' \otimes \{\!\{c, a\}\!\}''. \end{aligned}$$

Thus we find

$$\begin{aligned} \tau_{(123)}\{\!\{b, \{\!\{cd, a\}\!\}\}\!\}_L &= c\tau_{(123)}\{\!\{b, \{\!\{d, a\}\!\}\}\!\}_L \\ &\quad + \tau_{(123)}\{\!\{b, \{\!\{c, a\}\!\}\}\!\}_L d - \{\!\{a, c\}\!\}\{\!\{b, d\}\!\}. \end{aligned} \tag{2.8}$$

Finally

$$\begin{aligned} \{\!\{cd, \{\!\{a, b\}\!\}\}\!\}_L &= \{\!\{cd, \{\!\{a, b\}\!\}'\}\!\} \otimes \{\!\{a, b\}\!\}'' \\ &= c * \{\!\{d, \{\!\{a, b\}\!\}'\}\!\} \otimes \{\!\{a, b\}\!\}'' + \{\!\{c, \{\!\{a, b\}\!\}'\}\!\} * d \otimes \{\!\{a, b\}\!\}'' \\ &= \{\!\{d, \{\!\{a, b\}\!\}'\}\!\}' \otimes c\{\!\{d, \{\!\{a, b\}\!\}'\}\!\}'' \otimes \{\!\{a, b\}\!\}'' \\ &\quad + \{\!\{c, \{\!\{a, b\}\!\}'\}\!\}'d \otimes \{\!\{c, \{\!\{a, b\}\!\}'\}\!\}'' \otimes \{\!\{a, b\}\!\}'' \end{aligned}$$

which yields

$$\tau_{(132)}\{\!\{cd, \{\!\{a, b\}\!\}\}\!\}_L = c\tau_{(132)}\{\!\{d, \{\!\{a, b\}\!\}\}\!\}_L + \tau_{(132)}\{\!\{c, \{\!\{a, b\}\!\}\}\!\}_L d. \tag{2.9}$$

Taking the sum of (2.7), (2.8), and (2.9) yields the desired result. □

**Definition 2.3.2.** A double bracket $\{\!\{-,-\}\!\}$ on $A$ is *a double Poisson* bracket if $\{\!\{-,-,-\}\!\} = 0$. An algebra with a double Poisson bracket is *a double Poisson* algebra.

We will call the identity $\{\!\{-,-,-\}\!\} = 0$ *the double Jacobi identity.*

**Example 2.3.3.** Put $A = k[t]$. It is easy to check that up to automorphisms of $A$ the only double Poisson brackets on $A$ are given by

$$\{\!\{t,t\}\!\} = t \otimes 1 - 1 \otimes t \tag{2.10}$$

and

$$\{\!\{t,t\}\!\} = t^2 \otimes t - t \otimes t^2. \tag{2.11}$$

These two brackets are related. Extend (2.10) to $k[t,t^{-1}]$ (this is possible by Proposition 2.5.3 below). It turns out that the resulting double bracket preserves $k[t^{-1}]$, and the corresponding restriction is precisely (2.11) up to changing the sign and replacing $t$ by $t^{-1}$.

Assume that (2.10) holds. An easy computation shows

$$\{\!\{u(t),v(t)\}\!\} = \frac{(u(t_1)-u(t_2))(v(t_1)-v(t_2))}{t_1-t_2}$$

where $t_1 = t\otimes 1$, $t_2 = 1 \otimes t$. From this formula it follows that any quotient of $k[t]$ has an induced double Poisson bracket. This is for example the case for $k[t]/(t^n)$.

2.4. **Brackets associated to double brackets.** If $\{\!\{-,\cdots,-\}\!\}$ is an $n$-bracket, then we put $\{-,\cdots,-\} = m \circ \{\!\{-,\cdots,-\}\!\}$. If $n=2$, then we call $\{-,-\}$ the *bracket* associated to $\{\!\{-,-\}\!\}$. By definition $\{a,b\} = \{\!\{a,b\}\!\}' \cdot \{\!\{a,b\}\!\}''$. It is clear that $\{-,-\}$ is a derivation in its second argument. I.e.

$$\{a,bc\} = \{a,b\}c + b\{a,c\} \tag{2.12}$$

and furthermore by (2.2),

$$\{b,a\} \cong -\{a,b\} \mod [A,A]. \tag{2.13}$$

Finally an easy computation shows

$$\{bc,a\} = \{cb,a\}. \tag{2.14}$$

**Lemma 2.4.1.** $\{-,-\}$ *induces well-defined maps*

$$A/[A,A] \times A \to A \tag{2.15}$$

*and*

$$A/[A,A] \times A/[A,A] \to A/[A,A] \tag{2.16}$$

*where the latter one is anti-symmetric.*

*Proof.* The map (2.15) is well defined by (2.14). From (2.14) together with (2.13) it follows that $\{a,bc\}$ is symmetric in $b,c$ modulo commutators. Thus (2.16) is well defined as well. Its anti-symmetry follows also from (2.13). □

**Proposition 2.4.2.** *If $\{\!\{-,-\}\!\}$ is a double bracket on $A$, then the following identity holds in $A\otimes A$:*

$$\{a,\{\!\{b,c\}\!\}\} - \{\!\{\{a,b\},c\}\!\} - \{\!\{b,\{a,c\}\}\!\} = (m\otimes 1)\{\!\{a,b,c\}\!\} - (1\otimes m)\{\!\{b,a,c\}\!\} \tag{2.17}$$

*where $m$ is the multiplication map and $\{a,-\}$ acts on tensors by $\{a,u\otimes v\} = \{a,u\}\otimes v + u \otimes \{a,v\}$.*

*Proof.* First we record a useful identity:

$$
\begin{aligned}
\{\!\!\{a, \{\!\!\{c,b\}\!\!\}\}\!\!\}_R &= -\{\!\!\{a, \{\!\!\{b,c\}\!\!\}'' \otimes \{\!\!\{b,c\}\!\!\}'\}\!\!\}_R \\
&= -\{\!\!\{b,c\}\!\!\}'' \otimes \{\!\!\{a, \{\!\!\{b,c\}\!\!\}'\}\!\!\} \\
&= -\tau_{(123)}\Big(\{\!\!\{a, \{\!\!\{b,c\}\!\!\}'\}\!\!\} \otimes \{\!\!\{b,c\}\!\!\}''\Big) \\
&= -\tau_{(123)}\{\!\!\{a, \{\!\!\{b,c\}\!\!\}\}\!\!\}_L.
\end{aligned}
\tag{2.18}
$$

We now compute

$$
\begin{aligned}
\{a, \{\!\!\{b,c\}\!\!\}\} &= \{a, \{\!\!\{b,c,\}\!\!\}'\} \otimes \{\!\!\{b,c\}\!\!\}'' + \{\!\!\{b,c\}\!\!\}' \otimes \{a, \{\!\!\{b,c\}\!\!\}''\} \\
&= (m \otimes 1)\{\!\!\{a, \{\!\!\{b,c\}\!\!\}\}\!\!\}_L + (1 \otimes m)\{\!\!\{a, \{\!\!\{b,c\}\!\!\}\}\!\!\}_R \\
&= (m \otimes 1)\{\!\!\{a, \{\!\!\{b,c\}\!\!\}\}\!\!\}_L - (1 \otimes m)\tau_{(123)}\{\!\!\{a, \{\!\!\{c,b\}\!\!\}\}\!\!\}_L, \\
\{\!\!\{\{a,b\},c\}\!\!\} &= -\{\!\!\{c, \{a,b\}\}\!\!\}^\circ \\
&= -\tau_{(12)}(\{\!\!\{c, \{\!\!\{a,b\}\!\!\}'\}\!\!\}\{\!\!\{a,b\}\!\!\}'' + \{\!\!\{a,b\}\!\!\}' \{\!\!\{c, \{\!\!\{a,b\}\!\!\}''\}\!\!\}) \\
&= -(m \otimes 1)\tau_{(132)}\{\!\!\{c, \{\!\!\{a,b\}\!\!\}\}\!\!\}_L - (1 \otimes m)\tau_{(123)}\{\!\!\{c, \{\!\!\{a,b\}\!\!\}\}\!\!\}_R \\
&= -(m \otimes 1)\tau_{(132)}\{\!\!\{c, \{\!\!\{a,b\}\!\!\}\}\!\!\}_L + (1 \otimes m)\tau_{(132)}\{\!\!\{c, \{\!\!\{b,a\}\!\!\}\}\!\!\}_L, \\
\{\!\!\{b, \{a,c\}\!\!\}\} &= \{\!\!\{b, \{\!\!\{a,c\}\!\!\}'\}\!\!\}\{\!\!\{a,c\}\!\!\}'' + \{\!\!\{a,c\}\!\!\}'\{\!\!\{b, \{\!\!\{a,c\}\!\!\}''\}\!\!\} \\
&= (1 \otimes m)\{\!\!\{b, \{\!\!\{a,c\}\!\!\}\}\!\!\}_L + (m \otimes 1)\{\!\!\{b, \{\!\!\{a,c\}\!\!\}\}\!\!\}_R \\
&= (1 \otimes m)\{\!\!\{b, \{\!\!\{a,c\}\!\!\}\}\!\!\}_L - (m \otimes 1)\tau_{(123)}\{\!\!\{b, \{\!\!\{c,a\}\!\!\}\}\!\!\}_L.
\end{aligned}
$$

Collecting everything we obtain the desired result. □

**Definition 2.4.3.** A *left Loday (or Leibniz) algebra* [13, 16] is a vector space equipped with a bilinear operation $[-,-]$ such that the following version of the Jacobi identity is satisfied:

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]].$$

**Corollary 2.4.4.** *Assume* $\{\!\!\{-,-\}\!\!\}$ *is a double bracket on* $A$*. Then the following identity holds in* $A$*:*

$$\{a,\{b,c\}\} - \{\{a,b\},c\} - \{b,\{a,c\}\!\!\}. = \{a,b,c\} - \{b,a,c\}. \tag{2.19}$$

*If* $\{-,-,-\} = 0$ *(e.g. when* $A$ *is a double Poisson algebra), then* $A$ *becomes a left Loday algebra.*

*Proof.* Applying the multiplication map to (2.17) we obtain (2.19) which in case $\{-,-,-\} = 0$ yields

$$\{a,\{b,c\}\} = \{\{a,b\},c\} + \{b,\{a,c\}\!\!\},$$

i.e. the defining equation for a left Loday algebra. □

*Remark* 2.4.5. Jean-Louis Loday asks if the relations between $\{-,-\}$ and $\{-,-,-\}$ can be explained by some kind of Leibniz-brace-algebra structure on $A$.

**Corollary 2.4.6.** *If* $\{\!\!\{-,-\}\!\!\}$ *is a double bracket on* $A$ *such that* $\{-,-,-\} = 0$*, then* $A/[A,A]$ *equipped with the bracket* $\{-,-\}$ *is a Lie algebra.*

2.5. **Induced brackets.** In this section we discuss the compatibility of double brackets and double Poisson brackets with some natural constructions.

**Proposition 2.5.1.** *Assume that* $A, \{\!\{-,-\}\!\}$, $A', \{\!\{-,-\}\!\}$ *are double brackets over* $B$. *Then there is a unique double bracket on* $A *_B A'$ *extending the double brackets on* $A$ *and* $A'$ *with the additional property*

$$\forall a \in A, \forall a' \in A' : \{\!\{a, a'\}\!\} = 0.$$

*If* $A$, $A'$ *are double Poisson, then so is* $A *_B A'$.

*Proof.* Easy. □

A special case is when the bracket on $A'$ is trivial. In that case we obtain an $A'$ linear bracket on $A *_B A'$. Hence we obtain:

**Corollary 2.5.2.** *Double (Poisson) brackets are compatible with base change.*

Quite similarly we have

**Proposition 2.5.3.** *Double (Poisson) brackets are compatible with universal localization. I.e. if* $S \subset A$ *and* $\{\!\{-,-\}\!\}$ *is a double bracket on* $A$, *then there is a unique extended double bracket on* $A_S$. *If* $A$ *is double Poisson, then so is* $A_S$.

*Proof.* Left to the reader. □

**Proposition 2.5.4.** *Assume that* $e \in B$ *is an idempotent. Then a* $B$*-linear double bracket* $\{\!\{-,-\}\!\}$ *on* $A$ *induces an* $eBe$*-linear double bracket on* $eAe$. *If* $\{\!\{-,-\}\!\}$ *is double Poisson, then so is the induced bracket on* $eAe$.

*Proof.* This follows from

$$\{\!\{eae, ebe\}\!\} = e\{\!\{a,b\}\!\}'e \otimes e\{\!\{a,b\}\!\}''e.$$ □

**Example 2.5.5.** In [15] Lieven Le Bruyn and Geert Van de Weyer define $\sqrt[n]{A}$ as the $B$-algebra which represents the functor of $B$-algebras to sets given by

$$\operatorname{Hom}_B(A, M_n(-)).$$

A concrete realization of $\sqrt[n]{A}$ is $e(A *_B M_n(B))e$ where $e$ is the upper left corner idempotent of $M_n(k)$. So we obtain that if $A, \{\!\{-,-\}\!\}$ is a double Poisson algebra over $B$, then there is an induced double Poisson structure on $\sqrt[n]{A}$.

Another realization of $\sqrt[n]{A}$ is the algebra generated by symbols $a_{ij}$ for $a \in A$ and $i, j = 1, \ldots, n$ which are linear in $a$ and which satisfy in addition the following relations:

$$\begin{aligned} a_{ij}b_{jk} &= (ab)_{ik}, \\ b_{ij} &= \delta_{ij} b \qquad \text{if } b \in B, \end{aligned}$$

where we sum over repeated indices. In this realization the double bracket on $\sqrt[n]{A}$ is given by the formula

$$\{\!\{a_{ij}, b_{uv}\}\!\} = \{\!\{a,b\}\!\}'_{uj} \otimes \{\!\{a,b\}\!\}''_{iv}. \tag{2.20}$$

Now we discuss "fusion". This is a procedure which allows one to collapse two idempotents into one. In the case of quivers it amounts to gluing vertices. This is explained in more detail in §6.

Assume that $e_1, e_2 \in B$ are orthogonal idempotents. Construct $\bar{A}$ from $A$ by formally adjoining two variables $e_{12}$, $e_{21}$ satisfying the usual matrix relations $e_{uv}e_{wt} = \delta_{wv}e_{ut}$ (with $e_{ii} = e_i$). We have

$$\bar{A} = A *_{ke_1 \oplus ke_2 \oplus k\mu} (M_2(k) \oplus k\mu) = A *_B \bar{B}$$

where $\mu = 1 - e_1 - e_2$. The fusion algebra of $A$ along $e_1, e_2$ is defined as

$$A^f = \epsilon \bar{A} \epsilon$$

where $\epsilon = 1 - e_2$. Clearly $\bar{A}$ is a $\bar{B}$-algebra and $A^f$ is a $B^f$-algebra. Combining Corollary 2.5.2 and Proposition 2.5.4 we obtain:

**Corollary 2.5.6.** *If $A, \{\!\{-,-\}\!\}$ is a double Poisson algebra over $B$, then there are associated double Poisson algebras $\bar{A}$ and $A^f$ over $\bar{B}$ and $B^f$ respectively.*

We recall the definition of the trace map. Let $e \in B$ be an idempotent such that $BeB = B$. Write $1 = \sum_i p_i e q_i$. Then we put

$$\mathrm{tr} : A \to eAe : a \mapsto \sum_i e q_i a p_i e.$$

The trace map depends on the chosen decomposition $1 = \sum_i p_i e q_i$. However it gives a uniquely defined isomorphism

$$A/[A,A] \to eAe/[eAe, eAe]$$

which is an inverse to the obvious map

$$eAe/[eAe, eAe] \to A/[A,A].$$

**Proposition 2.5.7.** *We have for $a, b \in A$,*

$$\mathrm{tr}\{a,b\} = \{\mathrm{tr}(a), \mathrm{tr}(b)\}.$$

*Proof.* This is a simple computation:

$$\begin{aligned}
\{\!\{\mathrm{tr}(a), \mathrm{tr}(b)\}\!\} &= \{\!\{\sum_i e q_i a p_i e, \sum_j e q_j b p_j e\}\!\} \\
&= \sum_{i,j} e q_j (e q_i * \{\!\{a,b\}\!\} * p_i e) p_j e \\
&= \sum_{i,j} e q_j \{\!\{a,b\}\!\}' p_i e \otimes e q_i \{\!\{a,b\}\!\}'' p_j e
\end{aligned}$$

and hence

$$\begin{aligned}
\{\mathrm{tr}(a), \mathrm{tr}(b)\} &= \sum_{i,j} e q_j \{\!\{a,b\}\!\}' p_i e q_i \{\!\{a,b\}\!\}'' p_j e \\
&= \sum_j e q_j \{a,b\} p_j e \\
&= \mathrm{tr}\{a,b\}.
\end{aligned}$$

2.6. **Poisson structures and moment maps.** By $\mathrm{Der}_B(A, A)$ we denote the $B$-derivations $A \to A$ and by $\mathrm{Inn}_B(A, A) \subset \mathrm{Der}_B(A, A)$ the subvector space of inner derivations (i.e. those derivations which are of the form $[a, -]$ for $a$ in the centralizer of $B$). For an arbitrary linear map

$$p : A/[A, A] \to \mathrm{Der}_B(A, A)/\,\mathrm{Inn}_B(A, A) \tag{2.21}$$

and for $\bar{a}, \bar{b} \in A/[A, A]$ put

$$\{\bar{a}, \bar{b}\}_p = \overline{p(\bar{a})\tilde{}\,(b)} \in A/[A, A] \tag{2.22}$$

where $p(\bar{a})\tilde{}$ is an arbitrary lift of $p(\bar{a})$. It is easy to show that this is well-defined. Following Crawley-Boevey [5, Rem. 1.3] we define

**Definition 2.6.1** ([5])**.** If $p$ is as in (2.21), then we say that $p$ is a *Poisson structure* on $A$ over $B$ if $\{\bar{a}, \bar{b}\}_p$ is a Lie bracket on $A/[A, A]$.

We then have the following result.

**Lemma 2.6.2.** *If $\{\!\{-, -\}\!\}$ is a double Poisson bracket on $A$, then the map*

$$p : A/[A, A] \to \mathrm{Der}_B(A)/\,\mathrm{Inn}_B(A) : \bar{a} \mapsto \overline{\{a, -\}}$$

*defines a Poisson bracket of $A$ over $B$.*

*Proof.* This is a combination of (2.15) and Corollary 2.4.6. □

*Remark* 2.6.3. Note that a Poisson structure is in fact a map

$$HH_0(A) \to HH^1(A)$$

where "$HH$" denotes Hochschild (co)homology.

The following definition will be motivated afterward. We assume that $B = ke_1 \oplus \cdots \oplus ke_n$ is semi-simple.

**Definition 2.6.4.** Let $A, \{\!\{-, -\}\!\}$ be a double Poisson algebra. A *moment map* for $A$ is an element $\mu = (\mu_i)_i \in \bigoplus_i e_i A e_i$ such that for all $a \in A$ we have

$$\{\!\{\mu_i, a\}\!\} = ae_i \otimes e_i - e_i \otimes e_i a.$$

A double Poisson algebra equipped with a moment map is said to be a *Hamiltonian algebra*.

One application of a moment map is the following.

**Proposition 2.6.5.** *Let $A, \{\!\{-, -\}\!\}, \mu$ be a Hamiltonian algebra. Fix $\lambda \in B$ and put $\bar{A} = A/(\mu - \lambda)$. Then the associated Poisson structure*

$$p : A/[A, A] \to \mathrm{Der}_B(A, A)/\,\mathrm{Inn}_B(A, A)$$

*descends to a Poisson structure on $\bar{A}/[\bar{A}, \bar{A}]$.*

*Proof.* Left to the reader. □

It is easy to verify that the existence of a moment map is compatible with the induction procedures described in §2.5. For further reference we record the following.

**Proposition 2.6.6** (Fusion)**.** *Assume that $A, \{\!\{-,-\}\!\}$ is a double Poisson algebra over $B$ with moment map $\mu$. Then $A^f$ considered as a double Poisson algebra over $B^f = ke_1 \oplus ke_3 \oplus \cdots \oplus ke_n$ has a moment map given by*

$$\mu_i^f = \begin{cases} \mu_1 + e_{12}\mu_2 e_{21} & \text{if } i = 1, \\ \mu_i & \text{if } i \geq 3. \end{cases}$$

*Proof.* Left to the reader. □

2.7. **Super version.** As usual it is possible to define $\mathbb{Z}$-graded super versions of double Poisson algebras. As usual the signs are determined by the Koszul convention. We write $|a|$ for the degree of a homogeneous element $a$ of a graded vector space.

If $V_i$, $i = 1, \ldots, n$ are graded vector spaces and $a = a_1 \otimes \cdots \otimes a_n$ is a homogeneous element of $V_1 \otimes \cdots \otimes V_n$ and $s \in S_n$, then

$$\sigma_s(a) = (-1)^t a_{s^{-1}(1)} \otimes \cdots \otimes a_{s^{-1}(n)}$$

where

$$t = \sum_{\substack{i<j \\ s^{-1}(i)>s^{-1}(j)}} |a_{s^{-1}(i)}||a_{s^{-1}(j)}|.$$

Let $D$ be a graded algebra. We will call $D$ *a double Gerstenhaber algebra* if it is equipped with a graded bilinear map

$$\{\!\{-,-\}\!\} : D \otimes D \to D \otimes D$$

of degree $-1$ such that the following identities hold:

$$\begin{aligned} \{\!\{a, bc\}\!\} &= (-1)^{(|a|-1)|b|} b \{\!\{a, c\}\!\} + \{\!\{a, b\}\!\} c, \\ \{\!\{a, b\}\!\} &= -\sigma_{(12)}(-1)^{(|a|-1)(|b|-1)} \{\!\{b, a\}\!\}, \\ 0 &= \{\!\{a, \{\!\{b, c\}\!\}\}\!\}_L + (-1)^{(|a|-1)(|b|+|c|)} \sigma_{(123)} \{\!\{b, \{\!\{c, a\}\!\}\}\!\}_L \\ &\quad + (-1)^{(|c|-1)(|a|+|b|)} \sigma_{(132)} \{\!\{c, \{\!\{a, b\}\!\}\}\!\}_L. \end{aligned}$$

We will omit the routine verifications of graded generalizations of the ungraded statements we have proved. In particular if $\{\!\{a, b\}\!\} = \{\!\{a, b\}\!\}' \{\!\{a, b\}\!\}''$, then as in the ungraded case one proves that if $D, \{\!\{-,-\}\!\}$ is a double Gerstenhaber algebra, then $D/[D, D][1]$ equipped with $\{-,-\}$ is a graded Lie algebra.

## 3. Poly-vector fields and the double Schouten-Nijenhuis bracket

3.1. **Generalities.** In this section we assume that $A$ is a finitely generated $B$-algebra. Following [6] we define

$$D_{A/B} = \mathrm{Hom}_{A^e}(\Omega_{A/B}, A \otimes A) = \mathrm{Der}_B(A, A \otimes A).$$

The bimodule structure on $A \otimes A$ is the outer structure. The surviving inner bimodule structure on $A^{\otimes 2}$ makes $D_{A/B}$ into an $A$-bimodule. Put $D_B A = T_A D_{A/B}$.

3.2. **The double Schouten-Nijenhuis bracket.** Our aim is to define the structure of a double Gerstenhaber algebra on $D_B A$.

**Proposition 3.2.1.** *Let $\delta, \Delta \in D_{A/B}$. Then*

$$\{\!\{\delta,\Delta\}\!\}\tilde{}_l = (\delta\otimes 1)\Delta - (1\otimes\Delta)\delta,$$
$$\{\!\{\delta,\Delta\}\!\}\tilde{}_r = (1\otimes\delta)\Delta - (\Delta\otimes 1)\delta = -\{\!\{\Delta,\delta\}\!\}\tilde{}_l$$

*define $B$-derivations $A \to A^{\otimes 3}$, where the bimodule structure on $A^{\otimes 3}$ is the outer structure.*

*Proof.* Left to the reader. □

Since $\Omega_{A/B}$ is finitely generated we obtain

$$\mathrm{Der}_B(A, A^{\otimes 3}) \cong \mathrm{Hom}_{A^e}(\Omega_{A/B}, A\otimes A)\otimes A.$$

We will view $\{\!\{\delta,\Delta\}\!\}\tilde{}_l$ and $\{\!\{\delta,\Delta\}\!\}\tilde{}_r$ as elements of $D_{A/B}\otimes_k A$ and $A\otimes_k D_{A/B}$ respectively. To this end we define

$$\{\!\{\delta,\Delta\}\!\}_l = \tau_{(23)}\circ\{\!\{\delta,\Delta\}\!\}\tilde{}_l,$$
$$\{\!\{\delta,\Delta\}\!\}_r = \tau_{(12)}\circ\{\!\{\delta,\Delta\}\!\}\tilde{}_r$$

and we write

$$\{\!\{\delta,\Delta\}\!\}_l = \{\!\{\delta,\Delta\}\!\}_l'\otimes\{\!\{\delta,\Delta\}\!\}_l'',$$
$$\{\!\{\delta,\Delta\}\!\}_r = \{\!\{\delta,\Delta\}\!\}_r'\otimes\{\!\{\delta,\Delta\}\!\}_r'' \tag{3.1}$$

with $\{\!\{\delta,\Delta\}\!\}_l'', \{\!\{\delta,\Delta\}\!\}_r' \in A$, $\{\!\{\delta,\Delta\}\!\}_l', \{\!\{\delta,\Delta\}\!\}_r''$ in $D_{A/B}$. An easy verification shows that

$$\{\!\{\delta,\Delta\}\!\}_r = -\{\!\{\Delta,\delta\}\!\}_l^\circ. \tag{3.2}$$

For $a,b\in A$, $\delta,\Delta\in D_{A/B}$ we put

$$\{\!\{a,b\}\!\} = 0,$$
$$\{\!\{\delta,a\}\!\} = \delta(a),$$
$$\{\!\{\delta,\Delta\}\!\} = \{\!\{\delta,\Delta\}\!\}_l + \{\!\{\delta,\Delta\}\!\}_r. \tag{3.3}$$

Here we consider the righthand sides of (3.3) as elements of $D_B A$.

**Theorem 3.2.2.** *The definitions in* (3.3) *define a unique structure of a double Gerstenhaber algebra on $D_B A$.*

*Proof.* Uniqueness is clear. Furthermore it is easy to see that the derivation property and anti-symmetry of $\{\!\{-,-\}\!\}$ have to be checked only on generators. Using (a graded version of) Proposition 2.3.1 and (2.6) it follows that we have to check the double Jacobi identity only on generators also. Thus we need to check the following list of identities. For $a\in A$, $\alpha,\beta,\gamma\in D_{A/B}$ we need

$$\{\!\{\alpha,\beta\}\!\} = -\sigma_{(12)}\{\!\{\beta,\alpha\}\!\}, \tag{3.4}$$
$$\{\!\{\alpha,a\beta\}\!\} = a\,\{\!\{\alpha,\beta\}\!\} + \{\!\{\alpha,a\}\!\}\beta, \tag{3.5}$$
$$\{\!\{\alpha,\beta a\}\!\} = \{\!\{\alpha,\beta\}\!\}a + \beta\{\!\{\alpha,a\}\!\}, \tag{3.6}$$
$$0 = \{\!\{a,\{\!\{\alpha,\beta\}\!\}\}\!\}_L + \sigma_{(123)}\{\!\{\alpha,\{\!\{\beta,a\}\!\}\}\!\}_L + \sigma_{(132)}\{\!\{\beta,\{\!\{a,\alpha\}\!\}\}\!\}_L, \tag{3.7}$$
$$0 = \{\!\{\alpha,\{\!\{\beta,\gamma\}\!\}\}\!\}_L + \sigma_{(123)}\{\!\{\beta,\{\!\{\gamma,\alpha\}\!\}\}\!\}_L + \sigma_{(132)}\{\!\{\gamma,\{\!\{\alpha,\beta\}\!\}\}\!\}_L. \tag{3.8}$$

Identities (3.4)-(3.6) take place in $D_{A/B}\otimes A\oplus A\otimes D_{A/B}$. (3.7) takes place in $A^{\otimes 3}$ and (3.8) takes place in $D_{A/B}\otimes A\otimes A\oplus A\otimes D_{A/B}\otimes A\oplus A\otimes A\otimes D_{A/B}$. Taking into account cyclic symmetry it is sufficient to prove (3.4) and (3.8) after projection on the first factor. So it is sufficient to prove the following identities:

(3.4-1) $$\{\!\{\alpha,\beta\}\!\}_l = -\sigma_{(12)}\{\!\{\beta,\alpha\}\!\}_r,$$

(3.5-1) $$\{\!\{\alpha,a\beta\}\!\}_l = a\,\{\!\{\alpha,\beta\}\!\}_l,$$

(3.5-2) $$\{\!\{\alpha,a\beta\}\!\}_r = a\,\{\!\{\alpha,\beta\}\!\}_r + \{\!\{\alpha,a\}\!\}\beta,$$

(3.6-1) $$\{\!\{\alpha,\beta a\}\!\}_r = \{\!\{\alpha,\beta\}\!\}_r a,$$

(3.6-2) $$\{\!\{\alpha,\beta a\}\!\}_l = \{\!\{\alpha,\beta\}\!\}_l a + \beta\,\{\!\{\alpha,a\}\!\},$$

(3.7-1) $$0 = \{\!\{a,\{\!\{\alpha,\beta\}\!\}_l\}\!\}_L + \sigma_{(123)}\{\!\{\alpha,\{\!\{\beta,a\}\!\}\}\!\}_L + \sigma_{(132)}\{\!\{\beta,\{\!\{a,\alpha\}\!\}\}\!\}_L,$$

(3.8-1) $$0 = \{\!\{\alpha,\{\!\{\beta,\gamma\}\!\}_l\}\!\}_{l,L} + \sigma_{(123)}\{\!\{\beta,\{\!\{\gamma,\alpha\}\!\}_r\}\!\}_L + \sigma_{(132)}\{\!\{\gamma,\{\!\{\alpha,\beta\}\!\}_l\}\!\}_{r,L}.$$

We now check these identities systematically. By convention $\sigma$ permutes factors in tensor products of $D_BA$ and $\tau$ permutes factors in tensor products of $A$ (so no signs occur in $\tau$).

(3.4-1) This is (3.2).
(3.5-1) We compute

$$\begin{aligned}\{\!\{\alpha,a\beta\}\!\}_l &= \tau_{(23)}((\alpha\otimes 1)(a\beta)-(1\otimes a\beta)\alpha),\\ &= \tau_{(23)}((1\otimes 1\otimes a\cdot -)((\alpha\otimes 1)(\beta)-(1\otimes\beta)\alpha)),\\ &= (1\otimes a\cdot -\otimes 1)\tau_{(23)}((\alpha\otimes 1)(\beta)-(1\otimes\beta)\alpha),\\ &= a\,\{\!\{\alpha,\beta\}\!\}_l.\end{aligned}$$

(3.5-2) We compute

$$\begin{aligned}\{\!\{\alpha,a\beta\}\!\}_r &= \tau_{(12)}((1\otimes\alpha)(a\beta)-(a\beta\otimes 1)\alpha)\\ &= \tau_{(12)}((1\otimes a\cdot -\otimes 1)((1\otimes\alpha)(\beta)-(\beta\otimes 1)\alpha)) + \tau_{(12)}\epsilon\\ &= (a\cdot -\otimes 1\otimes 1)\tau_{(12)}((1\otimes\alpha)(\beta)-(\beta\otimes 1)\alpha) + \tau_{(12)}\epsilon\end{aligned}$$

where $\epsilon$ is a map $A\to A^{\otimes 3}$ satisfying for $c\in A$:

$$\epsilon(c) = \beta(c)'\otimes\alpha(a)'\otimes\alpha(a)''\beta(c)''$$

and thus

$$\tau_{(12)}\epsilon(c) = (\alpha(a)\beta)(c).$$

Here $\alpha(a)$ is to be interpreted as an element of $A\otimes A\subset D_BA\otimes D_BA$, and $\beta\in D_BA$ acts on $D_BA\otimes D_BA$ through the outer bimodule structure.

Thus we obtain

$$\begin{aligned}\{\!\{\alpha,a\beta\}\!\}_r &= (a\cdot -\otimes 1\otimes 1)\{\!\{\alpha,\beta\}\!\}_r + \alpha(a)\beta\\ &= a\,\{\!\{\alpha,\beta\}\!\}_r + \alpha(a)\beta\\ &= a\,\{\!\{\alpha,\beta\}\!\}_r + \{\!\{\alpha,a\}\!\}\beta.\end{aligned}$$

(3.6-1)(3.6-2) These are similar to (3.5-1) and (3.5-2).
(3.7-1) We compute the individual terms:

$$\begin{aligned}
\{\!\{a,\{\!\{\alpha,\beta\}\!\}_l\}\!\}_L &= \{\!\{a,\{\!\{\alpha,\beta\}\!\}_l'\}\!\}\otimes\{\!\{\alpha,\beta\}\!\}_l'' \\
&= -\tau_{(12)}\{\!\{\alpha,\beta\}\!\}_l'(a)\otimes\{\!\{\alpha,\beta\}\!\}_l'' \\
&= -\tau_{(12)}\tau_{(23)}((\alpha\otimes 1)\beta-(1\otimes\beta)\alpha)(a) \\
&= -\tau_{(123)}((\alpha\otimes 1)\beta-(1\otimes\beta)\alpha)(a), \\
\sigma_{(123)}\{\!\{\alpha,\{\!\{\beta,a\}\!\}\}\!\}_L &= \tau_{(123)}(\alpha\otimes 1)\beta(a), \\
\sigma_{(132)}\{\!\{\beta,\{\!\{a,\alpha\}\!\}\}\!\}_L &= -\tau_{(132)}(\beta\otimes 1)\tau_{(12)}\alpha(a) \\
&= -\tau_{(132)}\tau_{(132)}(1\otimes\beta)\alpha(a) \\
&= -\tau_{(123)}(1\otimes\beta)\alpha(a).
\end{aligned}$$

The sum of these three terms is indeed zero.
(3.8-1) This is the most tedious computation. We compute again the individual terms:

$$\begin{aligned}
&\{\!\{\alpha,\{\!\{\beta,\gamma\}\!\}_l\}\!\}_{l,L} = \{\!\{\alpha,\{\!\{\beta,\gamma\}\!\}_l'\}\!\}_l\otimes\{\!\{\beta,\gamma\}\!\}'' \\
&\quad= \tau_{(23)}((\alpha\otimes 1)\{\!\{\beta,\gamma\}\!\}_l'-(1\otimes\{\!\{\beta,\gamma\}\!\}_l')\alpha)\otimes\{\!\{\beta,\gamma\}\!\}_l'' \\
&\quad= \tau_{(23)}((\alpha\otimes 1)\{\!\{\beta,\gamma\}\!\}_l-(1\otimes\{\!\{\beta,\gamma\}\!\}_l)\alpha) \\
&\quad= \tau_{(23)}((\alpha\otimes 1\otimes 1)\tau_{(23)}((\beta\otimes 1)\gamma-(1\otimes\gamma)\beta)-1\otimes\tau_{(23)}((\beta\otimes 1)\gamma-(1\otimes\gamma)\beta)\alpha) \\
&\quad= \tau_{(23)}\tau_{(34)}((\alpha\otimes 1\otimes 1)((\beta\otimes 1)\gamma-(1\otimes\gamma)\beta)-1\otimes((\beta\otimes 1)\gamma-(1\otimes\gamma)\beta)\alpha) \\
&\quad= \tau_{(234)}((\alpha\otimes 1\otimes 1)(\beta\otimes 1)\gamma-(\alpha\otimes 1\otimes 1)(1\otimes\gamma)\beta-(1\otimes\beta\otimes 1)(1\otimes\gamma)\alpha \\
&\qquad\quad +(1\otimes 1\otimes\gamma)(1\otimes\beta)\alpha),
\end{aligned}$$

$$\begin{aligned}
\sigma_{(132)}\{\!\{\beta,\{\!\{\gamma,\alpha\}\!\}_r\}\!\}_L &= \tau_{(13)(24)}(\beta\{\!\{\gamma,\alpha\}\!\}_r'\otimes\{\!\{\gamma,\alpha\}\!\}_r'') \\
&= \tau_{(13)(24)}((\beta\otimes 1\otimes 1)\{\!\{\gamma,\alpha\}\!\}_r) \\
&= \tau_{(13)(24)}(\beta\otimes 1\otimes 1)\tau_{(12)}((1\otimes\gamma)\alpha-(\alpha\otimes 1)\gamma) \\
&= \tau_{(13)(24)}\tau_{(132)}(1\otimes\beta\otimes 1)((1\otimes\gamma)\alpha-(\alpha\otimes 1)\gamma) \\
&= \tau_{(234)}((1\otimes\beta\otimes 1)(1\otimes\gamma)\alpha-(1\otimes\beta\otimes 1)(\alpha\otimes 1)\gamma),
\end{aligned}$$

$$\begin{aligned}
&\sigma_{(132)}\{\!\{\gamma,\{\!\{\alpha,\beta\}\!\}_l\}\!\}_{r,L} \\
&\quad= \tau_{(1432)}(\tau_{(12)}((1\otimes\gamma)\{\!\{\alpha,\beta\}\!\}_l'-(\{\!\{\alpha,\beta\}\!\}_l'\otimes 1)\gamma)\otimes\{\!\{\alpha,\beta\}\!\}_l'') \\
&\quad= \tau_{(1432)}\tau_{(12)}((1\otimes\gamma\otimes 1)\{\!\{\alpha,\beta\}\!\}_l-\tau_{(34)}(\{\!\{\alpha,\beta\}\!\}_l\otimes 1)\gamma) \\
&\quad= \tau_{(1432)}\tau_{(12)}((1\otimes\gamma\otimes 1)\tau_{(23)}((\alpha\otimes 1)\beta-(1\otimes\beta)\alpha) \\
&\qquad\quad -\tau_{(34)}\tau_{(23)}(((\alpha\otimes 1)\beta-(1\otimes\beta)\alpha)\otimes 1)\gamma) \\
&\quad= \tau_{(1432)}\tau_{(12)}\tau_{(243)}((1\otimes 1\otimes\gamma)((\alpha\otimes 1)\beta-(1\otimes\beta)\alpha) \\
&\qquad\quad -(((\alpha\otimes 1)\beta-(1\otimes\beta)\alpha)\otimes 1)\gamma) \\
&\quad= \tau_{(234)}((1\otimes 1\otimes\gamma)(\alpha\otimes 1)\beta-(1\otimes 1\otimes\gamma)(1\otimes\beta)\alpha-(\alpha\otimes 1\otimes 1)(\beta\otimes 1)\gamma \\
&\qquad\quad +(1\otimes\beta\otimes 1)(\alpha\otimes 1)\gamma),
\end{aligned}$$

and again the sum of the three terms is zero. □

*Remark* 3.2.3. If we equip $D_B A$ with the associated single bracket $\{-,-\}$, then it follows from the above theorem and Corollary 2.4.4 that $D_B A$ is a Loday algebra.

It is easy to see that the Loday bracket on $D_BA$ is compatible with the map $D_BA \to \operatorname{Der}_B A$ when we equip $\operatorname{Der}_B A$ with the usual commutator bracket.

3.3. **Gauge elements.** We now assume in addition that $B$ is commutative semi-simple, i.e.

$$B = ke_1 \oplus \cdots \oplus ke_n$$

such that $e_i^2 = e_i$. We will define some special elements $E_i$ of $D_{A/B}$ which we call "gauge elements". This terminology will be explained in §7.9. We have

$$E_i(a) = ae_i \otimes e_i - e_i \otimes e_i a.$$

We will also put $E = \sum_i E_i$. Clearly $E_i = e_i E e_i$.

**Proposition 3.3.1.** *For $D \in D_BA$ we have*

$$\{\!\{E_i, D\}\!\} = De_i \otimes e_i - e_i \otimes e_i D. \tag{3.9}$$

*Proof.* Since $D_BA \to D_BA \otimes D_BA : D \to De_i \otimes e_i - e_i \otimes e_i D$ is a graded derivation (of degree zero) it suffices to prove (3.9) for $D = a$, $a \in A$ and $D = \delta$, $\delta \in D_{A/B}$.

We compute

$$\{\!\{E_i, a\}\!\} = E_i(a) = ae_i \otimes e_i - e_i \otimes e_i a$$

and

$$\begin{aligned}
\{\!\{E_i, \delta\}\!\}_l(a) &= \tau_{(23)}((E_i \otimes 1)\delta(a) - (1 \otimes \delta)E_i(a)) \\
&= \tau_{(23)}(\delta(a)' e_i \otimes e_i \otimes \delta(a)'' - e_i \otimes e_i \delta(a)' \otimes \delta(a)'' \\
&\qquad + e_i \otimes \delta(e_i a)' \otimes \delta(e_i a)'') \\
&= (\delta e_i)(a) \otimes e_i \\
&= (\delta e_i \otimes e_i)(a)
\end{aligned}$$

(where in the third line we use the $B$-linearity of $\delta$),

$$\begin{aligned}
\{\!\{E_i, \delta\}\!\}_r(a) &= \tau_{(12)}((1 \otimes E_i)\delta(a) - (\delta \otimes 1)E_i(a)) \\
&= \tau_{(12)}(\delta(a)' \otimes \delta(a)'' e_i \otimes e_i \\
&\qquad - \delta(a)' \otimes e_i \otimes e_i \delta(a)'' - \delta(ae_i)' \otimes \delta(ae_i)'' \otimes e_i) \\
&= -e_i \otimes (e_i \delta)(a) \\
&= -(e_i \otimes e_i \delta)(a).
\end{aligned}$$

Taking the sum of these two expressions and letting $a$ vary we obtain

$$\{\!\{E_i, \delta\}\!\} = \delta e_i \otimes e_i - e_i \otimes e_i \delta.$$ □

3.4. **Morita invariance.** In this section we show that $D_BA/[D_BA, D_BA]$, with its Schouten bracket, is invariant under Morita equivalence. The fact that $D_BA$ is invariant under Morita equivalence was already proved in [6], but for the convenience of the reader we restate the proof. We will only consider the case when the Morita equivalence is given by an idempotent (since this is the only case we will need). It is well known that this implies the general case.

**Lemma 3.4.1.** *Let $M$ be an $A$ bimodule and let $e \in A$ be an idempotent such that $AeA = A$. Then*

$$e(T_AM)e = T_{eAe}(eMe) \tag{3.10}$$

*and furthermore $T_AMeT_AM = T_AM$. Hence $T_{eAe}(eMe)$ is Morita equivalent to $T_AM$.*

*Proof.* Since $AeA = A$ we have $Ae \otimes_{eAe} eA \cong A$. Thus for $A$-bimodules $M$, $N$ we obtain

$$\begin{aligned} e(M \otimes_A N)e &= eA \otimes_A M \otimes_A Ae \otimes_{eAe} eA \otimes_A M \otimes_A Ae \\ &= eMe \otimes_{eAe} eNe. \end{aligned}$$

This implies (3.10). The second assertion is clear since $T_A MeT_A M$ contains $A = AeA$. □

**Lemma 3.4.2.** *Assume $e \in B$ is an idempotent such that $BeB = B$. Then*

$$eD_{A/B}e = D_{eAe/eBe}.$$

*Proof.* There is an obvious map

$$c : eD_{A/B}e \to D_{eAe/eBe}.$$

We have to construct its inverse $c^{-1}$. Write

$$1 = \sum_i p_i e q_i$$

with $p_i, q_i \in B$. Then any element $a \in A$ can be written as

$$\sum_{i,j,k} p_i(eq_i a p_j e)q_j.$$

Let $\delta \in D_{eAe/eBe}$. We put

$$c^{-1}(\delta)(a) = \sum_{i,j,k} p_i \delta(eq_i a p_j e)q_j.$$

It is easy to see that this is a well-defined element of $eD_{A/B}e$ and that $c^{-1}$ is indeed a two sided inverse to $c$. □

Using Lemma 3.4.1 we obtain that there is an isomorphism

$$D_{eBe}(eAe) = eD_B Ae.$$

**Proposition 3.4.3.** *Assume $e \in B$ is an idempotent such that $BeB = B$. Then the Schouten bracket on $D_B A$ restricted to $e(D_B A)e = D_{eBe}(eAe)$ coincides with the Schouten bracket on $D_{eBe}(eAe)$.*

*Proof.* Since $e(D_B A)e = T_{eAe}(eD_{A/B}e)$ it suffices to check that the Schouten bracket on $D_{eAe/eBe}$ and the restricted Schouten bracket on $eD_{A/B}e$ coincide. Since $\delta \in eD_{A/B}e$ restricts to a derivation $eAe \to eAe \otimes eAe$ it is easy to see that both Schouten brackets are given by the same formulas. □

3.5. **Hamiltonian vector fields.** Assume that $A$ is equipped with a $B$-linear double bracket. If $a \in A$, then we write $H_a = \{\!\{a, -\}\!\}$. We call $H_a$ the *Hamiltonian vector field corresponding to $a$.* Using this notation we may write

$$\{\!\{a, b\}\!\} = H_a(b). \tag{3.11}$$

**Proposition 3.5.1.** *The following are equivalent:*

(1) *$\{\!\{-, -\}\!\}$ is a double Poisson bracket.*
(2) *The following identity holds for all $a, b \in A$:*

$$\{\!\{H_a, H_b\}\!\}_l = H_{\{\!\{a,b\}\!\}'} \otimes \{\!\{a, b\}\!\}''.$$

(3) *The following identity holds for all $a, b \in A$:*

$$\{\!\!\{H_a, H_b\}\!\!\}_r = \{\!\!\{a, b\}\!\!\}' \otimes H_{\{\!\!\{a,b\}\!\!\}'''}.$$

(4) *The following identity holds for all $a, b \in A$:*

$$\{\!\!\{H_a, H_b\}\!\!\} = H_{\{\!\!\{a,b\}\!\!\}}$$

*where we use the convention $H_{x'\otimes x''} = H_{x'} \otimes x'' + x' \otimes H_{x''}$.*

*Proof.* We first prove the equivalence of (1) and (2). We have to rewrite the expression for the associated triple bracket:

$$\{\!\!\{a, b, c\}\!\!\} = \{\!\!\{a, \{\!\!\{b, c\}\!\!\}\}\!\!\}_L + \tau_{123}\{\!\!\{b, \{\!\!\{c, a\}\!\!\}\}\!\!\}_L + \tau_{132}\{\!\!\{c, \{\!\!\{a, b\}\!\!\}\}\!\!\}_L. \tag{3.12}$$

For the first term we use

$$\{\!\!\{a, \{\!\!\{b, c\}\!\!\}\}\!\!\}_L = (H_a \otimes 1)H_b(c).$$

For the second term we use

$$\{\!\!\{b, \{\!\!\{c, a\}\!\!\}\}\!\!\}_L = -(H_b \otimes 1)((H_a(c))^\circ)$$

and hence

$$\begin{aligned}\tau_{123}\{\!\!\{b, \{\!\!\{c, a\}\!\!\}\}\!\!\}_L &= -\tau_{123}(H_b \otimes 1)\tau_{12}(H_a(c))\\ &= -\tau_{123}\tau_{132}(1 \otimes H_b)H_a(c)\\ &= -(1 \otimes H_b)H_a(c).\end{aligned}$$

For the third term we use

$$\begin{aligned}\{\!\!\{h, x' \otimes x''\}\!\!\}_L &= \{\!\!\{h, x'\}\!\!\} \otimes x''\\ &= -\{\!\!\{x', h\}\!\!\}^\circ \otimes x''\\ &= -(H_{x'}h)^\circ \otimes x''\end{aligned}$$

and thus

$$\{\!\!\{c, \{\!\!\{a, b\}\!\!\}\}\!\!\}_L = -(H_{\{\!\!\{a,b\}\!\!\}'}c)^\circ \otimes \{\!\!\{a, b\}\!\!\}''$$

and hence

$$\begin{aligned}\tau_{132}\{\!\!\{c, \{\!\!\{a, b\}\!\!\}\}\!\!\}_L &= -\tau_{132}\tau_{12}(H_{\{\!\!\{a,b\}\!\!\}'}c) \otimes \{\!\!\{a, b\}\!\!\}''\\ &= -\tau_{23}(H_{\{\!\!\{a,b\}\!\!\}'}c \otimes \{\!\!\{a, b\}\!\!\}'').\end{aligned}$$

So we get that $\{\!\!\{a, b, c\}\!\!\}$ is equal to

$$\begin{aligned}&(H_a \otimes 1)H_b(c) - (1 \otimes H_b)H_a(c) - \tau_{23}(H_{\{\!\!\{a,b\}\!\!\}'}c \otimes \{\!\!\{a, b\}\!\!\}'')\\ &= \{\!\!\{H_a, H_b\}\!\!\}_{\tilde{l}} - \tau_{23}(H_{\{\!\!\{a,b\}\!\!\}'}c \otimes \{\!\!\{a, b\}\!\!\}'')\\ &= \tau_{23}(\{\!\!\{H_a, H_b\}\!\!\}_l - H_{\{\!\!\{a,b\}\!\!\}'}(-) \otimes \{\!\!\{a, b\}\!\!\}'')(c),\end{aligned}$$

which completes the proof of the equivalence of (1) and (2).

To prove the implication (2)⇒(3) we use (3.2). Assuming (2) we obtain

$$\begin{aligned}\{\!\!\{H_a, H_b\}\!\!\}_r &= -\{\!\!\{H_b, H_a\}\!\!\}_l^\circ\\ &= -\{\!\!\{b, a\}\!\!\}'' \otimes H_{\{\!\!\{b,a\}\!\!\}'}\\ &= \{\!\!\{a, b\}\!\!\}' \otimes H_{\{\!\!\{a,b\}\!\!\}''}.\end{aligned}$$

The implication (3)⇒(2) is proved in the same way. (4) is the sum of (2) and (3). To go back we regard (4) as an identity in $D_B A \otimes A \oplus A \otimes D_B A$. The projection on the two terms yields (2) and (3). □

## 4. The relation between poly-vector fields and brackets

4.1. **Generalities.** We assume that $A$ is a finitely generated $B$-algebra.

**Proposition 4.1.1.** *There is a well-defined linear map*

$$\mu : (D_B A)_n \to \{B\text{-linear } n\text{-brackets on } A\} : Q \mapsto \{\!\{-, \cdots, -\}\!\}_Q, \tag{4.1}$$

*which on* $Q = \delta_1 \cdots \delta_n$ *is given by*

$$\{\!\{-, \cdots, -\}\!\}_Q = \sum_{i=0}^{n-1} (-1)^{(n-1)i} \tau_{(1\cdots n)}^{i} \circ \{\!\{-, \cdots, -\}\!\}\tilde{}_Q \circ \tau_{(1\cdots n)}^{-i}$$

*where*

$$\{\!\{a_1, \cdots, a_n\}\!\}\tilde{}_Q = \delta_n(a_n)'\delta_1(a_1)'' \otimes \delta_1(a_1)'\delta_2(a_2)'' \otimes \cdots \otimes \delta_{n-1}(a_{n-1})'\delta_n(a_n)''. \tag{4.2}$$

*This map factors through* $D_B A/[D_B A, D_B A]$.

*Proof.* This follows easily from the following alternative formula:

$$\{\!\{a_1, \cdots, a_n\}\!\}_{\delta_1\cdots\delta_n} = \sum_{i=0}^{n-1} (-1)^{(n-1)i} \{\!\{a_1, \cdots, a_n\}\!\}\tilde{}_{\delta_{n-i+1}\cdots\delta_n\cdots\delta_1\cdots\delta_{n-i}}. \qquad \square$$

Slightly generalizing [10] let us say that $A/B$ is *smooth* if $A/B$ is left and right flat and $\Omega_{A/B}$ is a projective $A$-bimodule (in addition to $A/B$ being finitely generated).

**Proposition 4.1.2.** *If* $A/B$ *is smooth, then* $\mu$ *is an isomorphism.*

*Proof.* To prove this it will be convenient to work in slightly greater generality.

Let $M$ be an $A$-bimodule. We put $M^* = \operatorname{Hom}_{A^e}(M, A \otimes A)$ where we use the outer bimodule structure on $A^{\otimes 2}$. We view $M^*$ as an $A$-bimodule through the inner bimodule structure on $A^{\otimes 2}$.

We will consider $M^{\otimes n}$ as an $(A^e)^{\otimes n}$-module where the the $i$-th copy of $A^e$ acts on the $i$-th copy of $M$. We will also consider an $(A^e)^{\otimes n}$-module $[A^{\otimes(n+1)}]$ which is equal to $A^{\otimes n+1}$ as a vector space and where the $i$-th copy of $A^e$ act as follows:

$$(a' \otimes a'')(a_1 \otimes \cdots \otimes a_{n+1}) = a_1 \otimes \cdots \otimes a_i a'' \otimes a' a_{i+1} \otimes \cdots \otimes a_{n+1}.$$

All these bimodule structures commute with the outer bimodule structure on $[A^{\otimes(n+1)}]$. There is a morphism of $A$-bimodules

$$\Psi : (M^*)^{\otimes_A n} \to \operatorname{Hom}_{(A^e)^{\otimes n}}(M^{\otimes n}, [A^{\otimes(n+1)}])$$

given by

$$\phi_1 \otimes \cdots \otimes \phi_n \mapsto \big(m_1 \otimes \cdots \otimes m_n \mapsto \phi_1(m_1)'' \otimes \phi_1(m_1)'\phi_2(m_2)'' \otimes \cdots \\ \otimes \phi_{n-1}(m_{n-1})'\phi_n(m_n)'' \otimes \phi_n(m_n)'\big).$$

In case $M$ is a finitely generated projective bimodule then this is an isomorphism. To prove this one may assume $M = A \otimes_k A$, in which case it is easy.

Let $\{A^{\otimes n}\}$ be the $(A^e)^{\otimes n}$-module which is $A^{\otimes n}$ as a vector space and where the $i$-th copy of $A^e$ for $i = 1, \ldots, n-1$ acts as on $[A^{\otimes n+1}]$, but where the $n$-th copy acts by the outer bimodule structure.

If $N$ is an $A$-bimodule, then we have that $N \otimes_{A^e} A = N/[A,N]$. Therefore we denote an element of $N \otimes_{A^e} A$ by $\bar{n}$ where $n \in N$.

The map

$$\overline{a_1 \otimes \cdots \otimes a_{n+1}} \to a_{n+1}a_1 \otimes \cdots \otimes a_n$$

gives an isomorphism

$$[A^{\otimes(n+1)}] \otimes_{A^e} A \cong \{A^{\otimes n}\}.$$

We define $\psi$ as the composition

$$\begin{aligned}(M^*)^{\otimes_A n} \otimes_{A^e} A &\xrightarrow{\Psi\otimes 1} \operatorname{Hom}_{(A^e)^{\otimes n}}(M^{\otimes n}, [A^{\otimes(n+1)}]) \otimes_{A^e} A \\ &\xrightarrow{\text{can.}} \operatorname{Hom}_{(A^e)^{\otimes n}}(M^{\otimes n}, [A^{\otimes(n+1)}] \otimes_{A^e} A) \cong \operatorname{Hom}_{(A^e)^{\otimes n}}(M^{\otimes n}, \{A^{\otimes n}\}).\end{aligned}$$

Explicitly:

$$\begin{aligned}\overline{\phi_1 \otimes \cdots \otimes \phi_n} \mapsto \big(m_1 \otimes \cdots \otimes m_n \mapsto \phi_n(m_n)'\phi_1(m_1)'' \otimes \phi_1(m_1)'\phi_2(m_2)'' \otimes \cdots \\ \otimes \phi_{n-1}(m_{n-1})'\phi_n(m_n)''\big).\end{aligned}$$

It is clear that $\psi$ will also be an isomorphism if $M$ is finitely generated projective. The cyclic group $C_n$ acts on $(M^*)^{\otimes_A n} \otimes_{A^e} A$ by

$$\sigma_{(1\cdots n)}(\overline{\phi_1 \otimes \cdots \otimes \phi_n}) = (-1)^{n-1}\overline{\phi_n \otimes \phi_1 \otimes \cdots \phi_{n-1}}.$$

An easy verification shows that the following diagram is commutative:

$$\begin{array}{ccc} (M^*)^{\otimes_A n} \otimes_{A^e} A & \xrightarrow{\psi} & \operatorname{Hom}_{(A^e)^{\otimes n}}(M^{\otimes n}, \{A^{\otimes n}\}) \\ {\scriptstyle \sigma_{(1\cdots n)}}\downarrow & & \downarrow {\scriptstyle (-1)^{n-1}\tau_{(1\cdots n)}\circ - \circ \tau_{(1\cdots n)}^{-1}} \\ (M^*)^{\otimes_A n} \otimes_{A^e} A & \xrightarrow{\psi} & \operatorname{Hom}_{(A^e)^{\otimes n}}(M^{\otimes n}, \{A^{\otimes n}\}). \end{array}$$

Let inv and coinv denote respectively the signed invariants and coinvariants for the action of $C_n$. We view $T_A M^*$ as a graded ring with $M^*$ in degree 1. In that case we have

$$(T_A M^*/[T_A M^*, T_A M^*])_n = \operatorname{coinv}\big((M^*)^{\otimes_A n} \otimes_{A^e} A\big)$$

where $[-,-]$ means signed commutators.

We define $\mu$ as the composition of the maps

$$\begin{aligned}(T_A M^*/[T_A M^*, T_A M^*])_n &= \operatorname{coinv}\big((M^*)^{\otimes_A n} \otimes_{A^e} A\big) \\ &\xrightarrow{\psi} \operatorname{coinv}\operatorname{Hom}_{(A^e)^{\otimes n}}(M^{\otimes n}, \{A^{\otimes n}\}) \xrightarrow[\cong]{\text{trace}} \operatorname{inv}\operatorname{Hom}_{(A^e)^{\otimes n}}(M^{\otimes n}, \{A^{\otimes n}\}).\end{aligned}$$

If $M$ is finitely generated projective, then $\mu$ is an isomorphism. $\overline{\phi_1 \otimes_A \cdots \otimes_A \phi_n}$ is mapped under $\mu$ to

$$\sum_i (-1)^{(n-1)i} \tau_{(1\cdots n)}^{i} \circ \Phi \circ \tau_{(1\cdots n)}^{-i} \tag{4.3}$$

where

$$\Phi(m_1 \otimes \cdots \otimes m_n) = \phi_n(m_n)'\phi_1(m_1)'' \otimes \phi_1(m_1)'\phi_2(m_2)'' \otimes \cdots \otimes \phi_{n-1}(m_{n-1})'\phi_n(m_n)''.$$

Now consider the case $M = \Omega_{A/B}$. In that case there is an isomorphism

$$\operatorname{inv}\operatorname{Hom}_{(A^e)^{\otimes n}}(\Omega^{\otimes n}, \{A^{\otimes n}\}) \cong \{B\text{-linear } n\text{-brackets on } A\},$$

which maps $\theta \in \operatorname{Hom}_{(A^e)^{\otimes n}}(\Omega^{\otimes n}, \{A^{\otimes n}\})$ to the bracket

$$\{\!\{a_1, \dots, a_n\}\!\} = \theta(da_1 \otimes \cdots \otimes da_n).$$

Composing this identification with the map $\mu$ defined by (4.3) gives us precisely (4.1). $\square$

4.2. **Compatibility.**

**Proposition 4.2.1.** *For $Q \in (D_B A)_n$ the following identity holds:*

$$\{\!\{a_1, \dots, a_n\}\!\}_Q = (-1)^{\frac{n(n-1)}{2}} \{\!\{a_1, \dots, \{\!\{a_{n-1}, \{Q, a_n\}\}\!\}_L \cdots\}\!\}_L.$$

*Proof.* It suffices to prove this for $Q = \delta_1 \cdots \delta_n$ with $\delta_i \in D_B A$. We compute

$$\{\delta_1 \cdots \delta_n, a_n\} = (-1)^{n-1}\delta_1(a_n)'\delta_2 \cdots \delta_n \delta_1(a_n)'' + (-1)^{n-2}\delta_2(a_n)'\delta_3 \cdots \delta_n\delta_1\delta_2(a_n)'' \\ + \cdots + \delta_n(a_n)'\delta_1 \cdots \delta_{n-1}\delta_n(a_n)''.$$

We concentrate on the last term. The other terms are obtained by cyclically permuting the $\delta$'s. We find

$$\begin{aligned} &\{\!\{a_1, \dots, \{\!\{a_{n-1}, \delta_n(a_n)'\delta_1 \cdots \delta_{n-1}\delta_n(a_n)''\}\!\}_L \cdots\}\!\}_L \\ &\quad = (-1)^{\frac{n(n-1)}{2}} \delta_n(a_n)'\delta_1(a_1)'' \otimes \delta_1(a_1)'\delta_2(a_2)'' \otimes \delta_2(a_2)' \\ &\qquad\qquad \cdots \delta_{n-1}(a_{n-1})'' \otimes \delta_{n-1}(a_{n-1})'\delta_n(a_n)'' \\ &\quad = (-1)^{\frac{n(n-1)}{2}} \{\!\{a_1, \dots, a_n\}\!\}^{\sim}_{\delta_1 \cdots \delta_n \delta_n}. \end{aligned}$$

We find

$$\{\!\{a_1, \dots, \{\!\{a_{n-1}, \{\delta_1 \cdots \delta_n, a_n\}\}\!\}_L \cdots\}\!\}_L = (-1)^{\frac{n(n-1)}{2}} \{\!\{a_1, \dots, a_n\}\!\}_{\delta_1 \cdots \delta_n},$$

which is what we want. $\square$

**Proposition 4.2.2.** *Let $P \in (D_B A)_2$. We have the following identity for $a, b, c \in A$:*

$$-(1/2)\{\!\{a, \{\!\{b, \{\!\{P, P\}, c\}\!\}\}\!\}\}_L = \{\!\{a, \{\!\{b, c\}\!\}_P\}\!\}_{P,L} + \tau_{(123)}\{\!\{b, \{\!\{c, a\}\!\}_P\}\!\}_{P,L} \\ + \tau_{(132)}\{\!\{c, \{\!\{a, b\}\!\}_P\}\!\}_{P,L}.$$

*Proof.* By (the graded version of) (2.19) we have

$$\{\!\{, P\}, c\} = 2\{P, \{P, c\}\!\}$$

and by (2.17),

$$\{\!\{b, \{P, \{P, c\}\!\}\}\!\} = -\{P, \{\!\{b, \{P, c\}\!\}\}\!\} + \{\!\{\{P, b\}, \{P, c\}\!\}\}.$$

We now apply $\{\!\{a, -\}\!\}_L$ to the individual terms:

$$\begin{aligned} \{\!\{a, \{P, \{\!\{b, \{P, c\}\!\}\}\!\}\}\!\}_L &= \{\!\{a, \{P, \{\!\{b, \{P, c\}\!\}'\}\!\}\} \otimes \{\!\{b, \{P, c\}\!\}''\\ &= \{\!\{\{P, a\}, \{\!\{b, \{P, c\}\!\}'\}\!\} \otimes \{\!\{b, \{P, c\}\!\}''\\ &= \{\!\{a, \{\!\{b, c\}\!\}_P\}\!\}_{P,L}, \end{aligned}$$

where in the second line we have used the graded version of (2.17) in the form of the formula

$$\{\!\{\{P, a\}, x\} = \{\!\{a, \{P, x\}\!\}\}. \tag{4.4}$$

In the third line we have once more used this identity.

Recall that the double bracket on $D_BA$ is odd. This explains the signs.

A similar computation yields

$$\begin{aligned}\{\!\{a, \{\!\{\{P,b\},\{P,c\}\}\!\}\}\!\}_L &= -\sigma_{(123)}\{\!\{\{P,b\},\{\!\{\{P,c\},a\}\!\}\}\!\}_L \\ &\quad - \sigma_{(132)}\{\!\{\{P,c\},\{\!\{a,\{P,b\}\}\!\}\}\!\}_L \\ &= -\tau_{(123)}\{\!\{b,\{\!\{c,a\}\!\}_P\}\!\}_{P,L} - \tau_{(132)}\{\!\{c,\{\!\{a,b\}\!\}_P\}\!\}_{P,L}.\end{aligned}$$

Collecting everything proves the proposition. □

Summarizing we obtain

**Theorem 4.2.3.** *Let $P \in (D_BA)_2$. Then one has*

$$\begin{aligned}\{\!\{a,b,c\}\!\}_{\frac{1}{2}\{P,P\}} = \{\!\{a,\{\!\{b,c\}\!\}_P\}\!\}_{P,L} + \tau_{(123)}\{\!\{b,\{\!\{c,a\}\!\}_P\}\!\}_{P,L} \\ + \tau_{(132)}\{\!\{c,\{\!\{a,b\}\!\}_P\}\!\}_{P,L}.\end{aligned}$$

4.3. **The trace map.** As above let $e \in B$ be an idempotent such that $BeB = B$. We have an associated trace map

$$\begin{aligned}\mathrm{tr} : D_BA/[D_BA, D_BA] &\to eD_BAe/[eD_BAe, eD_BAe] \\ &= D_{eBe}(eAe)/[D_{eBe}(eAe), D_{eBe}(eAe)]\end{aligned}$$

respecting $\{-,-\}$ by Propositions 2.5.7 and 3.4.3.

Furthermore one has

$$\{\!\{-,\cdots,-\}\!\}_Q = \{\!\{-,\cdots,-\}\!\}_{\mathrm{tr}(Q)}$$

since $\mathrm{tr}(Q) = \sum_i eq_iQp_ie$ (with the notation of §2.5) and hence

$$\begin{aligned}\{\!\{-,\cdots,-\}\!\}_{\mathrm{tr}(Q)} &= \{\!\{-,\cdots,-\}\!\}_{\sum_i eq_iQp_ie} \\ &= \{\!\{-,\cdots,-\}\!\}_{\sum_i p_ieq_iQ} \\ &= \{\!\{-,\cdots,-\}\!\}_Q.\end{aligned}$$

4.4. **Differential double Poisson algebras.**

**Definition 4.4.1.** We say that $A$ is a *differential double Poisson* algebra (DDP) over $B$ if it is equipped with an element $P \in (D_BA)_2$ (a *differential double Poisson bracket*) such that

$$\{P,P\} = 0 \mod [D_BA, D_BA]. \tag{4.5}$$

If $A, P$ is a differential double Poisson algebra, then by Theorem 4.2.3 $A$ is a double Poisson algebra with double bracket $\{\!\{-,-\}\!\}_P$. From Proposition 4.1.2 it follows that in the smooth case the notions of differential double Poisson algebra and double Poisson algebra are equivalent. This is not true in the non-smooth case as the following example shows.

**Example 4.4.2.** Let $A = k[\epsilon]/(\epsilon^2)$, $B = k$. According to Example 2.3.3 $A$ has a double Poisson bracket given by $\{\!\{\epsilon,\epsilon\}\!\} = \epsilon \otimes 1 - 1 \otimes \epsilon$.

On the other hand it is easy to check that every element of $\mathrm{Der}(A, A\otimes A)$ sends $\epsilon$ to $k\epsilon\otimes\epsilon$. Using (4.2) we deduce that if $P \in (D_BA)_2$, then $\{\!\{\epsilon,\epsilon\}\!\}_P = 0$. So $\{\!\{-,-\}\!\}$ is not differential.

**Example 4.4.3.** Assume that $A = k[t]$ and $B = k$. Then the double Poisson bracket $\{\!\{t, t\}\!\} = t \otimes 1 - 1 \otimes t$ is obtained from

$$P = t\frac{\partial}{\partial t}\frac{\partial}{\partial t}$$

where $\partial/\partial t$ is the double derivation defined by $\partial t/\partial t = 1 \otimes 1$.

**Proposition 4.4.4.** *If $P \in (D_B A)_2$ is a differential double Poisson bracket, then $\mu \in \bigoplus_i e_i A e_i$ is a moment map (cf. Definition 2.6.4) for $\{\!\{-,-\}\!\}_P$ if and only if*

$$\{P, \mu_i\} = -E_i.$$

*Proof.* By Proposition 4.2.1 and (4.4) we have

$$\{\!\{\mu_i, a\}\!\}_P = -\{\!\{\{P, \mu_i\}, a\}\!\} = -\{P, \mu_i\}(a).$$

Thus $\mu$ is indeed a moment map if and only if $\{P, \mu_i\} = -E_i$. □

It seems logical to call a differential double Poisson algebra equipped with a moment map a differential Hamiltonian algebra.

## 5. Double quasi-Poisson algebras

We now introduce a twisted version of double Poisson algebras. For simplicity we assume throughout that $B = ke_1 \oplus \cdots \oplus ke_n$ is semi-simple.

### 5.1. General definitions.

**Definition 5.1.1.** A *double quasi-Poisson bracket* on $A$ (over $B$) is a $B$-linear bracket $\{\!\{-,-\}\!\}$ such that

$$\{\!\{-,-,-\}\!\} = \frac{1}{12}\sum_i \{\!\{-,-,-\}\!\}_{E_i^3}.$$

We say that $A$ is a *double quasi-Poisson algebra* over $A$ if $A$ is equipped with a double quasi-Poisson bracket.

**Proposition 5.1.2.** *If $A, \{\!\{-,-\}\!\}$ is a double quasi-Poisson algebra, then $A$, $\{-,-\}$ is a left Loday algebra.*

*Proof.* According to Corollary 2.4.4 we have to show

$$\{-,-,-\}_{E_i^3} = 0.$$

This identity is immediate from the definition. □

In a similar way we obtain

**Lemma 5.1.3.** *If $A, \{\!\{-,-\}\!\}$ is a double quasi-Poisson algebra, then $\{-,-\}$ induces a Poisson structure on $A$.*

*Proof.* This is proved as Lemma 2.6.2. □

**Definition 5.1.4.** Let $A, \{\!\{-,-\}\!\}$ be a double quasi-Poisson algebra. A *multiplicative moment map* for $A$ is an element $\Phi = (\Phi_i)_i \in \bigoplus_i e_i A e_i$ such that for all $a \in A$ we have

$$\{\!\{\Phi_i, a\}\!\} = \frac{1}{2}(\Phi_i E_i + E_i \Phi_i)(a).$$

A double quasi-Poisson algebra equipped with a moment map is said to be a *quasi-Hamiltonian algebra.*

**Proposition 5.1.5.** *Let* $A, \{\!\{-,-\}\!\}, \Phi$ *be a quasi-Hamiltonian algebra. Fix* $q \in B^*$ *and put* $\bar{A} = A/(\Phi - q)$*. Then the associated Poisson structure*

$$p : A/[A,A] \to \operatorname{Der}_B(A,A)/\operatorname{Inn}_B(A,A)$$

*descends to a Poisson structure on* $\bar{A}/[\bar{A},\bar{A}]$*.*

*Proof.* Left to the reader. □

5.2. **Differential versions.**

**Definition 5.2.1.** We say that $A$ is a *differential double quasi-Poisson* algebra (DDQP-algebra) over a $B$-algebra $A$ if $A$ is equipped with an element $P \in (D_B A)_2$ (a *differential double quasi-Poisson bracket*) such that

$$\{P,P\} = \frac{1}{6}\sum_{i=1}^{n} E_i^3 \mod [D_B A, D_B A]. \tag{5.1}$$

It follows from Theorem 4.2.3 that a DDQP-algebra is a double quasi-Poisson algebra. For smooth algebras the two notions are equivalent by Proposition 4.1.2.

**Proposition 5.2.2.** *If* $P \in (D_B A)_2$ *is a double quasi-Poisson bracket, then* $\Phi \in \bigoplus_i e_i A e_i$ *is a multiplicative moment map for* $\{\!\{-,-\}\!\}_P$ *if and only if*

$$\{P, \Phi_i\} = -\frac{1}{2}(E_i\Phi_i + \Phi_i E_i)$$

*in* $D_{A/B}$*.*

*Proof.* This is similar to the proof of Proposition 4.4.4. □

A differential quasi-Hamiltonian algebra is a quasi-Hamiltonian algebra where the double bracket comes from an element of $(D_B A)_2$.

5.3. **Calculus on fusion algebras.** In this section the notation is as in §2.5. Our aim is to show that if $A$ is a double quasi-Poisson algebra or a quasi-Hamiltonian algebra over $B$, then the same is true for the fused algebra $A^f$. Why this is to be expected will be explained in §7.10. The methods in this section are basically translations of the methods in [1, §5].

The non-quasi-versions of these methods are easy and have been treated in Corollary 2.5.6 and Proposition 2.6.6. The quasi-case is more tricky notationwise. For this reason we will restrict ourselves to the differential case. I have no doubt that the general case also works but I have not checked it.

Extending derivations yields a canonical map

$$D_{A/B} \to D_{\bar{A}/\bar{B}}$$

and hence a corresponding map

$$(\bar{-}) : D_B A \to D_{\bar{B}}\bar{A}.$$

We will often identify $D_B A$ with its image in $D_{\bar{B}}\bar{A}$. It is easy to see that $(\bar{-})$ is compatible with the Schouten bracket.

By composition we define a map

$$(-)^f : D_B A \to D_{B^f}(A^f) : P \mapsto \operatorname{tr}(\bar{P})$$

where we compute tr using the decomposition $1 = 1 \cdot \epsilon \cdot 1 + e_{21}\epsilon e_{12}$. It follows from §4.3 that $(-)^f$ is compatible with Schouten brackets.

For convenience we now define some operators in $D_{\bar{A}/\bar{B}}$. In order to avoid confusing notation we define $F_i \in D_{\epsilon B\epsilon}(\epsilon\bar{A}\epsilon)$ for $i \neq 2$ by $F_i(a) = ae_i \otimes e_i - e_i \otimes e_i a$. Note that $E_i^f = F_i$ for $i > 2$, but this is not the case for $i = 1$.

In this section we prove the following two results.

**Theorem 5.3.1.** *Assume that $A, P$ is a differential double quasi-Poisson algebra over $B$. Then $A^f, P^{ff}$ with*

$$P^{ff} = P^f - \frac{1}{2}E_1^f E_2^f \tag{5.2}$$

*is a differential double quasi-Poisson algebra.*

**Theorem 5.3.2.** *Assume that $A, P, \Phi$ is a differential quasi-Hamiltonian algebra over $B$. Then $A^f, P^{ff}, \Phi^{ff}$ with $P^{ff}$ as in (5.2) and with*

$$\Phi_i^{ff} = \begin{cases} \Phi_1^f \Phi_2^f & \text{if } i = 1, \\ \Phi_i^f & \text{if } i > 2 \end{cases}$$

*is a differential quasi-Hamiltonian algebra over $B^f$.*

The proof of these theorems needs some preparation. We put

$$E = \bar{E}_1,$$
$$\hat{E} = e_{12}\bar{E}_2 e_{21}.$$

**Lemma 5.3.3.** *We have for $a \in \epsilon\bar{A}\epsilon$,*

$$(E + \hat{E})(a) = ae_1 \otimes e_1 - e_1 \otimes e_1 a.$$

*Proof.* Both sides of the equation are derivations in $a$. Hence it suffices to check the identity on generators for $\epsilon\bar{A}\epsilon$. It is easy to check that these generators are given by

$$\begin{aligned} t & \quad \text{for } t \in \epsilon A\epsilon, \\ e_{12}u & \quad \text{for } u \in e_2 A\epsilon, \\ ve_{21} & \quad \text{for } v \in \epsilon A e_2, \\ e_{12}we_{21} & \quad \text{for } w \in e_2 A e_2. \end{aligned}$$

We compute

$$\begin{aligned} (E + \hat{E})(t) &= \bar{E}_1(t) + (e_{12}\bar{E}_2 e_{21})(t) \\ &= te_1 \otimes e_1 - e_1 \otimes e_1 t, \\ (E + \hat{E})(e_{12}u) &= e_{12}\bar{E}_1(u) + e_{12}(e_{12}\bar{E}_2 e_{21})(u) \\ &= e_{12}ue_1 \otimes e_1 - e_1 \otimes e_{12}u, \\ (E + \hat{E})(ve_{21}) &= \bar{E}_1(v)e_{21} + (e_{12}\bar{E}_2 e_{21})(v)e_{21} \\ &= -e_1 \otimes e_1 ve_{21} + ve_{21} \otimes e_1, \\ (E + \hat{E})(e_{12}we_{21}) &= e_{12}\bar{E}_1(w)e_{21} + e_{12}(e_{12}\bar{E}_2 e_{21})(w)e_{21} \\ &= e_{12}we_{21} \otimes e_1 - e_1 \otimes e_{12}we_{21}. \end{aligned}$$

In each of the cases we find the correct result. $\square$

We now compute the Schouten brackets between the operators $E, \hat{E}$:

$$\begin{aligned}
\{\!\{E,E\}\!\} &= \{\!\{\bar{E}_1,\bar{E}_1\}\!\}\\
&= \overline{\{\!\{E_1,E_1\}\!\}}\\
&= \overline{E_1\otimes e_1 - e_1\otimes E_1} \qquad \text{(by (3.9))}\\
&= E\otimes e_1 - e_1\otimes E,
\end{aligned}$$

$$\begin{aligned}
\{\!\{E,\hat{E}\}\!\} &= e_{12}\{\!\{\bar{E}_1,\bar{E}_2\}\!\}e_{21}\\
&= e_{12}\,\overline{\{\!\{E_1,E_2\}\!\}}\,e_{21}\\
&= e_{12}\,\overline{(E_2e_1\otimes e_1 - e_1\otimes e_1E_2)}\,e_{21}\\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\{\!\{\hat{E},\hat{E}\}\!\} &= e_{12}(e_{12} * \{\!\{\bar{E}_2,\bar{E}_2\}\!\} * e_{21})e_{21}\\
&= e_{12}(e_{12} * \overline{\{\!\{E_2,E_2\}\!\}} * e_{21})e_{21}\\
&= e_{12}(e_{12} * (\bar{E}_2\otimes e_2 - \bar{E}_2\otimes e_2) * e_{21})e_{21}\\
&= e_{12}\bar{E}_2e_{21}\otimes e_1 - e_1\otimes e_{12}\bar{E}_2e_{21}\\
&= \hat{E}\otimes e_1 - e_1\otimes\hat{E}.
\end{aligned}$$

We also need the following Schouten bracket:

$$\begin{aligned}
\{\!\{E\hat{E},E\hat{E}\}\!\} &= \{\!\{E\hat{E},E\}\!\}\bar{E} - E\{\!\{E\hat{E},\hat{E}\}\!\}\\
&= (\{\!\{E,E\}\!\} * \hat{E})\hat{E} - E(E * \{\!\{\hat{E},\hat{E}\}\!\})\\
&= E\hat{E}\otimes\hat{E} + \hat{E}\otimes E\hat{E} + E\hat{E}\otimes E + E\otimes E\hat{E}.
\end{aligned}$$

Hence

$$\{E\hat{E},E\hat{E}\} = 2E\hat{E}^2 + 2E^2\hat{E} \mod [D_{\bar{B}}\bar{A}, D_{\bar{B}}\bar{A}].$$

For $P \in (D_BA)_2$ we compute

$$\begin{aligned}
\{\!\{E\hat{E},\bar{P}\}\!\} &= -\{\!\{E,\bar{P}\}\!\} * \hat{E} + E * \{\!\{\hat{E},\bar{P}\}\!\}\\
&= -\overline{\{\!\{E_1,P\}\!\}} * \hat{E} + E * (e_{12} * \overline{\{\!\{E_2,P\}\!\}} * e_{21})\\
&= -(\bar{P}e_1\otimes e_1 - e_1\otimes e_1\bar{P}) * \hat{E} + E * (e_{12} * (\bar{P}e_2\otimes e_2 - e_2\otimes e_2\bar{P}) * e_{21})\\
&= -\bar{P}\hat{E}\otimes e_1 + \hat{E}\otimes e_1\bar{P} + \bar{P}e_{21}\otimes Ee_{12} - e_{21}\otimes Ee_{12}\bar{P}.
\end{aligned}$$

Hence

$$\{E\hat{E},\bar{P}\} = 0 \mod [D_{\bar{B}}\bar{A}, D_{\bar{B}}\bar{A}].$$

*Proof of Theorem* 5.3.1. Note that $E_1^f = E$, $E_2^f = \hat{E}$ and $E_1^fE_2^f = E\hat{E}$ (here and below we view $E, \hat{E}, E\hat{E}$ as elements of $D_{\epsilon B\epsilon}(\epsilon\overline{A}\epsilon)$).

The result of Lemma 5.3.3 may be rewritten as

$$E_1^f + E_2^f = F_1. \tag{5.3}$$

Using the fact that $\epsilon\bar{E}_2 = 0$ we have

$$(E_2^n)^f = \operatorname{tr}(\bar{E}_2^n) = e_{12}\bar{E}_2^ne_{21} = \hat{E}^n.$$

A similar computation yields

$$(E_1^n)^f = \operatorname{tr}(\bar{E}_1^n) = E^n.$$

Finally we have for $i > 2$,

$$(E_i^n)^f = F_i^n.$$

Applying $(-)^f$ to the identity

$$\{P, P\} = \frac{1}{6}\sum_i E_i^3$$

yields

$$\{P^f, P^f\} = \frac{1}{6}E^3 + \frac{1}{6}\hat{E}^3 + \frac{1}{6}\sum_{i>2} F_i^3.$$

We compute (modulo commutators)

$$\begin{aligned}
\left\{P^f - \frac{1}{2}E_1^f E_2^f, P^f - \frac{1}{2}E_1^f E_2^f\right\} &= \left\{P^f - \frac{1}{2}E\hat{E}, P^f - \frac{1}{2}E\hat{E}\right\}\\
&= \operatorname{tr}\left\{\bar{P} - \frac{1}{2}E\hat{E}, \bar{P} - \frac{1}{2}E\hat{E}\right\}\\
&= \operatorname{tr}\{\bar{P}, \bar{P}\} + \frac{1}{2}\operatorname{tr}(E\hat{E}^2) + \frac{1}{2}\operatorname{tr}(E^2\hat{E})\\
&= \{P^f, P^f\} + \frac{1}{2}E\hat{E}^2 + \frac{1}{2}E^2\hat{E}\\
&= \frac{1}{6}(E + \hat{E})^3 + \frac{1}{6}\sum_{i>2} F_i^3\\
&= \frac{1}{6}F_1^3 + \frac{1}{6}\sum_{i>2} F_i^3\\
&= \frac{1}{6}\sum_{i\neq 2} F_i^3.
\end{aligned}$$

This finishes the proof. □

*Proof of Theorem* 5.3.2. We need to prove

$$\{P^{f\!f}, \Phi_i^{f\!f}\} = -\frac{1}{2}(F_i\Phi_i + \Phi_i F_i).$$

Since the case $i > 2$ is easy we assume $i = 1$. In that case we have to prove

$$\left\{P^f - \frac{1}{2}E_1^f E_2^f, \Phi_1^f \Phi_2^f\right\} = -\frac{1}{2}(F_1\Phi_1^f\Phi_2^f + \Phi_1^f\Phi_2^f F_1). \tag{5.4}$$

We compute the left hand side of this equation. We have

$$\left\{P^f - \frac{1}{2}E_1^f E_2^f, \Phi_1^f \Phi_2^f\right\} = \operatorname{tr}\left\{\bar{P} - \frac{1}{2}E\hat{E}, \Phi\hat{\Phi}\right\}$$

where $\Phi = \bar{\Phi}_1$, $\hat{\Phi} = e_{12}\bar{\Phi}_2 e_{21}$.

We compute

$$\{\bar{P}, \Phi\hat{\Phi}\} = \{\bar{P}, \Phi\}\hat{\Phi} + \Phi\{\bar{P}, \hat{\Phi}\}$$

where

$$\begin{aligned}\{\bar{P},\Phi\} &= \overline{\{P,\Phi_1\}}\\ &= -\frac{1}{2}\overline{(E_1\Phi_1+\Phi_1E_1)}\\ &= -\frac{1}{2}(E\Phi+\Phi E)\end{aligned}$$

and

$$\begin{aligned}\{\bar{P},\hat{\Phi}\} &= \{\bar{P},e_{12}\bar{\Phi}_2e_{21}\}\\ &= e_{12}\,\overline{\{P,\Phi_2\}}\,e_{21}\\ &= -\frac{1}{2}e_{12}\overline{(E_2\Phi_2+\Phi_2E_2)}\,e_{21}\\ &= -\frac{1}{2}(\hat{E}\hat{\Phi}+\hat{\Phi}\hat{E}).\end{aligned}$$

Taking things together we find

$$\begin{aligned}\{\bar{P},\Phi\hat{\Phi}\} &= -\frac{1}{2}(E\Phi+\Phi E)\hat{\Phi}-\frac{1}{2}\Phi(\hat{E}\hat{\Phi}+\hat{\Phi}\hat{E})\\ &= -\frac{1}{2}E\Phi\hat{\Phi}-\frac{1}{2}\Phi E\hat{\Phi}-\frac{1}{2}\Phi\hat{E}\hat{\Phi}-\frac{1}{2}\Phi\hat{\Phi}\hat{E}.\end{aligned}\tag{5.5}$$

Next we compute $\{E\hat{E},\Phi\hat{\Phi}\}$. We need the following preliminary results:

$$\begin{aligned}\{\!\{E,\Phi\}\!\} &= \overline{\{\!\{E_1,\Phi_1\}\!\}}\\ &= \Phi\otimes e_1-e_1\otimes\Phi,\\ \{\!\{E,\hat{\Phi}\}\!\} &= e_{12}\,\overline{\{\!\{E_1,\Phi_2\}\!\}}\,e_{21}\\ &= e_{12}\,\overline{(\Phi_2\otimes e_1-e_1\otimes\Phi_2)}\,e_{21}\\ &= 0,\\ \{\!\{\hat{E},\Phi\}\!\} &= e_{12}*\overline{\{\!\{E_2,\Phi_1\}\!\}}*e_{21}\\ &= e_{12}*\overline{(\Phi_1\otimes e_2-e_2\otimes\Phi_1)}*e_{21}\\ &= 0,\\ \{\!\{\hat{E},\hat{\Phi}\}\!\} &= e_{12}(e_{12}*\overline{\{\!\{E_2,\Phi_2\}\!\}}*e_{21})e_{21}\\ &= e_{12}(e_{12}*\overline{(\Phi_2\otimes e_2-e_2\otimes\Phi_2)}*e_{21})e_{21}\\ &= \hat{\Phi}\otimes e_1-e_1\otimes\hat{\Phi}.\end{aligned}$$

We then compute

$$\begin{aligned}\{\!\{E\hat{E},\Phi\hat{\Phi}\}\!\} &= \Phi\{\!\{E\hat{E},\hat{\Phi}\}\!\}+\{\!\{E\hat{E},\Phi\}\!\}\hat{\Phi}\\ &= \Phi(E*\{\!\{\hat{E},\hat{\Phi}\}\!\}-\{\!\{E,\hat{\Phi}\}\!\}*\hat{E})+(E*\{\!\{\hat{E},\Phi\}\!\}-\{\!\{E,\Phi\}\!\}*\hat{E})\hat{\Phi}\\ &= \Phi(E*(\hat{\Phi}\otimes e_1-e_1\otimes\hat{\Phi}))-((\Phi\otimes e_1-e_1\otimes\Phi)*\hat{E})\hat{\Phi}\\ &= \Phi\hat{\Phi}\otimes E-\Phi\otimes E\hat{\Phi}-\Phi\hat{E}\otimes\hat{\Phi}+\hat{E}\otimes\Phi\hat{\Phi}\end{aligned}$$

and hence

$$\{E\hat{E},\Phi\hat{\Phi}\}=\Phi\hat{\Phi}E-\Phi E\hat{\Phi}-\Phi\hat{E}\hat{\Phi}+\hat{E}\Phi\hat{\Phi}.\tag{5.6}$$

Combining (5.5) and (5.6) we obtain

$$\begin{aligned}\left\{\bar{P}-\frac{1}{2}E\hat{E},\Phi\hat{\Phi}\right\} &= -\frac{1}{2}E\Phi\hat{\Phi}-\frac{1}{2}\Phi E\hat{\Phi}-\frac{1}{2}\Phi\hat{E}\hat{\Phi}-\frac{1}{2}\Phi\hat{\Phi}\hat{E}\\ &\quad -\frac{1}{2}\Phi\hat{\Phi}E+\frac{1}{2}\Phi E\hat{\Phi}+\frac{1}{2}\Phi\hat{E}\hat{\Phi}-\frac{1}{2}\hat{E}\Phi\hat{\Phi}\\ &= -\frac{1}{2}E\Phi\hat{\Phi}-\frac{1}{2}\Phi\hat{\Phi}E-\frac{1}{2}\hat{E}\Phi\hat{\Phi}-\frac{1}{2}\Phi\hat{\Phi}\hat{E}\end{aligned}$$

and hence we obtain

$$\begin{aligned}\text{LHS}\,(5.4) &= \operatorname{tr}\left\{\bar{P}-\frac{1}{2}E\hat{E},\Phi\hat{\Phi}\right\}\\ &= -\frac{1}{2}E_1^f\Phi_1^f\Phi_2^f-\frac{1}{2}\Phi_1^f\Phi_2^fE_1^f-\frac{1}{2}E_2^f\Phi_1^f\Phi_2^f-\frac{1}{2}\Phi_1^f\Phi_2^fE_2^f\\ &= -\frac{1}{2}F_1\Phi_1^f\Phi_2^f-\frac{1}{2}\Phi_1^f\Phi_2^fF_1\\ &= \text{RHS}\,(5.4). \qquad \square\end{aligned}$$

## 6. Quivers

6.1. **Generalities.** Below $Q=(Q,I,h,t)$ is a finite quiver with vertex set $I=\{1,\dots,n\}$ and edge set $Q$. The maps $t,h:Q\to I$ associate with every edge its start and end. We extend the definitions of $h,t$ to paths in $Q$. By $e_i$ we denote the idempotent associated to the vertex $i$ and we put $B=\bigoplus_i ke_i$. We let $\bar{Q}$ be the double of $Q$. $\bar{Q}$ is obtained from $Q$ by adjoining for every arrow $a$ an opposite arrow $a^*$. We define $\epsilon:\bar{Q}\to\{\pm 1\}$ as the function which is 1 on $Q$ and $-1$ on $\bar{Q}-Q$. By $kQ$ we denote the path algebra of $Q$ (with multiplication given by concatenation of paths). Note that $kQ/B$ is smooth, so we don't have to distinguish between differential and ordinary notions (see §5.2) in the case of quivers.

6.2. **Vector fields and the Schouten bracket.** Let $A=kQ$. For $a\in Q$ we define the element $\frac{\partial}{\partial a}\in D_BA$ which on $b\in Q$ acts as

$$\frac{\partial b}{\partial a}=\begin{cases}e_{t(a)}\otimes e_{h(a)} & \text{if } a=b,\\ 0 & \text{otherwise.}\end{cases}$$

It is clear that $D_{A/B}$ is generated by $\left(\frac{\partial}{\partial a}\right)_{a\in Q}$ as an $A$-bimodule. Hence $D_BA$ is the tensor algebra over $A$ generated by $\left(\frac{\partial}{\partial a}\right)_a$.

**Proposition 6.2.1.** *Let $a,b\in Q$. Then*

$$\begin{aligned}\{\!\{a,b\}\!\} &= 0,\\ \{\!\{\frac{\partial}{\partial a},b\}\!\} &= \begin{cases}e_{t(a)}\otimes e_{h(a)} & \textit{if } a=b,\\ 0 & \textit{otherwise,}\end{cases}\\ \{\!\{\frac{\partial}{\partial a},\frac{\partial}{\partial b}\}\!\} &= 0.\end{aligned}$$

*Proof.* Only the third equality is not immediately obvious. But a quick check of the definitions reveals that $\{\!\{\frac{\partial}{\partial a},\frac{\partial}{\partial b}\}\!\}(c)=0$ for any $c\in Q$. $\square$

**Proposition 6.2.2.** (1) *For $\delta \in D_{A/B}$ we have the equality in $D_B A$:*

$$\delta = \sum_{a \in Q} \delta(a)'' \frac{\partial}{\partial a} \delta(a)'. \tag{6.1}$$

(2) *For $i = 1, \ldots, n$ we have the equality:*

$$E_i = \sum_{a \in Q, h(a)=i} \frac{\partial}{\partial a} a - \sum_{a \in Q, t(a)=i} a \frac{\partial}{\partial a}. \tag{6.2}$$

*Proof.* (1) Let $b \in Q$. Evaluated on $b$, (6.1) can be rewritten as

$$\delta(b) = \delta(b)'' * (e_{t(b)} \otimes e_{h(b)}) * \delta(b)'.$$

The right hand side of this equation is equal to $e_{t(b)}\delta(b)' \otimes \delta(b)'' e_{h(b)} = \delta(e_{t(b)} b e_{h(b)}) = \delta(b)$.

(2) If we substitute $\delta = E_i$ in (6.1), then we obtain

$$\begin{aligned} E_i &= \sum_{a \in Q} e_i \frac{\partial}{\partial a} a e_i - e_i a \frac{\partial}{\partial a} e_i \\ &= \sum_{a \in Q, h(a)=i} \frac{\partial}{\partial a} a - \sum_{a \in Q, t(a)=i} a \frac{\partial}{\partial a}. \end{aligned}$$

□

*Remark* 6.2.3. The expression for $E_i$ can be conveniently rewritten as follows. Put $E = \sum_i E_i$. Then

$$E = \sum_{a \in Q} \left[ \frac{\partial}{\partial a}, a \right].$$

## 6.3. **Hamiltonian structure.**

**Theorem 6.3.1.** *$A = k\bar{Q}$ has a Hamiltonian structure given by*

$$\begin{aligned} P &= \sum_{a \in Q} \frac{\partial}{\partial a}, \frac{\partial}{\partial a^*}, \\ \mu &= \sum_{a \in Q} [a, a^*]. \end{aligned} \tag{6.3}$$

*Proof.* The fact that $\{P, P\} = 0$ is trivial. For the moment map property we compute

$$\begin{aligned} \{\!\{P, a\}\!\} &= -(e_{t(a)} \otimes e_{h(a)}) * \frac{\partial}{\partial a^*}, \\ \{\!\{P, a^*\}\!\} &= \frac{\partial}{\partial a} * (e_{h(a)} \otimes e_{t(a)}), \end{aligned}$$

whence

$$\begin{aligned} \{P, a\} &= -\frac{\partial}{\partial a^*}, \\ \{P, a^*\} &= \frac{\partial}{\partial a}. \end{aligned}$$

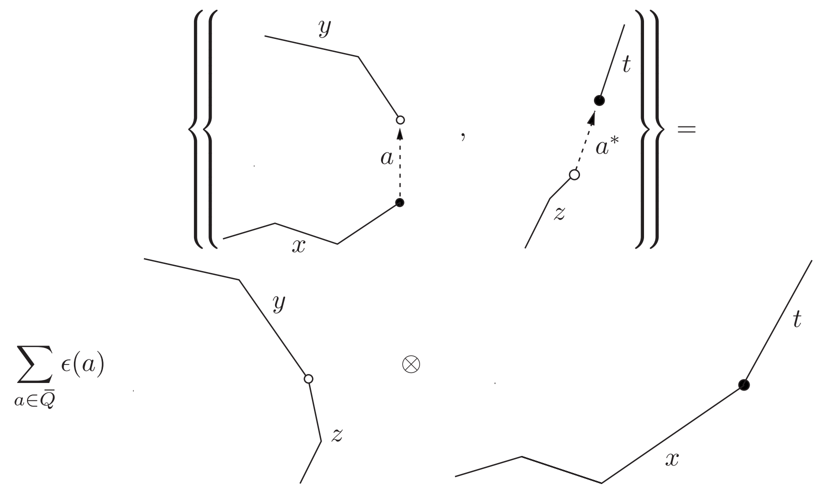


FIGURE 1

Thus

$$
\begin{aligned}
\{P,\mu\} &= \sum_{a\in Q}[\{P,a\},a^*]+[a,\{P,a^*\}]\\
&= \sum_{a\in Q} -\left[\frac{\partial}{\partial a^*},a^*\right]+\left[a,\frac{\partial}{\partial a}\right]\\
&= -E.
\end{aligned}
$$

□

6.4. **The necklace Loday algebra.** A simple computation yields the following formula for the induced double Poisson bracket on $k\bar{Q}$. Let $a\in Q$. Then

$$
\begin{aligned}
\{\!\{a,a^*\}\!\}_P &= e_{h(a)}\otimes e_{t(a)},\\
\{\!\{a^*,a\}\!\}_P &= -e_{t(a)}\otimes e_{h(a)}
\end{aligned}
$$

and all other double brackets are zero.

The double bracket on paths is pictorially given in Figure 1, where the black and white dots represent identical vertices.

The double Poisson structure on $k\bar{Q}$ induces a left Loday algebra structure $\{-,-\}$ on $k\bar{Q}$.

**Proposition 6.4.1.** *For oriented paths $x,y$ in $\bar{Q}$ we have $\{x,y\}=0$ if $x$ is not closed. Otherwise the bracket can be pictorially represented as in Figure 2.*

If we restrict this bracket to closed paths, we obtain the so-called *necklace Lie algebra* structure on $k\bar{Q}/[k\bar{Q},k\bar{Q}]$ [4, 11, 12].

6.5. **Quasi-Hamiltonian structure for a very simple quiver.** In this section we consider the quiver $Q$ given by

$$1 \underset{a^*}{\overset{a}{\rightleftarrows}} 2\,.$$

We let $A$ be the path algebra of $kQ$ with $e_1+aa^*$ and $e_2+a^*a$ inverted. By inverted we mean that we introduce elements $I$, $J$ such that $I=Ie_1=e_1I$, $J=Je_2=e_2J$

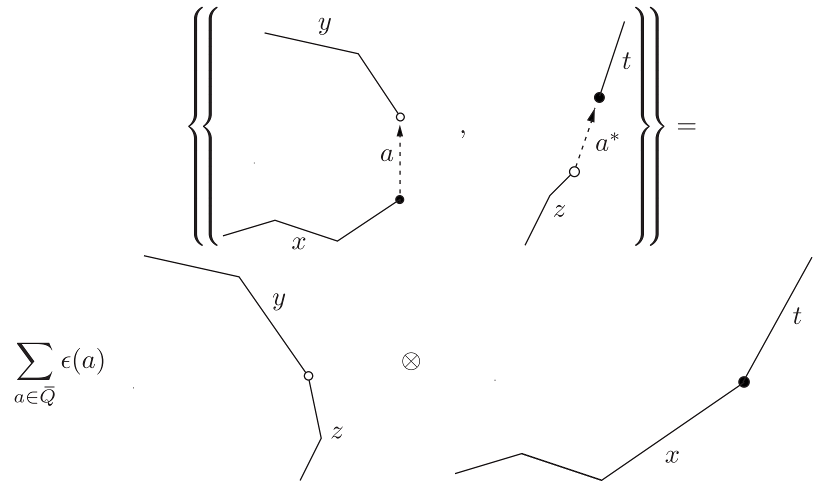


FIGURE 2

and $I(e_1 + aa^*) = (e_1 + aa^*)I = e_1$ and $J(e_2 + a^*a) = (e_2 + a^*a)J = e_2$. Below we use the notation $(e_1 + aa^*)^{-1}$ and $(e_2 + a^*a)^{-1}$ for $I$ and $J$ (commiting an abuse of notation).

**Theorem 6.5.1.** *A has a quasi-Hamiltonian structure given by*

$$P = \frac{\partial}{\partial a}\frac{\partial}{\partial a^*} + \frac{1}{2}\left(a\frac{\partial}{\partial a}\frac{\partial}{\partial a^*}a^* - a^*\frac{\partial}{\partial a^*}\frac{\partial}{\partial a}a\right) \tag{6.4}$$

$$= \frac{1}{2}(1 + a^*a)\frac{\partial}{\partial a}\frac{\partial}{\partial a^*} - \frac{1}{2}(1 + aa^*)\frac{\partial}{\partial a^*}\frac{\partial}{\partial a} \qquad \operatorname{mod} [-,-], \tag{6.5}$$

$$\Phi = (1 + aa^*)(1 + a^*a)^{-1}. \tag{6.6}$$

Note that the partial derivatives have odd degree. So their commutator has a plus sign. This explains why the 1's in (6.5) do not cancel.

*Proof.* We first consider the quasi-Poisson structure. For simplicity we introduce the following elements of $D_{A/B}$:

$$U = a\frac{\partial}{\partial a},$$
$$V = \frac{\partial}{\partial a}a,$$
$$U^* = a^*\frac{\partial}{\partial a^*},$$
$$V^* = \frac{\partial}{\partial a^*}a^*.$$

Then $P$ becomes

$$P = \frac{\partial}{\partial a}\frac{\partial}{\partial a^*} + \frac{1}{2}(UV^* - U^*V).$$

We have to prove

$$\{P, P\} = \frac{1}{6}(E_1^3 + E_2^3) \qquad \operatorname{mod} [-,-]. \tag{6.7}$$

By (6.2) we have

$$E_1 = \frac{\partial}{\partial a^*}a^* - a\frac{\partial}{\partial a} = V^* - U,$$
$$E_2 = \frac{\partial}{\partial a}a - a^*\frac{\partial}{\partial a^*} = V - U^*.$$

We compute

$$\begin{aligned}
\{\!\{UV^*, UV^*\}\!\} &= \{\!\{UV^*, U\}\!\}V^* - U\{\!\{UV^*, V^*\}\!\} \\
&= (\{\!\{U, U\}\!\} * V^*)V^* - U(U * \{\!\{V^*, V^*\}\!\}) \\
&= ((e_1 \otimes U - U \otimes e_1) * V^*)V^* - U(U * (-e_1 \otimes V^* + V^* \otimes e_1)) \\
&= -V^* \otimes UV^* - UV^* \otimes V^* + U \otimes UV^* + UV^* \otimes U.
\end{aligned}$$

Here $\{\!\{U, U\}\!\}$ and $\{\!\{V^*, V^*\}\!\}$ have been computed using Lemma 6.5.2 below. Hence we obtain

$$\{UV^*, UV^*\} = -2U(V^*)^2 + 2U^2V^* \qquad \text{mod } [-,-].$$

Similarly

$$\{U^*V, U^*V\} = -2U^*V^2 + 2(U^*)^2V \qquad \text{mod } [-,-].$$

It is also clear from Lemma 6.5.2 that $\{\!\{UV^*, U^*V\}\!\} = 0$. Hence

$$\{UV^*, U^*V\} = 0.$$

We need some more computations:

$$\begin{aligned}
\{\!\{UV^*, \frac{\partial}{\partial a}\frac{\partial}{\partial a^*}\}\!\} &= \{\!\{UV^*, \frac{\partial}{\partial a}\}\!\}\frac{\partial}{\partial a^*} - \frac{\partial}{\partial a}\{\!\{UV^*, \frac{\partial}{\partial a^*}\}\!\} \\
&= \Big(\{\!\{U, \frac{\partial}{\partial a}\}\!\} * V^*\Big)\frac{\partial}{\partial a^*} - \frac{\partial}{\partial a}\Big(U * \{\!\{V^*, \frac{\partial}{\partial a^*}\}\!\}\Big) \\
&= \Big(\Big(-\frac{\partial}{\partial a} \otimes e_1\Big) * V^*\Big)\frac{\partial}{\partial a^*} - \frac{\partial}{\partial a}\Big(U * \Big(-e_1 \otimes \frac{\partial}{\partial a^*}\Big)\Big) \\
&= -\frac{\partial}{\partial a}V^* \otimes \frac{\partial}{\partial a^*} + \frac{\partial}{\partial a} \otimes U\frac{\partial}{\partial a^*}.
\end{aligned}$$

Hence

$$\begin{aligned}
\Big\{UV^*, \frac{\partial}{\partial a}\frac{\partial}{\partial a^*}\Big\} &= -\frac{\partial}{\partial a}V^*\frac{\partial}{\partial a^*} + \frac{\partial}{\partial a}U\frac{\partial}{\partial a^*} \\
&= -\frac{\partial}{\partial a}\frac{\partial}{\partial a^*}a^*\frac{\partial}{\partial a^*} + \frac{\partial}{\partial a}a\frac{\partial}{\partial a}\frac{\partial}{\partial a^*}.
\end{aligned}$$

Similarly, computing modulo commutators we obtain

$$\begin{aligned}
\Big\{U^*V, \frac{\partial}{\partial a}\frac{\partial}{\partial a^*}\Big\} &= -\Big\{U^*V, \frac{\partial}{\partial a^*}\frac{\partial}{\partial a}\Big\} \\
&= \frac{\partial}{\partial a^*}\frac{\partial}{\partial a}a\frac{\partial}{\partial a} - \frac{\partial}{\partial a^*}a^*\frac{\partial}{\partial a^*}\frac{\partial}{\partial a}.
\end{aligned}$$

If follows that

$$\Big\{UV^* - U^*V, \frac{\partial}{\partial a}\frac{\partial}{\partial a^*}\Big\} = 0 \qquad \text{mod } [-,-].$$

Combining everything we find

$$\{P, P\} = \frac{1}{2}(-U(V^*)^2 + U^2V^* - U^*V^2 + (U^*)^2V) \qquad \text{mod } [-,-].$$

On the other hand we have

$$\begin{aligned} E_1^3 &= (V^*)^3 - U^3 - 3U(V^*)^2 + 3U^2V^* \\ E_2^3 &= V^3 - (U^*)^3 - 3U^*V^2 + 3(U^*)^2V \end{aligned} \qquad \text{mod } [-,-]$$

and also

$$\begin{aligned} U^3 &= V^3 \\ (U^*)^3 &= (V^*)^3 \end{aligned} \qquad \text{mod } [-,-].$$

It follows that

$$\frac{1}{6}(E_1^3 + E_2^3) = \frac{1}{2}(-U(V^*)^2 + U^2V^* - U^*V^2 + (U^*)^2V) \qquad \text{mod } [-,-].$$

So it follows that (6.7) is indeed true.

Now we prove that $\Phi$ is a multiplicative moment map. We have $\Phi = \Phi_1 + \Phi_2$ where

$$\begin{aligned} \Phi_1 &= e_1 + aa^*, \\ \Phi_2 &= (e_2 + a^*a)^{-1}. \end{aligned}$$

We have to prove

$$\{P, \Phi_i\} = -\frac{1}{2}(E_i\Phi_i + E_i\Phi_i).$$

We will first consider $\Phi_1$. We compute

$$\begin{aligned} \{\!\{UV^*, aa^*\}\!\} &= \{\!\{UV^*, a\}\!\}a^* + a\{\!\{UV^*, a^*\}\!\} \\ &= -(\{\!\{U, a\}\!\} * V^*)a^* + a(U * \{\!\{V^*, a^*\}\!\}) \\ &= -((e_1 \otimes a) * V^*)a^* + a(U * (a^* \otimes e_1)) \\ &= -V^* \otimes aa^* + aa^* \otimes U. \end{aligned}$$

So

$$\begin{aligned} \{UV^*, aa^*\} &= -V^*aa^* + aa^*U \\ &= -\frac{\partial}{\partial a^*}a^*aa^* + aa^*a\frac{\partial}{\partial a}. \end{aligned}$$

Similarly

$$\begin{aligned} \{\!\{U^*V, aa^*\}\!\} &= \{\!\{U^*V, a\}\!\}a^* + a\{\!\{U^*V, a^*\}\!\} \\ &= (U^* * \{\!\{V, a\}\!\})a^* - a(\{\!\{U^*, a^*\}\!\} * V) \\ &= (U^* * (a \otimes e_2))a^* - a((e_2 \otimes a^*) * V) \\ &= a \otimes U^*a^* - aV \otimes a^*, \end{aligned}$$

which yields

$$\begin{aligned} \{U^*V, aa^*\} &= aU^*a^* - aVa^* \\ &= aa^*\frac{\partial}{\partial a^*}a^* - a\frac{\partial}{\partial a}aa^*. \end{aligned}$$

We obtain

$$\begin{aligned} \{UV^* - U^*V, aa^*\} &= -\frac{\partial}{\partial a^*}a^*aa^* + aa^*a\frac{\partial}{\partial a} - aa^*\frac{\partial}{\partial a^*}a^* + a\frac{\partial}{\partial a}aa^* \\ &= -E_1aa^* - aa^*E_1. \end{aligned}$$

We also have

$$\begin{aligned}\left\{\!\!\left\{\frac{\partial}{\partial a}\frac{\partial}{\partial a^*}, aa^*\right\}\!\!\right\} &= \left\{\!\!\left\{\frac{\partial}{\partial a}\frac{\partial}{\partial a^*}, a\right\}\!\!\right\}a^* + a\left\{\!\!\left\{\frac{\partial}{\partial a}\frac{\partial}{\partial a^*}, a^*\right\}\!\!\right\} \\ &= -\Big((e_1 \otimes e_2) * \frac{\partial}{\partial a^*}\Big)a^* + a\Big(\frac{\partial}{\partial a} * (e_2 \otimes e_1)\Big) \\ &= -\frac{\partial}{\partial a^*} \otimes a^* + a \otimes \frac{\partial}{\partial a},\end{aligned}$$

which yields

$$\left\{\frac{\partial}{\partial a}\frac{\partial}{\partial a^*}, aa^*\right\} = -E_1.$$

Taking everything together we obtain

$$\begin{aligned}\{P, \Phi_1\} &= \{P, aa^*\} \\ &= -E_1 - \frac{1}{2}(E_1 aa^* + aa^* E_1) \\ &= -\frac{1}{2}(E_1\Phi_1 + \Phi_1 E_1).\end{aligned}$$

Now we consider $\Phi_2$. Applying the automorphism of order two $e_1 \leftrightarrow e_2$, $a \leftrightarrow a^*$ has the effect $P \mapsto -P$ (up to commutators) and $E_1 \mapsto E_2$, $E_2 \mapsto E_1$.

Hence we obtain the following identity for $\Psi = e_2 + a^*a$:

$$-\{P, \Psi\} = -\frac{1}{2}(E_2\Psi + \Psi E_2).$$

Since $\Phi_2 = \Psi^{-1}$ and $\{P, -\}$ is a derivation we obtain

$$\begin{aligned}\{P, \Phi_2\} &= -\Psi^{-1}\{P, \Psi\}\Psi^{-1} \\ &= -\frac{1}{2}(\Psi^{-1}E_2 + E_2\Psi^{-1}) \\ &= -\frac{1}{2}(\Phi_2 E_2 + E_2\Phi_2).\end{aligned}$$ □

**Lemma 6.5.2.** *Let $Q$ be an arbitrary quiver, $a \in Q$ and $h(a) \neq t(a)$ (i.e. $a$ is not a loop). Put $e = e_{t(a)}$, $f = e_{h(a)}$. If $X$ is in the subring of $D_B A$ generated by $a$ and $\frac{\partial}{\partial a}$, then*

$$\begin{aligned}\{\!\!\{U, X\}\!\!\} &= -Xe \otimes e + e \otimes eX, \\ \{\!\!\{V, X\}\!\!\} &= Xf \otimes f - f \otimes fX.\end{aligned}$$

*Proof.* By (6.2) we obtain

$$\begin{aligned}\{\!\!\{E_{t(a)}, X\}\!\!\} &= -\{\!\!\{U, X\}\!\!\}, \\ \{\!\!\{E_{h(a)}, X\}\!\!\} &= \{\!\!\{V, X\}\!\!\}.\end{aligned}$$

Using (3.9) this implies what we want. □

6.6. **Fusion for quivers.** We now discuss what happens if we perform fusion on path algebras. This will be used in the next section. Put $A = kQ$. It is clear that $\bar{A}$ is generated over $B$ by $a \in Q$ and $e_{12}$, $e_{21}$, subject to the relations $e_{12}e_{21} = e_1$,

$e_{21}e_{12} = e_2$. Then it is not hard to see that $A^f$ is freely generated over $B^f$ by

$$\begin{array}{rl} a & h(a) \neq 2,\ t(a) \neq 2, \\ ae_{21} & h(a) = 2,\ t(a) \neq 2, \\ e_{12}a & h(a) \neq 2,\ t(a) = 2, \\ e_{12}ae_{21} & h(a) = 2,\ t(a) = 2. \end{array}$$

Now let $Q^f$ be the quiver obtained from $Q$ by "fusing" vertices 1 and 2. I.e. $Q^f$ has the same edges as $Q$ and vertices $I^f = I - \{2\}$. The maps $t, h$ are redefined as follows:

$$h^f(a) = \begin{cases} 1 & \text{if } h(a) = 2, \\ h(a) & \text{otherwise,} \end{cases}$$

$$t^f(a) = \begin{cases} 1 & \text{if } t(a) = 2, \\ t(a) & \text{otherwise.} \end{cases}$$

The following result is easy to prove.

**Proposition 6.6.1.** *The map*

$$(kQ)^f \to k(Q^f),$$

*which is defined by (for $a \in Q$)*

$$\begin{array}{rl} a \mapsto a & h(a) \neq 2,\ t(a) \neq 2, \\ ae_{21} \mapsto a & h(a) = 2,\ t(a) \neq 2, \\ e_{12}a \mapsto a & h(a) \neq 2,\ t(a) = 2, \\ e_{12}ae_{21} \mapsto a & h(a) = 2,\ t(a) = 2, \end{array}$$

*is an isomorphism.*

We need a slight extension of this result.

**Proposition 6.6.2.** *Let $S \subset \bigcup_{i,j} e_i A e_j$ and let $A_S$ be the algebra obtained from $A$ by formally adjoining for all $s \in S$ an element $s^{-1}$ which satisfies the relations $s^{-1}s = e_{h(s)}$, $ss^{-1} = e_{t(s)}$. Then one has*

$$(A_S)^f = A^f_{S^f}$$

*where*

$$S^f = \{s^f \mid s \in S\}.$$

*Proof.* Left to the reader. □

6.7. **Quasi-Hamiltonian structure for general quivers.** In this section we prove the following result.

**Theorem 6.7.1.** *Let $A$ be obtained from $k\bar{Q}$ by inverting all elements $(1+aa^*)_{a\in\bar{Q}}$. Fix an arbitrary total ordering on $\bar{Q}$. Then $A$ has a quasi-Hamiltonian structure* *given by*

$$P = \frac{1}{2}\left(\sum_{a\in\bar{Q}}\left(\epsilon(a)(1+a^*a)\frac{\partial}{\partial a}\frac{\partial}{\partial a^*}\right) - \sum_{a<b\in\bar{Q}}\left(\frac{\partial}{\partial a^*}a^* - a\frac{\partial}{\partial a}\right)\left(\frac{\partial}{\partial b^*}b^* - b\frac{\partial}{\partial b}\right)\right),$$

$$\Phi = \prod_{a\in\bar{Q}}(1+aa^*)^{\epsilon(a)}.$$

*In the definition of* $\Phi$ *the product is taken with respect to the chosen ordering on* $\bar{Q}$*.*

*Proof.* We will deduce this result from Theorem 6.5.1 using fusion (as discussed in §6.6).

Let $Q^{\text{sep}}$ be the quiver with the same edges as $\bar{Q}$ but with vertices $(v_a)_{a\in\bar{Q}}$. The head and tail of an edge are defined by

$$t(a) = v_a,$$
$$h(a) = t(a^*) = v_{a^*}.$$

Thus $Q^{\text{sep}}$ is a disjoint union of little quivers of the form

$$v_a \underset{a^*}{\overset{a}{\rightleftarrows}} v_{a^*} .$$

Let $A^{\text{sep}}$ be the algebra obtained from $kQ^{\text{sep}}$ by inverting all $(1+aa^*)_{a\in Q^{\text{sep}}}$. By Theorem 6.5.1, $kQ^{\text{sep}}$ has a quasi-Hamiltonian structure $(P^{\text{sep}}, \Phi^{\text{sep}})$ where

$$P^{\text{sep}} = \frac{1}{2}\sum_{a\in Q^{\text{sep}}}\epsilon(a)(1+a^*a)\frac{\partial}{\partial a}\frac{\partial}{\partial a^*}$$

and

$$\Phi^{\text{sep}}_{v_a} = (e_{v_a} + aa^*)^{\epsilon(a)}.$$

To obtain $k\bar{Q}$ from $kQ^{\text{sep}}$ we need to fuse certain vertices. More precisely for a vertex $i\in I$ we need to fuse the vertices $v_a$ such that $t(a) = i$. The fusing process depends on the order in which we perform it. To fix this we fix a total ordering of all edges in $\bar{Q}$. We put the same total ordering on the vertices $v_a$.

By Theorems 5.3.1, 5.3.2 and (6.2) we see that fusing the vertices $v_a$ with $t(a) = i$ has the effect of adding

$$-\frac{1}{2}\sum_{a<b\in\bar{Q},t(a)=t(b)=i} F_aF_b$$

to $P^{\text{sep}}$ where

$$F_a = \frac{\partial}{\partial a^*}a^* - a\frac{\partial}{\partial a}$$

and to replace $(\Phi^{\text{sep}}_{v_a})_{t(a)=i}$ by the product

$$\Phi_i = \prod_{a\in\bar{Q},t(a)=i}(e_i + aa^*)^{\epsilon(a)}$$

where the order on the product is given by the ordering of the edges. Performing this for all vertices in $\bar{Q}$ proves the theorem. □

*Remark* 6.7.2. The total ordering on the edges of $\bar{Q}$ actually contains too much information. It is sufficient to order for every vertex $i$ the edges starting in $i$.

*Remark* 6.7.3. It follows from the formulas for $P$ and $\Phi$ that $k\bar{Q}$ has always double quasi-Poisson structure. However in order to have a quasi-Hamiltonian structure we need to invert the elements $(1+aa^*)_{a\in\bar{Q}}$.

6.8. **Preprojective algebras and multiplicative preprojective algebras.** Fix $\lambda \in B$. The algebra

$$\Pi^\lambda = k\bar{Q}/\Big(\sum_{a\in Q}[a,a^*]-\lambda\Big)$$

is the so-called "deformed preprojective algebra". It was introduced by Crawley-Boevey and Holland in [9]. A multiplicative version was introduced by Crawley-Boevey and Shaw in [8]. Fix $q \in B^*$ and put

$$\Lambda^q = k\bar{Q}_{(1+aa^*)_{a\in\bar{Q}}}\Big/\Big(\prod_{a\in\bar{Q}}(1+aa^*)^{\epsilon(a)}-q\Big).$$

The product is taken with respect to an arbitrary ordering of $\bar{Q}$, but it is shown in [8] that the resulting algebra is independent of this ordering.

Combining Propositions 2.6.5, 5.1.5, 6.3.1 and Theorem 6.5.1 we obtain:

**Proposition 6.8.1.** *Both the ordinary deformed preprojective algebra and the deformed multiplicative preprojective algebra have a Poisson structure (as in Definition* 2.6.1*).*

## 7. Representation spaces

7.1. **General principles.** We put $I=\{1,\dots,n\}$. Let $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$ and put $|\alpha|=\sum_i \alpha_i$. Define the function

$$\phi : [\,1\,..\,|\alpha|\,] \to I$$

by the property

$$\phi(p)=i \iff \alpha_1+\cdots+\alpha_{i-1}+1\le p\le \alpha_1+\cdots+\alpha_i.$$

Throughout we assume that $B=ke_1\oplus\cdots\oplus ke_n$ is semi-simple. As usual $A$ is a finitely generated $B$-algebra.

We view an element $X$ of $M_{|\alpha|}(k)$ as a block matrix $(X_{uv})_{uv}$ with $u,v=1,\dots,n$ and $X_{uv}\in M_{\alpha_u\times\alpha_v}(k)$. We will also consider $B$ as being diagonally embedded in $M_{|\alpha|}(k)$ where $e_i$ is the identity matrix in $M_{\alpha_i\times\alpha_i}(k)$.

We define $\operatorname{Rep}(A,\alpha)$ as the affine scheme representing the functor

$$R\mapsto \operatorname{Hom}_B(A, M_{|\alpha|}(R))$$

from commutative $k$-algebras to sets. The coordinate ring of $\operatorname{Rep}(A,\alpha)$ is generated by symbols $a_{pq}$ for $a\in A$, $p,q=1,\dots,|\alpha|$ which are linear in $a$ and satisfy the relations

$$a_{pq}b_{qr}=(ab)_{pr},$$
$$(e_i)_{pq}=\delta_{pq}\delta_{\phi(p),i}.$$

A map $f\in\operatorname{Hom}_B(A,M_{|\alpha|}(R))$ corresponds to the point $x\in\operatorname{Rep}(A,\alpha)(R)$ if the following relation holds for $a\in A$, $p,q=1,\dots,|\alpha|$:

$$a_{pq}(x)=f(a)_{pq}.$$

Below we identify $\operatorname{Rep}(A,\alpha)(R)$ and $\operatorname{Hom}_B(A,M_{|\alpha|}(R))$.

For $a \in A$ it will be convenient to introduce the $M_{|\alpha|}(k)$-valued function $X(a)$ on $\mathrm{Rep}(A,\alpha)$ by the rule $X(a)_{ij} = a_{ij}$. The defining relations on $\mathrm{Rep}(A,\alpha)$ may then be written as

$$\begin{aligned} X(ab) &= X(a)X(b), \\ X(e_i) &= e_i. \end{aligned}$$

Put $\mathrm{Gl}_\alpha = \prod_i \mathrm{Gl}_{\alpha_i}$. $\mathrm{Gl}_\alpha$ acts by conjugation on $M_{|\alpha|}$. This induces an action on $\mathrm{Rep}(A,\alpha)$. To work out what this action is let $x \in \mathrm{Rep}(A,\alpha)(R) = \mathrm{Hom}_B(A, M_{|\alpha|}(R))$. We have for $a \in A$,

$$a_{ij}(x) = x(a)_{ij}$$

and hence for $g \in \mathrm{Gl}_\alpha(R)$,

$$\begin{aligned} (g \cdot a_{ij})(x) &= a_{ij}((g^{-1} - g) \circ x) \\ &= (g^{-1}x(a)g)_{ij} \\ &= (g^{-1})_{iu} x(a)_{uv} g_{vj} \\ &= (g^{-1})_{iu} a_{uv}(x) g_{vj}. \end{aligned}$$

In terms of the $X(a)$ we may write:

$$g \cdot X(a) = g^{-1} X(a) g$$

where the "$\cdot$" means that we apply the action of $g$ entrywise.

Let $M_\alpha = \prod_i M_{\alpha_i}$. We consider $M_\alpha$ as being diagonally embedded in $M_{|\alpha|}$. $M_\alpha$ is the Lie algebra of $\mathrm{Gl}_\alpha$. The derivative of the $\mathrm{Gl}_\alpha$-action on $\mathrm{Rep}(A,\alpha)$ yields an $M_\alpha$ action which has the following formula for $v \in M_\alpha(k)$:

$$v \cdot X(a) = [X(a), v]. \tag{7.1}$$

We now indicate how some of the possible properties of $A$ we have introduced induce standard geometrical properties on $\mathrm{Rep}(A,\alpha)$.

7.2. **Functions.** We have already seen that $a \in A$ induces functions $(a_{ij})_{ij}$ on $\mathrm{Rep}(A,\alpha)$.

7.3. **Differential forms.** If $\omega = f_1 df_2 \cdots df_n \in (\Omega_B A)_n$, then we define

$$\omega_{ij} = f_{1,ia_1} df_{2,a_1a_2} \cdots df_{n,a_{n-1}j}. \tag{7.2}$$

$(\omega_{ij})_{ij}$ is a matrix valued differential form on $\mathrm{Rep}(A,\alpha)$. If we write it as $X(\omega)$, then (7.2) may be rewritten as

$$X(\omega) = X(f_1) dX(f_2) \cdots dX(f_n).$$

7.4. **Poly-vector fields.** If $\delta \in D_{A/B}$, then we define corresponding vector fields $\delta_{ij} \in \mathrm{Rep}(A,\alpha)$ by the rule

$$\delta_{ij}(a_{uv}) = \delta(a)'_{uj}\delta(a)''_{iv}. \tag{7.3}$$

If $\delta = \delta_1 \cdots \delta_n \in (D_B A)_n$, then we put

$$\delta_{ij} = \delta_{1,ia_1}\delta_{2,a_1a_2}\cdots\delta_{n,a_{n-1}j} \in \bigwedge_{\mathcal{O}(\mathrm{Rep}(A,\alpha))} \mathrm{Der}(\mathcal{O}(\mathrm{Rep}(A,\alpha)))$$

or in the standard matrix notation,

$$X(\delta) = X(\delta_1)\cdots X(\delta_n).$$

7.5. **Brackets.** We have the following result.

**Proposition 7.5.1.** *Assume that* $\{\!\{-,-\}\!\} : A \times A \to A \otimes A$ *is a* $B$*-linear double bracket on* $A$*. Then there is a unique antisymmetric biderivation*

$$\{-,-\} : \mathcal{O}(\mathrm{Rep}(A,\alpha)) \times \mathcal{O}(\mathrm{Rep}(A,\alpha)) \to \mathcal{O}(\mathrm{Rep}(A,\alpha))$$

*with the property*

$$\{a_{ij}, b_{uv}\} = \{\!\{a,b\}\!\}'_{uj}\{\!\{a,b\}\!\}''_{iv} \tag{7.4}$$

*for* $a, b \in A$*.*

*Proof.* It is a routine verification that (7.4) is compatible with the defining relations of $\mathcal{O}(\mathrm{Rep}(A,\alpha))$. The antisymmetry of $\{-,-\}$ may be checked on the generators $(a_{ij})_{ij}$ where it follows from the corresponding property of $\{\!\{-,-\}\!\}$. □

The following proposition gives the connection between the double Jacobi identity in $A$ and the Jacobi identity on $\mathrm{Rep}(A,\alpha)$.

**Proposition 7.5.2.** *The following identity holds for* $a, b, c \in A$*:*

$$\begin{aligned} &\{a_{pq}, \{b_{rs}, c_{uv}\}\!\} + \{b_{rs}, \{c_{uv}, a_{pq}\}\!\} + \{c_{uv}, \{a_{pq}, b_{rs}\}\!\} \\ &\quad = \{\!\{a,b,c\}\!\}'_{uq}\{\!\{a,b,c\}\!\}''_{ps}\{\!\{a,b,c\}\!\}'''_{rv} - \{\!\{a,c,b\}\!\}'_{rq}\{\!\{a,c,b\}\!\}''_{pv}\{\!\{a,c,b\}\!\}'''_{us}. \end{aligned} \tag{7.5}$$

*In particular, if* $A, \{\!\{-,-\}\!\}$ *is a double Poisson algebra, then* $\mathcal{O}(\mathrm{Rep}(A,\alpha)), \{-,-\}$ *is a Poisson algebra.*

*Proof.* We compute

$$\begin{aligned} \{a_{pq}, \{b_{rs}, c_{uv}\}\!\} &= \{a_{pq}, \{\!\{b,c\}\!\}'_{us}\{\!\{b,c\}\!\}''_{rv}\} \\ &= \{a_{pq}, \{\!\{b,c\}\!\}'_{us}\}\{\!\{b,c\}\!\}''_{rv} + \{\!\{b,c\}\!\}'_{us}\{a_{pq}, \{\!\{b,c\}\!\}''_{rv}\} \\ &= \{\!\{a, \{\!\{b,c\}\!\}'\}\!\}'_{uq}\{\!\{a, \{\!\{b,c\}\!\}'\}\!\}''_{ps}\{\!\{b,c\}\!\}''_{rv} \\ &\qquad + \{\!\{b,c\}\!\}'_{us}\{\!\{a, \{\!\{b,c\}\!\}''\}\!\}'_{rq}\{\!\{a, \{\!\{b,c\}\!\}''\}\!\}''_{pv} \\ &= \{\!\{a, \{\!\{b,c\}\!\}'\}\!\}'_{uq}\{\!\{a, \{\!\{b,c\}\!\}'\}\!\}''_{ps}\{\!\{b,c\}\!\}''_{rv} \\ &\qquad - \{\!\{a, \{\!\{c,b\}\!\}'\}\!\}'_{rq}\{\!\{a, \{\!\{c,b\}\!\}'\}\!\}''_{pv}\{\!\{c,b\}\!\}''_{us} \end{aligned}$$

and hence

$$\begin{aligned} \{b_{rs}, \{c_{uv}, a_{pq}\}\!\} &= \{\!\{b, \{\!\{c,a\}\!\}'\}\!\}'_{ps}\{\!\{b, \{\!\{c,a\}\!\}'\}\!\}''_{rv}\{\!\{c,a\}\!\}''_{uq} \\ &\qquad - \{\!\{b, \{\!\{a,c\}\!\}'\}\!\}'_{us}\{\!\{b, \{\!\{a,c\}\!\}'\}\!\}''_{rq}\{\!\{a,c\}\!\}''_{pv}, \\ \{c_{uv}, \{a_{pq}, b_{rs}\}\!\} &= \{\!\{c, \{\!\{a,b\}\!\}'\}\!\}'_{rv}\{\!\{c, \{\!\{a,b\}\!\}'\}\!\}''_{uq}\{\!\{a,b\}\!\}''_{ps} \\ &\qquad - \{\!\{c, \{\!\{b,a\}\!\}'\}\!\}'_{pv}\{\!\{c, \{\!\{b,a\}\!\}'\}\!\}''_{us}\{\!\{b,a\}\!\}''_{rq}. \end{aligned}$$

Taking the sum yields (7.5). □

**Example 7.5.3.** Recall that if $\mathfrak{g}$ is a Lie algebra, then the functions on $\mathfrak{g}^*$ carry a Poisson bracket defined by

$$\{\mathrm{ev}_v, \mathrm{ev}_w\} = \mathrm{ev}_{[v,w]}$$

where $v, w \in \mathfrak{g}$ and $\mathrm{ev}_v$ is the evaluation of an element of $\mathfrak{g}^*$ at $v$. Clearly $\mathrm{ev}_v$ defines a set generating functions for $\mathcal{O}(\mathfrak{g}^*)$.

Since $M_n(k)$ can be identified with its dual through the trace pairing it follows that the functions on $M_n$ have a canonical Poisson bracket. On the other hand

$M_n(k) = \mathrm{Rep}(k[t], M_n(k))$. It is then easy to show that this Poisson bracket on $\mathcal{O}(M_n(k))$ comes from the double Poisson bracket on $k[t]$ given by

$$\{\!\{t,t\}\!\} = t\otimes 1 - 1\otimes t,$$

which we considered in Example 2.3.3.

7.6. **The Schouten bracket.** The idea is that constructions in $A$ are compatible with the corresponding constructions on $\mathrm{Rep}(A,\alpha)$. This is usually clear. For the Schouten bracket it requires some work.

**Proposition 7.6.1.** *Let $P, Q \in D_B A$. Then*

$$\{P_{ij}, Q_{uv}\} = \{\!\{P,Q\}\!\}'_{uj}\{\!\{P,Q\}\!\}''_{iv} \tag{7.6}$$

*where $\{\!\{-,-\}\!\}$ denotes the Schouten bracket on $D_BA$ and $\{-,-\}$ is the Schouten bracket between poly-vector fields on* $\mathrm{Rep}(A,\alpha)$.

*Proof.* We claim that the correctness of (7.6) is multiplicative in both arguments. To check this put first $Q = RS$ and assume that $P$, $R$, $S$ are homogeneous. Assume that (7.6) holds for $Q = R, S$. We compute

$$\begin{aligned}
\{P_{ij}, (RS)_{uv}\} &= \{P_{ij}, R_{uw}S_{wv}\}\\
&= \{P_{ij}, R_{uw}\}S_{wv} + (-1)^{|R|(|P|-1)}R_{uw}\{P_{ij}, S_{wv}\}\\
&= \{\!\{P,R\}\!\}'_{uj}\{\!\{P,R\}\!\}''_{iw}S_{wv} + (-1)^{|R|(|P|-1)}R_{uw}\{\!\{P,S\}\!\}'_{wj}\{\!\{P,S\}\!\}''_{iv}\\
&= \{\!\{P,R\}\!\}'_{uj}(\{\!\{P,R\}\!\}''S)_{iv} + (-1)^{|R|(|P|-1)}(R\{\!\{P,S\}\!\}')_{uj}\{\!\{P,S\}\!\}''_{iv}\\
&= \{\!\{P,RS\}\!\}'_{uj}\{\!\{P,RS\}\!\}''_{iv}.
\end{aligned}$$

We now check multiplicativity in the other argument. Put $P = UV$ and assume that (7.6) holds with $P = U, V$. Then

$$\begin{aligned}
\{(UV)_{ij}, Q_{uv}\} &= \{U_{ik}V_{kj}, Q_{uv}\}\\
&= U_{ik}\{V_{kj}, Q_{uv}\} + (-1)^{|V|(|Q|-1)}\{U_{ik}, Q_{uv}\}V_{kj}\\
&= U_{ik}\{\!\{V,Q\}\!\}'_{uj}\{\!\{V,Q\}\!\}''_{kv} + (-1)^{|V|(|Q|-1)}\{\!\{U,Q\}\!\}'_{uk}\{\!\{U,Q\}\!\}''_{iv}V_{kj}\\
&= \{\!\{V,Q\}\!\}'_{uj}(U\{\!\{V,Q\}\!\}'')_{iv} + (-1)^{|V|(|Q|-1)}(\{\!\{U,Q\}\!\}'V)_{uj}\{\!\{U,Q\}\!\}''_{iv}\\
&= \{\!\{UV,Q\}\!\}'_{uj}\{\!\{UV,Q\}\!\}''_{iv}.
\end{aligned}$$

It follows that we have to check (7.6) only on elements of $(D_BA)_i$ with $i = 0, 1$.

If $P, Q \in A$, then there is nothing to prove. So assume $P = \delta \in D_{A/B}$ and $Q = a \in A$. Then we need to prove

$$\delta_{ij}(a_{uv}) = \delta(a)'_{uj}\delta(a)''_{iv},$$

but this is precisely (7.3).

The case $P \in A$ and $Q \in D_BA$ follows from the previous case by antisymmetry of both $\{-,-\}$ and $\{\!\{-,-\}\!\}$. Hence we concentrate on the final case $P = \delta \in D_{A/B}$ and $Q = \Delta \in D_{A/B}$. Let $a$ be an arbitrary element of $A$. We will show

$$\{\delta_{ij}, \Delta_{uv}\}(a_{pq}) = (\{\!\{\delta,\Delta\}\!\}'_{uj}\{\!\{\delta,\Delta\}\!\}''_{iv})(a_{pq}).$$

This equation translates into

$$\begin{aligned}
\delta_{ij}\Delta_{uv}(a_{pq}) - \Delta_{uv}\delta_{ij}(a_{pq}) &= \{\!\{\delta,\Delta\}\!\}'_{l,uj}(a_{pq})\{\!\{\delta,\Delta\}\!\}''_{l,iv}\\
&\quad + \{\!\{\delta,\Delta\}\!\}'_{r,uj}\{\!\{\delta,\Delta\}\!\}''_{r,iv}(a_{pq}).
\end{aligned} \tag{7.7}$$

The lefthand side of this equation is obtained from the fact that the Schouten bracket of vector fields is the commutator. The righthand side is obtained by writing $\{\!\{-,-\}\!\} = \{\!\{-,-\}\!\}_l + \{\!\{-,-\}\!\}_r$ and observing that $\{\!\{-,-\}\!\}_l$ takes values in $D_B A \otimes A$ and $\{\!\{-,-\}\!\}_r$ takes values in $A \otimes D_B A$.

We compute

$$\begin{aligned}\delta_{ij}\Delta_{uv}(a_{pq}) &= \delta_{ij}(\Delta(a)'_{pv}\Delta(a)''_{uq})\\ &= \delta_{ij}(\Delta(a)'_{pv})\Delta(a)''_{uq} + \Delta(a)'_{pv}\delta_{ij}(\Delta(a)''_{uq})\\ &= \delta(\Delta(a)')'_{pj}\delta(\Delta(a)')''_{iv}\Delta(a)''_{uq} + \Delta(a)'_{pv}\delta(\Delta(a)'')'_{uj}\delta(\Delta(a)'')''_{iq}\end{aligned}$$

and in the same way,

$$\Delta_{uv}\delta_{ij}(a_{pq}) = \Delta(\delta(a)')'_{pv}\Delta(\delta(a)')''_{uj}\delta(a)''_{iq} + \delta(a)'_{pj}\Delta(\delta(a)'')'_{iv}\Delta(\delta(a)'')''_{uq}.$$

We deduce

$$\begin{aligned}&\delta_{ij}\Delta_{uv}(a_{pq}) - \Delta_{uv}\delta_{ij}(a_{pq})\\ &\quad = [\delta,\Delta]\tilde{}_l(a)'_{pj}[\delta,\Delta]\tilde{}_l(a)''_{iv}[\delta,\Delta]\tilde{}_l(a)'''_{uq} + [\delta,\Delta]\tilde{}_r(a)'_{pv}[\delta,\Delta]\tilde{}_r(a)''_{uj}[\delta,\Delta]\tilde{}_r(a)'''_{iq}.\end{aligned}$$

Now we look at the righthand side of (7.7):

$$\begin{aligned}\{\!\{\delta,\Delta\}\!\}'_{l,uj}(a_{pq})\{\!\{\delta,\Delta\}\!\}''_{l,iv} &= \{\!\{\delta,\Delta\}\!\}'_l(a)'_{pj}\{\!\{\delta,\Delta\}\!\}'_l(a)''_{uq}\{\!\{\delta,\Delta\}\!\}''_{l,iv}\\ &= [\delta,\Delta]\tilde{}_l(a)'_{pj}[\delta,\Delta]\tilde{}_l(a)''_{iv}[\delta,\Delta]\tilde{}_l(a)'''_{uq}\end{aligned}$$

and similarly

$$\begin{aligned}\{\!\{\delta,\Delta\}\!\}'_{r,uj}\{\!\{\delta,\Delta\}\!\}''_{r,iv}(a_{pq}) &= \{\!\{\delta,\Delta\}\!\}'_{r,uj}\{\!\{\delta,\Delta\}\!\}''_r(a)'_{pv}\{\!\{\delta,\Delta\}\!\}''_r(a)''_{iq}\\ &= [\delta,\Delta]\tilde{}_r(a)'_{pv}[\delta,\Delta]\tilde{}_r(a)''_{uj}[\delta,\Delta]\tilde{}_r(a)'''_{iq},\end{aligned}$$

which finishes the proof. □

7.7. **Invariant functions.** We leave it to the reader to check the following property. Let $a \in A$, $\omega \in (\Omega_B A)_n$, $\delta \in (D_B A)_n$. Then $\mathrm{tr}X(a)$, $\mathrm{tr}X(\omega)$, $\mathrm{tr}X(\delta)$ depend only on the value of $a, \omega, \delta$, modulo commutators. For simplicity we write $\mathrm{tr}(-) = \mathrm{tr}X(-)$.

The famous Artin, Le Bruyn, Procesi theorem reformulated in this language reads:

**Theorem 7.7.1.** *$\mathcal{O}(\mathrm{Rep}(A,\alpha))^{\mathrm{Gl}_\alpha}$ is the ring generated by the functions $\mathrm{tr}(a)$ for $a \in A$.*

The following result was proved by Crawley-Boevey [5].

**Proposition 7.7.2.** *If $A$ is equipped with a Poisson structure (see 2.6) with associated Lie bracket $\{-,-\}$, then $\mathcal{O}(\mathrm{Rep}(A,\alpha))^{\mathrm{Gl}_\alpha}$ has a unique Poisson structure with the property*

$$\{\mathrm{tr}(a), \mathrm{tr}(b)\} = \mathrm{tr}\{\bar{a}, \bar{b}\}.$$

Traces are also compatible with the Schouten bracket.

**Proposition 7.7.3.** *For $P, Q \in D_B A$ one has*

$$\{\mathrm{tr}(P), \mathrm{tr}(Q)\} = \mathrm{tr}\{P, Q\}.$$

*Proof.* This is an easy computation:

$$\begin{aligned}\{\operatorname{tr}(P), \operatorname{tr}(Q)\} &= \{P_{ii}, Q_{jj}\} \\ &= \{\!\{P, Q\}\!\}'_{ji} \{\!\{P, Q\}\!\}''_{ij} \\ &= \{P, Q\}_{ii} \\ &= \operatorname{tr}\{P, Q\}.\end{aligned}$$

□

**Corollary 7.7.4.** *The map*

$$\operatorname{tr} : D_B A/[D_B A, D_B A] \to \bigwedge_{\mathcal{O}(\operatorname{Rep}(A,a))} \operatorname{Der}(\mathcal{O}(\operatorname{Rep}(A, a)))$$

*is a Lie algebra homomorphism if both sides are equipped with the Schouten bracket.*

7.8. **Compatibility.** Assume that $P \in (D_B A)_2$. Then $P$ induces a double bracket $\{\!\{-,-\}\!\}_P$ on $A$ and hence a corresponding antisymmetric $\{-,-\}_P$ biderivation on $\mathcal{O}(\operatorname{Rep}(A,\alpha))$. On the other hand $\operatorname{tr}(P)$ also induces an antisymmetric biderivation on $\mathcal{O}(\operatorname{Rep}(A,\alpha))$. We claim that these are the same. More precisely we want to show for $f, g \in \mathcal{O}(\operatorname{Rep}(A,\alpha))$ that

$$\{f, g\}_P = \operatorname{tr}(P)(f, g).$$

It suffices to check this for $P = \delta\Delta$ with $\delta, \Delta \in D_{A/B}$. Recall that we have for $a, b \in A$,

$$\begin{aligned}\{\!\{a, b\}\!\}_P &= -\{\!\{a, \{\delta\Delta, b\}\!\}\} \\ &= -(\delta b)'(\Delta a)'' \otimes (\Delta a)'(\delta b)'' + (\Delta b)'(\delta a)'' \otimes (\delta a)'(\Delta b)''.\end{aligned}$$

Hence we compute

$$\begin{aligned}\{a_{ij}, b_{uv}\}_P &= \{\!\{a, b\}\!\}'_{uj} \{\!\{a, b\}\!\}''_{iv} \\ &= -(\delta b)'_{uw}(\Delta a)''_{wj}(\Delta a)'_{iz}(\delta b)''_{zv} + (\Delta b)'_{uw}(\delta a)''_{wj}(\delta a)'_{iz}(\Delta b)''_{zw} \\ &= -\delta_{zw}(b_{uv})\Delta_{wz}(a_{ij}) + \Delta_{zw}(b_{uv})\delta_{wz}(a_{ij}) \\ &= (\delta_{wz} \wedge \Delta_{zw})(a_{ij}, b_{uv}) \\ &= \operatorname{tr}(\delta\Delta)(a_{ij}, b_{uv}).\end{aligned}$$

7.9. **Base change.**

**Proposition 7.9.1.** *Let $f_{ij} \in M_\alpha = \operatorname{Lie}(\operatorname{Gl}_\alpha)$ be the elementary matrix which is 1 in the $(i,j)$-entry and zero everywhere else. Then $(E_p)_{ij}$ acts as $f_{ji}$ on $\mathcal{O}(\operatorname{Rep}(A,\alpha))$ if $\phi(i) = \phi(j) = p$ and else as zero.*

*Proof.* Consider first the case $\phi(i) = \phi(j) = p$. The formula (7.1) yields

$$f_{ji} a_{uv} = [X(a), f_{ji}]_{uv} = a_{uj}\delta_{iv} - \delta_{uj}a_{iv}$$

(here $\delta$ is the Kronecker delta) and

$$\begin{aligned}(E_p)_{ij}(a_{uv}) &= E_p(a)'_{uj} E_p(a)''_{iv} \\ &= (ae_p)_{uj}(e_p)_{iv} - (e_p)_{uj}(e_p a)_{iv} \\ &= a_{uj}\delta_{iv} - \delta_{uj}a_{iv}\end{aligned}$$

where we have used $(ae_p)_{uj} = a_{uw}(e_p)_{wj} = a_{uw}\delta_{wj} = a_{uj}$. If it is not true that $\phi(i) = \phi(j) = p$, then a similar computation shows that $(E_p)_{ij}$ acts as zero. □

*Remark* 7.9.2. The previous proposition explains why we have called the elements $(E_p)_p \in (D_B A)_1$ "gauge elements" in §3.3. They correspond to gauge transformations on $\mathcal{O}(\mathrm{Rep}(A,\alpha))$.

7.10. **Fusion.** In this section the notation is as in §2.5.

**Lemma 7.10.1.** *Assume that $\alpha_1 = \alpha_2$ and let $\alpha^f = (\alpha_1, \alpha_3, \ldots, \alpha_n)$. Consider $\mathrm{Gl}_{\alpha^f}$ as being embedded in $\mathrm{Gl}_\alpha$ where the embedding on the first factor is diagonal and on the other factors is the identity.*

*There is a canonical isomorphism between $\mathrm{Rep}(A,\alpha)$ and $\mathrm{Rep}(A^f,\alpha^f)$ such that the induced $\mathrm{Gl}_{\alpha^f}$-action on $\mathrm{Rep}(A,\alpha)$ is obtained by restriction from the $\mathrm{Gl}_\alpha$-action.*

*Proof.* Left to the reader. □

7.11. **Hamiltonian structure.** We have shown in Proposition 7.5.2 that if $A$ is a double Poisson algebra, then $\mathcal{O}(\mathrm{Rep}(A,\alpha))$ is a Poisson algebra. In this section we discuss the Hamiltonian structure.

If $G$ is an algebraic group, with Lie algebra $\mathfrak{g}$, acting on an (affine) Poisson variety $X$, then a moment map for this action is by definition an invariant map $\psi : X \to \mathfrak{g}^*$ such that for all $v \in \mathfrak{g}$ and $f \in \mathcal{O}(X)$ we have

$$\{\langle v, -\rangle \circ \psi, f\} = v(f). \tag{7.8}$$

**Proposition 7.11.1.** *Let $A, \{\!\{-,-\}\!\}, \mu$ be a Hamiltonian algebra. Then*

$$X(\mu_p)_p : \mathrm{Rep}(A,\alpha) \to M_\alpha$$

*is a moment map for $\mathrm{Rep}(A,\alpha)$ equipped with the associated bracket $\{-,-\}$ (as in §7.5).*

*Proof.* We verify this (7.8) in the case $X = \mathrm{Rep}(A,\alpha)$ and $\psi = X(\mu_p)_p$. It suffices to check (7.8) with $v = f_{ji}$ with $\phi(i) = \phi(j) = p$ and $f = a_{uv}$, $a \in A$. Then (7.8) becomes

$$\sum_p \{\mathrm{tr}(f_{ji} X(\mu_p)_p), a_{uv}\} = (E_p)_{ij}(a_{uv}).$$

We compute the left hand side of this equation:

$$\begin{aligned}\sum_p \{\mathrm{tr}(f_{ji} X(\mu_p)_p), a_{uv}\} &= \{\mu_{p,ij}, a_{uv}\} \\ &= \{\!\{\mu_p, a\}\!\}'_{uj} \{\!\{\mu_p, a\}\!\}''_{iv} \\ &= E_p(a)'_{uj} E_p(a)''_{iv} \\ &= (E_p)_{ij}(a_{uv}).\end{aligned}$$

This finishes the proof. □

7.12. **Quasi-Poisson structure.** Let $\mathfrak{g}$ be a Lie algebra equipped with an invariant non-degenerate symmetric bilinear form $(-,-)$. Let $(f_a)_a$, $(f^a)_a$ be dual bases of $\mathfrak{g}$. Then there is a canonical invariant element $\phi \in \wedge^3 \mathfrak{g}$ given by

$$\frac{1}{12} c^{abc} f_a \wedge f_b \wedge f_c$$

where

$$c^{abc} = (f^a, [f^b, f^c]).$$

If $G$ acts on an affine variety $X$, then we have an induced 3-vector field $\phi_X$ on $X$. Following [1] an element $P \in \bigwedge^2_{\mathcal{O}(X)} \operatorname{Der}(\mathcal{O}(X))$ is said to be a *quasi-Poisson bracket* if

$$\{P, P\} = \phi_X.$$

Now we compute $\phi$ for $M_\alpha$ with the trace pairing. In that case $(f_a)_a = (f_{ij})_{ij}$, $(f^a)_a = (f_{ji})_{ij}$. Hence

$$\begin{aligned} c^{ij,kl,mn} &= \operatorname{tr}(f_{ji}[f_{lk}, f_{nm}]) \\ &= \operatorname{tr}(f_{ji} f_{lk} f_{nm} - f_{ji} f_{nm} f_{lk}) \\ &= \delta_{il}\delta_{kn}\delta_{jm} - \delta_{in}\delta_{lm}\delta_{jk}. \end{aligned}$$

We can now compute $\phi$:

$$\begin{aligned} 12\phi &= (\delta_{il}\delta_{kn}\delta_{jm} - \delta_{in}\delta_{lm}\delta_{jk}) f_{ij} \wedge f_{kl} \wedge f_{mn} \\ &= f_{ij} \wedge f_{ki} \wedge f_{jk} - f_{ij} \wedge f_{jl} \wedge f_{li} \\ &= 2 f_{ij} \wedge f_{ki} \wedge f_{jk}. \end{aligned}$$

From Proposition 7.9.1 we obtain

**Proposition 7.12.1.** *The three vector field on* $\operatorname{Rep}(A, \alpha)$ *induced by* $\phi$ *is given by*

$$\frac{1}{6} \sum_i \operatorname{tr}(E_i^3).$$

We obtain

**Theorem 7.12.2.** *Assume that* $A, P$ *is a differential double quasi-Poisson algebra. Then* $\operatorname{tr}(P)$ *is a quasi-Poisson bracket on* $\operatorname{Rep}(A, \alpha)$.

*Proof.* This follows by taking the trace of the defining property

$$\{P, P\} = \frac{1}{6} \sum_{i=1}^{n} E_i^3 \mod [D_B A, D_B A]$$

(see §5.2) together with Propositions 7.7.3 and 7.12.1. □

*Remark* 7.12.3. By a somewhat tedious verification using Proposition 7.5.2 it follows that Theorem 7.12.2 is also true in the non-differential case. We omit this.

7.13. **Quasi-Hamiltonian structure.** Let $G, X, \mathfrak{g}, (-,-)$ be as in the beginning of the previous section.

For $v \in \mathfrak{g}$ let $v^L$, $v^R$ be the associated left and right invariant vector fields. According to the conventions in [1, pp. 2,3], if $g$ is a function on $G$, then

$$v^L(g)(z) = \frac{d}{dt} g(z \exp(tv))_{t=0}, \tag{7.9}$$

$$v^R(g)(z) = \frac{d}{dt} g(\exp(tv) z)_{t=0}. \tag{7.10}$$

If $v \in \mathfrak{g}$, then $v_X$ is by definition the vector field on $X$ defined by

$$v_X(g)(x) = \frac{d}{dt} g(\exp(-tv)x)_{t=0}$$

for a function $g$ on $X$.

**Definition 7.13.1** ([1])**.** Assume that $\mathcal{O}(X), P$ is a quasi-Poisson algebra. Let $(f^a)_a$ and $(f_a)_a$ be a pair of dual bases for $\mathfrak{g}$. An Ad-equivariant map

$$\Phi : X \to G$$

is a multiplicative moment map if for all functions $g$ on $G$ we have

$$\{g \circ \Phi, -\} = \frac{1}{2} f^a_X \left((f^L_a + f^R_a)(g) \circ \Phi\right).$$

We can now prove the following result.

**Proposition 7.13.2.** *Let $A, P$ be a double differential quasi-Poisson algebra and let $\Phi = (\Phi_p)_p \in \bigoplus_p e_p A e_p$ be a multiplicative moment map. Then*

$$X(\Phi_p)_p : \operatorname{Rep}(A, \alpha) \to M_\alpha$$

*is a multiplicative moment map for* $\operatorname{Rep}(A, \alpha)$ *equipped with the Poisson bracket* $\operatorname{tr}(P)$.

*Proof.* As dual bases (for the trace pairing on $M_\alpha$) we choose $(f_{ij})_{ij}$ and $(f_{ji})_{ij}$.

We apply (7.9) with $v = f_{ij}$ and $g = g_{uv}$ where $g_{uv}$ is the projection on the $uv$-th entry of $M_\alpha$ and $u, v$ are such that $\phi(u) = \phi(v) = q$. This yields

$$\begin{aligned} f^L_{ij}(g_{uv})(z) &= g_{uv}(z f_{ij}) \\ &= \delta_{jv} z_{ui} \end{aligned}$$

and hence

$$f^L_{ij}(g_{uv}) = \delta_{jv} g_{ui}.$$

Similarly

$$\begin{aligned} f^R_{ij}(g_{uv})(z) &= g_{uv}(f_{ij} z) \\ &= \delta_{iu} z_{jv} \end{aligned}$$

and hence

$$f^R_{ij}(g_{uv}) = \delta_{iu} g_{jv}.$$

From this computation we obtain (with $X = \operatorname{Rep}(A, \alpha)$)

$$\begin{aligned} \frac{1}{2}(f_{ji})_X \left(((f_{ij})^L + (f_{ij})^R)(g_{uv}) \circ \Phi\right) &= \frac{1}{2}(f_{ji})_X((\delta_{jv} g_{ui} + \delta_{iu} g_{jv}) \circ \Phi) \\ &= \frac{1}{2}((f_{vi})_X \Phi_{q,ui} + (f_{ju})_X \Phi_{q,jv}) \\ &= \frac{1}{2}(E_{q,iv} \Phi_{q,ui} + E_{q,uj} \Phi_{q,jv}) \\ &= \frac{1}{2}(\Phi_q E_q + E_q \Phi_q)_{uv}. \end{aligned}$$

Thus for $a \in A$:

$$\begin{aligned} \frac{1}{2}(f_{ji})_X \left(((f_{ij})^L + (f_{ij})^R)(g_{uv}) \circ \Phi\right)(a_{rs}) \\ = \frac{1}{2}(\Phi_q E_q + E_q \Phi_q)(a)'_{rv} (\Phi_q E_q + E_q \Phi_q)(a)''_{us}. \end{aligned} \tag{7.11}$$

On the other hand,

$$\{g_{uv} \circ X(\Phi), a_{rs}\} = \{\Phi_{q,uv}, a_{rs}\} = \{\!\{\Phi_q, a\}\!\}'_{rv} \{\!\{\Phi_q, a\}\!\}''_{us}. \tag{7.12}$$

We obtain that (7.11) is indeed equal to (7.12) from the defining identity for a multiplicative moment map:

$$\{\!\{\Phi_q, a\}\!\} = \frac{1}{2}(\Phi_q E_q + E_q \Phi_q)(a). \qquad \square$$

7.14. **Interpretation for quivers.** It follows from Proposition 6.8.1 together with Proposition 7.7.2 that if $A$ is either a deformed preprojective algebra or a deformed multiplicative preprojective algebra, then $\mathcal{O}(\operatorname{Rep}(A,\alpha))^{\operatorname{Gl}(\alpha)}$ has a Poisson structure. The explicit formulas for the Poisson bracket may be obtained from (6.3) and (6.4) provided we can interpret the partial derivatives that occur.

It is easy to see that $\operatorname{Rep}(kQ,\alpha)$ is the polynomial algebra with generators $a_{ij}$ for $a \in Q$ and $\phi(i) = h(a)$, $\phi(j) = t(a)$. It is convenient to set $a_{ij} = 0$ if this last condition is not satisfied.

**Lemma 7.14.1.** *We have*

$$\left(\frac{\partial}{\partial a}\right)_{ij} = \begin{cases} \dfrac{\partial}{\partial a_{ji}} & \text{if } \phi(i) = h(a),\ \phi(j) = t(a), \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* By the definition of $\frac{\partial}{\partial a}$ it follows that $e_p \frac{\partial}{\partial a} = 0$ for $p \neq h(a)$ and $\frac{\partial}{\partial a} e_q = 0$ for $q \neq t(a)$. From this it follows that if $\phi(i) \neq h(a)$ or $\phi(j) \neq t(a)$, then $\left(\frac{\partial}{\partial a}\right)_{ij} = 0$. So let us assume that $\phi(i) = h(a)$ and $\phi(j) = t(a)$.

We have for $a, b \in Q$,

$$\left(\frac{\partial}{\partial a}\right)_{ij}(b_{uv}) = \left(\frac{\partial b}{\partial a}\right)'_{uj}\left(\frac{\partial b}{\partial a}\right)''_{iv}.$$

If $a \neq b$, then we obtain

$$\left(\frac{\partial}{\partial a}\right)_{ij}(b_{uv}) = 0.$$

So assume $b = a$. Then

$$\left(\frac{\partial}{\partial a}\right)_{ij}(a_{uv}) = (e_{t(a)})_{uj}(e_{h(a)})_{iv}. \tag{7.13}$$

If $\phi(u) \neq t(a)$ or $\phi(v) \neq h(a)$, then both sides of (7.13) are zero. So let us assume that $\phi(u) = t(a)$ or $\phi(v) = h(a)$. Then (7.13) becomes

$$\begin{aligned}\left(\frac{\partial}{\partial a}\right)_{ij}(a_{uv}) &= \delta_{uj}\delta_{iv} \\ &= \frac{\partial a_{uv}}{\partial a_{ji}}. \qquad \square\end{aligned}$$

In the case of the deformed preprojective algebra we obtain the classical result that the Poisson bracket corresponds to the bi-vector field

$$\sum_{a\in Q} \frac{\partial}{\partial a_{ij}} \frac{\partial}{\partial a^*_{ji}}.$$

For the deformed multiplicative preprojective algebra we obtain (using (6.4)) a similar but more complicated Poisson bracket.

## Appendix A. Relation to the theory of bi-symplectic forms

In this appendix we relate our theory of double Poisson brackets to the theory of bi-symplectic forms introduced in [7]. The analogous, but more involved theory for double quasi-Poisson brackets will be deferred to a separate note.

We assume as usual that $A/B$ is finitely generated. Let $\Omega_B A$ be the tensor algebra over $A$ of $\Omega_{A/B}$. This is a DG-algebra. Assume that $A$ is equipped with a $B$-linear bi-symplectic form $\omega$ (see Definition A.3.1 below). We prove:

(1) The Lie bracket on $A/[A, A]$ associated to $\omega$ [7, Prop. 4.4.1] comes from a $B$-linear double differential Poisson bracket $P$ on $A$ (and hence, by Proposition 2.4.4, from the structure of a left Loday algebra on $A$).
(2) The algebras $\Omega_B A$ and $D_B A$ become isomorphic DG-algebras if we equip $D_B A$ with the differential $-\{P, -\}$.

The formalism we will outline is remarkably similar to the commutative case. For example in Theorem A.6.1 below we prove that the condition $d\omega = 0$ for a bi-symplectic form is precisely equivalent to the condition $\{P, P\} = 0$ for the corresponding double Poisson bracket.

A.1. **Differentials and double derivations.** We recall some definitions from [7]. We also give some properties which we will need afterward.

Let $\delta \in D_{A/B}$. Then we may define double derivations

$$i_\delta : \Omega_B A \to \Omega_B A \otimes \Omega_B A,$$
$$L_\delta : \Omega_B A \to \Omega_B A \otimes \Omega_B A$$

in the usual way: for $a \in A$ define

$$i_\delta(a) = 0, \qquad i_\delta(da) = \delta(a),$$
$$L_\delta(a) = \delta(a), \qquad L_\delta(da) = d(\delta(a)),$$

where here and below we use the convention that $d$ acts on tensor products by means of the usual Leibniz rule.

If $C$ is a graded $k$-algebra and $c = c_1 \otimes c_2$, then we put

$$^\circ c = (-1)^{|c_1||c_2|} c_2 c_1$$

and if $\phi : C \to C^{\otimes 2}$ is a linear map, then we define

$$^\circ\phi : C \to C : c \mapsto {}^\circ(\phi(c)).$$

If $\delta$ is a double derivation, then $^\circ\delta$ vanishes on commutators.

We apply this with $C = \Omega_B A$. Following [7] we put

$$\imath_\delta = {}^\circ i_\delta,$$
$$\mathcal{L}_\delta = {}^\circ L_\delta.$$

Now we discuss some commutation relations between these operators. Checking on the generators $a \in A$ and $da \in \Omega_{A/B}$ of $\Omega_B A$ we find the usual Cartan formula [7, eq. (2.7.2)]

$$L_\delta = d i_\delta + i_\delta d$$

from which one obtains by applying the operation $^\circ(-)$ [7, Lemma 2.8.8(i)]

$$\mathcal{L}_\delta = d\imath_\delta + \imath_\delta d. \tag{A.1}$$

As the $(i_\delta)_\delta$ are double derivations one can take their Schouten brackets. One has for $\delta,\Delta\in D_{A/B}$,

$$\{\!\{i_\delta,i_\Delta\}\!\}_l=\{\!\{i_\delta,i_\Delta\}\!\}_r=0. \tag{A.2}$$

To see this note that both $\{\!\{i_\delta,i_\Delta\}\!\}_l^\sim$ and $\{\!\{i_\delta,i_\Delta\}\!\}_r^\sim$ are derivations $\Omega_B A\to(\Omega_B)^{\otimes_B A}$ of degree $-2$. Hence they must vanish on $A$ and $\Omega_{A/B}$. This means that they must vanish on the whole of $\Omega_B A$.

We will also use the following identities:

$$\begin{aligned}\{\!\{i_\delta,L_\Delta\}\!\}_l&=i_{\{\!\{\delta,\Delta\}\!\}_l'}\otimes{}^{\circ}\{\!\{\delta,\Delta\}\!\}_l'',\\ \{\!\{i_\delta,L_\Delta\}\!\}_r&={}^{\circ}\{\!\{\delta,\Delta\}\!\}_r'\otimes i_{\{\!\{\delta,\Delta\}\!\}_r''},\end{aligned}\tag{A.3}$$

which are proved by checking them on the generators of $\Omega_B A$.

Let $C$ be a $k$-algebra and $\delta,\Delta$ be double $C$-derivations. The straightforward proof of the next formula is left to the reader:

$$\delta\circ{}^{\circ}\Delta-\tau_{12}\circ\Delta\circ{}^{\circ}\delta={}^{\circ,l}\{\!\{\delta,\Delta\}\!\}_l+{}^{\circ,r}\{\!\{\Delta,\delta\}\!\}_r, \tag{A.4}$$

where ${}^{\circ,l}(\epsilon'\otimes\epsilon'')={}^{\circ}\epsilon'\otimes\epsilon''$ and ${}^{\circ,r}(\mu'\otimes\mu'')=\mu'\otimes{}^{\circ}\mu''$.

From the graded version of (A.4) together with (A.2) we obtain the following formula:

$$i_\delta\imath_\Delta+\sigma_{12}i_\Delta\imath_\delta=0. \tag{A.5}$$

Combining the graded version of (A.4) with (A.3) we find

$$i_\delta\mathcal{L}_\Delta-\sigma_{12}L_\Delta\imath_\delta=\imath_{\{\!\{\delta,\Delta\}\!\}_l'}\otimes{}^{\circ}\{\!\{\delta,\Delta\}\!\}_l''+{}^{\circ}\{\!\{\delta,\Delta\}\!\}_r'\otimes\imath_{\{\!\{\delta,\Delta\}\!\}_r''}. \tag{A.6}$$

Finally assume that $\delta$ is an inner double $A$-derivation of the form $[b,-]$ for $b\in B\otimes_k B$. Then for all $\eta\in\Omega_B A$ one has

$$\mathcal{L}_\delta\eta=0. \tag{A.7}$$

To prove this note first that

$$L_\delta\eta=b\eta-\eta b. \tag{A.8}$$

As usual this is checked on generators. (A.7) follows immediately from (A.8).

A.2. **The Koszul bracket.** Assume that $\{\!\{-,-\}\!\}$ is a $B$-linear double bracket on $A$. One has the following result.

**Proposition A.2.1.** *There is a unique double bracket $\{\!\{-,-\}\!\}^{\Omega_B A}$ of degree $-1$ on $\Omega_B A$ commuting with $d$ which satisfies for $a,b\in A$,*

$$\{\!\{da,b\}\!\}^{\Omega_B A}=\{\!\{a,b\}\!\}.$$

*If $\{\!\{-,-\}\!\}$ is Poisson, then so is $\{\!\{-,-\}\!\}^{\Omega_B A}$.*

*Proof.* The various asserted properties may be checked on generators. We leave the full proof to the reader. □

Following the commutative case we call $\{\!\{-,-\}\!\}^{\Omega_B A}$ the *Koszul bracket* associated to $\{\!\{-,-\}\!\}$.

For use below we give some formulas which are easy consequences of the definition:

$$\begin{aligned} \{\!\{a,b\}\!\}^{\Omega_B A} &= 0, \\ \{\!\{da,b\}\!\}^{\Omega_B A} &= \{\!\{a,db\}\!\}^{\Omega_B A} = \{\!\{a,b\}\!\}, \\ \{\!\{da,db\}\!\}^{\Omega_B A} &= d\{\!\{a,b\}\!\}. \end{aligned} \tag{A.9}$$

**Proposition A.2.2.** *Assume that $\{\!\{-,-\}\!\}$ is a double bracket on $A$. Then there is a well-defined map of graded $A$-algebras*

$$\Sigma : \Omega_B A \to D_B A : da \mapsto H_a$$

*(see §3.5 for notation). If $\{\!\{-,-\}\!\}$ is Poisson, then this map intertwines the Koszul bracket on the left with the Schouten bracket on the right.*

*Proof.* That $\Sigma$ is well defined is easy to see by checking on generators. To prove that $\Sigma$ is compatible with the brackets we have to show that the following analogues of (A.9) hold in $D_B A$:

$$\begin{aligned} \{\!\{a,b\}\!\}^{D_B A} &= 0, \\ \{\!\{H_a,b\}\!\}^{D_B A} &= \{\!\{a,H_b\}\!\}^{D_B A} = \{\!\{a,b\}\!\}, \\ \{\!\{H_a,H_b\}\!\}^{D_B A} &= H_{\{\!\{a,b\}\!\}}, \end{aligned} \tag{A.10}$$

where to avoid confusion we have denoted the Schouten bracket by $\{\!\{-,-\}\!\}^{D_B A}$. The first equation is obvious. The second equation follows from (3.11). The third equation follows from Proposition 3.5.1. □

A.3. **Bi-symplectic forms.** Following [7] we put $\mathrm{DR}_B(A) = \Omega_B A/[\Omega_B A, \Omega_B A]$. The differential on $\Omega_B A$ descends to a differential on $\mathrm{DR}_B(A)$.

**Definition A.3.1** ([7])**.** An element $\omega \in \mathrm{DR}_B(A)_2$ is *bi-non-degenerate* if the map of $A$-bimodules

$$\imath(\omega) : D_{A/B} \to \Omega_{A/B} : \delta \mapsto \imath_\delta \omega$$

is an isomorphism. If in addition $\omega$ is closed in $\mathrm{DR}_B(A)$, then we say that $\omega$ is *bi-symplectic*.

Assume that $\omega \in \mathrm{DR}_B(A)_2$ is bi-non-degenerate. Let $a \in A$. Following [7] we define the Hamiltonian vector field $H_a \in D_{A/B}$ corresponding to $a$ via

$$\imath_{H_a} \omega = da,$$

and we put

$$\{\!\{a,b\}\!\}_\omega = H_a(b).$$

Since $H_a(b) = i_{H_a}(db)$ we may also write

$$\{\!\{a,b\}\!\}_\omega = i_{H_a} \imath_{H_b}(\omega). \tag{A.11}$$

**Lemma A.3.2.** *$\{\!\{a,b\}\!\}_\omega$ is a double bracket on $A$.*

*Proof.* It is clear that $\{\!\{a,b\}\!\}_\omega$ is a derivation in its second argument. So we only need to prove anti-symmetry. This follows immediately from (A.11) and (A.5). □

According to [7, Prop. 4.4.1] if $\omega$ is bi-symplectic, then the associated simple bracket $\{-,-\}_\omega$ induces a Lie algebra structure on $A/[A,A]$. We will prove the following stronger result.

**Proposition A.3.3.** *Assume that $\omega \in (\Omega_B A)_2$ is bi-symplectic. Then $\{\!\{-,-\}\!\}_\omega$ is a double Poisson bracket on $A$.*

*Proof.* To simplify the notation we write $\{\!\{-,-\}\!\}$ for $\{\!\{-,-\}\!\}_\omega$. According to Proposition 3.5.1 we have to prove

$$\{\!\{H_a, H_b\}\!\}_l = H_{\{\!\{a,b\}\!\}'} \otimes \{\!\{a,b\}\!\}''.$$

Appying $\imath(\omega) \otimes 1$, this is equivalent to

$$\imath_{\{\!\{H_a,H_b\}\!\}_l'}(\omega) \otimes \{\!\{H_a, H_b\}\!\}_l'' = d\{\!\{a,b\}\!\}_l' \otimes \{\!\{a,b\}\!\}_l''.$$

This would follow by projecting the following identity in $\Omega_{A/B} \otimes A \oplus A \otimes \Omega_{A/B}$ onto the first factor:

$$\imath_{\{\!\{H_a,H_b\}\!\}_l'}(\omega) \otimes \{\!\{H_a, H_b\}\!\}_l'' + \{\!\{H_a, H_b\}\!\}_r' \otimes \imath_{\{\!\{H_a,H_b\}\!\}_r''}(\omega) = d\{\!\{a,b\}\!\}.$$

Applying (A.6) we find that this is equivalent to

$$(i_{H_a}\mathcal{L}_{H_b} - \tau_{12} L_{H_b}\imath_{H_a})(\omega) = d\{\!\{a,b\}\!\}. \tag{A.12}$$

Now

$$\imath_{H_a}(\omega) = da$$

and

$$\mathcal{L}_{H_b}(\omega) = d\imath_{H_b}(\omega) + \imath_{H_b} d\omega = ddg = 0$$

(this is the place where the assumption that $\omega$ is closed is used).

It follows that the left hand side of (A.12) is equal to

$$-\tau_{12} L_{H_b}(da) = -\tau_{12} d\{\!\{b,a\}\!\} = d\{\!\{a,b\}\!\},$$

which is indeed equal to the righthand side of (A.12). □

A.4. **Duality yoga for bimodules.** Below $A$ is a $k$-algebra and $M$ is an $A$-bimodule. Later we take $M = \Omega_B A$, but in this section this is not necessary.

If $m \in M$, then there is a double derivation of degree $-1$,

$$i_m : T_A(M^*) \to T_A(M^*) \otimes T_A(M^*),$$

which on $\sigma \in M^*$ is given by

$$i_m(\sigma) = \sigma(m)'' \otimes \sigma(m)'.$$

Similarly for $\sigma \in M^*$ there is an associated double derivation of degree $-1$,

$$i_\sigma : T_A(M) \to T_A(M) \otimes T_A(M)$$

with the property

$$i_\sigma(m) = \sigma(m)$$

for $m \in M$. As before we put $\imath_m = {}^\circ i_m$ and $\imath_\sigma = {}^\circ i_\sigma$.

**Proposition A.4.1.** *Assume there is an element $\omega \in M \otimes_A M$ such that the induced map*

$$\imath(\omega) : M^* \to M : \sigma \mapsto \imath_\sigma \omega$$

*is an isomorphism. Define $P = -(\imath(\omega)^{-1} \otimes \imath(\omega)^{-1})(\omega)$. Then one has for $m \in M$,*

$$\imath(P) = \imath(\omega)^{-1}$$

*where $\imath(P)$ is the map*

$$\imath(P) : M \to M^* : m \mapsto \imath_m P.$$

*Proof.* Put $\theta = \imath(\omega)$ and $\psi = \imath(\omega)^{-1}$. We have to prove $\psi = \imath(P)$. We have explicitly for $\sigma \in M^*$,

$$\theta(\sigma) = \sigma(\omega')''\omega''\sigma(\omega')' - \sigma(\omega'')''\omega'\sigma(\omega'')' \tag{A.13}$$

and hence for $\tau \in M^*$,

$$\tau(\theta(\sigma)) = \sigma(\omega')''\tau(\omega'')\sigma(\omega')' - \sigma(\omega'')''\tau(\omega')\sigma(\omega'')'.$$

We deduce

$$\tau(\theta(\sigma)) = -\sigma(\theta(\tau))^\circ. \tag{A.14}$$

Put $\tau = \psi(n)$ and $\sigma = \psi(m)$. We deduce from (A.14):

$$\psi(n)(m) = -\psi(m)(n)^\circ. \tag{A.15}$$

Taking into account that $P = -\psi(\omega') \otimes \psi(\omega'')$ we find

$$\imath_m P = -\psi(\omega')(m)'\psi(\omega'')\psi(\omega')(m)'' + \psi(\omega'')(m)'\psi(\omega')\psi(\omega'')(m)''.$$

Hence

$$\begin{aligned}\theta(\imath_m P) &= -\psi(\omega')(m)'\omega''\psi(\omega')(m)'' + \psi(\omega'')(m)'\omega'\psi(\omega'')(m)'' \\ &= -\theta(\psi(-)(m)^\circ),\end{aligned}$$

where in the last line we have used (A.13). Hence for $n \in M$,

$$(\imath_m P)(n) = -\psi(n)(m)^\circ = \psi(m)(n),$$

where we have used (A.15). □

A.5. **Compatibility of brackets.** Let $\omega \in (\Omega_B A)_2$ be a bi-non-degenerate form. Putting $M = \Omega_{A/B}$ and $M^* = D_{A/B}$ we define $P \in (D_B A)_2$ by

$$P = -(\imath(\omega)^{-1} \otimes \imath(\omega)^{-1})(\omega)$$

as in the previous section.

**Proposition A.5.1.** *One has*

$$\{\!\{-,-\}\!\}_P = \{\!\{-,-\}\!\}_\omega.$$

*Proof.* Using Proposition A.4.1 one has

$$\{\!\{f,g\}\!\}_\omega = H_f(g) = \imath(P)(df)(g) = (\imath_{df} P)(g).$$

The result now follows from Lemma A.5.2 below. □

**Lemma A.5.2.** *Let $P$ be an arbitrary element of $(D_B A)_2$. Then*

$$\{\!\{f,g\}\!\}_P = (\imath_{df} P)(g).$$

*Proof.* It is sufficient to check this for $P = \delta\Delta$ with $\delta, \Delta \in D_{A/B}$. We have

$$\{\!\{f,g\}\!\}_{\delta\Delta} = \Delta(g)'\delta(f)'' \otimes \delta(f)'\Delta(g)'' - \delta(g)'\Delta(f)'' \otimes \Delta(f)'\delta(g)''$$

and

$$\begin{aligned}i_{df}(\delta\Delta) &= i_{df}(\delta)\Delta - \delta i_{df}(\Delta) \\ &= \delta(f)'' \otimes \delta(f)'\Delta - \delta\Delta(f)'' \otimes \Delta(f)'.\end{aligned}$$

Thus

$$\imath_{df}(\delta\Delta) = \delta(f)'\Delta\delta(f)'' - \Delta(f)'\delta\Delta(f)''$$

and hence

$$\imath_{df}(\delta\Delta)(g) = \Delta(g)'\delta(f)'' \otimes \delta(f)'\Delta(g)'' - \delta(g)'\Delta(f)'' \otimes \Delta(f)'\delta(g)'',$$

finishing the proof. □

The following result will be used below.

**Lemma A.5.3.** *Let $\{\!\{-,-\}\!\}$ be the double bracket associated to a bi-non-degenerate form $\omega$ and let $P$ be the element of $(D_BA)_2$ associated to $\omega$ (as in* §A.5*). Let $\Sigma : \Omega_B A \to D_B A$ be as in Proposition* A.2.2*. Then $\Sigma$ is an isomorphism of $A$-algebras and furthermore*

$$\Sigma(\omega) = -P. \tag{A.16}$$

*Proof.* Let $\overline{\imath(P)}$ be the $A$-algebra morphism obtained by extending $\imath(P)$. We claim that $\Sigma = \imath(P)$. For $a \in A$ we need $\Sigma(da) = \imath(P)(da)$. By definition $\Sigma(da) = H_a$ and $\imath(P)(da) = \imath_{da}(P) = H_a$ by Lemma A.5.2. In particular $\Sigma$ is an isomorphism of algebras. We also deduce

$$\Sigma(\omega) = (\imath(P) \otimes \imath(P))(\omega) = (\imath(\omega)^{-1} \otimes \imath(\omega)^{-1})(\omega) = -P. \qquad \square$$

A.6. **The relation between Poisson brackets and bi-symplectic forms.** In this section we prove the following result.

**Theorem A.6.1.** *Assume that $\omega \in (\Omega_B A)_2$ is a bi-non-degenerate form and let $P$ be the corresponding element of $D_BA$. Then the following are equivalent:*

(1) *$\omega$ is bi-symplectic, i.e.*

$$d\omega = 0$$

*in* $\mathrm{DR}_B(A)$.

(2) *$P$ is differential Poisson, i.e.*

$$\{P, P\} = 0$$

*in $D_BA/[D_BA, D_BA]$.*

*Proof.* Let $\{\!\{-,-\}\!\} = \{\!\{-,-\}\!\}_\omega = \{\!\{-,-\}\!\}_P$ be the corresponding bracket on $A$ and let

$$\Sigma : \Omega_B A \to D_B A$$

be as in Proposition A.2.2. Below we assume that either $d\omega = 0$ or $\{P, P\} = 0$. It follows by Proposition A.3.3 and Theorem 4.2.3 that in both these cases $\{\!\{-,-\}\!\}$ is Poisson. Hence by Proposition A.2.2, $\Sigma$ intertwines the Koszul bracket and the Schouten bracket.

Assume first that $P$ is Poisson. We claim that the following diagram is commutative:

$$\begin{array}{ccc} \Omega_B A & \xrightarrow{\Sigma} & D_B A \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle -\{P,-\}} \\ \Omega_B A & \xrightarrow{\Sigma} & D_B A. \end{array} \tag{A.17}$$

It is sufficient to check this on generators. Let $a \in A$. We have

$$(\Sigma \circ d)(a) = \Sigma(da) = H_a$$

and

$$-(\{P, -\} \circ \Sigma)(a) = -\{P, a\}.$$

We need to see $\{P,a\} = -H_a$. Evaluating on $b \in A$ it is sufficient to prove

$$\{P,a\}(b) = -\{\!\{a,b\}\!\}_P. \tag{A.18}$$

Using Proposition 4.2.1 we find that the left hand side of (A.18) is equal to

$$-\{\!\{b,\{P,a\}\}\!\}^\circ = \{\!\{b,a\}\!\}_P^\circ = -\{\!\{a,b\}\!\}_P.$$

This is equal to the right hand side of (A.18).

Now we consider the generators of the type $da$, $a \in A$. We have

$$(\Sigma \circ d)(da) = 0$$

and

$$-(\{P,-\} \circ \Sigma)(da) = -\{P,H_a\} = -\{P,\{P,a\}\!\} = -(1/2)\{\!\{,P\},a\}\!\} = 0.$$

Applying the identity $\Sigma \circ d = -\{P,-\} \circ d$ to $\omega$ we conclude

$$\Sigma(d\omega) = -\{P,\Sigma(\omega)\} = \{P,P\} = 0,$$

where we have used Lemma A.5.3. Since $\Sigma$ is an isomorphism it follows that $d\omega = 0$.

Now assume that $d\omega = 0$. We claim $d = \{\omega,-\}^{\Omega_B A}$. Let $a \in A$. First we need to check $da = \{\omega,a\}^{\Omega_B A}$. Applying $\Sigma$ this is equivalent to $H_a = -\{P,a\}$. This we have checked above.

Now we need to check $\{\omega,da\}^{\Omega_B A} = 0$. This is the following verification:

$$\{\omega,da\}^{\Omega_B A} = d\{\omega,a\}^{\Omega_B A} - \{d\omega,a\}^{\Omega_B A} = dda = 0.$$

From $d = \{\omega,-\}^{\Omega_B A}$ we deduce $0 = d\omega = \{\omega,\omega\}^{\Omega_B A}$. Applying $\Sigma$ we obtain $\{P,P\} = 0$ (using Lemma A.5.3 again), finishing the proof. □

We obtain the following consequence of (A.17).

**Corollary A.6.2.** *Assume that $A$ has a bi-symplectic form $\omega$ with corresponding differential Poisson bracket $P$. If we equip $D_B A$ with the differential $-\{P,-\}$, then $\Omega_B A$ and $D_B A$ become isomorphic double DG-Gerstenhaber algebras.*

The *De Rham cohomology* of $A$ is the cohomology of $\mathrm{DR}_B(A)$ for the differential $d$. Likewise the *Poisson cohomology* of $(A,P)$ [17] is the cohomology of the complex $D_B/[D_B A, D_B A]$ for the differential $\{P,-\}$. We obtain the following corollary.

**Corollary A.6.3.** *If $A$ is equipped with a bi-symplectic form, then its Poisson cohomology coincides with its De Rham cohomology.*

This is an analogue of a well-known commutative result.

A.7. **The moment map.** We keep the assumptions of the previous section. We assume in addition that $B = ke_1 + \cdots + ke_n$.[2] In [7] the very beautiful observation is made that if $A$ is equipped with a bi-symplectic form $\omega$, then $A$ is automatically Hamiltonian in a suitable sense (and hence this is true for all representation spaces). For completeness we give the construction of the moment map in our present context. No originality is intended.

According to Definition 2.6.4 we need to find $\mu_i$ such that for all $a$ one has

$$\{\!\{\mu_i,a\}\!\}_\omega = E_i(a).$$

[2]In [7] it is only assumed that $B$ is semi-simple. The role of the $(e_i)_i$ is played by a separability idempotent.

Or in other words,

$$H_{\mu_i} = E_i.$$

Applying $\imath(\omega)$ this is equivalent to

$$d\mu_i = \imath_{E_i}(\omega).$$

Since $\Omega_B A$ is acyclic in degrees $\geq 1$ (see [7, §2.5]) the existence of $\mu_i$ follows from $d\imath_{E_i}(\omega) = 0$. By (A.1) and the fact that $\omega$ is closed we have

$$d\imath_{E_i}(\omega) = \mathcal{L}_{E_i}(\omega) = 0$$

where we have also used (A.7).

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## References

[1] A. Alekseev, Y. Kosmann-Schwarzbach, and E. Meinrenken, *Quasi-Poisson manifolds*, Canad. J. Math. **54** (2002), no. 1, 3–29. MR1880957 (2002k:53157)

[2] A. Alekseev, A. Malkin, and E. Meinrenken, *Lie group valued moment maps*, J. Differential Geom. **48** (1998), no. 3, 445–495. MR1638045 (99k:58062)

[3] M. F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), no. 1505, 523–615. MR702806 (85k:14006)

[4] R. Bocklandt and L. Le Bruyn, *Necklace Lie algebras and noncommutative symplectic geometry*, Math. Z. **240** (2002), no. 1, 141–167. MR1906711 (2003g:16013)

[5] W. Crawley-Boevey, *A note on non-commutative Poisson structures*, math.QA/0506268.

[6] ———, *Preprojective algebras, differential operators and a Conze embedding for deformations of Kleinian singularities*, Comment. Math. Helv. **74** (1999), no. 4, 548–574. MR1730657 (2001c:16028)

[7] W. Crawley-Boevey, P. Etingof, and V. Ginzburg, *Noncommutative geometry and quiver algebras*, Advances in Mathematics **209** (2007), 274–336. MR2294224 (2008a:14006)

[8] W. Crawley-Boevey and P. Shaw, *Multiplicative preprojective algebras, middle convolution and the Deligne-Simpson problem*, Adv. Math. **201** (2006), 180–208. MR2204754 (2006m:16015)

[9] W. Crawley-Boevey and M. P. Holland, *Non-commutative deformations of Kleinian singularities*, Duke Math. J. **92** (1998), 605–635. MR1620538 (99f:14003)

[10] J. Cuntz and D. Quillen, *Algebra extensions and nonsingularity*, J. Amer. Math. Soc. **8** (1995), no. 2, 251–289. MR1303029 (96c:19002)

[11] V. Ginzburg, *Non-commutative symplectic geometry, quiver varieties, and operads*, Math. Res. Lett. **8** (2001), no. 3, 377–400. MR1839485 (2002m:17020)

[12] M. Kontsevich, *Formal (non)commutative symplectic geometry*, The Gel′fand Mathematical Seminars, 1990–1992 (Boston, MA), Birkhäuser Boston, Boston, MA, 1993, pp. 173–187. MR1247289 (94i:58212)

[13] Y. Kosmann-Schwarzbach, *Loday algebras (Leibniz algebras)*, available as http://math.polytechnique.fr/cmat/kosmann/lod.ps.

[14] L. Le Bruyn, *Noncommutative geometry@n, volumes 1/2*, neverendingbooks.org, 2005.

[15] L. Le Bruyn and G. Van de Weyer, *Formal structures and representation spaces*, J. Algebra **247** (2002), no. 2, 616–635. MR1877866 (2002m:16028)

[16] J.-L. Loday, *Une version non commutative des algèbres de Lie: Les algèbres de Leibniz*, Enseign. Math. (2) **39** (1993), no. 3-4, 269–293. MR1252069 (95a:19004)

[17] G. Van de Weyer, *Double Poisson structures on finite dimensional semi-simple algebras*, math.AG/0603533.

Departement WNI, Limburgs Universitair Centrum, 3590 Diepenbeek, Belgium
*E-mail address*: michel.vandenbergh@uhasselt.be