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STANDARD GRADED VERTEX COVER ALGEBRAS, CYCLES AND LEAVES

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ABSTRACT. The aim of this paper is to characterize simplicial complexes which have standard graded vertex cover algebras. This property has several nice consequences for the squarefree monomial ideals defining these algebras. It turns out that such simplicial complexes are closely related to a range of hypergraphs which generalize bipartite graphs and trees. These relationships allow us to obtain very general results on standard graded vertex cover algebras which cover previous major results on Rees algebras of squarefree monomial ideals.

INTRODUCTION

Let Δ be a simplicial complex on the vertex set $[n] = \{1, ..., n\}$. Let $\mathcal{F}(\Delta)$ denote the set of the facets of Δ. An integer vector $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{N}^n$ is called a *cover of order* k or a k*-cover* of Δ if $\sum_{i\in F} c_i \geq k$ for all facets F of Δ. If $\mathbf{c}$ happens to be a $(0,1)$-vector, then $\mathbf{c}$ may be identified with the subset $C = \{i \in [n] : \ c_i \neq 0\}$ of $[n]$. It is clear that $\mathbf{c}$ is a 1-cover if and only if C is a *vertex cover* of Δ in the classical sense, that is, $C \cap F \neq \emptyset$ for all $F \in \mathcal{F}(\Delta)$.

Let $S = K[x_1, \dots, x_n]$ be a polynomial ring in n variables over a field K. Let $A_k(\Delta)$ denote the K-vector space generated by all monomials $x_1^{c_1} \cdots x_n^{c_n} t^k$ such that $(c_1, \dots, c_n) \in \mathbb{N}^n$ is a k-cover of Δ, where t is a new variable. Then

$$A(\Delta) := \bigoplus_{k\geq 0} A_k(\Delta)$$

is a graded S-algebra. We call $A(\Delta)$ the *vertex cover algebra* of Δ [14].

Vertex cover algebras have an interesting algebraic interpretation. For any subset $F \subset [n]$ let P_F denote the ideal of S generated by the variables x_i with $i \in F$. Set

$$I^*(\Delta) := \bigcap_{F\in\mathcal{F}(\Delta)} P_F.$$

Then $I^*(\Delta)$ is the ideal generated by the squarefree monomials $x_{i_1} \cdots x_{i_j}$ such that $\{i_1, \dots, i_j\}$ is a vertex cover of Δ and $A(\Delta)$ is the symbolic Rees algebra of $I^*(\Delta)$. It is shown in [14] that $A(\Delta)$ is a finitely generated, graded and normal Cohen-Macaulay S-algebra.

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Ideals of the form $I^*(\Delta)$ are exactly the squarefree monomial ideals. A similar notion is the *facet ideal* $I(\Delta)$ which is generated by the squarefree monomials $x_{i_1}\cdots x_{i_j}$ such that $\{i_1,\dots,i_j\}$ is a facet of Δ. If we denote by Δ^* the simplicial complex whose facets are the minimal vertex covers of Δ, then $I^*(\Delta^*) = I(\Delta)$. Thus, one can use vertex cover algebras as a tool for the study of facet ideals.

It is of great interest to know when $A(\Delta)$ is a standard graded algebra, that is, when $A(\Delta)$ is generated over S by homogeneous elements of degree 1. Combinatorially, that means every k-covers of Δ can be written as a sum of k 1-covers for all $k \geq 1$. Ideal-theoretically, that means the symbolic powers of $I^*(\Delta)$ coincide with the ordinary powers. In particular, this condition implies that the Rees algebra of $I(\Delta^*)$ is a Cohen-Macaulay normal domain and that the associated graded ring is Cohen-Macaulay and reduced.

In case Δ is pure and 1-dimensional, we identify Δ with the graph whose edges are the facets of Δ. For such a simplicial complex it is shown in [14] that $A(\Delta)$ is a standard graded algebra if and only if Δ is a bipartite graph. This result has led us to study more generally the relationship between standard graded vertex cover algebras and the combinatorics of the corresponding simplicial complexes.

In the more general situation it proves to be favorable to identify a given simplicial complex Δ with the hypergraph whose edges are the facets of Δ. Recall that a hypergraph H is a collection of subsets of a finite set of vertices. The elements of H are called the edges of H. Of course, the hypergraphs of the form $\mathcal{F}(\Delta)$ are special, because there are no inclusions between the sets representing its edges. Up to the order of the vertices and the order of the edges, a hypergraph is determined by its incidence matrix and vice versa. This matrix is defined as follows: let $x_1,\dots,x_n$ be the vertices and $C_1,\dots,C_m$ be the edges of the hypergraph H. Then the incidence matrix $M = (e_{ij})$ is an $n \times m$ matrix with $e_{ij} = 1$ if $x_i \in C_j$ and $e_{ij} = 0$ if $x_i \notin C_j$.

We say that Δ is a Mengerian simplicial complex if the incidence matrix of $\mathcal{F}(\Delta)$ satisfies a certain min-max equation, which is known as the Mengerian property in hypergraph theory or as the max-flow min-cut property in integer programming. As a main result in Section 1 we show in Theorem 1.4 that $A(\Delta)$ is a standard graded algebra if and only if Δ^* is a Mengerian simplicial complex. The proof of this result is based on the observation that $A(\Delta)$ is standard graded if and only if the vertex set of every polarized simplicial complex of Δ can be decomposed into k vertex covers, where k is the minimum number of vertices of the facets. As a consequence, Δ is a Mengerian simplicial complex if and only if the symbolic powers of the facet ideal $I(\Delta)$ coincide with the ordinary powers. This result recovers two recent results of Escobar, Villarreal and Yoshino [8] and Gitler, Valencia and Villarreal [13] on the normality of the Rees algebra of $I(\Delta)$.

It suggests asking whether the fact mentioned above that each bipartite graph has a standard graded vertex cover algebra may be extended to higher dimensions. This is indeed the case and is discussed in Section 2. But what is the analogue of a bipartite graph in higher dimensions? Recall that a bipartite graph is characterized by the property that it has no odd cycles. It is natural to call an alternating sequence of distinct vertices and edges $v_1, F_1, v_2, F_2, \dots, v_s, F_s, v_{s+1} = v_1$, $s \geq 2$, in a hypergraph a *cycle* if $v_i, v_{i+1} \in F_i$ for all i. Such a cycle is called *special* if no edge contains more than two vertices of the cycle.

We show in Theorem 2.2 that for a given simplicial complex Δ, the vertex cover algebra $A(\Gamma)$ is standard graded for all subcomplexes $\Gamma \subseteq \Delta$ if and only if Δ has no special odd cycles. Hypergraphs having no special odd cycle are called balanced, and our proof uses the well-known result of Berge [1] that the vertices of a balanced hypergraph H can be colored by $\min\{|F| : F \in H\}$ colors such that every color occurs in every edge of H.

Fulkerson, Hoffman and Oppenheim [11] showed that a balanced hypergraph is Mengerian. This implies at once that $A(\Delta^*)$ is standard graded if Δ has no special odd cycle. This, in turn, immediately yields the result of Simis, Vasconcelos and Villarreal [23] according to which the edge ideal of a bipartite graph is normally torsionfree.

A well-known class of balanced hypergraphs is the class of unimodular hypergraphs which are defined by the property that all determinants of the incidence matrix equal $0, \pm 1$. We say that Δ is unimodular if $\mathcal{F}(\Delta)$ is unimodular. In this case, the above-mentioned results imply that $A(\Delta)$ and $A(\Delta^*)$ are standard graded algebras.

A facet F of a simplicial complex of Δ is called a *leaf* if either F is the only facet of Δ, or there exists $G \in \mathcal{F}(\Delta)$, $G \neq F$ such that $H \cap F \subset G \cap F$ for each $H \in \mathcal{F}(\Delta)$ with $H \neq F$. If each subcomplex Γ of Δ has a leaf, then Δ is called a forest. If, in addition, Δ is connected, then Δ is called a tree. These notions were introduced by Faridi [10] who proved, among other things, that the Rees algebra of the facet ideal of a tree is a Cohen-Macaulay domain.

In Theorem 3.2 we show that Δ is a forest if and only if Δ has no special cycle of length ≥ 3. Hypergraphs having no special cycle of length ≥ 3 are called totally balanced and our proof is based on the characterization of totally balanced hypergraphs in terms of the so-called greedy matrix found by Hoffman, Kolen and Sakarovitch [18] and Lubiw [21]. The results in Section 2 imply that $A(\Delta)$ and $A(\Delta^*)$ are standard graded if Δ is a forest. This then implies the above-mentioned result of Faridi and stronger assertions on the facet ideal of a tree.

There is another interesting property of forests, shown in Corollary 3.4, namely that each forest has a good leaf, that is, a facet which is a leaf of each subcomplex to which it belongs. The notion of good leaf was introduced in the thesis of Zheng [26]. The existence of good leaves implies immediately that each forest has a good leaf order, that is to say, that the facets $F_1, \ldots, F_m$ of Δ can be ordered in such a way that F_i is a good leaf of the subcomplex whose facets are $F_1, \ldots, F_i$. It was recently shown by Pelsmajer, Tokaz and West [22] that totally balanced hypergraphs have good leaf orders. Their proof compared with the arguments given here is relatively complicated.

Using good leaf orders and an algebraic result of Conca and De Negri [5] we show in Corollary 3.5 that if I is a graded ideal in a polynomial ring, whose initial ideal with respect to a suitable term order is the facet ideal of a forest, then it has the property that the Rees algebra of I is Cohen-Macaulay and that the associated graded ring of I is reduced.

We close Section 3 by showing in Theorem 3.6 that if Δ is a simplicial complex, F is a good leaf of Δ and Γ is the simplicial complex obtained from Δ by removing F, then the highest degree of the generators of $A(\Delta)$ is the same as the highest degree of the generators of $A(\Gamma)$. We use this result in Section 4 to study vertex cover algebras of quasi-forests.

A quasi-forest is a simplicial complex whose facets can be ordered $F_1, \ldots, F_m$ such that for all i, F_i is a leaf of the simplicial complex with the facets $F_1, \ldots, F_i$.

It is clear that any forest is a quasi-forest. The significance of quasi-forests results from a theorem of Dirac [6] who proved that (translated into the language of quasi-forests) a simplicial complex is a quasi-forest if and only if its 1-skeleton is a chordal graph.

There is a subclass of the chordal graphs, called strongly chordal. Farber [9] showed that strongly chordal graphs can be described by forbidden induced subgraphs, the so-called trampolines, and that a simplicial complex is a forest if and only if its 1-skeleton is a strongly chordal graph. Comparing this result with Dirac's theorem the difference between forests and quasi-forests becomes apparent and one would expect that unlike forests the quasi-forests have nonstandard graded vertex cover algebras. However, this is not always the case as can be shown by examples. On the other hand, we show in our final Theorem 4.2 that a quasi-forest with the property that it is connected in codimension 1 and that each face of codimension 1 belongs to at most two facets is a forest if and only if $A(\Delta)$ is standard graded.

We have seen in this paper several applications of hypergraph theory on vertex cover algebras and facet ideals. It would be of interest to use algebraic methods to solve problems in hypergraph theory. Moreover, the notion of vertex cover algebras has been introduced for weighted simplicial complexes. It remains to see whether one can extend some of the results of this paper for general vertex cover algebras.

1. Standard vertex cover algebras

We adhere to the notions of the introduction.

Let Δ be a simplicial complex on the vertex set $[n]$. We say that a k-cover $\mathbf{c}$ of Δ is *decomposable* if there exists an i-cover $\mathbf{a}$ and a j-cover $\mathbf{b}$ such that $\mathbf{c} = \mathbf{a} + \mathbf{b}$, $k = i+j$ and $a, b \neq 0$. If $\mathbf{c}$ is not decomposable, we call it *indecomposable*. It is clear that the indecomposable covers of order ≥ 1 correspond to a minimal homogeneous set of generators of the S-algebra $A(\Delta)$.

An obvious vertex cover of Δ is the *full cover* $[n]$, which corresponds to the vector $(1, ..., 1)$. The maximal order of this cover is obviously given by the number

$$s(\Delta) = \min\{|F| : \; F \in \mathcal{F}(\Delta)\}.$$

If $A(\Delta)$ is a standard graded S-algebra, then $[n]$ can be decomposed into $s(\Delta)$ vertex covers of Δ. In this case, Δ is called *totally decomposable*. We shall use this property to give a characterization of standard graded vertex cover algebras.

Let $\mathbf{c} = (c_1, ..., c_n) \in \mathbb{N}^n$ be an arbitrary integral vector. We associate with $\mathbf{c}$ a new set of vertices

$$X^{\mathbf{c}} = \{x_{ij} : \; i = 1, ..., n, j = 1, ..., c_i\}.$$

To each subset F of $[n]$ we associate the subset

$$F^{\mathbf{c}} := \{x_{ij} : \; i \in F, j = 1, ..., c_i\}$$

of X^c. Let $\Delta^{\mathbf{c}}$ denote the simplicial complex on $X^{\mathbf{c}}$ whose facets are the minimal sets of the form $F^{\mathbf{c}}$, $F \in \mathcal{F}(\Delta)$. Following the method of polarization of monomials (see e.g. [4]) we call $\Delta^{\mathbf{c}}$ the *polarization* of Δ with respect to $\mathbf{c}$.

The following result allows us to reduce the decomposition of arbitrary covers of Δ to the decomposition of the full cover of their polarizations.

Lemma 1.1. *An integer vector $\mathbf{c} \in \mathbb{N}^n$ can be written as a sum of k 1-covers if and only if $X^{\mathbf{c}}$ can be decomposed into k vertex covers of $\Delta^{\mathbf{c}}$.*

Proof. Assume that $\mathbf{c} = \mathbf{c}_1 + \cdots + \mathbf{c}_k$, where $\mathbf{c}_1, ..., \mathbf{c}_k$ are 1-covers of Δ. For $h = 1, ..., k$, let $\mathbf{c}_h = (c_{h1}, ..., c_{hn})$ and

$$C_h := \{x_{ij} : i \in 1, ..., n,\ j = c_{1i} + \cdots + c_{h-1i} + 1, ..., c_{1i} + \cdots + c_{h-1i} + c_{hi}\}.$$

It is easily seen that $X^{\mathbf{c}} = C_1 \cup \cdots \cup C_k$ and that $C_1, ..., C_k$ are disjoint vertex covers of $\Delta^{\mathbf{c}}$.

Conversely, assume that $X^{\mathbf{c}} = C_1 \cup \cdots \cup C_k$ and that $C_1, ..., C_k$ are disjoint vertex covers of $\Delta^{\mathbf{c}}$. For $h = 1, ..., k$ let $\mathbf{c}_h = (c_{h1}, ..., c_{hn})$ with $c_{hi} := |x_{ij} :\ x_{ij} \in C_h|$, $i = 1, ..., n$. It is obvious that $\mathbf{c} = \mathbf{c}_1 + \cdots + \mathbf{c}_k$. Each $\mathbf{c}_h$ is a 1-cover of Δ because for all facets F of Δ, $F^{\mathbf{c}}$ contains a facet of $\Delta^{\mathbf{c}}$. But C_h contains a vertex, say x_{ij}, of this facet of $\Delta^{\mathbf{c}}$. Therefore, $i \in F$ and $c_{hi} \geq 1$, which implies $\sum_{i \in F} c_{hi} \geq 1$. $\square$

Proposition 1.2. *The vertex cover algebra $A(\Delta)$ is standard graded if and only if $\Delta^{\mathbf{c}}$ is totally decomposable for all $\mathbf{c} \in \mathbb{N}^n$.*

Proof. $A(\Delta)$ is standard graded if and only if every k-cover $\mathbf{c}$ of Δ can be written as a sum of k 1-covers for all $k \geq 1$. By Lemma 1.1, this is equivalent to the condition that $X^{\mathbf{c}}$ can be decomposed into k vertex covers of $\Delta^{\mathbf{c}}$. It is easily seen that

$$s(\Delta^{\mathbf{c}}) = \min\{|F^{\mathbf{c}}| :\ F \in \mathcal{F}(\Delta)\} = \min\{\sum_{i \in F} c_i :\ F \in \mathcal{F}(\Delta)\}$$

is the maximal order of $\mathbf{c}$. Therefore, $A(\Delta)$ is standard graded if and only if $X^{\mathbf{c}}$ can be decomposed into $s(\Delta^{\mathbf{c}})$ vertex covers of $\Delta^{\mathbf{c}}$. $\square$

Let $o(\mathbf{c})$ denote the maximal order of $\mathbf{c}$ and $\sigma(\mathbf{c})$ the maximum number k such that $\mathbf{c}$ can be written as a sum of k 1-covers of Δ. Then $o(\mathbf{c}) = s(\Delta^{\mathbf{c}})$ and $\sigma(\mathbf{c})$ is the maximum number k such that $X^{\mathbf{c}}$ can be decomposed into k vertex covers of $\Delta^{\mathbf{c}}$. Thus, $\sigma(\mathbf{c}) \leq o(\mathbf{c})$ and $\Delta^{\mathbf{c}}$ is totally decomposable means $\sigma(\mathbf{c}) = o(\mathbf{c})$.

The above notions have the following meanings in hypergraph theory. Recall that a hypergraph H is a collection of subsets of a vertex set. The elements of H are called the edges of H. One calls a set of vertices meeting all edges of H a *transversal* of H. If H has a partition into k transversals, where $k = \min\{|F| :\ F \in H\}$, then H is said to have the Gupta property [7, 3.1]. Therefore, $\Delta^{\mathbf{c}}$ is totally decomposable if and only if the hypergraph $\mathcal{F}(\Delta^{\mathbf{c}})$ has the Gupta property.

Berge [2, Chapter 5, Lemma, p. 207] already studied hypergraphs H such that every polarization of H has the Gupta property, using different notation. Following his approach we can give a characterization of standard graded algebras by means of a min-max property.

Let $C_1, ..., C_m \subset X$ be the minimal vertex covers of Δ. Let Δ^* denote the simplicial complexes whose facets are $C_1, ..., C_m$. It is well known that

$$(\Delta^*)^* = \Delta.$$

Let M be the facet-vertex incidence matrix of Δ^*. Then we have the following formulas for $o(\mathbf{c})$ and $\sigma(\mathbf{c})$ in terms of M.

Lemma 1.3. *Let $\mathbf{1}$ denote the vector $(1, ..., 1)$ of $\mathbb{N}^m$. Then*
(i) $o(\mathbf{c}) = \min\{\mathbf{a} \cdot \mathbf{c} :\ \mathbf{a} \in \mathbb{N}^n,\ M \cdot \mathbf{a} \geq \mathbf{1}\}$,
(ii) $\sigma(\mathbf{c}) = \max\{\mathbf{b} \cdot \mathbf{1} :\ \mathbf{b} \in \mathbb{N}^m,\ M^T \cdot \mathbf{b} \leq \mathbf{c}\}$.

Proof. The vectors $\mathbf{a} \in \mathbb{N}^n$ with $M \cdot \mathbf{a} \geq \mathbf{1}$ are the 1-covers of Δ^*. Since the minimal 1-covers of Δ^* correspond to the facets of Δ, we have

$$\begin{aligned} o(\mathbf{c}) &= \min\{\sum_{i \in F} c_i : \ F \in \mathcal{F}(\Delta)\} \\ &= \min\{\mathbf{a} \cdot \mathbf{c} : \ \mathbf{a} \in \{0,1\}^n, \ M \cdot \mathbf{a} \geq \mathbf{1}\} \\ &= \min\{\mathbf{a} \cdot \mathbf{c} : \ \mathbf{a} \in \mathbb{N}^n, \ M \cdot \mathbf{a} \geq \mathbf{1}\}. \end{aligned}$$

Let $\mathbf{c}_1, ..., \mathbf{c}_m$ be the (0,1) vectors of $C_1, ..., C_m$. As $\mathbf{c}_1, ..., \mathbf{c}_m$ are the minimum 1-covers of Δ, $\mathbf{c}$ can be written as a sum of k 1-covers of Δ if there exist b_1 copies of $\mathbf{c}_1$,..., b_m copies of $\mathbf{c}_m$ such that $k = b_1 + \cdots + b_m$ and $b_1\mathbf{c}_1 + \cdots + b_m\mathbf{c}_m \leq \mathbf{c}$. Let $\mathbf{b} = (b_1, ..., b_m)$. Then $\mathbf{b} \cdot \mathbf{1} = b_1 + \cdots + b_m$ and $M^T \cdot \mathbf{b} = b_1\mathbf{c}_1 + \cdots + b_m\mathbf{c}_m$. Therefore,

$$\begin{aligned} \sigma(\mathbf{c}) &= \max\{k : \ \mathbf{c} \text{ can be written as a sum of } k \text{ 1-covers of } \Delta\} \\ &= \max\{\mathbf{b} \cdot \mathbf{1} : \ \mathbf{b} \in \mathbb{N}^m, \ M^T \cdot \mathbf{b} \leq \mathbf{c}\}. \end{aligned}$$

□

Following the terminology of hypergraph theory [2, Chapter 7], we call Δ^* a *Mengerian simplicial complex* if for all $\mathbf{c} \in \mathbb{N}^n$,

$$\min\{\mathbf{a} \cdot \mathbf{c} : \ \mathbf{a} \in \mathbb{N}^n, \ M \cdot \mathbf{a} \geq \mathbf{1}\} = \max\{\mathbf{b} \cdot \mathbf{1} : \ \mathbf{b} \in \mathbb{N}^m, \ M^T \cdot \mathbf{b} \leq \mathbf{c}\}.$$

This definition applies to an arbitrary simplicial complex or a hypergraph with suitably adopted notation. The Mengerian property is also known under the name max-flow min-cut property in integer linear programming.

Lemma 1.3 immediately yields the following characterization of standard graded vertex cover algebras.

Theorem 1.4. *The vertex cover algebra $A(\Delta)$ is standard graded if and only if Δ^* is a Mengerian simplicial complex.*

As $\Delta = (\Delta^*)^*$, this result can be reformulated as follows.

Corollary 1.5. *Δ is a Mengerian simplicial complex if and only if $A(\Delta^*)$ is a standard graded algebra.*

Let $I(\Delta)$ be the *facet ideal* of Δ, the ideal generated by the monomials of the variables of the facets of Δ. Then $I(\Delta) = I^*(\Delta^*)$. This interpretation of $I(\Delta)$ leads to interesting relationships between the Mengerian property and properties of the facet ideal.

Corollary 1.6. *The following conditions are equivalent:*
(i) *Δ is a Mengerian simplicial complex,*
(ii) *$I(\Delta)^{(k)} = I(\Delta)^k$ for all $k \geq 0$,*
(iii) *the associated graded ring of $I(\Delta)$ is reduced,*
(iv) *$I(\Delta)$ is normally torsionfree (i.e., all powers of $I(\Delta)$ have the same associated prime ideals).*
Moreover, if one of these conditions is satisfied, then the Rees algebra of $I(\Delta)$ is a normal Cohen-Macaulay domain and the associated graded ring is Cohen-Macaulay.

Proof. The equivalence of (i) and (ii) follows from Corollary 1.5 because $A(\Delta^*)$ is the symbolic Rees algebra of $I(\Delta)$. The equivalence of (ii) to (iii) and (iv) is well known [20]. These conditions imply that the Rees algebra of $I(\Delta)$ coincides with $A(\Delta^*)$. By [14, Theorem 4.3], $A(\Delta^*)$ is a normal Cohen-Macaulay domain. It is known that the Cohen-Macaulay property of the Rees algebra implies the Cohen-Macaulay property of the associated graded ring [19]. $\square$

Corollary 1.7. *Assume that Δ is pure and Mengerian. Let $k[I(\Delta)]$ denote the toric ring generated by the monomial generators of $I(\Delta)$. Then $k[I(\Delta)]$ is a normal Cohen-Macaulay domain.*

Proof. Since Δ is pure, $I(\Delta)$ is generated by monomials of the same degree, say d. Therefore, we may view the Rees algebra $\mathcal{R}$ of $I(\Delta)$ as a bigraded k-algebra with $\mathcal{R}_{(h,k)} = (I(\Delta)^k)_{dh}t^k$ for $h, k \geq 0$ and $k[I(\Delta)]$ as the subalgebra $\bigoplus_{k\geq 0} \mathcal{R}_{(k,k)}$. Since Δ is Mengerian, $A(\Delta)$ is a normal domain by Corollary 1.6. Hence, $k[I(\Delta)]$ is also a normal domain. By [17], this implies that $k[I(\Delta)]$ is Cohen-Macaulay. $\square$

The relationship between facet ideals and the Mengerian simplicial complexes was already studied in a recent paper of Gitler, Valencia and Villarreal [13, Theorem 3.5]. They proved that Δ is Mengerian if and only if the polyhedron

$$Q(\Delta) := \{\mathbf{a} \in \mathbb{R}^n : \ \mathbf{a} \geq \mathbf{0}, \ M \cdot \mathbf{a} \geq \mathbf{1}\}$$

(M is now the incidence matrix of Δ and $\mathbf{0}$ is the vector of zero components) has integral vertices and the Rees algebra of $I(\Delta)$ is a normal domain. On the other hand, Escobar, Villarreal and Yoshino [8, Proposition 3.4] showed that the latter conditions are satisfied if and only if the associated graded ring of $I(\Delta)$ is reduced. Hence, Corollary 1.6 can be deduced from their results.

It is not hard to see that $Q(\Delta)$ has integral vertices if and only if the normalization of the Rees algebra of $I(\Delta)$ coincides with the symbolic Rees algebra. Therefore, one can also recover the two aforementioned results from Corollary 1.6.

Example 1.8. Let Δ be the simplicial complex of Figure 1 which has the facets

$$\{1,2,3\}, \{1,5,6\}, \{2,4,6\}, \{3,4,5\}.$$

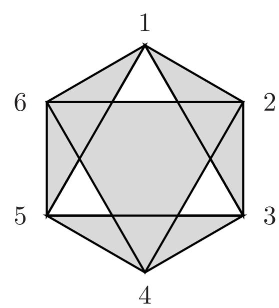

FIGURE 1

Then Δ^* is the simplicial complex with the facets

$$\{1,4\}, \{2,5\}, \{3,6\}, \{1,2,3\}, \{1,5,6\}, \{2,4,6\}, \{3,4,5\}.$$

It is known that Δ is not Mengerian but Δ^* is Mengerian [2, p. 198 and p. 209]. We will use Proposition 1.4 to establish this fact.

Δ is not Mengerian because $A(\Delta^*)$ is not standard graded. In fact, the full cover of Δ^* corresponds to an indecomposable 2-cover of Δ, which implies that $A(\Delta)$ has a generator in degree 2.

Δ^* is Mengerian because $A(\Delta)$ is standard graded. Let $\mathbf{c} = (c_1, ..., c_6)$ be an arbitrary k-cover of Δ of maximal order $k \geq 2$. It suffices to show that $\mathbf{c}$ can written as a sum of a 1-cover and a $(k-1)$-cover. Let $f = x_1^{c_1} \cdots x_6^{c_6}$. If f is divisible by monomials g of the forms x_1x_4, x_2x_5, x_3x_6, then f/g corresponds to a cover of order $k-1$ because g corresponds to a 1-cover of Δ which meets every edge at only one vertex. Therefore, the decomposition $f = (f/g)g$ corresponds to a partition of $\mathbf{c}$ into a 1-cover and a $(k-1)$-cover of Δ. If f is not divisible by the monomials x_1x_4, x_2x_5, x_3x_6, then the support of f must be one of the sets $\{x_1, x_2, x_3\}$, $\{x_1, x_5, x_6\}$, $\{x_2, x_4, x_6\}$, $\{x_3, x_4, x_5\}$. Without restriction we may assume that the support of f is $\{x_1, x_2, x_3\}$. Using the fact

$$f \in (x_1, x_2, x_3)^k \cap (x_1, x_5, x_6)^k \cap (x_2, x_4, x_6)^k \cap (x_3x_4x_5)^k$$

we can show that f is divisible by $x_1^k x_2^k x_3^k$. Since $x_1x_2x_3$ and $x_1^{k-1}x_2^{k-1}x_3^{k-1}$ correspond to a 1-cover and a $(k-1)$-cover of Δ, $\mathbf{c}$ can be written as a sum of a 1-cover and a $(k-1)$-cover of Δ.

2. Cycles of simplicial complexes

Let Δ be a simplicial complex. A subcomplex of Δ is a simplicial complex Γ such that the facets of Γ are facets of Δ, denoted by $\Gamma \subseteq \Delta$. The aim of this section is to study simplicial complexes for which the vertex cover algebra of every subcomplex is standard graded.

We are inspired by the result of [14, Theorem 5.1] (see also [12, Corollary 2.6]) that the vertex cover algebra of a simple graph is standard graded if and only if the graph is bipartite. This implies that the vertex cover algebra of every subgraph of a bipartite graph is standard graded because subgraphs of a bipartite graph are also bipartite. It is well known that a simple graph is bipartite if and only if it has no odd cycle. It turns out that the notion of cycle is the right tool to characterize the above kind of simplicial complexes.

A *cycle* or, more precisely, an s-cycle of Δ ($s \geq 2$) is an alternating sequence of distinct vertices and facets $v_1, F_1, ..., v_s, F_s, v_{s+1} = v_1$ such that $v_i, v_{i+1} \in F_i$ for $i = 1, ..., s$. A cycle is *special* if it has no facet containing more than two vertices of the cycle. Notice that a cycle of a graph is always special.

In the simplicial complex of Figure 2, the cycle $1, \{1,2,4\}, 2, \{2,3,4\}, 3, \{1,3,4\}, 1$ is special, whereas the cycle $1, \{1,2,4\}, 2, \{2,3,4\}, 4, \{1,3,4\}, 1$ is not.

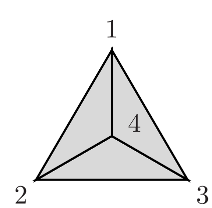

Figure 2

Our starting point is the following simple observation.

Lemma 2.1. *Assume that $A(\Gamma)$ has no generators in degree 2 for all $\Gamma \subseteq \Delta$. Then Δ has no special odd cycle.*

Proof. Assume to the contrary that Δ has a special cycle $v_1, F_1, ..., v_s, F_s, v_1$, where s is an odd number. Let Γ be the subcomplex of Δ with the facets $F_1, ..., F_s$. Then $C = \{v_1, ..., v_s\}$ corresponds to a 2-cover of Γ. By assumption, there are two disjoint vertex covers C_1 and C_2 of Γ such that $C = C_1 \cup C_2$. Since the cycle is special, every facet of Γ has exactly two vertices in C. Therefore, one of the two vertices belongs to C_1 and the other to C_2. It follows that the numbers $|C_1|$ and $|C_2|$ of vertices of C_1 and C_2 are equal. Hence $s = |C_1| + |C_2|$ is an even number, a contradiction. $\square$

The above observation led us to the following result.

Theorem 2.2. *The following conditions are equivalent:*
(i) *The vertex cover algebra $A(\Gamma)$ is standard graded for all $\Gamma \subseteq \Delta$.*
(ii) *The vertex cover algebra $A(\Gamma)$ has no generator of degree 2 for all $\Gamma \subseteq \Delta$.*
(iii) *Δ has no special odd cycle.*

It suffices to show that if Δ has no special odd cycle, then $A(\Delta)$ is a standard graded algebra. In fact, since the assumption implies that all $\Gamma \subseteq \Delta$ have no special odd cycle, it also implies that $A(\Gamma)$ is a standard graded algebra.[1]

By Proposition 1.2 we have to show that the polarization $\Delta^{\mathbf{c}}$ is totally decomposable for all $\mathbf{c} \in \mathbb{N}^n$.

Lemma 2.3. *If Δ has no special odd cycle, then neither does $\Delta^{\mathbf{c}}$.*

Proof. It suffices to show that every special cycle of $\Delta^{\mathbf{c}}$ of length ≥ 3 corresponds to a special cycle of Δ of the same length. Let $z_1, F_1^*, ..., z_s, F_s^*, z_1$ be a special cycle of $\Delta^{\mathbf{c}}$. For $k = 1, ..., s$ let $v_k = i$ if $z_k = x_{ij}$. Let Q denote the sequence $v_1, F_1, ..., v_s, F_s, v_1$.

We first show that $v_1, ..., v_s$ are different vertices. Assume to the contrary that $v_h = v_k$ for some indices $h < k \leq s$. By the defintion of $\Delta^{\mathbf{c}}$, $z_h, z_{k-1}, z_k \in F_{k-1}^{\mathbf{c}}$ and $z_h, z_k, z_{k+1} \in F_k^{\mathbf{c}}$. Since $z_h \neq z_k$ and since z_h can coincide with only one of the vertices z_{k-1}, z_{k+1}, one of the facets $F_{k-1}^{\mathbf{c}}, F_k^{\mathbf{c}}$ must contain three different vertices of P, a contradiction.

It is obvious from the definition of $\Delta^{\mathbf{c}}$ that $F_1, ..., F_s$ are different, $v_i, v_{i+1} \in F_i$ for $i = 1, ..., s$, and no facet of Q contains three different vertices of Q. Therefore, Q is a special s-cycle of Δ. $\square$

By the above discussion, Theorem 2.2 now follows from the following well-known result ([1] and [2, Corollary 2, p. 177]) of Berge in hypergraph theory.

Theorem 2.4. *Assume that Δ has no special odd cycle. Then Δ is totally decomposable.*

Notice that a hypergraph H is called *balanced* if H has no special odd cycle. By this definition, Δ has no special odd cycle means the hypergraph $\mathcal{F}(\Delta)$ is balanced.

The notion of balanced simplicial complex has been used for another property. Stanley [24, 4.1] called Δ a *balanced simplicial complex* if Δ has a coloring of the vertices by $\dim \Delta + 1$ colors such that the vertices of every facet have different colors. According to [1], [2, Corollary 1, p.177], Δ has no special odd cycle if and only if every subcomplex of Δ is balanced in the sense of Stanley. The balanced simplicial complex of Figure 3 has a special 3-cycle.

[1] R. Villarreal informed the authors that Theorem 2.2 can also be deduced from Theorem 1.4 by using a result in A. Schrijver, Combinatorial Optimization, Springer, 2003 (Corollary 83.1a (i) $\Leftrightarrow$ (v), p. 1441).

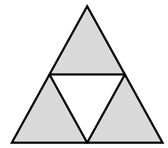

Figure 3

Balanced hypergraphs have several interesting characterizations. For instance, a hypergraph is balanced if and only if every partial subgraph satisfies the König property; that is, the minimum number of vertices in a transversal is equal to the maximum number of disjoint edges (Berge and Las Vergnas [3]).

In particular, Fulkerson, Hoffman and Oppenheim [11] showed that balanced hypergraphs are Mengerian. As a consequence, simplicial complexes without a special odd cycle are Mengerian. By Corollary 1.5, this implies the following result on the simplicial complex Δ^* of the minimal vertex covers of Δ.

Theorem 2.5. *Assume that Δ has no special odd cycle. Then $A(\Delta^*)$ is a standard graded algebra.*

Notice that by Corollary 1.6, there are other interesting consequences on the facet ideal $I(\Delta)$. As bipartite graphs are exactly graphs without an odd cycle, we immediately obtain the following result [23] of Simis, Vasconcelos and Villarreal.

Corollary 2.6. *Let G be a bipartite graph. Then the edge ideal $I(G)$ is normally torsionfree.*

Example 2.7. Let Δ be the simplicial complex with the facets

$$\{1,2,3\},\{1,5,6\},\{2,4,6\},\{3,4,5\}.$$

We have seen in Example 1.8 that $A(\Delta)$ is standard graded. But $A(\Gamma)$ is not standard graded for the subcomplex $\Gamma \subseteq \Delta$ with the facets $\{1,2,3\},\{1,5,6\},\{3,4,5\}$ which has the special odd cycle $1,\{1,2,3\},3,\{3,4,5\},5,\{1,5,6\},1$. This cycle is also a special odd cycle of the simplicial complex Δ^* which has the facets

$$\{1,4\},\{2,5\},\{3,6\},\{1,2,3\},\{1,5,6\},\{2,4,6\},\{3,4,5\}.$$

Since Δ^* is Mengerian, this shows that the class of simplicial complexes without a special odd cycle is properly contained in the class of Mengerian simplicial complexes.

A simplicial complex without a special odd cycle can also be characterized in terms of its incidence matrix. In fact, a special cycle corresponds to an $s \times s$ submatrix of the form

$$\begin{pmatrix}
1 & 0 & 0 & \cdot & \cdot & 0 & 1\\
1 & 1 & 0 & \cdot & \cdot & 0 & 0\\
0 & 1 & 1 & \cdot & \cdot & 0 & 0\\
\cdot & & & & & & \cdot\\
\cdot & & & & & & \cdot\\
\cdot & & & & & 1 & 0\\
0 & \cdot & \cdot & \cdot & 0 & 1 & 1
\end{pmatrix}$$

with $s \geq 2$. Therefore, Δ has no special odd cycle if and only if its incidence matrix has no such $s \times s$ submatrix with odd s even after a permutation of rows and columns.

We say that Δ is a *unimodular* simplicial complex if every square submatrix of its incidence matrix has determinant equal to $0, \pm 1$.

The above matrix has determinant equal to 2 if s is odd. Therefore, a unimodular simplicial complex has no special odd cycle. By Theorem 2.2 and Theorem 2.5 we obtain the following consequence.

Corollary 2.8. *Let Δ be a unimodular simplicial complex. Then $A(\Delta)$ and $A(\Delta^*)$ are standard graded algebras.*

Notice that simplicial complexes without an odd cycle are unimodular [2, Theorem 5, p. 164] and that a simplicial complex of dimension ≤ 2 is unimodular if and only if it has no special odd cycle [2, Corollary, p. 176]. Therefore, bipartite graphs are exactly unimodular graphs.

3. Leaves of simplicial complexes

Let Δ be a simplicial complex. A facet F of Δ is called a *leaf* if either F is the only facet of Δ, or there exists $G \in \mathcal{F}(\Delta)$, $G \neq F$ such that $H \cap F \subset G \cap F$ for each $H \in \mathcal{F}(\Delta)$ with $H \neq F$.

The simplicial complex in Figure 4 has two leaves, namely $\{1, 2, 3\}$ and $\{4, 5\}$.

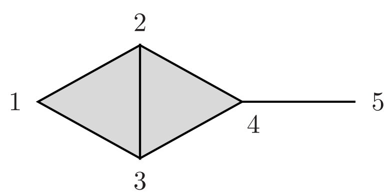

Figure 4

A simplicial complex Δ is called a *forest* if each subcomplex Γ of Δ has a leaf. A forest is called a *tree* if it is connected [10].

The simplicial complex in Figure 4 is a tree. However the simplicial complex in Figure 2 is not a tree.

We will characterize forests by means of the notion of special cycle introduced in the preceding section. For that we need the following notation.

Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors of integers of the same length. We define $\mathbf{a} \prec \mathbf{b}$ if the right-most nonzero component of $\mathbf{a} - \mathbf{b}$ is negative. Let M be a matrix of integers with rows $\mathbf{a}_1, ..., \mathbf{a}_m$ and columns $\mathbf{b}_1, ..., \mathbf{b}_n$. We say that M is *canonical* if $\mathbf{a}_1 \prec \cdots \prec \mathbf{a}_m$ and $\mathbf{b}_1 \prec \cdots \prec \mathbf{b}_n$. A *canonical form* of M is a canonical matrix which can be obtained by permuting the rows and the columns of M. The following result is well known in hypergraph theory and linear algebra.

Lemma 3.1. *Every matrix of integers has a canonical form.*

Proof. Let $M = (a_{ij})$ be an $m \times n$ matrix of integers. Set $\delta_k = \sum_{i+j=k} a_{ij}$ and consider the vector $\delta(M) = (\delta_2, \ldots, \delta_{m+n})$. If M has two rows or columns $\mathbf{a}_{i_1}$ and $\mathbf{a}_{i_2}$, $i_1 < i_2$, in the wrong order $\mathbf{a}_{i_2} \prec \mathbf{a}_{i_1}$ and if M' is the matrix obtained by permuting $\mathbf{a}_{i_1}$ and $\mathbf{a}_{i_2}$, then $\delta(M) \prec \delta(M')$. Hence permuting the rows and the columns of M to maximize $\delta(M)$ will yield a canonical form of M. $\square$

For example, the incidence matrix of a special cycle has the canonical form

$$\begin{pmatrix} 1 & 1 & 0 & \cdot & \cdot & 0 & 0 \\ 1 & 0 & 1 & \cdot & \cdot & 0 & 0 \\ 0 & 1 & 0 & \cdot & \cdot & 0 & 0 \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & 0 & 1 \\ 0 & \cdot & \cdot & \cdot & 0 & 1 & 1 \end{pmatrix}.$$

Theorem 3.2. *The following conditions are equivalent:*

(i) Δ *is a forest,*

(ii) Δ *has no special cycle of length* ≥ 3*,*

(iii) *the incidence matrix of* Δ *has a canonical form which contains no submatrix of the form*

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proof. (i) $\Rightarrow$ (ii). Assume that Δ has a special cycle $v_1, F_1, ..., v_s, F_s, v_{s+1} = v_1$ with $s \geq 3$. Let Γ be the subcomplex with the facets $F_1, ..., F_s$ and F_1 a leaf of Γ. Then there exists a facet $F_i \neq F_1$ such that $F_i \cap F_1 \neq \emptyset$ and $F_j \cap F_1 \subseteq F_i \cap F_1$ for all $j \neq 1$. Therefore, $v_1, v_2 \in F_i$. Since F_1 is the only facet of the cycle which contains v_1, v_2, we get $F_i = F_1$, a contradiction.

(ii) $\Rightarrow$ (iii). Let $M = (a_{ij})$ be a canonical form of the incidence matrix of Δ. Assume to the contrary that M has a submatrix of the form B. Since M is canonical, this matrix is contained in a submatrix of M of the form

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & a \end{pmatrix}.$$

If $a = 1$, this matrix corresponds to a special cycle of length 3. Since Δ has no special cycle of length ≥ 3, we must have $a = 0$. Again, since M is canonical, the new submatrix is contained in a submatrix of M of the form

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & b \end{pmatrix}.$$

We can argue as above to see that $b = 0$. Since we cannot go on infinitely, we get a contradiction.

(iii) $\Rightarrow$ (i). Since the incidence matrix of every subcomplex of Δ is a submatrix of M, they are also canonical and have no submatrix of the form B. Therefore, it suffices to show that Δ has a leaf. Let $M = (a_{ij})$. For each $j = 2, ..., m$ let q_j be the smallest integer for which $a_{q_j 1} = a_{q_j j} = 1$. Let $\mathcal{F}(\Delta) = \{F_1, \dots, F_m\}$, where the order comes from the ordering of the rows of M. Since M has no matrix of the form B, $F_1 \cap F_j = F_1 \cap \{x_{q_j}, x_{q_j+1}, \dots, x_n\}$. Let $j_2, \dots, j_m$ be a permutation of $2, \dots, s$ such that $q_{j_2} \leq \cdots \leq q_{j_m}$. Then

$$F_1 \cap F_{j_m} \subset \cdots \subset F_1 \cap F_{j_2}.$$

Hence F_1 is a leaf of Δ. $\square$

The equivalence of (ii) and (iii) is well known in hypergraph theory (Hoffman, Kolen and Sakarovitch [18], Lubiw [21]) where a hypergraph without a simple cycle of length ≥ 3 is called *totally balanced* and a matrix of the form in (iii) is called greedy. They play an essential role in the study of Gröbner bases of toric ideals arising from bipartite graphs [16]. That these conditions are equivalent to (i) seems to be not known.

Corollary 3.3. *Assume that Δ is a forest. Then $A(\Delta)$ and $A(\Delta^*)$ are standard graded algebras.*

Proof. This follows from Theorem 2.2 and Theorem 2.5. □

In [10, Corollary 3.12] Faridi proved that the Rees algebra of the facet ideal $I(\Delta)$ of a tree Δ is a normal Cohen-Macaulay domain. By Corollary 1.6, this is only a consequence of the fact that $A(\Delta^*)$ is a standard graded algebra, and we even obtain stronger properties such as $I(\Delta)$ is normally torsionfree.

The proof of Theorem 3.2 also reveals an interesting relationship between forests and a special kind of leaves.

A leaf F of a simplicial complex Δ is called a *good leaf* if F is a leaf of each subcomplex of Δ to which it belongs. Equivalently, F is a good leaf if the collection of sets $F\cap G$ with $G \in \mathcal{F}(\Delta)$ is totally ordered with respect to inclusion. This notion was introduced in the thesis of Zheng [26].

Corollary 3.4. *Every forest has a good leaf.*

Proof. This follows immediately from the proof of Theorem 3.2 (iii) $\Rightarrow$ (i) (F_1 is in fact a good leaf). □

If the facets $F_1, \dots, F_m$ of a simplicial complex Δ can be ordered in such a way that F_i is a good leaf of the subcomplex with the facets $F_1, \dots, F_i$ for $i = 2, \dots, m$, we say that $F_1, \dots, F_m$ is a *good leaf order* of Δ. Using Corollary 3.4 it follows immediately that a simplicial complex Δ admits a good leaf order if and only if Δ is a forest.

It was shown recently by Pelsmajer, Tokaz and D.B. West [22, Theorem 3.3] that totally balanced hypergraphs have good leaf orders, where a good leaf is named a simple edge. However, their proof is complicated and needs some background.

Now we give some nice properties of good leaves.

Given a monomial $m = \prod_{j=1}^{r} x_{i_j}^{a_i}$, we say this product presentation of m is standard if $i_1 < i_2 < \cdots < i_r$ and $a_1 > 0, \dots, a_r > 0$. Of course, if we change the numbering of the variables the standard presentation of m also changes. In the following, unless otherwise stated, we always write the monomials in standard form. A sequence of monomials $m_1, \dots, m_s$ is said to be an *M-sequence* if for all $1 \leq i \leq s$ there exists a numbering of the variables such that if $m_i = x_{i_1}^{a_1} \cdots x_{i_r}^{a_r}$ and whenever $x_{i_k} | m_j$ for some $1 \leq k \leq r$ and $i < j$, then $x_{i_k}^{a_k} \cdots x_{i_r}^{a_r} | m_j$. Notice that the numbering of the variables may depend on the index i.

It was shown in [26, Proposition 3.11] that if $F_1, \dots, F_s$ is a good leaf order of a forest, then $x^{F_s}, x^{F_{s-1}}, \dots, x^{F_1}$ is an M-sequence. Combining this fact with the result [5, Theorem 2.5] of Conca and De Negri we obtain

Corollary 3.5. *Let $I \subset S$ be a graded ideal. Suppose there exists a term order such that the initial ideal* $\mathrm{in}(I)$ *is the facet ideal of a forest. Then*

(i) *the Rees algebra of I is Cohen–Macaulay,*
(ii) *the associated graded ring of I is reduced.*

Let $d(A(\Delta))$ denote the maximal degree of the generators of the vertex cover algebra $A(\Delta)$. The following result shows that $d(A(\Delta))$ does not change by removing a good leaf.

Theorem 3.6. *Let Δ be a simplicial complex and F a good leaf of Δ. Let Γ be the subcomplex obtained from Δ by removing F. Then $d(A(\Delta)) \leq d(A(\Gamma))$.*

Proof. By the definition of a good leaf, F must contain a vertex which does not belong to any facet of Γ. Let $1, ..., m$ be the vertices of the facets of Γ. For all $k \geq 1$, if $(c_1, \ldots, c_m, 0, \ldots, 0)$ is an indecomposable k-cover of Γ, then $(c_1, \ldots, c_m, k, \ldots, 0)$ is an indecomposable k-cover of Δ'. Therefore, $d(A(\Delta)) \geq d(A(\Gamma))$.

Now let $\mathbf{a} = (a_1, \ldots, a_n)$ be a cover of Δ of order $r > d(A(\Delta))$. We want to find a vertex cover $\mathbf{b}$ of Δ of order $s < r$ such that $\mathbf{a} - \mathbf{b}$ is an $(r-s)$-cover. Then this implies $d(A(\Delta)) \leq d(A(\Gamma))$.

Let Δ be a simplicial complex on the vertex set $[n]$. We may assume that $F = \{\ell, \ldots, n\}$ with free vertices $m, m+1, \ldots, n$ in Δ. Since F is a good leaf of Δ, the set $\{F \cap G : \ G \in \mathcal{F}(\Gamma)\}$ is totally ordered with respect to inclusion, and we may assume that this set is the set $\{\{\ell, \ldots, s_1\}, \{\ell, \ldots, s_2\}, \ldots, \{\ell, \ldots, s_t\}\}$ with $m - 1 = s_1 > s_2 > \cdots > s_t \geq \ell - 1$.

If $\ell = m$, then $F \cap G = \emptyset$ for all $G \in \mathcal{F}(\Gamma)$, and the assertion is trivial. We now may assume that $\ell < m$.

The vector $\mathbf{c} = (a_1, \ldots, a_{m-1}, 0, \ldots, 0)$ is an r-cover of Γ. Since $r > d(A(\Delta))$ there exists an s-cover $\mathbf{e} = (e_1, \ldots, e_{m-1}, 0, \ldots, 0)$ of Γ with $s < r$ such that $\mathbf{c} - \mathbf{e}$ is an $(r-s)$-cover of Γ.

We are going to construct the desired s-cover $\mathbf{b}$ of Δ by modifying the vertex cover $\mathbf{e}$ of Γ. We set $\rho = \sum_{i=\ell}^{m-1} e_i$ and distinguish two cases.

Case 1: $\rho < s$. Let $p = \max\{j \in \{\ell, \ldots, n\} : \ \sum_{i=\ell}^{j-1} e_i + \sum_{i=j}^{n} a_i \geq s\}$. Since $\sum_{i=\ell}^{n} a_i \geq r > s$, such a j exists. We define $\mathbf{b} = (b_1, \ldots, b_n)$ to be the vector with

$$b_i = \begin{cases} e_i, & \text{if} \quad i < p, \\ a_i, & \text{if} \quad i > p, \end{cases}$$

and $b_p = s - \sum_{i=\ell, i \neq p}^{n} b_i$. Then $\mathbf{b}$ is an s-cover of Δ. We claim $\mathbf{a} - \mathbf{b}$ is an $(r-s)$-cover of Δ. Since $\sum_{i \in F} a_i \geq r$ and since $\sum_{i \in F} b_i = s$, it is clear that $\sum_{i \in F}(a_i - b_i) \geq r - s$. Thus, in particular, since $\sum_{i \in F}(a_i - b_i) = \sum_{i=\ell}^{p}(a_i - b_i)$, it follows that $\sum_{i=\ell}^{p}(a_i - b_i) \geq r - s$. Let G be a facet of Γ and $F \cap G = \{\ell, \ldots, s_j\}$. If $s_j < p$, then $\sum_{i \in G}(a_i - b_i) = \sum_{i \in G}(a_i - e_i) \geq r - s$. If $s_j \geq p$, then $\sum_{i \in G}(a_i - b_i) \geq \sum_{i=\ell}^{p}(a_i - b_i) \geq r - s$, as desired.

Case 2: $\rho \geq s$. Let $p = \min\{j \in \{\ell, \ldots, m\} : \ \sum_{i=\ell}^{j} e_i \geq s\}$. We define $\mathbf{b} = (b_1, \ldots, b_n)$ to be the vector with

$$b_i = \begin{cases} e_i, & \text{if} \quad i < p, \\ 0, & \text{if} \quad i > p, \end{cases}$$

and $b_p = s - \sum_{i=\ell}^{p-1} b_i$. Then $\mathbf{a} - \mathbf{b}$ is an $(r-s)$-cover of Δ. We claim $\mathbf{b}$ is an s-cover of Δ. One has $\sum_{i \in F} b_i = s$. Thus, in particular, since $\sum_{i \in F} b_i = \sum_{i=\ell}^{p} b_i$, it follows that $\sum_{i=\ell}^{p} b_i = s$. Let G be a facet of Γ and $F \cap G = \{\ell, \ldots, s_j\}$. If $s_j < p$, then $\sum_{i \in G} b_i = \sum_{i \in G} e_i \geq s$. Let $s_j \geq p$. Then $\sum_{i \in G} b_i \geq \sum_{i=\ell}^{p} b_i = s$, as desired. $\square$

4. Quasi-forests

A simplicial complex Δ is called a *quasi-forest*, if the facets of Δ can be ordered $F_1, \ldots, F_m$ such that F_i is a leaf of the simplicial complex with facets $F_1, \ldots, F_i$ for $i = 2, \ldots, m$. Such an order of the leaves is called a *leaf order.* If, in addition, Δ is connected, then Δ is called a *quasi-tree*.

The name quasi-forest for this kind of simplicial complex was introduced by Zheng [27]. It is clear that any forest (tree) is a quasi-forest (quasi-tree). The converse is not true, as the simplicial complex Δ in Figure 5 demonstrates. Indeed, Δ is a quasi-tree but not a tree, because if we remove the inside facet, then the remaining simplicial complex no longer has leaves. In particular, none of the leaves of Δ is a good leaf.

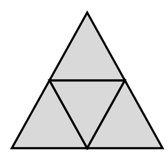

Figure 5

In hypergraph theory, quasi-forests are known under the name co-arboreal hypergraphs or hyperforests. We refer to [2, Chap. 5, §4] and [25] for more information on this class of hypergraphs.

The significance of quasi-trees results from Dirac's characterization of chordal graphs. Let G be a finite graph on $[n]$ without loops, multiple edges and isolated vertices, and $E(G)$ its edge set. Dirac proved that G is chordal (i.e., every cycle in the graph of length > 3 has a chord) if and only if G has a perfect elimination ordering on its vertices. Recall that a *perfect elimination ordering* (or a *simplicial elimination ordering*) is an ordering $v_n, ..., v_2, v_1$ on the vertices of G such that v_i is a simplicial vertex in the graph induced on vertices $\{v_1, ..., v_i\}$. Here a *simplicial vertex* in a graph is one whose neighbors form a clique, and a clique is a subset F of $[n]$ such that $\{v, w\} \in E(G)$ for all $v, w \in F$ with $v \neq w$.

In the terminology of quasi-forests, Dirac's theorem [6] can be phrased as follows. The following conditions are equivalent:

(a) G is a chordal graph,
(b) G is the 1-skeleton of a quasi-forest.

In fact, the quasi-forest belonging to the chordal graph G is the clique complex $\Delta(G)$ of G, that is, the simplicial complex of all cliques of G. In [15] there is also given a higher-dimensional version of Dirac's theorem.

A perfect elimination ordering of a graph G is a *strong perfect elimination ordering* if for all $i < j < k < l$ such that $\{v_i, v_k\}, \{v_i, v_l\}, \{v_j, v_k\}$ are edges, then $\{v_j, v_l\}$ is an edge. A graph is *strongly chordal* if it has a strong perfect elimination ordering.

In analogy to Dirac's theorem one has that the following conditions are equivalent:

(a) G is a strongly chordal graph,
(b) G is the 1-skeleton of a forest.

This was shown by Farber [9, Theorem 5.3]. He also showed that strongly chordal graphs can be described by forbidden induced subgraphs, called trampolines [9, Theorem 4.1′]. For example the 1-skeleton of the simplicial complex in Figure 4 is a trampoline.

While the vertex cover algebra of a forest is standard graded, as we have seen in Corollary 3.3, this is not the case for quasi-forests. Indeed, the vertex cover of the simplicial complex in Figure 5 which assigns to each nonfree vertex the value 1 is an indecomposable vertex cover of order 2. So one might expect that a quasi-forest which is not a forest always has indecomposable vertex covers of higher order. But again this is not the case. Figure 6 shows a quasi-tree which is not a tree but whose vertex cover algebra is nevertheless standard graded. Its only nonleaf is the inside 3-dimensional simplex.

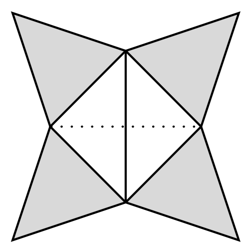

FIGURE 6

The main goal of this section will be to show that for a certain restricted class of quasi-forests the vertex cover algebra is standard graded if and only if the quasi-forest is a forest. For this purpose we first prove

Proposition 4.1. *Let Δ be a quasi-forest on the vertex set $[n]$, and suppose that Δ contains the subcomplex Γ with facets*

$$B = \{i_1, \dots, i_k\}, \quad F_a = \{a, i_2, \dots, i_k\},$$
$$F_b = \{i_1, b, i_3, \dots, i_k\} \ \textit{ and } \ F_c = \{i_1, i_2, c, i_4, \dots, i_k\}.$$

Then $d(A(\Delta)) > 1$.

Proof. Let $a = (a_1, \dots, a_n)$ be the integer vector with

$$a_i = \begin{cases} 0, & \text{if} \quad i \in \{a, b, c, i_4, \dots, i_k\}, \\ 1, & \text{if} \quad i \in \{i_1, i_2, i_3\}, \\ 2, & \text{otherwise.} \end{cases}$$

We show by induction on $\delta = |\Delta| - |\Gamma|$ that $d(A(\Delta)) > 1$. If $\delta = 0$, then $\Delta = \Gamma$, and a is obviously an indecomposable 2-cover of Γ.

Now suppose $\delta > 0$, and that there is a leaf F of Δ belonging to Δ but not to Γ. Let Σ be the subcomplex of Δ which is obtained from Δ by removing F. It is shown in [15, Corollary 3.4] that Σ is again a quasi-forest, and of course $\Gamma \subset \Sigma$. By the induction hypothesis we have $d(A(\Sigma)) > 1$. Applying Theorem 3.6 we conclude that $d(A(\Delta)) > 1$.

On the other hand, if all leaves of Δ belong to Γ we will show that $\Delta = \Gamma$, and we obtain again the desired conclusion.

To see why $\Gamma = \Delta$, we consider a relation forest $T(\Delta)$ of Δ introduced in [15]. The vertices of $T(\Delta)$ are the facets of Δ. The edges are obtained recursively as follows: Choose a leaf F of Δ and a branch G of F. Then $\{F, G\}$ is an edge of $T(\Delta)$. Remove F from Δ and proceed with the remaining quasi-forest as before to find the other edges of Δ. The resulting graph may depend on the order of how one chooses the leaves, but in any case it is a forest. Moreover, each free vertex of $T(\Delta)$ is a leaf of Δ.

Suppose now that all the leaves of Δ belong to Γ. Since Δ has at least two leaves and B is obviously not a leaf, we may assume that F_a and F_b are leaves of Δ. Since F_a and F_b have a face of codimension 1 in common with B, it follows that B is a branch of F_a and F_b in Δ. Hence we may construct a relation forest $T(\Delta)$ of Δ with edges $\{F_a, B\}$ and $\{F_b, B\}$, and free vertices F_a and F_b.

Since $F_c \notin \{F_a, F_b, B\}$, the relation forest $T(\Delta)$ must have a free vertex $G \neq F_a, F_b$. If $G \neq F_c$, then Δ has a leaf which does not belong to Γ, a contradiction. So $G = F_c$, and hence F_c is a leaf of Δ and B a branch of F_c in Δ. Thus we add the edge $\{F_c, B\}$. Therefore $T(\Delta)$ contains the tree T_0 given by Figure 7.

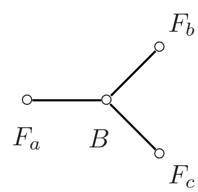

FIGURE 7

Since $T(\Delta)$ has no other free vertices, we must have that $T(\Delta) = T_0$, and hence $\Delta = \Gamma$. □

Now we have

Theorem 4.2. *Let Δ be a quasi-forest satisfying*

(i) *each connected component of Δ is connected in codimension 1;*
(ii) *each face of codimension 1 belongs to at most two facets.*

Then Δ is a forest if and only if $A(\Delta)$ is standard graded.

Proof. In view of Corollary 3.3 it suffices to show that the vertex cover algebra of a quasi-forest Δ satisfying the conditions (i) and (ii) is not standard graded, unless it is a forest. We may assume that Δ is connected.

Let $F_1, \ldots, F_m$ be a leaf order of Δ, and set $\Delta_j = \langle F_1, \ldots, F_j \rangle$ for $j = 1, \ldots, m$. Notice that Δ_j is connected in codimension 1, since this property is preserved by removing a leaf.

Assuming that Δ is not a tree, there exists an integer i such that F_i is not a good leaf of Δ_i. In other words, there exist facets G and H of Δ_i such that the faces $G \cap F_i$ and $H \cap F_i$ are not contained in each other. In particular, $G \neq H$. Since Δ_{i-1} is connected in codimension 1 there exists a sequence of facets $G = G_0, G_1, \ldots, G_p = B$ in Δ_{i-1} where B is the branch of F_i such that $G_k \cap G_{k+1}$ is a face of codimension 1 for $k = 0, \ldots, p-1$. The sequence can be chosen such that $G_s \neq G_t$ for $s \neq t$.

Similarly there exists a chain $H = H_0, H_1, \ldots, H_q = B$ in Δ_{i-1} with $H_s \neq H_t$ if $s \neq t$ and such that $H_k \cap H_{k+1}$ is a face of codimension 1 for $k = 0, \ldots, q-1$.

It is clear that G_{p-1} and H_{q-1} are different from F_i. We set $G_{p+1} = H_{q+1} = F_i$. Then there exists an integer $r \geq 0$ such that $G_{p-j} = H_{q-j}$ for all $0 \leq j \leq r$ and $G_{p-r-1} \neq H_{q-r-1}$. Then by assumption (ii) the simplicial complex Γ with facets $G_{p-r+1}, G_{p-r}, G_{p-r-1}$ and H_{q-r-1} is of the form as described in Proposition 4.1, and hence $A(\Delta)$ is not standard graded. $\square$

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