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SEMI-COMPLETE VECTOR FIELDS OF SADDLE-NODE TYPE IN $\mathbb{C}^n$

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ABSTRACT. We classify the foliations associated to codimension 1 saddle-node vector fields on $\mathbb{C}^n$, with an isolated singularity, admitting a semi-complete representative. This will be done under some further assumptions that are generic in dimension 3. These singularities play an essential role in the program to classify semi-complete vector fields in dimension 3.

1. INTRODUCTION

The definition of a semi-complete vector field relatively to an open set U was introduced in [7]. Semi-complete vector fields are essentially the local version of complete vector fields in the sense that restriction of a complete holomorphic vector field to every open set $U \subseteq M$ is semi-complete in U [7]. The advantage of using semi-complete vector fields in that they are, indeed, stable under restriction of the domain: the restriction of a semi-complete vector field to a smaller domain is still semi-complete. Clearly this does not hold for complete vector fields.

In particular, we can talk about semi-complete singularities of vector fields, i.e. those that are given by a semi-complete vector field on some sufficiently small neighborhood of the origin in $\mathbb{C}$. What precedes then implies that a singularity that *is not* semi-complete cannot be realized by a complete vector field on any manifold (in particular by a holomorphic vector field defined on a compact manifold).

The germs of the associated singular foliations that are tangent to semi-complete vector fields in dimension 2 were totally classified by J. Rebelo and E. Ghys [3, 8]. In particular, obstructions to the completeness of vector fields were found in that case:

Theorem 1 ([7]). *If X is a holomorphic vector field on $\mathbb{C}^2$ with an isolated singularity at p and such that $J_p^2 X = 0$, then X is not semi-complete on arbitrarily small neighbourhoods of the singularity.*

E. Ghys conjectures that this result still holds in dimension 3. One of the main difficulties in extending the above result to dimension 3 lies in the absence of separatrices for these vector fields. A detailed analysis of these singularities then becomes unavoidable. The general program consists of first reducing the singularities to special "simple" models by performing some birational transformations (this is the analogue of Seidenberg's theorem in dimension 2). The "simple" models mentioned

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above are "irreducible" in the sense that they cannot be further simplified by means of birational transformations and normalizations. The understanding of these models, as it comes so their possible semi-complete character, is therefore the next step in the program. Saddle-node singularites correspond exactly to the possible nontrivial irreducible models. This work is, in fact, devoted to the classification of codimension 1 saddle nodes under some mild further assumptions that are, in particular, almost always verified when the dimension is 3. More specifically, the linear part at the singularity is supposed to be diagonalizable and the nonzero eigenvalues belong to the Poincaré domain and are nonresonant.

By a codimension 1 saddle-node vector field, we mean a vector field whose linear part at the singularity has exactly 1 eigenvalue equal to zero. As mentioned, these singularities are irreducible by blow-ups, and so their understanding is indispensable to approach Ghys's conjecture. For simplicity, from now on we shall refer to these singularities simply as saddle-nodes.

The characterization of semi-complete saddle-node foliations is based on the analytic classification of saddle-nodes presented by Canille Martins [1] which, in turn, was strongly inspired by the algebro-geometric methods of Martinet and Ramis [6].

First, using the properties of semi-complete vector fields and the study of the sectorial isotropy, we prove that a semi-complete vector field of saddle-node type verifies special properties such as: the convergence of the weak separatrix and the fact that the holonomy relative to the weak separatrix is trivial, i.e., the holonomy application is the identity map. Secondly we prove that those two properties imply on the one hand that the $n-1$ special formal invariants are integers and, on the other hand, that the maps, conjugating the given vector field to its formal normal form by sectors, fit together as a holomorphic conjugacy. This leads us to the main result of this paper:

Theorem 2. *Let $\mathcal{F}$ be a saddle-node foliation defined on a neighbourhood of the origin. Then $\mathcal{F}$ is associated to a semi-complete vector field if and only if it admits the normal form*

$$\begin{cases} \dot{x_1} = x_1^2 \\ \dot{x_i} = x_i(\lambda_i + \alpha_i x_1), \quad i = 2, \dots, n \end{cases}$$

with $(\alpha_2, \dots, \alpha_n) \in \mathbb{Z}^{n-1}$.

Summarizing, it is proved that the normal form of foliations, associated to semi-complete saddle-nodes, corresponds to the analytic equivalence class of their formal normal forms, where $(\alpha_2, \dots, \alpha_n) \in \mathbb{Z}^{n-1}$.

The characterization of saddle-nodes admitting holomorphic central manifold (also called weak invariant manifold, or weak separatrix) presented here is identical to the one obtained by Martinet and Ramis [6] for dimension 2.

At the end of the article, saddle-nodes whose set of singularities coincides with the invariant manifold of the singularity (whose existence is guaranteed in [1] and which is transverse to the weak separatrix) are also considered:

Proposition. *Let X be a holomorphic vector field of saddle-node type, with an isolated singularity at the origin, and let M be the invariant hypersurface transverse to the weak direction of X. If F is a holomorphic function such that $F(x) = 0 \Leftrightarrow x \in M$, then FX is not semi-complete in arbitrarily small neighbourhoods of the origin.*

Finally we would like to point out that, in dimension 3, the only case not covered by our results occurs when the two eigenvalues different from zero are aligned with the origin of $\mathbb{C}$. By considering the results of L. Stolovitch [11] in place of those of Canille Martins [1], it is likely that our methods can yield the classification of these singularities as well.

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2. Preliminaries - definitions and basic results

Let $X : U \subseteq \mathbb{C}^n \to \mathbb{C}^n$ and $Y : V \subseteq \mathbb{C}^n \to \mathbb{C}^n$ be holomorphic vector fields with a singularity at the origin. We say that X is analytically (formally, C^∞, C^k) conjugated to Y in a neighbourhood of the origin if there exists a holomorphic (formal, C^∞, C^k) diffeomorphism $H : V_1 \to U_1$, where $0 \in U_1 \subseteq U$, $0 \in V_1 \subseteq V$, such that $H(0) = 0$ and $Y = (DH)^{-1}(X \circ H)$. We say that X and Y are analytically (formally, C^∞, C^k) equivalent if X is analytically (formally, C^∞, C^k) conjugated to fY, for some holomorphic function f verifying $f(0) \neq 0$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the vector of the eigenvalues of $DX(0)$. We say that the eigenvalues are resonant if, for some i, there exists $I = (i_1, \ldots, i_n) \in \mathbb{N}_0^n$ with $\sum_{j=1}^n i_j \geq 2$ such that

$$\lambda_i = (I, \lambda) = i_1\lambda_1 + \ldots + i_n\lambda_n.$$

Finally, we say that λ is in the Poincaré domain if the origin is not in the convex hull of the points $\{\lambda_i : i = 1, \ldots, n\}$. Otherwise we say that it is in the Siegel domain.

Definition 1. Let X be a holomorphic vector field defined on a complex manifold M and $U \subseteq M$ an open subset of M. We say that X is semi-complete in U if there exists a holomorphic application

$$\Phi : \Omega \subseteq \mathbb{C} \times U \to U$$

where Ω is an open set containing $\{0\} \times U$ such that

a) $\Phi(0, x) = x \qquad \forall x \in M$;
b) $X(x) = \frac{d}{dT}|_{T=0}\Phi(T, x)$;
c) $\Phi(T_1 + T_2, x) = \Phi(T_2, \Phi(T_1, x))$, when the two members are defined;
d) $(T_i, x) \in \Omega$ and $(T_i, x) \to \partial\Omega \quad \Rightarrow \quad \Phi(T_i, x)$ escapes from any compact subset of U.

We call Φ the semi-complete flow associated to the vector field X.

We say that X is complete if there is a holomorphic application $\Phi : \mathbb{C} \times M \to M$ satisfying a), b) and c).

The orbits of a complete vector field are topologically the complex plane $\mathbb{C}$, the cylinder $\mathbb{C}/\mathbb{Z}$ or the torus $\mathbb{C}/\Lambda$. The orbits of a noncomplete vector field also define a singular foliation of M, where each leaf is also a Riemann surface, but its topology can be much more complex.

In [7] and [8], Rebelo presents sufficient and necessary conditions for a vector field to be semi-complete in an open set U. The regular orbits of a vector field X ($X \not\equiv 0$) are Riemann surfaces. To each one of its orbits (leaves), L, we can associate a holomorphic differential 1-form, called time form and denoted by dT_L, such that $dT_L(X) = 1$.

Proposition 1 ([7, 8])**.** *Let X be a semi-complete vector field in an open set U. If L is a regular leaf of X, then the integral of dT_L over any one-to-one embedded curve in L is nonzero. In the contrary direction, if the integral of dT_L over c is nonzero for all regular orbits L of X and every $c : [0,1] \to L$ such that $c(0) \neq c(1)$, then X is semi-complete in U.*

It is important to remark that we can have two representatives of the same foliation, one being semi-complete but not the other. However, if X is semi-complete and f is a first integral, then fX is also semi-complete [3].

The 1-dimensional semi-complete holomorphic vector fields were completely characterized in [7]. In [9], this characterization is extended to meromorphic vector fields:

Lemma 1 ([9])**.** *Suppose that the 1-dimensional meromorphic vector field $X = f(x)\frac{\partial}{\partial x}$ ($f \not\equiv 0$) is semi-complete in a punctured neighbourhood of the origin. Then f admits a holomorphic extension to the origin and $J_0^2 X \not\equiv 0$. Moreover, if $J_0^1 X \equiv 0$, then X is analytically conjugated to $x^2\partial/\partial x$.*

Remark 1. The first part of the last lemma expresses that if $k \geq 3$ or $k \leq -1$, then $X = x^k g(x)\partial/\partial x$ is not semi-complete in any small neighbourhood of the origin.

To prove this we need the following lemma:

Lemma 2. *Let $f_\lambda(x) = x^k(a + g(x,\lambda))$, $\lambda \in \mathbb{C}^n$, where $k \in \mathbb{Z}$, $a \in \mathbb{C} \setminus \{0\}$ and g is a holomorphic function such that $g(x, 0) = 0$. Let W be a simply connected open set of D_f such that $0 \notin W$ and f is never zero in W for any λ such that $\|\lambda\| \leq \varepsilon_0$. Consider the function*

$$I_\lambda : W \to \mathbb{C}$$
$$p \mapsto \int_{c_p} \frac{dx}{x^k(a + g(x,\lambda))}$$

where $c_p \subseteq W$ is a curve joining a fixed $x_0 \in W$ to p. Then, there exist real and positive numbers θ and λ_0 such that

$$\forall \lambda : \|\lambda\| \leq \lambda_0, \quad B(0,\theta) \subseteq I_\lambda(W).$$

Remark 2. Since $0 \notin W$ and f is nonzero in W the integral does not depend on the chosen curve.

Proof of the Lemma 2. We have $I_\lambda(x_0) = 0$, $\forall \lambda \in \mathbb{C}$; in particular, $I_0(x_0) = 0$. Since $g(x, 0) = 0$, $I_0'(x_0) = \frac{1}{ax_0^k} \neq 0$. Thus I_0 has a zero of order 1 at x_0. Moreover, as I_λ' is a continuous function of λ, there exists $0 < \lambda_0 \leq \varepsilon_0$ such that

$$I_\lambda'(x_0) \in B\left(\frac{1}{ax_0^k}, \frac{1}{2|ax_0^k|}\right) \qquad \forall \lambda : \|\lambda\| \leq \lambda_0.$$

In particular, $I_\lambda'(x_0) \neq 0$, $\forall \lambda : \|\lambda\| \leq \lambda_0$, and so I_λ is a zero of order 1 at x_0, $\forall \lambda : \|\lambda\| \leq \lambda_0$. Thus, by the Inverse Function Theorem, for each $\lambda \in B(0, \lambda_0) \subseteq \mathbb{C}^n$ there exists $\eta_\lambda > 0$ such that $\forall \tau \in]0, \eta_\lambda]$ there exists $\delta_\lambda^\tau > 0$ such that if $|w-0| < \delta_\lambda^\tau$, then $I_\lambda(p) = w$ has exactly one solution in the disc $|p - x_0| < \tau$.

Consider the map $T_\eta : D(0, \lambda_0) \to \mathbb{R}$ given by $T_\eta(\lambda) = \eta_\lambda$. Since 0 does not belong to the image of $D(0, \lambda_0)$ by T_η and $D(0, \lambda_0)$ is compact, its image has a minimum μ. Then $0 < \mu \leq \eta_\lambda$, $\forall \lambda : \|\lambda\| \leq \lambda_0$.

Now, the map $T_\delta : D(0,\lambda_0) \to \mathbb{R}$ given by $T_\delta(\lambda) = \delta^\mu_\lambda$ also has a minimum: $\theta > 0$. Since μ is the minimum of T_η we conclude that

$$B(0,\theta) \subseteq D_\lambda(W), \quad \forall \lambda : \|\lambda\| \leq \lambda_0$$

□

Proof of Remark 1. Let $f(x) = x^k g(x)$, where g is a holomorphic function such that $g(0) \neq 0$. Suppose that $k \geq 3$ or $k \leq -1$ and that X is semi-complete in $B(0,\varepsilon) \subseteq \mathbb{C}$. We can assume ε so small that there are not singularities in $B(0,\varepsilon) \setminus \{0\}$.

Fixing $\lambda \in \mathbb{C} \setminus \{0\}$, the vector field X is semi-complete in $B(0,\varepsilon)$ if and only if $Y_\lambda = \lambda^{-1} X(\lambda x)$ is semi-complete in $B(0, \frac{\varepsilon}{|\lambda|})$. For $\lambda = \varepsilon$ it is sufficient to check whether Y_λ is semi-complete in $B(0,1)$ or, equivalently, whether $Z^\lambda = \frac{Y^\lambda}{\lambda^{k-1}}$ is semi-complete in the same ball.

The time form is given by $dT^X = \dfrac{dx}{x^k g(x)}$ and so

$$dT^{Z_\lambda} = \lambda^{k-1} H^*_\lambda (dT^X) = \frac{dx}{x^k g(\lambda x)}.$$

Consider the curve $c(t) = re^{2\pi i t/(k-1)}$, $t \in [0,1]$ and $0 < r < 1$, which is embedded in the case where $|k-1| \geq 2$.

Let $W \subseteq B(0,1)$ be a simply connected neighbourhood of $c(1)$ containing neither the origin nor $c(0)$. Denote by

$$\begin{aligned} I_\lambda : W &\to \mathbb{C} \\ p &\mapsto \int_{c_p} \frac{dx}{x^k g(\lambda x)} \end{aligned}$$

the application that associates to every point $p \in W$ the integral of the time form dT^{Z^λ} through a curve c_p joining $c(1)$ to p, inside W. Again, the value of the integral does not depend on the chosen curve.

However, $x^k g(\lambda x) = a_0 x^k + h(x,\lambda)$, where $h(x,\lambda)$ is holomorphic in W satisfying $h(x,0) = 0$. By Lemma 2 there exist real and positive numbers θ and λ_0 such that $B(0,\theta) \subseteq I_\lambda(W)$, $\forall \lambda : |\lambda| \leq \lambda_0$.

But, since

$$\int_c \frac{dx}{x^k g(\lambda x)} = \int_c \frac{dx}{a_0 x^k + h(x,\lambda)} \xrightarrow{\lambda \to 0} \int_c \frac{dx}{a_0 x^k} = 0$$

we can choose λ so small (in particular, λ smaller than λ_0) such that

$$\int_c \frac{dx}{x^k g(\lambda x)} = \alpha$$

with $|\alpha| < \theta$. Let $p \in W$ be such that $I_\lambda(p) = -\alpha$. Obviously $p \neq c(0)$, because $c(0) \notin W$. If $p \cap \{c(t)\} = \emptyset$ the curve $\tilde{c}$ joining $c(0)$ to p obtained by concatenating c to c_p is a one-to-one embedded curve such that

$$\int_{\tilde{c}} \frac{dx}{x^k g(\lambda x)} = 0.$$

If $p = c(t)$, for some $t \in [0,1]$, we consider $\tilde{c}$ as the subset of c joining $c(0)$ to p. Thus, the vector field X is not semi-complete relatively to any neighbourhood of the origin. □

Remark 3. The proof of Remark 1 also implies that X is not semi-complete in any sector with vertex at the origin and angle greater than $\dfrac{2\pi}{|k-1|}$.

3. A necessary condition for the semi-completeness of a saddle-node in $\mathbb{C}^n$

We will consider saddle-node foliations whose representatives, up to a linear change of coordinates, belong to $\mathfrak{X}$:

$$\begin{aligned}\mathfrak{X} = \{X : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0), &\text{holomorphic} : 0 \text{ is an isolated singularity},\\ &DX(0) = \operatorname{diag}(\lambda_1, \ldots, \lambda_n), \lambda_1 = 0, 0 \notin \mathcal{H}(\lambda_2, \ldots, \lambda_n),\\ &\text{there are no resonance relations between the non-}\\ &\text{vanishing eigenvalues}\},\end{aligned}$$

where $\mathcal{H}(\lambda_2, \ldots, \lambda_n)$ denotes the convex hull of $\{\lambda_i : i = 2, \ldots, n\}$. Remark that for $n = 3$ the considered foliations are generic under the codimension 1 saddle-node foliations. The elements of $\mathfrak{X}$ are analytically equivalent to a vector field of the form

$$Y_p : \begin{cases} \dot{x_1} = x_1^{p+1} \\ \dot{x_i} = \lambda_i x_i + x_1 a_i(x), \quad i = 2, \ldots, n \end{cases} \tag{1}$$

where $x = (x_1, \ldots, x_n)$ and the a_i are holomorphic functions such that $a_i(0) = 0$, $\forall i = 2, \ldots, n$ [1], that is, they can be written in the form fY_p for some $p \in \mathbb{N}$ (where p denotes the multiplicity of the singularity) and some holomorphic function f ($f(0) \neq 0$). This is the Dulac normal form for a saddle-node in $\mathbb{C}^n$ [1].

The main objective of this paper is to prove:

Theorem 3. *Let $\mathcal{F}$ be a saddle-node foliation defined in a neighbourhood of the origin. Then $\mathcal{F}$ is associated to a semi-complete vector field if and only if it admits the normal form*

$$\begin{cases} \dot{x_1} = x_1^2 \\ \dot{x_i} = x_i(\lambda_i + \alpha_i x_1), \quad i = 2, \ldots, n \end{cases}$$

with $(\alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^{n-1}$.

In this section we exhibit a necessary condition for a saddle-node to be semi-complete.

Proposition 2. *Assume $X \in \mathfrak{X}$ is semi-complete. Then the multiplicity of the singularity is equal to* 1.

Proof. The vector field X can be written in the form fY_p for some holomorphic function f such that $f(0) \neq 0$. Suppose that $p \geq 2$.

Denote by Π_1 the projection of U on the first axis ($\Pi_1(x) = x_1$) and by $\mathcal{F}$ the foliation associated to X. Since f is nonzero in a sufficiently small neighbourhood of the origin, the fibres of Π_1 are transverse to the leaves of the foliation $\mathcal{F}$, in that neighbourhood, except to those contained in the invariant manifold $\{x_1 = 0\}$: $D\Pi_1(x).X(x) = f(x)x_1^{p+1}$.

Fix $\varepsilon > 0$ such that X is semi-complete in $B(0, \varepsilon) \subseteq V$.

Assume first that $f \equiv k$ ($k \in \mathbb{C}$). Consider the curve $c(t) = (re^{2\pi i t/p}, 0, \ldots, 0) \subseteq B(0, \varepsilon)$, $t \in [0,1]$. For $p \geq 2$, c is a one-to-one embedded curve. For each $(r, x_2^0, \ldots, x_n^0)$ sufficiently close to $0 \in \mathbb{C}^n$, we can lift c to a curve c_L contained in $L \cap B(0, \varepsilon)$, where L is the leaf through $(r, x_2^0, \ldots, x_n^0)$. Since c is a one-to-one embedded curve so is c_L and

$$\int_{c_L} dT_L = \int_c \frac{dx_1}{kx_1^{p+1}} = 0,$$

contradicting the semi-completeness of X.

Assume now that $f \not\equiv \mathit{const}$. The time form no longer depends only on x_1 and the integral over c_L is not necessarily equal to zero. Let $S \subseteq \mathbb{C}$ be an angular sector with radius sufficiently small, vertex at the origin and angle greater than $\frac{2\pi}{p}$ and less than 2π. Since, in a neighbourhood of the origin, Π_1 is transverse to the leaves not contained in the invariant manifold $\{x_1 = 0\}$, for each leaf L in $S \setminus \{0\} \times (\mathbb{C}^{n-1}, 0)$, we can write $x_j = x_j^L(x_1)$, for all $j = 2, \ldots, n$, univocally.

Let c_L be a one-to-one embedded curve in L. We have:

$$\int_{c_L} dT_L = \int_{c_L} \frac{dx_1}{x_1^{p+1} f(x)} = \int_{\Pi_1(c_L)} \frac{dx_1}{x_1^{p+1} f(x_1, x_2^L(x_1), \ldots, x_n^L(x_1))}$$

where $\Pi_1(c_L)$ is also embedded.

The 1-form presented in the last integral is the time form associated to a 1-dimensional vector field defined on a sector of amplitude greater than $\frac{2\pi}{p}$ and of order equal to $p+1$. Such a 1-dimensional vector field is not semi-complete (Remark 3). So, X is not semi-complete in any neighbourhood of the origin. $\square$

From now on we assume $p = 1$.

Notation 1. We denote by $Y_{1,\alpha}$ a vector field of the type

$$Y_{1,\alpha} : \begin{cases} \dot{x_1} = x_1^2 \\ \dot{x_i} = x_i(\gamma_i + \alpha_i x_1) + x_1 h_i(x), \quad i = 2, \ldots, n \end{cases}$$

where $\alpha = (\alpha_2, \ldots, \alpha_n)$ and h_i are holomorphic functions such that $\dfrac{\partial h_i}{\partial x_i}|_0 = 0$, $\forall i = 2, \ldots, n$. We can obviously assume $\gamma_2 = 1$.

In [1], it is proved that there exists a formal change of coordinates of the form

$$\hat{H}(x) = (x_1, x_2 + \sum_{k=1}^{\infty} f_{2k}(\bar{x}) x_1^k, \ldots, x_n + \sum_{k=1}^{\infty} f_{nk}(\bar{x}) x_1^k) \tag{2}$$

with $\bar{x} = (x_2, \ldots, x_n)$ and f_{ik} holomorphic in a neighbourhood of $0 \in \mathbb{C}^{n-1}$ such that $f_{i1}(0) = 0$ for all $i \in \{2, \ldots, n\}$, conjugating $Y_{1,\alpha}$ and

$$Z_\alpha : \begin{cases} \dot{x_1} = x_1^2 \\ \dot{x_i} = x_i(\gamma_i + \alpha_i x_1), \quad i = 2, \ldots, n \end{cases} \tag{3}$$

Nevertheless, the formal change of coordinates is, in general, divergent. Denote by $\hat{G}_0$ the set of formal maps of type (2).

Although $Y_{1,\alpha}$ and Z_α are, in general, not analytically conjugated, there are sectors $S \subseteq \mathbb{C}$ of angle less than 2π and with vertex at $0 \in \mathbb{C}$, covering $B(0,r)\setminus\{0\}$, such that $Y_{1,\alpha}$ and Z_α are analytically conjugated in $S \times (\mathbb{C}^{n-1},0)$:

Theorem of Malmquist ([4, 1])**.** *Let $\hat{H}$ be the unique formal transformation of type* (2) *conjugating $Y_{1,\alpha}$ and Z_α. Then there exists a holomorphic transformation H defined in $S \times (\mathbb{C}^{n-1},0)$, S a sector as described before, such that*

a) $dH(Y_{1,\alpha}) = Z_\alpha(H)$, *in* $S \times (\mathbb{C}^{n-1},0)$,
b) $H \tilde{\to} \hat{H}$ *in* S, *as* $x_1 \to 0$.

The vector fields are analytically conjugated if and only if $H_i = H_j$ in $S_i \cap S_j$, $\forall i \neq j$. Each holomorphic transformation H_i is called a normalizing application.

Definition 2 ([6])**.** Let f be a holomorphic function in $S \times K$, where $S \subseteq \mathbb{C}$ is a sector with vertex at the origin and K is a (compact) subset of $\mathbb{C}^{n-1}$. We say that f is asymptotic to $\hat{f}(x) = \sum_{r=0}^{\infty} a_r(\bar{x})x_1^r$, with $a_r(\bar{x})$ holomorphic in K, as $x_1 \to 0$ and $x_1 \in S$ if

$$\forall \bar{x} \in K, m \in \mathbb{N}, \exists A_m(\bar{x}) > 0: \quad |f(x) - \sum_{r=0}^{m-1} a_r(\bar{x})x_1^r| \leq A_m(\bar{x})x_1^m. \tag{4}$$

In this case we write $f \tilde{\to} \hat{f}$ in S, as $x_1 \to 0$.

Equation (4) is equivalent to

$$f(x) = \sum_{r=0}^{m} a_r(\bar{x})x_1^r + x_1^m \varepsilon_m(x), \quad \lim_{\substack{x_1 \to 0 \\ x_1 \in U}} \varepsilon_m(x) = 0. \tag{5}$$

4. Sectorial Isotropy of the formal normal form

The study of the Sectorial Isotropy for the saddle-node foliations considered here was done in [1]. In that work they deduce the properties of $(H_i \circ H_j^{-1})$ on the overlaps $(S_i \cap S_j) \times (\mathbb{C}^{n-1},0)$, which lead them towards a classification of saddle-nodes. That study is used in the proof of the main result of this chapter. In this way, we will recall the essential results of that work.

The solutions of the formal normal form (3) out of $\{0\} \times (\mathbb{C}^{n-1},0)$ are parameterized by

$$x_j(x_1) = c_j x_1^{\alpha_j} e^{-\frac{\gamma_j}{x}}, \quad j \geq 2$$

with $(c_2,\ldots,c_n) \in \mathbb{C}^{n-1}$. Our objective in this subsection is to relate the solutions of Z_α with the solutions of $Y_{1,\alpha}$, by sectors.

Denote by φ_i the argument of the eigenvalue γ_i, for $i = 2,\ldots,n$. The behavior of $x_j(x_1)$ along the curve $x_1 = re^{i\theta}$ as $r \to 0$, for a fixed θ, is given by the term $\frac{|\gamma_j|}{r}\cos(\varphi_j - \theta)$:

$$x_j(re^{i\theta}) = c_j r^{\alpha_j} e^{i\theta\alpha_j} e^{-\frac{|\gamma_j|}{r}(\cos(\varphi_j-\theta)+i\sin(\varphi_j-\theta))}.$$

A sector such that $(x_2(x_1),\ldots,x_n(x_1)) \to (0,\ldots,0)$ as $r \to 0$ is called attractor. This sector corresponds exactly to the points such that $\cos(\varphi_j - \theta) > 0$, $\forall j = 2,\ldots,n$. The sector where $\cos(\varphi_j - \theta) < 0$, $\forall j = 2,\ldots,n$, is called saddle (in this case $|x_j(x_1)| \to \infty$, $\forall j = 2,\ldots,n$).

Contrary to the case of the saddle-node in $\mathbb{C}^2$ there are, in general, sectors that are neither attractors nor saddles: they are called mixed and are characterized by the condition $\cos(\varphi_i - \theta)\cos(\varphi_j - \theta) < 0$, for some $i \neq j$.

The directions for which there exists j such that $\cos(\varphi_j - \theta) = 0$ are called singular directions of the solution and are given by $\theta = \varphi_j \pm \frac{\pi}{2}$, $j = 2, \ldots, n$.

Remark 4. For simplicity in the notation we sometimes say that $\theta \in S$ in the sense that $x = re^{i\theta} \in S$.

4.1. **The sectors where the Theorem of Malmquist is valid.** Let S be a sector as described before. Denote by $\Lambda_{Z_\alpha}(S)$ the group of holomorphic transformations $H : S \times (\mathbb{C}^{n-1}, 0) \to S \times (\mathbb{C}^{n-1}, 0)$ such that:

a) $dH(Z_\alpha) = Z_\alpha(H)$,

b) H is asymptotic to the identity of $\{0\} \times \mathbb{C}^{n-1}$, as $x_1 \to 0$, $x_1 \in S$.

$\Lambda_{Z_\alpha}(S)$ is a presheaf and we denote by Λ_{Z_α} the sheaf associated to that presheaf.

An element H tangent to the identity and preserving the first component can be written in the form

$$H(x) = (x_1, x_2 + a_{20}(x_1) + \sum_{|Q|\geq 1} a_{2Q}(x_1)\bar{x}^Q, \ldots, \\ x_n + a_{n0}(x_1) + \sum_{|Q|\geq 1} a_{nQ}(x_1)\bar{x}^Q), \tag{6}$$

where $Q = (q_2, \ldots, q_n)$, $|Q| = q_2 + \ldots + q_n$ and $\bar{x}^Q = x_2^{q_2} \ldots x_n^{q_n}$. We look for conditions under which $H \in \Lambda_{Z_\alpha}(S)$.

Condition a) expresses that the leaves of $Z_\alpha|_{S\times(\mathbb{C}^{n-1},0)}$ are preserved by the elements of $\Lambda_{Z_\alpha}(S)$. It follows that

$$a_{jQ}(x_1) = a_{jQ} x_1^{-((Q,\alpha)-\alpha_j)} e^{\frac{(Q,\gamma)-\gamma_j}{x_1}}, \tag{7}$$

where $\gamma = (\gamma_2, \ldots, \gamma_n)$ [1].

Condition b) says that $H \tilde{\to} Id|_{\{0\}\times\mathbb{C}^{n-1}}$ for $x_1 \to 0$, $x_1 \in S$. This is simply equivalent to saying that $a_{jQ}(x_1) \tilde{\to} 0$, for $x_1 \in S$, $x_1 \to 0$, $\forall j \in \{2, \ldots, n\}$, $\forall Q \in \mathbb{N}_0^{n-1}$.

Denoting by φ_{jQ} the argument of the complex number $(Q, \gamma) - \gamma_j$, the behavior of $a_{jQ}(x_1)$ along $x_1 = re^{i\theta}$, as $r \to 0$, is given by $\cos(\varphi_{jQ} - \theta)$. The directions θ for which $\cos(\varphi_{jQ} - \theta) = 0$ for some j and Q, are called singular directions of the sheaf Λ_{Z_α} and are given by $\theta_{jQ}^{\pm} = \varphi_{jQ} \pm \frac{\pi}{2}$, $j = 2, \ldots, n$. The singular directions of the solution are always singular directions of the sheaf. They correspond to $Q = 0 \in \mathbb{C}^{n-1}$.

To study the behavior of the arguments of $(Q, \gamma) - \gamma_j$, $Q \in \mathbb{N}_0^{n-1}$, we represent all these points in the complex plane (Figure 1). We can easily observe that the singular directions of the sheaf are dense in the mixed sectors, while they are discrete in the attractor and saddle sectors. The singular directions of the solution are points of accumulation. In the case $\gamma \in \mathbb{R}^+ \setminus (\mathbb{N} \cup 1/\mathbb{N})$ there is no mixed sector.

Let us consider a direction φ_0 in the attractor sector that is not a singular direction of the sheaf Λ_{Z_α}: the sectors where the Theorem of Malmquist is valid are the sectors obtained by extending the sectors between the angles φ_0 and $\varphi_0 \pm \pi$ until reaching a singular direction of the sheaf Λ_{Z_α}. Both sectors have amplitude greater than π. Denote each one of this sectors by S_1 and S_2. They are well defined

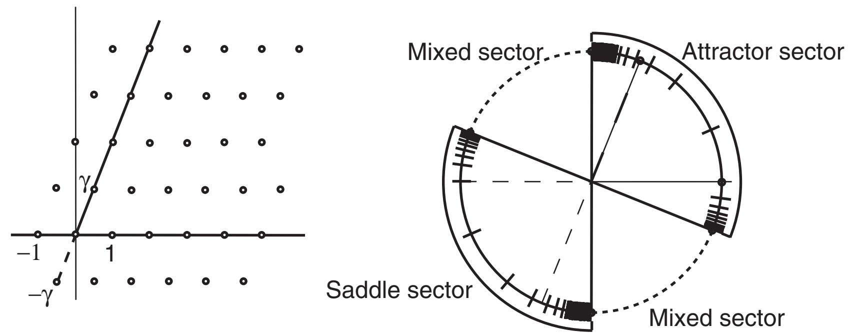

FIGURE 1. On the left, the complex numbers $(Q, \gamma) - \gamma_j$. On the right, the singular directions of the sheaf Λ_{Z_α} which are dense in the mixed sectors.

because the singular directions of the sheaf are discrete in the attractor and in the saddle sectors.

By definition, $S_1 \cap S_2$ is the union of two open sets, S_+ and S_-, contained in the attractor sector and in the saddle sector, respectively (and so $S_+ \cap S_- = \emptyset$). The saddle sector and the attractor sector, denoted by S_s and A_s respectively, are antipodes (that is, $S_s = A_s + \pi = \{e^{\pi i} a : a \in A_s\}$), and so are S_+ and S_-.

4.2. **The importance of the pre-sheaves $\Lambda_{Z_\alpha}(S_+)$ and $\Lambda_{Z_\alpha}(S_-)$.** As we have already said, there exists only one element $\hat{H} \in \mathbb{C}\{\bar{x}\}[[x_1]]$ conjugating $Y_{1,\alpha}$ and Z_α.

Proposition 3 ([1])**.** *Let S_1 and S_2 be the sectors where the Theorem of Malmquist is valid. Let H_1 and H_2 be the normalizing applications defined on S_1 and S_2, respectively. Then, $H_j \circ H_i^{-1}|_{S_+}$ and $H_j \circ H_i^{-1}|_{S_-}$ belong to $\Lambda_{Z_\alpha}(S_+)$ and $\Lambda_{Z_\alpha}(S_-)$, respectively.*

Let $g_\pm = H_2 \circ H_1^{-1}|_{S_\pm}$. Those maps represent the gluing of the leaves and $Y_{1,\alpha}$ is analytically conjugated to its formal normal form Z_α if and only if g_+ and g_- are both the identity map.

4.3. **Gluing of the leaves.** To know how the gluing of the leaves is done, it is important to know the behavior of the applications (6) in $\Lambda_{Z_\alpha}(S_+)$ and $\Lambda_{Z_\alpha}(S_-)$. The next two propositions describe their fundamental properties:

Proposition 4. *If $a_{jQ} \neq 0$ in S, then $\cos(\varphi_{jQ} - \theta) < 0$, $\forall \theta : re^{i\theta} \in S$.*

Proof. We must have $a_{jQ}(x_1) \overset{\sim}{\longrightarrow} 0$ as $x_1 \to 0$. Writing $x_1 = re^{i\theta}$, for fixed θ, the behavior of $a_{jQ}(x_1)$, as $r \to 0$, is given by the real part of $\frac{(Q,\gamma)-\gamma_j}{x_1}$, that is, by $\dfrac{|(Q,\gamma) - \gamma_j|}{r} \cos(\varphi_{jQ} - \theta)$.

Suppose that $a_{jQ} \neq 0$. Then

$$\exists \theta \in S : \cos(\varphi_{jQ} - \theta) > 0 \Rightarrow \frac{|(Q,\gamma) - \gamma_j|}{r} \cos(\varphi_{jQ} - \theta) \overset{r \to 0}{\longrightarrow} +\infty$$
$$\Rightarrow a_{jQ}(x) \overset{\sim}{\not\longrightarrow} 0.$$

□

Proposition 5 ([1]). *There exists a duality between $\Lambda_{Z_\alpha}(S)$ and $\Lambda_{Z_\alpha}(S+\pi)$ in the following sense: if $a_{jQ} \neq 0$ in S, then $a_{jQ} = 0$ in $S+\pi$.*

In particular there exists a duality between $\Lambda_{Z_\alpha}(S_+)$ and $\Lambda_{Z_\alpha}(S_-)$. So, we only need to know the constants that can be nonzero in $\Lambda_{Z_\alpha}(S_+)$, i.e., we need to know for which (j, Q) we have $\cos(\varphi_{jQ} - \theta) < 0$, $\forall \theta : re^{i\theta} \in S_+$ $(r \in \mathbb{R}^+)$.

The next result expresses how the gluing of the leaves is done.

Proposition 6 ([1]). *Take an element H of the sheaf Λ_{Z_α}. Thus H transforms the solution of the differential equation associated to the formal normal form given by*

$$x_j(x_1) = c_j x_1^{\alpha_j} e^{-\frac{\gamma_j}{x_1}}, \quad j \geq 2$$

into the solution of the same equation given by

$$x_j(x_1) = (c_j + a_{j0} + \sum_{|Q|\geq 1} a_{jQ} c^Q) x_1^{\alpha_j} e^{-\frac{\gamma_j}{x_1}}, \quad j \geq 2,$$

where $c^Q = c_2^{q_2} \dots c_n^{q_n}$.

In the sector S_+, each $c = (c_2, \dots, c_n) \in (\mathbb{C}^{n-1}, 0)$ represents a leaf of the foliation associated to the restriction of Z_α to $S_+ \times (\mathbb{C}^{n-1}, 0)$, i.e., c works like a parametrization of the leaves in S_+. Since S_+ does not contain singular directions of the sheaf, we can identify $\Lambda_{Z_\alpha}(S_+)$ with the set of transformations in the space of the leaves given by

$$\{c \mapsto (c_2 + a_{20} + \sum_{|Q|\geq 1} a_{2Q} c^Q, \dots, c_n + a_{n0} + \sum_{|Q|\geq 1} a_{nQ} c^Q)\},$$

also denoted by $\Lambda_{Z_\alpha}(S_+)$. The pre-sheaf $\Lambda_{Z_\alpha}(S_+)$ is a formalization of Proposition 6 in the following sense: $\Lambda_{Z_\alpha}(S_+)$ expresses that the leaf of $Z_\alpha|_{U_+ \times (\mathbb{C}^{n-1},0)}$ parameterized by $(c_2, \dots, c_n)$ is taken into the leaf parameterized by $(c_2 + a_{20} + \sum_{|Q|\geq 1} a_{2Q} c^Q, \dots, c_n + a_{n0} + \sum_{|Q|\geq 1} a_{nQ} c^Q)$. The same interpretation can be given to the pre-sheaf $\Lambda_{Z_\alpha}(S_-)$.

Before proceeding, I am going to explain how to determine $\Lambda_{Z_\alpha}(S_+)$, geometrically, for a given sector S_+. The first step is to choose the sectors S_1 and S_2 or, equivalently, the sectors S_+ and S_- we are going to work on.

We consider the set of complex numbers $C = \{(Q, \gamma) - \gamma_i : i = 2, \dots, n, Q \in \mathbb{N}_0^{n-1}\}$. We assume that $0 = \arg(\gamma_2) \leq \dots \leq \arg(\gamma_n) < \pi$.

Denote by K the sector with vertex at the origin whose elements have arguments between $0 = \arg(\gamma_2)$ and $\arg(\gamma_n)$. Then we choose two straight lines not contained in K and such that the two sectors defined by those straight lines do not contain any element of C in its interior (Figure 2).

Fix one of those sectors: S. Then, if $S + \frac{\pi}{2}$ is contained in the attractor sector we take $S_+ = S + \frac{\pi}{2}$, otherwise we take $S_+ = S - \frac{\pi}{2}$. Remark that S can be chosen as close to the real axis as we want.

The constants a_{jQ} that can be nonzero in $\Lambda_{Z_\alpha}(S_+)$ are those for which $(Q, \gamma) - \gamma_j$ is in the half plane, defined by the bisectrix of S, not containing S_+: R. For example, if we look at Figure 2, on the left, $\Lambda_{Z_\alpha}(S_+)$ is given by

$$\{(y, z) \mapsto (y + a_{200} + a_{201} z, z + a_{300})\}$$

while on the right side, $\Lambda_{Z_\alpha}(S_+)$ is given by

$$\{(y, z) \mapsto (y + a_{200}, z + a_{300} + a_{310} y + a_{320} y^2)\}.$$

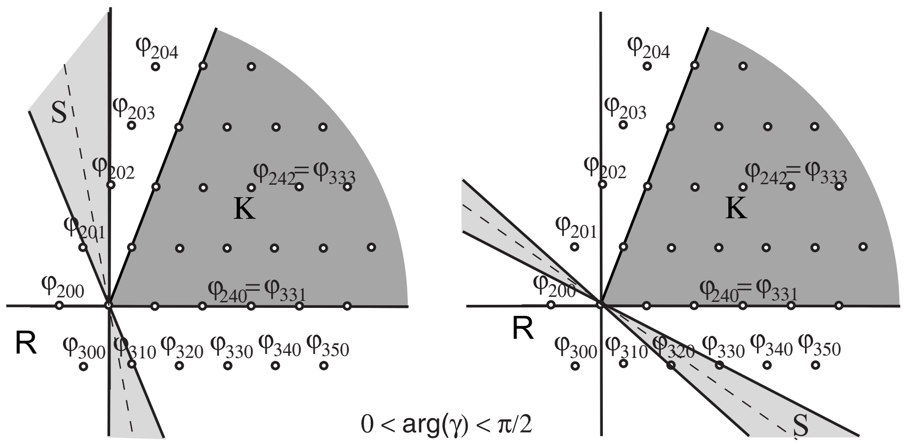

FIGURE 2. Each point represents an element of the form $(Q,\gamma)-\gamma_i$ for some $i=2,\dots,n$ and $Q\in\mathbb{N}^{n-1}$.

So, the monomial c^Q appears on the $(i-1)^{th}$-component of g_+ if $(Q_{ij},\gamma)-\gamma_i\in R$. We should remark that the elements of $\Lambda_{Z_\alpha}(S_+)$ are always polynomial while the elements $\Lambda_{Z_\alpha}(S_-)$ are always tangent to the identity, i.e, a_{i0} must be equal to 0 in S_- for all $i=2,\dots,n$.

In the case $0=\arg(\gamma_2)=\ldots=\arg(\gamma_n)$, the constants a_{jQ} that can be nonzero in $\Lambda_{Z_\alpha}(S_+)$ are those such that $\arg((Q,\gamma)-\gamma_j)=\pi$.

5. Semi-complete saddle-node foliations in $\mathbb{C}^n$

We first consider the case $f\equiv k\in\mathbb{C}$ (we can assume $f=1$). There exists a formal invariant manifold tangent to the x_1-axis (weak separatrix), but such a formal manifold is, in general, divergent. However we can deduce necessary and sufficient conditions for its convergence. In that case we call it a holomorphic central manifold.

Proposition 7. *A vector field belonging to the $\hat{G}_0$-orbit of $Y_{1,\alpha}$ has holomorphic central manifold if and only if the associated sheaf has no translation, i.e., if and only if $a_{i0\ldots0}=0$ for all $i=2,\dots,n$.*

Proof. Suppose that $a_{i0\ldots0}=0$ for all $i=2,\dots,n$. Let L be the leaf containing $H_1^{-1}(S_1\times\{0,\dots,0\})$, where H_1 is the normalizing application. Consider the curve $c(t)=(re^{2\pi it},0,\dots,0)$, where r is such that $\Pi_1(c(0))=r\in S_+$, and let c_L be its lift to L. By definition, the leaf L is parameterized by $(0,\dots,0)$.

Since $\Pi_1(c(\frac{1}{2}))\in S_-$ and the application g_- is tangent to the identity, $(0,\dots,0)$ is taken into $(0,\dots,0)$ by g_-, which means that $H_2(L\cap(S_2\times(\mathbb{C}^{n-1},0)))\subseteq\{x_i=0,i\geq 2\}$.

On the other hand, $\Pi_1(c_L(1))=\Pi_1(c_L(0))$ belongs to S_+. But g_+ takes $(0,\dots,0)$ into $(a_{20\ldots0},\dots,a_{n0\ldots0})=(0,\dots,0)$, by hypothesis. Thus the restriction of g_+ and g_- to the leaf $\{x_i=0,i\geq 2\}$ is given by the identity function: $(x,0,\dots,0)\mapsto(x,0,\dots,0)$. Since the x_1-axis is a holomorphic central manifold for the formal normal form, the leaf L is a holomorphic central manifold for $Y_{1,\alpha}$.

Suppose now that $Y_{1,\alpha}$ has a holomorphic central manifold. Denote this leaf by L and consider its image by the normalizing application H_1 defined on $S_1 \times (\mathbb{C}^{n-1}, 0)$.

The intersection of S_1 with the saddle sector is not empty. As in the formal normal form the only leaf in $S_1 \times (\mathbb{C}^{n-1}, 0)$ such that $x_i(x_1) \to 0$ as $x_1 \to 0$, for all $i \in \{2, \ldots, n\}$, is the invariant manifold $\{x_i = 0, i = 2, \ldots, n\}$ we have that

$$H_1(L \cap (S_1 \times (\mathbb{C}^{n-1}, 0))) = S_1 \times \{x_i = 0, i = 2, \ldots, n\}.$$

In the same way we can deduce that

$$H_2(L \cap (S_2 \times (\mathbb{C}^{n-1}, 0))) = S_2 \times \{x_i = 0, i = 2, \ldots, n\}.$$

We conclude that $(a_{20\ldots0}, \ldots, a_{n0\ldots0}) = (0, \ldots, 0)$. □

The next lemma enables us to guarantee that semi-completeness implies the convergence of the weak separatrix.

Lemma 3. *Let X be a field of type $Y_{1,\alpha}$ and suppose that X is semi-complete in a neighbourhood of the origin. Then there is no translation in $\Lambda_{Z_\alpha}(S_+)$, i.e., $a_{i0\ldots0} = 0$, $\forall i = 2, \ldots, n$.*

Proof. Let $\mathcal{F}$ be the foliation associated to X. In a neighbourhood of the origin, Π_1 is transverse to the leaves of $\mathcal{F}$, except to those contained in the invariant manifold $\{x_1 = 0\}$.

Consider the curve $c(t) = (re^{2\pi i t}, 0, \ldots, 0)$, $t \in [0, 1]$ and $r \in U_+$. Let L be the leaf containing $H_1^{-1}(S_1 \times \{(0, \ldots, 0)\})$ and c_L be the lift of the curve c to the leaf L. Then

$$\int_{c_L} dT_L = \int_c \frac{dx_1}{x_1^2} = 0.$$

As the vector field is semi-complete, the curve c_L is closed.

By definition of L, $H_1(L \cap (S_1 \times (\mathbb{C}^{n-1}, 0)))$ is parameterized by $(0, \ldots, 0)$. The application g_- is tangent to the identity, so $(0, \ldots, 0)$ is taken into $(0, \ldots, 0)$ by g_-. This means that $H_2(L \cap (S_2 \times (\mathbb{C}^{n-1}, 0)))$ is also parameterized by $(0, \ldots, 0)$. Finally $(0, \ldots, 0)$ is taken into $(a_{20\ldots0}, \ldots, a_{n0\ldots0})$ by g_+. But, since c_L is closed, $(a_{20\ldots0}, \ldots, a_{n0\ldots0}) = (0, \ldots, 0)$, i.e, there is no translation. □

In this result there is a great difference between the $\mathbb{C}^2$ and the $\mathbb{C}^n$ cases. In $\mathbb{C}^2$, g_+ is the identity plus a translation. So, the semi-completeness of X implies that g_+ is the identity. Here the best we can get, at this step, is g_+ of type $(y, z) \mapsto (y + a_{201}z, z)$.

Lemma 4. *Let X be a semi-complete vector field as in Lemma 3. Then the holonomy relative to the holomorphic central manifold is the identity.*

Proof. Consider the curve $c(t) = (re^{2\pi i t}, 0, \ldots, 0)$, r sufficiently close to 0, and let c_0 be the lift of c to the holomorphic central manifold (whose existence is guaranteed in the last lemma).

Let $\Sigma = \{c_0(0) + (0, \tau_2, \ldots, \tau_n) : 0 \leq \sum_{i=2}^n |\tau_i|^2 < \varepsilon\}$ be a transversal section to the curve c_0 at $c_0(0)$ and c_L be the lift of c to the leaf through the point $c_0(0) + (0, \tau_2, \ldots, \tau_n) \in \Sigma$. Then $\int_{c_L} dT_L = 0$. Since X is semi-complete we conclude that c_L is closed. But c_L is closed for all $(\tau_2, \ldots, \tau_n)$ with norm less then ε. This means that the holonomy is trivial. □

We will now describe the vector fields satisfying both properties.

Proposition 8. *Let X be a vector field of type $Y_{1,\alpha}$. Suppose that X has a holomorphic manifold and the holonomy relative to such invariant manifold is trivial. Then $\alpha \in \mathbb{Z}^{n-1}$ and X is analytically conjugated to its formal normal form.*

Proof. The idea of the proof is to use induction over the degree of Q, i.e., over $|Q|$. We will prove that if $a_{iQ} = 0$, $\forall i, Q : |Q| \leq q \wedge (Q, \gamma) - \gamma_i \in R$, then $a_{iQ} = 0$, $\forall i, Q : |Q| = q+1 \wedge (Q, \gamma) - \gamma_i \in R$. Remark that $(Q, \gamma) - \gamma_i \in R$ if and only if a_{iQ} can be nonzero in S_+. Abusing notation, we say that $a_{iQ} \in \Lambda_{Z_\alpha}(S_+)$ if a_{iQ} can be nonzero in S_+.

Since X has a holomorphic central manifold, $a_{j0\ldots0} = 0$ in S_+, for all j (Proposition 7). So the induction hypothesis is satisfied for $|Q| = 0$.

Suppose now that $a_{iQ} = 0$ for all Q such that $|Q| \leq q$ and $(Q, \gamma) - \gamma_i \in R$. Then, in terms of leaves, g_+ is given by

$$(c_2, \ldots, c_n) \mapsto (c_2 + \sum_{j=1}^{k_2} a_{2Q_{2j}} c^{Q_{2j}}, \ldots, c_n + \sum_{j=1}^{k_n} a_{nQ_{nj}} c^{Q_{nj}})$$

with $|Q_{ij}| \geq q+1$, $\forall i = 2, \ldots, n$ and $1 \leq j \leq k_i$. To say that the holonomy relative to the invariant manifold is the identity is equivalent to saying that $g_- \circ g_+ = id$. The $(l-1)^{th}$-component of $g_- \circ g_+ = id$ is given by

$$\left[c_l + \sum_{j=1}^{k_l} a_{lQ_{lj}} c^{Q_{lj}} + \sum_{\substack{Q \neq e_{l-1} \\ Q:(Q,\gamma)-\gamma_l \notin R}} a_{lQ} \prod_{i=2}^{n} (c_i + \sum_{j=1}^{k_i} a_{iQ_{ij}} c^{Q_{ij}})^{q_i} \right] e^{2\pi i \alpha_l}.$$

The exponentials $e^{2\pi i \alpha_l}$ appear in the composition since g_- is taken not at x_1 but at $e^{2\pi i} x_1$: remark that $c_l (e^{2\pi i} x_1)^{\alpha_l} e^{\frac{\gamma_l}{e^{2\pi i} x_1}} = c_l e^{2\pi i \alpha_l} x_1^{\alpha_l} e^{\frac{\gamma_l}{x_1}}$. But, since the holonomy is trivial, $c_l e^{2\pi i \alpha_l} = c_l$ for all l, which means that the formal invariants α_l are integers. Thus the system reduces to

$$(8) \quad \begin{cases} \sum_{j=1}^{k_2} a_{2Q_{2j}} c^{Q_{2j}} + \sum_{\substack{Q \neq e_1 \\ Q:(Q,\gamma)-\gamma_2 \notin R}} a_{2Q} \prod_{i=2}^{n} (c_i + \sum_{j=1}^{k_i} a_{iQ_{ij}} c^{Q_{ij}})^{q_i}) = 0 \\ \vdots \\ \sum_{j=1}^{k_n} a_{nQ_{nj}} c^{Q_{nj}} + \sum_{\substack{Q \neq e_{n-1} \\ Q:(Q,\gamma)-\gamma_n \notin R}} a_{nQ} \prod_{i=2}^{n} (c_i + \sum_{j=1}^{k_i} a_{iQ_{ij}} c^{Q_{ij}})^{q_i}) = 0 \end{cases}$$

Fix Q_0 such that $|Q_0| = q+1$ and $(Q_0, \gamma) - \gamma_i \in R$ for some $i \geq 2$. We look for the coefficient of c^{Q_0} in the $(i-1)^{th}$ equation of the system (8). The monomial c^{Q_0} appears in $\prod_{p=2}^{n} (c_p + \sum_{j=1}^{k_p} a_{pQ_{pj}} c^{Q_{pj}})^{q_p}$ if $(q_2, \ldots, q_n) = e_{i-1}$ or $(q_2, \ldots, q_n) = Q_0$. However, both $(n-1)$-tuples are forbidden in the second sum since $(Q_0, \gamma) - \gamma_i \in R$.

We can ask if c^{Q_0} can be obtained in $\prod_{p=2}^{n} (c_p + \sum_{j=1}^{k_p} a_{pQ_{pj}} c^{Q_{pj}})^{q_p}$ by a different way. Since $\sum_{j=1}^{k_p} a_{pQ_{pj}} c^{Q_{pj}}$ involves only monomials of order greater than or equal to $q+1$, the only chance is the existence of $j \neq i$ such that $(Q_0, \gamma) - \gamma_j \in R$. In this case the monomial is obtained for $Q = e_{j-1}$.

Suppose that $\{i : (Q_0, \gamma) - \gamma_i \in R\} = \{i_1, i_2, \ldots, i_k\}$ where $i_j < i_{j+1}$. Then c^{Q_0} appears only on the $(i_j - 1)^{th}$ equations, $j = 1, \ldots, k$. Its coefficients on the $(i_j - 1)^{th}$ equation

$$a_{i_j Q_0} + \sum_{\substack{l \neq j \\ l:(e_{i_l - 1}, \gamma) - \gamma_{i_j} \notin R}} a_{i_j e_{i_l - 1}} a_{i_l Q_0}$$

must vanish. So we have a system of k equations in k unknowns: $a_{i_j Q_0}$. Denote by $B = (b_{ij})$ the matrix associated to this new system.

We have that $(e_{i_j-1}, \gamma) - \gamma_{i_l} = \gamma_{i_j} - \gamma_{i_l}$ and $(e_{i_l-1}, \gamma) - \gamma_{i_j} = \gamma_{i_l} - \gamma_{i_j}$, so only one of them belongs to R. As $\gamma_{i_j} - \gamma_{i_l} \in R$ means that e_{i_j-1} does not belong to the second sum of the $(i_l - 1)^{th}$ equation of system (8), if $\gamma_{i_j} - \gamma_{i_l} \in R$ then $b_{lj} = 0$.

Assume, reordering the variables, that $\Im(\gamma_{i_j}) \leq \Im(\gamma_{i_{j+1}})$ and if $\Im(\gamma_{i_j}) = \Im(\gamma_{i_{j+1}})$ then $\Re(\gamma_{i_j}) < \Re(\gamma_{i_{j+1}})$, where $\Im(x)$ ($\Re(x)$) denotes the imaginary (real) part of x. If $i_l > i_j$ and $\Im(\gamma_{i_l}) > \Im(\gamma_{i_j})$, then $\Im(\gamma_{i_l} - \gamma_{i_j}) > 0$. But, if $\Im(\gamma_{i_l}) = \Im(\gamma_{i_j})$, we will have that $\Re(\gamma_{i_l} - \gamma_{i_j}) > 0$. Since we can choose S as close to the real axis as we want (Figure 2), we can say that there is sector for which if $i_l > i_j$ then $\gamma_{i_j} - \gamma_{i_l} \in R$, that is, $b_{lj} = 0$.

We have just proved that the matrix B is an upper triangular matrix such that $b_{ii} = 1$. So $a_{i_j Q_o} = 0$, $\forall j = 1, \ldots, k$.

The induction over $|Q|$ stops in a finite number of steps. So $g_+ = id$. Since $g_- \circ g_+ = id$ it follows that $g_- = id$ also. □

We can thus prove:

Proposition 9. *Let X be a vector field of type $Y_{1,\alpha}$. X is semi-complete if and only if $\alpha \in \mathbb{Z}^{n-1}$ and it is analytically conjugated to its formal normal form.*

Proof. Suppose that X is semi-complete. Then X has holomorphic central manifold (Lemma 3 and Proposition 7). Lemma 4 guarantees that the holonomy relative to that invariant manifold is the identity. So $\alpha \in \mathbb{Z}^{n-1}$ and X is analytically conjugated to its formal normal form (Proposition 8).

It remains to see if the formal normal forms Z_α are semi-complete for $\alpha \in \mathbb{Z}^{n-1}$. The solution of the differential equation associated to Z_α is given by

$$x_j(T) = x_j(0) \frac{e^{\gamma_j T}}{(1 - x_1(0)T)^{\alpha_j}}, \quad j = 1, \ldots, n. \tag{9}$$

For $\alpha \in \mathbb{Z}^{n-1}$, $(1 - x_1(0)T)^{\alpha_j}$ is well defined and univalued for all $T \in \mathbb{C} \setminus \{\frac{1}{x_1(0)}\}$ and $j \in \{2, \ldots, n\}$. So, the application $\Phi : \Omega = \{(T, x_1, \ldots, x_n) : T \neq \frac{1}{x_1}\} \subseteq \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$, given by

$$(T, x_1, x_2, \ldots, x_n) \mapsto \left(\frac{x_1}{1 - x_1 T}, x_2 \frac{e^T}{(1 - x_1 T)^{\alpha_2}}, \ldots, x_n \frac{e^{\gamma_n T}}{(1 - x_1 T)^{\alpha_n}} \right),$$

defines a semi-complete flow associated to X. □

Our objective is to prove Theorem 3. We have already proved that if $\mathcal{F}$ is the foliation associated to $Y_{1,\alpha}$, with $\alpha \in \mathbb{Z}^{n-1}$ and $Y_{1,\alpha}$ is analytically conjugated to its formal normal form Z_α, then $\mathcal{F}$ is associated to a semi-complete vector field in a neighbourhood of the origin. We are going to prove that there are no more foliations of saddle-node type associated to semi-complete vector fields.

Proposition 10. *Let $Y = fY_{1,\alpha}$ for some holomorphic function f, $f(0) \neq 0$. Suppose that Y is semi-complete in $B(0, \varepsilon)$ with f nonzero in this open set. Then Y has holomorphic central manifold.*

Proof. Suppose that Y does not have holomorphic central manifold. Y and $Y_{1,\alpha}$ represent exactly the same foliation in $B(0, \varepsilon)$, so neither has $Y_{1,\alpha}$.

Consider the curve $c(t) = (re^{2\pi i t}, 0, \ldots, 0)$, for r sufficiently small (in particular $|r| < \varepsilon$), contained in the central manifold of Z_α (the formal normal form of $Y_{1,\alpha}$). We choose r in such a manner that $\Pi_1(c(0)) \in S_+$.

Let c_L be the image of c by H_1^{-1} and H_2^{-1}, where L denotes the leaf containing $H_1^{-1}(c)$. This curve is continuous since the application g_- is tangent to the identity, so the diffeomorphisms join together in $S_-|_{\{(x_1,0,\ldots,0)\}}$. The projection of c_L onto the x_1-axis is the curve $(re^{2\pi i t}, 0, \ldots, 0)$, but c_L is not closed since there is no holomorphic central manifold, by hypothesis $((a_{20\ldots 0}, \ldots, a_{n0\ldots 0}) \neq (0, \ldots, 0)$ by Proposition 7). So $c_L(0) \neq c_L(1)$.

Consider the analytic conjugation given by the homothety $H_\lambda(x) = \lambda x$: $Y_\lambda = (DH_\lambda)^{-1}(Y \circ H_\lambda)$. If Y is semi-complete in $B(0, \varepsilon)$, then Y_λ is semi-complete in $B(0, \frac{\varepsilon}{|\lambda|})$. In particular, Y_ε is semi-complete in $B(0, 1)$.

We have that

$$dT^{Y_\lambda} = H_\lambda^*(dT^Y) = \frac{dx_1}{\lambda f(\lambda x) x_1^2}.$$

Consider the family of curves $c^\lambda(t) = (\frac{\lambda}{2}e^{2\pi i t}, 0, \ldots, 0)$ $(t \in [0,1])$, parameterized by λ, also contained in the central manifold of Z_α. Denote by c_L^λ their image by H_1^{-1} and H_2^{-1}. This is again a family of continuous curves in L. Since H_1 and H_2 are both asymptotic to the formal change of coordinates conjugating $Y_{1,\alpha}$ and Z_α, we have that

$$\begin{aligned} H_{i,1}^{-1}(x) &= x_1, \\ H_{i,j}^{-1}(x) &= x_j + \sum_{k=1}^{m} f_{ijk}(\bar{x}) x_1^k + x_1^m \varepsilon_{i,j}(x), \quad i = 1, 2, j = 2, \ldots, n, \end{aligned}$$

where $\bar{x} = (x_2, \ldots, x_n)$, $f_{ij1}(0) = 0$ for all i and j (2) and $\lim_{x_1 \to 0} \varepsilon_{i,j}(x) = 0$. Restricted to the invariant manifold $\{(x_1, 0, \ldots, 0) : x_1 \in \mathbb{C}\}$ they reduce to

$$\begin{aligned} H_{i,1}^{-1}(x) &= x_1, \\ H_{i,j}^{-1}(x) &= \sum_{k=2}^{m} f_{ijk} x_1^k + x_1^m \varepsilon_{i,j}(x_1, 0), \quad i = 1, 2, j = 2, \ldots, n, \end{aligned}$$

where f_{ijk} are constants. This allows us to prove that, for each $u \in \mathbb{S}^1 \subseteq \mathbb{C}$,

$$\frac{H_{i,j}^{-1}(\frac{u}{2}t, 0)}{\frac{u}{2}t} \underset{t \to 0}{\longrightarrow} 0, \quad j = 2, \ldots, n.$$

Since $\mathbb{S}^1$ is compact, there exists t_0 such that $c_L^\lambda \subseteq B(0, \lambda)$ for all λ such that $|\lambda| \leq t_0$. Moreover, while the first component of c_L^λ is of order of λ, the other components are of order of λ^2.

Let $c_\lambda = H_\lambda^{-1}(c_L^\lambda)$; $\lambda \mapsto c_\lambda$ is a continuous map. Since

a) $\Pi_1(c_\lambda) = \Pi_1(H_\lambda^{-1}(c_L^\lambda)) = H_\lambda^{-1}(\Pi_1(c_L^\lambda)) = H_\lambda^{-1}(\frac{\lambda}{2}e^{2\pi i t}) = \frac{1}{2}e^{2\pi i t}$,
b) the other components of c_λ are of order λ,
c) $c_\lambda \subseteq D(0, 1)$, which is a compact set,

there exists $\lim_{\lambda \to 0} c_\lambda$ and

$$\lim_{\lambda \to 0} \int_{c_\lambda} \frac{dx_1}{f(\lambda x) x_1^2} = \int_{\frac{1}{2}e^{2\pi i t}} \frac{dx_1}{f(0) x_1^2} = 0. \tag{10}$$

But c_L^λ is not closed and H_λ is a homothety. Therefore, c_λ is also not closed.

So, let $W \subseteq \mathbb{C}$ be a simply connected neighbourhood of $\Pi_1(c_\lambda(1)) = \frac{1}{2}$, not containing the origin. In this neighbourhood we can write x_i as a function of x_1 for all $i \geq 2$ (Π_1 is transverse to the leaves). Define

$$\begin{aligned} I_\lambda : W \to &\quad \mathbb{C} \\ p \mapsto &\int_{c_p} \frac{dx_1}{f(\lambda x)x_1^2}, \end{aligned}$$

where c_p is a curve, contained in the connected component of $L \cap \Pi_1^{-1}(W)$ containing $c_\lambda(1)$, such that $\Pi_1(c_p)$ joins $\frac{1}{2}$ to p. By Hadamard's lemma, f can be written in the form $f(x) = f(0) + \sum_{i=1}^n x_i g_i(x)$ and so we can rewrite the application I_λ as

$$I_\lambda(p) = \int_{c_p} \frac{dx_1}{x_1^2(f(0) + \lambda h(\lambda, x_1))}.$$

Lemma 2 guarantees the existence of real and positive numbers λ_0 and θ such that $B(0,\theta) \subseteq I_\lambda(W)$, $\forall \lambda : |\lambda| \leq \lambda_0$. But (10) implies that there exists λ_1, with $|\lambda_1| < \lambda_0$, such that

$$\int_{c_{\lambda_1}} \frac{dx_1}{f(\lambda x)x_1^2} = \alpha$$

with $|\alpha| < \theta$. Since $B(0,\theta) \subseteq I_{\lambda_1}(W)$, there exists $p \in W$ such that $I_{\lambda_1}(p) = -\alpha$. If $\Pi_1^{-1}(p) \notin c_{\lambda_1}([0,1])$ let $\tilde{c}$ be the curve joining $c_{\lambda_1}(0)$ to $\Pi_1^{-1}(p)$ obtained by concatenating c_{λ_1} to c_p. If $\Pi_1^{-1}(p) \in c_{\lambda_1}([0,1])$, i.e., $p = c_{\lambda_1}(t)$ for some $0 < t < 1$, let $\tilde{c} = c_{|[0,t]}$. Thus

$$\int_{\tilde{c}} \frac{dx_1}{f(\lambda x)x_1^2} = 0.$$

In both cases $\tilde{c}$ is a one-to-one embedded curve. This result contradicts the semi-completeness of the vector field Y_{λ_1} and, consequently, the semi-completeness of the vector field Y. $\square$

Proposition 11. *Let $Y = fY_{1,\alpha}$ for some holomorphic function f, $f(0) \neq 0$. If Y is semi-complete in $B(0,\varepsilon)$, f nonvanishing in such an open set, then the holonomy relative to the holomorphic central manifold is trivial.*

Proof. Suppose that Y is semi-complete in a neighbourhood of the origin. Then it has holomorphic central manifold (Proposition 10). We can suppose that the holomorphic central manifold is the x_1-axis. Thus the vector field can be written in the form

$$\begin{cases} \dot{x_1} = x_1^2 h(x) \\ \dot{x_i} = \lambda_i x_i + x_1 \sum_{j=2}^n x_j f_{ij}(x), \quad i = 2, \ldots, n \end{cases} \tag{11}$$

where $h(0) \neq 0$.

Since Y is semi-complete, so is its restriction to the x_1-axis. Its restriction is equivalent to the 1-dimensional vector field $x^2 h(x, 0, \ldots, 0)\partial/\partial x$, which must be analytically conjugated, by a diffeomorphism H, to $h(0)x^2\partial/\partial x$ (Lemma 1).

Consider the curve $c(t) = (re^{2\pi i t}, 0, \ldots, 0)$, $t \in [0,1]$, for r sufficiently close to 0 and let $c'(t) = (H(\Pi_1(c(t))), 0, \ldots, 0)$. Then

$$\int_c dT_{\{x_i=0, i=2,\ldots,n\}} = \int_{c'} \frac{dx_1}{h(0)x_1^2} = 0. \tag{12}$$

Suppose that the holonomy is not the identity. Then, for any neighbourhood of $0 \in \mathbb{C}^{n-1}$, there exists some $\bar{x}_0 = (x_2^0, \ldots, x_n^0)$ such that the lift c_L of c to the leaf L through $(r, x_2^0, \ldots, x_n^0)$ is not closed. Thus $c_L(0) \neq c_L(1)$ although $\Pi_1(c_L(0)) = \Pi_1(c_L(1))$. We choose r and $\bar{x}_0$ in such a manner that $c_L \subseteq B(0, \varepsilon)$ and is not closed.

By the transversality of Π_1, there is a simply connected neighbourhood of r in $\mathbb{C} \setminus \{0\}$, denoted by W, such that, in each leaf of $Y|_{W \times \mathbb{C}^{n-1}}$, we can write $x_j = x_j(x_1; \bar{x}_0)$, for all $j = 2, \ldots, n$. Substituting $x_j(x_1; \bar{x}_0)$ for x_j in the first equation of the differential system (11), we obtain the differential equation $\dot{x_1} = x_1^2 h(x_1, x_2(x_1; \bar{x}_0), \ldots, x_n(x_1; \bar{x}_0))$ where $\bar{x}_0$ is considered as a parameter.

Consider the application

$$\begin{aligned} I_{\bar{x}_0} &: W \to \mathbb{C} \\ p &\mapsto \int_{c_p} \frac{dx_1}{x_1^2 h(x_1, x_2(x_1; \bar{x}_0), \ldots, x_n(x_1, \bar{x}_0))}, \end{aligned}$$

where $c_p \subseteq W$ represents a curve joining r to p. We have $I_{\bar{x}_0}(r) = 0, \forall \bar{x}_0 : \|\bar{x}_0\| \leq \varepsilon_1$ and $I_0'(r) = \frac{1}{r^2 h(r, 0, \ldots, 0)} \neq 0$. By the same argument used in the proof of Lemma 2, there exist real and positive numbers θ and ε_0 such that $B(0, \theta) \subseteq I_{\bar{x}_0}(W)$, $\forall \bar{x}_0 : \|\bar{x}_0\| \leq \varepsilon_0$. Since

$$\int_{c_L} dT_L \overset{\bar{x}_0 \to 0}{\longrightarrow} 0$$

there exists $\bar{x}_0$, with $\|\bar{x}_0\| \leq \varepsilon_0$, for which c_L is not closed and such that $\int_{c_L} dT_L = \alpha$ with $|\alpha| < \theta$.

But, we have just proved the existence of $p \in W$ such that $I_{\bar{x}_0}(p) = -\alpha$. So, let $P = (p, x_2(p, x_0), \ldots, x_n(p, x_0))$. If $P \notin c_L([0, 1])$ we denote by $\tilde{c}$ the curve obtained by the concatenation of c_L to the lift of c_p to L. If $P = c_L(t_0)$ for some $0 < t_0 < 1$, we denote by $\tilde{c}$ the curve $c_L|_{[0, t_0]}$. In both cases $\tilde{c}$ is a one-to-one embedded curve such that $\int_{\tilde{c}} dT_L = 0$, contradicting the fact that Y is semi-complete. □

Finally we are going to prove Theorem 3.

Proof of Theorem 3. Let $\mathcal{F}$ be a foliation associated to a vector field in $\mathfrak{X}$. For each vector field X, whose foliation coincides with $\mathcal{F}$ (in a small neighbourhood of the origin), there exist p and a holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ ($f(0) \neq 0$) such that X is written in the form fY_p.

First, we proved that if $\mathcal{F}$ is associated to a semi-complete vector field, then $p = 1$ (Proposition 2). Thus Y can be written in the form $Y = fY_{1,\alpha}$. Secondly, we proved that Y has holomorphic central manifold (Proposition 10) and the holonomy relative to that invariant manifold is trivial (Proposition 11). Since Y and $Y_{1,\alpha}$ represent exactly the same foliation in a neighbourhood of the origin, $Y_{1,\alpha}$ has holomorphic central manifold and its holonomy is trivial also. Then $Y_{1,\alpha}$ is analytically conjugated to its formal normal form Z_α with $\alpha \in \mathbb{Z}^{n-1}$ (Proposition 8). The reciprocal is also valid (Proposition 9).

It remains to consider the foliations associated to vector fields X whose linear part is not diagonal, but is diagonalizable. If H is a linear application diagonalizing $DX(0)$, let $Y = (DH)^{-1}(X \circ H)$. $\mathcal{F}_X$ admits a semi-complete representative if and only if so does $\mathcal{F}_Y$. But $Y = fY_{1,\alpha}$ and the result follows. □

6. Saddle-node with a nonisolated singularity

In this section we classify the semi-complete vector fields associated to a saddle-node foliation whose set of singularities coincides with the holomorphic invariant manifold transverse to the weak separatrix (when the linear part at the singularity is diagonalizable). We assume that such invariant manifold is the hyperplane $\{x_1 = 0\}$; that is, $X = fx_1^{-k}Y_p$ for some f, $f(0) \neq 0$, and $k \in \mathbb{Z} \setminus \{0\}$. Abusing notation we still call such vector fields of saddle-node type, even not admitting an isolated singularity, because the foliations associated to $fx_1^{-k}Y_p$ and fY_p coincide outside the set.

The restriction of $fx_1^{-k}Y_p$ to $\{x_1 = 0\}$ is of Poincaré type if $k = 0$ (the set of eigenvalues belong to the Poincaré domain), a set of singular points if $k < 0$, or it is not defined if $k > 0$.

Lemma 5. *Let $X = fx_1^{-k}Y_p$, where f is a holomorphic function such that $f(0) \neq 0$ and $k \in \mathbb{Z}$. Suppose that X is semi-complete in an open neighbourhood of the origin. Then $k \in \{p-1, p, p+1\}$.*

Proof. Fix a disc $B(0,\varepsilon) \subseteq \mathbb{C}^n$ of center at the origin of $\mathbb{C}^n$ and radius $\varepsilon > 0$ relative to which X is semi-complete.

Let $S \subseteq \mathbb{C}_{x_1}$ be an angular sector with vertex at the origin and angle greater than $\frac{2\pi}{|-k+p|}$ and less than 2π. For each leaf L in $S \times (\mathbb{C}^{n-1}, 0)$, we can write $x_j = x_j^L(x_1)$, $\forall j = 2, \ldots, n$, univocally. Let c_L be a one-to-one embedded curve in such a leaf L. We have that

$$\int_{c_L} dT_L = \int_{c_L} \frac{dx_1}{x_1^{-k+p+1} f(x)} = \int_{\Pi_1(c_L)} \frac{dx_1}{x_1^{-k+p+1} f(x_1, x_2^L(x_1), \ldots, x_n^L(x_1))}.$$

By transversality, $\Pi_1(c_L)$ is also an embedded curve. So we have reduced the study of the semi-completeness of a vector field in $S \times (\mathbb{C}^{n-1}, 0)$ to the study of the semi-completeness of a vector field in S.

In our case, the vector field

$$Y: \quad \dot{x} = x^{-k+p+1} f(x, x_2^L(x), \ldots, x_n^L(x))$$

satisfies $f(0) \neq 0$ and $-k+p+1 \geq 3$ or $-k+p+1 \leq -1$. Since the sector S has amplitude greater than $\frac{2\pi}{|-k+p|}$, Y is not semi-complete in any set of the type $(S \times W) \cap B(0,\varepsilon)$ (Remark 3). Thus X is not semi-complete in any neighbourhood of the origin. □

Immediately, we conclude:

Proposition 12. *Let X be a holomorphic vector field of saddle-node type, with an isolated singularity at the origin, and M the invariant hypersurface transverse to the weak direction of X. If F is a holomorphic function such that $F(x) = 0 \Leftrightarrow x \in M$, then the holomorphic vector field FX is not semi-complete in any neighbourhood of the origin.*

Proof. Let H be the composition between the linear transformation that diagonalizes the linear part of $DX(0)$ and the application taking M into $\{x_1 = 0\}$. The vector field $Y = (DH)^{-1}((FX) \circ H)$ can be written in the form $(F \circ H)gY_p$ for some $p \geq 1$ and some holomorphic function g satisfying $g(0) \neq 0$.

On the other hand, since M is the invariant hypersurface transverse to the weak direction of X, $H^{-1}(M)$ is the invariant manifold transverse to the weak direction of gY_p: $\{x_1 = 0\}$. Since $F(x) = 0$ for $x \in M$, we have that $(F \circ H)(x) = 0$ for $x \in H^{-1}(M)$, i.e., for $x \in \{x_1 = 0\}$. Thus $F \circ H = x_1^k h(x)$ for some $k \in \mathbb{N}$ and h such that $h(0) \neq 0$. By Lemma 5, Y is not semi-complete in any neighbourhood of the origin.

Since FX is analytically conjugated to Y, FX is not semi-complete. $\square$

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