TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 360, Number 12, December 2008, Pages 6677–6693
S 0002-9947(08)04522-4
Article electronically published on July 24, 2008

CHARACTERIZATION OF LIL BEHAVIOR IN BANACH SPACE

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Abstract. In a recent paper by the authors a general result characterizing two-sided LIL behavior for real valued random variables has been established. In this paper we look at the corresponding problem in the Banach space setting. We show that there are analogous results in this more general setting. In particular, we provide a necessary and sufficient condition for LIL behavior with respect to sequences of the form $\sqrt{nh(n)}$, where h is from a suitable subclass of the positive, non-decreasing slowly varying functions. To prove these results we have to use a different method. One of our main tools is an improved Fuk-Nagaev type inequality in Banach space which should be of independent interest.

1. Introduction

Let $(B, \|\cdot\|)$ be a real separable Banach space with topological dual B^*. Let $\{X,\ X_n;\ n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) B-valued random variables. As usual, let $S_n = \sum_{i=1}^n X_i,\ n \geq 1$, and set $Lt = \log(t \vee e), LLt = L(Lt), t \geq 0$.

One of the classical results of probability is the Hartman-Wintner LIL, and the definitive version of this result in Banach space has been proven by Ledoux and Talagrand [12].

Theorem A. *A random variable $X : \Omega \to B$ satisfies the bounded LIL; that is,*

$$\limsup_{n\to\infty} \|S_n\|/\sqrt{nLLn} < \infty \ a.s. \tag{1.1}$$

if and only if the following three conditions are fulfilled:

$$\mathbb{E}\|X\|^2/LL\|X\| < \infty,\ \ \mathbb{E}X = 0, \tag{1.2}$$

$$\mathbb{E}f^2(X) < \infty,\ f \in B^*, \tag{1.3}$$

$$\{S_n/\sqrt{nLLn}\} \text{ is bounded in probability.} \tag{1.4}$$

Furthermore it is known that if one assumes instead of (1.4)

$$S_n/\sqrt{nLLn} \xrightarrow{\mathbb{P}} 0, \tag{1.5}$$

Received by the editors October 16, 2006 and, in revised form, April 1, 2007.
2000 *Mathematics Subject Classification.* Primary 60B12, 60F15; Secondary 60G50, 60J15.
Key words and phrases. Law of the iterated logarithm, LIL behavior, Banach space, regularly varying function, sums of i.i.d. random variables, exponential inequalities.
The first author's research was supported in part by an FWO Vlaanderen grant.
The second author's research was supported in part by an NSERC Canada grant.

one has

$$\limsup_{n\to\infty} \|S_n\|/\sqrt{2nLLn} = \sigma \text{ a.s.,} \tag{1.6}$$

where $\sigma^2 = \sup_{f\in B_1^*} \mathbb{E}f^2(X)$ and B_1^* is the unit ball of B^*. It is easy to see that σ^2 is finite under assumption (1.3).

If B is a type 2 space, then (1.2) implies (1.5) and the bounded LIL holds if and only if conditions (1.2) and (1.3) are satisfied. Moreover, in this case we also know the exact value of the limsup in (1.1).

Recall that we call a Banach space a type 2 space if we have for any sequence $\{Y_n\}$ of independent mean zero random variables with $\mathbb{E}\|Y_n\|^2 < \infty, n \geq 1:$

$$\mathbb{E}\|\sum_{i=1}^n Y_i\|^2 \leq C\sum_{i=1}^n \mathbb{E}\|Y_i\|^2, n \geq 2,$$

where $C > 0$ is a constant. It is well known that finite-dimensional spaces and Hilbert spaces are type 2 spaces.

Finding the precise value of $\limsup_{n\to\infty} \|S_n\|/\sqrt{2nLLn}$ in general seems to be a difficult problem (see, for instance, Problem 5 on page 457 in [13]). If one imposes the stronger assumption $\mathbb{E}\|X\|^2 < \infty$ and $\mathbb{E}X = 0$ instead of (1.2) and (1.3), de Acosta, Kuelbs and Ledoux [1] proved that with probability one,

$$\sigma \vee \beta_0 \leq \limsup_{n\to\infty} \|S_n\|/\sqrt{2nLLn} \leq \sigma + \beta_0, \tag{1.7}$$

where $\beta_0 = \limsup_{n\to\infty} \mathbb{E}\|S_n\|/\sqrt{2nLLn}$. Moreover, they showed that the lower bound $\sigma \vee \beta_0$ is sharp for random variables in c_0. It is still open whether this is the case in other Banach spaces as well. If $S_n/\sqrt{nLLn} \xrightarrow{\mathbb{P}} 0$, one has $\beta_0 = 0$, and one can re-obtain result (1.6) if $\mathbb{E}\|X\|^2 < \infty$. Also note that in all other cases one misses the "true" value of lim sup at most by a factor 2. So if $\mathbb{E}\|X\|^2 < \infty$, we have a fairly complete picture, and it is natural to ask whether it is possible to establish (1.7) under conditions (1.2) and (1.3). This has been shown in [1] for certain Banach spaces which satisfy a so-called upper Gaussian comparison principle, but the question of whether this is the case for general Banach spaces still seems to be open. As a by-product of our present work we will be able to answer this in the affirmative.

There are also extensions of the Hartman-Wintner LIL to real-valued random variables with possibly infinite variance. Feller [6] obtained an LIL for certain variables in the domain of attraction to the normal distribution, and this was further generalized by Klass [7, 8]. Kuelbs [10] and Einmahl [3] found versions of these results in the Banach space setting. In a recent paper Einmahl and Li [5] looked at the following problem for real-valued random variables:

Given a sequence $a_n = \sqrt{nh(n)}$, where h is a slowly varying non-decreasing function, when does one have with probability one, $0 < \limsup_{n\to\infty} |S_n|/a_n < \infty$?

Somewhat unexpectedly it turned out that the classical Hartman-Wintner LIL could be generalized to a "law of a very slowly varying function". It is the main purpose of the present paper to investigate whether there are also such results in the Banach space setting. In the process we will derive a very general result on almost sure convergence (see Theorem 4.1) which, specialized to the classical normalizing sequence $\sqrt{2nLLn}$, also gives result (1.7) under the weakest possible conditions.

2. Statement of main results

Let $\mathcal{H}$ be the set of all continuous, non-decreasing functions $h : [0,\infty) \to (0,\infty)$, which are slowly varying at infinity. To simplify notation we set $\Psi(x) = \sqrt{xh(x)}, x \geq 0$, for $h \in \mathcal{H}$ and let $a_n = \Psi(n), n \geq 1$.

Given a random variable $X : \Omega \to B$ we consider an infinite-dimensional truncated second moment function $H : [0,\infty) \to [0,\infty)$ defined by

$$H(t) := \sup_{f \in B_1^*} \mathbb{E} f^2(X) I\{\|X\| \leq t\}, t \geq 0.$$

The first theorem gives a characterization for having $\limsup_{n\to\infty} \|S_n\|/a_n < \infty$ a.s., where a_n is a normalizing sequence of the above form.

Theorem 2.1. *Let X be a B-valued random variable. Then we have*

$$\limsup_{n\to\infty} \frac{\|S_n\|}{a_n} < \infty \ \textit{a.s.} \tag{2.1}$$

if and only if

$$\mathbb{E}X = 0, \ \mathbb{E}\Psi^{-1}(\|X\|) < \infty, \tag{2.2}$$

$$\textit{the sequence } \{S_n/a_n;\ n \geq 1\} \textit{ is bounded in probability,} \tag{2.3}$$

and there exists $c \in [0,\infty)$ such that

$$\sum_{n=1}^{\infty} \frac{1}{n} \exp\left\{-\frac{c^2 h(n)}{2H(a_n)}\right\} < \infty. \tag{2.4}$$

By strengthening condition (2.3) we can find the exact lim sup value in (2.1).

Theorem 2.2. *Assume* (2.2) *holds and* (2.3) *is strengthened to*

$$S_n/a_n \xrightarrow{\mathbb{P}} 0. \tag{2.5}$$

Then

$$\limsup_{n\to\infty} \frac{\|S_n\|}{a_n} = \alpha_0 \ \textit{a.s.}, \tag{2.6}$$

where

$$\alpha_0 = \sup\left\{c \geq 0 : \sum_{n=1}^{\infty} \frac{1}{n} \exp\left(-\frac{c^2 h(n)}{2H(a_n)}\right) = \infty\right\}. \tag{2.7}$$

As in the classical case (when considering the sequence $a_n = \sqrt{2nLLn}$) one can show that in type 2 spaces (2.2) implies (2.5), so that in this case (2.1) holds if and only if conditions (2.2) and (2.4) are satisfied. Moreover, the value of the lim sup in (2.1) is then always equal to α_0.

In general, it can be difficult to determine this parameter. For this reason we now look at normalizing sequences $a_n = \sqrt{nh(n)}$ for functions h from certain subclasses of $\mathcal{H}$. Given $0 \leq q < 1$, let $\mathcal{H}_q \subset \mathcal{H}$ denote the class which contains all functions $h \in \mathcal{H}$ satisfying the condition

$$\lim_{t\to\infty} \frac{h(tf_\tau(t))}{h(t)} = 1, \ 0 < \tau < 1 - q,$$

where $f_\tau(t) = \exp((Lt)^\tau), \ 0 \leq \tau \leq 1$. Finally let $\mathcal{H}_1 = \mathcal{H}$. Clearly $\mathcal{H}_{q_1} \subset \mathcal{H}_{q_2}$ whenever $0 \leq q_1 < q_2 \leq 1$. We call the functions in the smallest subclass $\mathcal{H}_0$ "very slowly varying". From the following theorem it follows that under assumption (2.5) we have $\alpha_0 \leq \lambda$ for *any* $h \in \mathcal{H}$ where λ is a parameter which can be easily determined via the H-function. If we have $h \in \mathcal{H}_q$, then it also follows that $\alpha_0 \geq (1-q)^{1/2}\lambda$. Thus, if $h \in \mathcal{H}_0$, we have $\alpha_0 = \lambda$, and this way we can extend the classical LIL to a "law of a very slowly varying function". Possible choices for very slowly varying functions are for instance $(LLt)^p$, $p \geq 1$, and $(Lt)^r$, $r > 0$.

Theorem 2.3. *Let X be a B-valued random variable. Suppose now that $h \in \mathcal{H}_q$ where $0 \leq q \leq 1$. Assume* (2.2) *and* (2.5) *hold. Then*

$$(1-q)^{1/2}\lambda \leq \limsup_{n\to\infty} \frac{\|S_n\|}{a_n} \leq \lambda \text{ a.s.}, \tag{2.8}$$

where

$$\lambda^2 = \limsup_{x\to\infty} \frac{2\Psi^{-1}(xLLx)}{x^2 LLx} H(x). \tag{2.9}$$

Note the lim sup in condition (2.9). If this lim sup is actually a limit for some $h \in \mathcal{H}$, it follows after a small calculation, using the regular variation of the function Ψ^{-1} and the monotonicity of $H(\cdot)$, that given $\delta > 0$ one has for all n sufficiently large that

$$H(a_n) \geq (1-\delta)^2(\lambda^2/2)h(n)/LLn.$$

Thus Theorem 2.2 implies $\alpha_0 \geq (1-\delta)\lambda$, and hence one has the lim sup in (2.8) then equal to λ, as the upper bound there remains applicable.

The existence of a limit in (2.9), however, is not necessary. It is a special feature of the function class $\mathcal{H}_0$ that under condition (2.5) the lim sup in (2.9) being equal to λ^2 is necessary and also sufficient in combination with (2.2) for having $\limsup_{n\to\infty} \|S_n\|/a_n = \lambda$ a.s. Moreover, we have for $h \in \mathcal{H}_q$ and $0 \leq q < 1$ that $\limsup_{n\to\infty} \|S_n\|/a_n < \infty$ a.s. if and only if $\lambda < \infty$ and conditions (2.2) and (2.3) hold.

Theorem 2.3 gives us analogous corollaries as in the real-valued case. We state two of these. The formulation of the other ones, for instance, a law of the logarithm (see Corollary 2, [5]), should then be obvious.

Corollary 2.1. *Let X be a B-valued random variable. Let $p \geq 1$. Then we have*

$$\limsup_{n\to\infty} \frac{\|S_n\|}{\sqrt{2n(LLn)^p}} < \infty \text{ a.s.} \tag{2.10}$$

if and only if

$$\mathbb{E}X = 0,\ \mathbb{E}\|X\|^2/(LL\|X\|)^p < \infty, \tag{2.11}$$

$$\lambda^2 = \limsup_{x\to\infty} (LLx)^{1-p} H(x) < \infty, \tag{2.12}$$

and

(2.13) *the sequence* $\{S_n/\sqrt{2n(LLn)^p};\ n \geq 1\}$ *is bounded in probability.*

Furthermore,

$$\limsup_{n\to\infty} \frac{\|S_n\|}{\sqrt{2n(LLn)^p}} = \lambda \text{ a.s.} \tag{2.14}$$

whenever condition (2.13) *is strengthened to*

$$S_n/\sqrt{2n(LLn)^p} \xrightarrow{\mathbb{P}} 0. \tag{2.15}$$

If $p = 1$ we re-obtain Theorem A, but the above corollary actually shows that we have for any $p \geq 1$ an LIL. If $\lambda = 0$ in Theorem 2.3, we obtain the following useful stability result.

Corollary 2.2. *Assume that $X : \Omega \to B$ is a random variable satisfying*

$$\mathbb{E}X = 0, \ \mathbb{E}\Psi^{-1}(\|X\|) < \infty, \tag{2.16}$$

$$\lim_{x\to\infty} \frac{\Psi^{-1}(xLLx)}{x^2 LLx} H(x) = 0, \tag{2.17}$$

$$S_n/a_n \xrightarrow{\mathbb{P}} 0. \tag{2.18}$$

Then we have:

$$\lim_{n\to\infty} \frac{S_n}{a_n} = 0 \ a.s. \tag{2.19}$$

Conversely, if $h \in \mathcal{H}_q$ for some $q < 1$, then (2.19) *implies* (2.16)-(2.18).

The remaining part of the paper is organized as follows. In Section 3 we state and prove an infinite-dimensional version of the Fuk-Nagaev inequality improving an earlier version of this inequality given as Theorem 5 in [3]. Using a recent result of Klein and Rio [9], who obtained in some sense an optimal version of the classical Bernstein inequality in infinite-dimensional spaces, we can replace the constant 144 in the exponential term of the earlier version by $2 + \delta$ for any $\delta > 0$. Employing this improved version of the Fuk-Nagaev inequality one can give much more direct proofs for LIL results than in [3]. Especially it is no longer necessary to use randomization arguments and Sudakov minoration for obtaining the precise value of $\limsup_{n\to\infty} \|S_n\|/a_n$. Readers who are mainly interested in inequalities can read this part independently of the other parts of the present paper. In Section 4 we then use the improved Fuk-Nagaev inequality to establish the upper bound part of a general result on almost sure convergence for normalized sums S_n/c_n where $\{c_n; n \geq 1\}$ is a sufficiently regular normalizing sequence. This includes all sequences $a_n = \sqrt{nh(n)}$, where $h \in \mathcal{H}$. For proving the lower bound part we first use an extension of a method employed in the proof of Theorem 2 in [3] to get a first lower bound (see Section 4.2). In the classical case $c_n = \sqrt{2nLLn}$ this bound would be equal to σ. Our method is fairly elementary, and one only needs classical results such as a non-uniform bound on the convergence speed for the CLT on the real line. In Section 4.3 we obtain a second lower bound which, in the classical case, matches β_0 defined in (1.7). Here we use a modification of an argument based on Fatou's lemma which is due to [1]. In Section 5 we finally infer the results stated in Section 2 from our general almost sure convergence result (Theorem 4.1).

3. A Fuk-Nagaev type inequality

As mentioned in Section 2 we use an infinite-dimensional version of the Bernstein inequality which essentially goes back to Talagrand [18]. This inequality turned out to be extremely useful in many applications, but there was a shortcoming that there were no explicit numerical constants. Ledoux [11] found a different and very elegant method for proving such inequalities which is based on a log-Sobolev type argument in combination with a tensorization of the entropy. He was also able to provide concrete numerical constants for these inequalites. His method was subsequently refined by Massart [14] and Rio [17], among other authors. Bousquet [2] obtained optimal constants in the iid case. Finally, Klein and Rio [9] generalized this result to independent, not necessarily identically distributed, random variables. Their results are formulated for empirical processes, but using a standard argument one can easily obtain inequalities for sums of independent B-valued variables from the ones for empirical processes.

We need the following fact which follows from Lemma 3.4 in [9].

Fact A. *Let $Y_1, \ldots, Y_n$ be independent B-valued random variables with mean zero such that*

$$\|Y_i\| \le M \text{ a.s.}, \ 1 \le i \le n.$$

Then we have for $0 < s < 2/(3M)$:

$$\mathbb{E}\exp(s\|\sum_{i=1}^n Y_i\|) \le \exp\left(s\mathbb{E}\|\sum_{i=1}^n Y_i\| + \beta_n s^2/(2-3Ms)\right), \tag{3.1}$$

where $\beta_n = 2M\mathbb{E}\|\sum_{i=1}^n Y_i\| + \Lambda_n$ with $\Lambda_n = \sup\{\sum_{j=1}^n \mathbb{E}f^2(Y_j) : f \in B_1^*\}$ and B_1^* is equal to the unit ball of B^*.

To prove this inequality we set $Z_i = Y_i/M, 1 \le i \le n$. Recall that B is separable so that we have for any $z \in B$, $\|z\| = \sup_{f\in D} f(z)$, where D is a countable subset of B_1^*. Set in Theorem 1.1 of [9], $\mathcal{X} = B$, and consider the following countable class of functions from $\mathcal{X}$ to $[-1,1]^n$:

$$\mathcal{S} = \{(-1 \vee (f \wedge 1), \ldots, -1 \vee (f \wedge 1)) : f \in D\}.$$

Then we readily obtain that $\sup_{s\in\mathcal{S}}\{s^1(Z_1) + \ldots + s^n(Z_n)\} = \|Z_1 + \ldots + Z_n\|$ a.s., and we can infer from the aforementioned lemma that for $0 < t < 2/3$,

$$\mathbb{E}\exp(t\|\sum_{i=1}^n Z_i\|) \le \exp\left(t\mathbb{E}\|\sum_{i=1}^n Z_i\| + \gamma_n t^2/(2-3t)\right),$$

where $\gamma_n = 2\mathbb{E}\|\sum_{i=1} Z_i\| + V_n$ and $V_n = \sup\{\sum_{j=1}^n \mathbb{E}f^2(Z_j) : f \in B_1^*\}$. Replacing Z_i by Y_i/M and setting $s = t/M$, we obtain (3.1).

Using the well known fact that $\exp(s\|\sum_{i=1}^k Y_i\|), 1 \le k \le n$, is a submartingale if $s > 0$ (recall that we assume $\mathbb{E}Y_k = 0, 1 \le k \le n$), we can infer from Doob's maximal inequality for submartingales that for any $x > 0$ and $0 < s < 2/(3M)$,

$$\mathbb{P}\left\{\max_{1\le k\le n}\|\sum_{i=1}^k Y_i\| \ge \mathbb{E}\|\sum_{i=1}^n Y_i\| + x\right\} \le \exp(\beta_n s^2/(2-3Ms) - sx).$$

Choosing $s = 2x/(2\beta_n + 3Mx)$ we finally obtain that

$$\begin{aligned}&\mathbb{P}\left\{\max_{1\le k\le n}\|\sum_{i=1}^k Y_i\| \ge \mathbb{E}\|\sum_{i=1}^n Y_i\| + x\right\}\\ &\le \exp\left(-\frac{x^2}{2\Lambda_n + (4\mathbb{E}\|\sum_{i=1}^n Y_i\| + 3x)M}\right).\end{aligned} \tag{3.2}$$

Next note that we trivially have for any $\epsilon > 0$,

$$\begin{aligned}&\exp\left(-\frac{x^2}{2\Lambda_n + (4\mathbb{E}\|\sum_{i=1}^n Y_i\| + 3x)M}\right)\\ &\le \exp\left(-\frac{x^2}{(2+\epsilon)\Lambda_n}\right) + \exp\left(-\frac{x^2}{(1+2/\epsilon)(4\mathbb{E}\|\sum_{i=1}^n Y_i\| + 3x)M}\right).\end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3) and setting $x = \eta\mathbb{E}\|\sum_{i=1}^n Y_i\| + y$, where $0 < \eta \le 1$ and $y > 0$, we can conclude that for any $y > 0$,

$$\begin{aligned}\mathbb{P}\left\{\max_{1\le k\le n}\|\sum_{i=1}^k Y_i\| \ge (1+\eta)\mathbb{E}\|\sum_{i=1}^n Y_i\| + y\right\} \\ \le \exp\left(-\frac{y^2}{(2+\epsilon)\Lambda_n}\right) + \exp\left(-\frac{y}{D_{\epsilon,\eta}\, M}\right),\end{aligned} \tag{3.4}$$

where $D_{\epsilon,\eta} = (1+2/\epsilon)(3+4/\eta)$.

We are now ready to prove

Theorem 3.1. *Let $Z_1, \dots, Z_n$ be independent B-valued random variables with mean zero such that for some $s > 2$, $\mathbb{E}\|Z_i\|^s < \infty, 1 \le i \le n$.*

Then we have for $0 < \eta \le 1, \delta > 0$ and any $t > 0$,

$$\begin{aligned}\mathbb{P}\left\{\max_{1\le k\le n}\|\sum_{i=1}^k Z_i\| \ge (1+\eta)\mathbb{E}\|\sum_{i=1}^n Z_i\| + t\right\} \\ \le \exp\left(-\frac{t^2}{(2+\delta)\Lambda_n}\right) + C\sum_{i=1}^n \mathbb{E}\|Z_i\|^s/t^s,\end{aligned} \tag{3.5}$$

where $\Lambda_n = \sup\{\sum_{j=1}^n \mathbb{E}f^2(Z_j) : f \in B_1^*\}$ and C is a positive constant depending on η, δ and s.

Proof. To simplify notation we set for $y > 0$

$$\beta(y) = \beta_s(y) = \sum_{i=1}^n \mathbb{E}\|Z_i\|^s/y^s.$$

Assume that $\beta(y) < 1$. For $\epsilon > 0$ fixed we consider the following truncated variables:

$$Y_i := Z_i I\{\|Z_i\| \le \rho\epsilon y\},\ Y_i' = Y_i - \mathbb{E}Y_i, 1 \le i \le n,$$

where

$$\rho = \rho(\epsilon, \eta, y) = 1 \wedge \frac{1}{2\epsilon D_{\epsilon,\eta}\,\log(1/\beta(y))}.$$

Applying inequality (3.4) with $M = 2\rho\epsilon y$ we find that

$$\mathbb{P}\left\{\max_{1\le k\le n}\|\sum_{i=1}^k Y_i'\| \ge (1+\eta)\mathbb{E}\|\sum_{i=1}^n Y_i'\| + y\right\} \le \exp\left(-\frac{y^2}{(2+\epsilon)\Lambda_n}\right) + \beta(y). \tag{3.6}$$

Next consider the variables

$$\Delta_i := Z_i I\{\rho\epsilon y < \|Z_i\| \le \epsilon y\}, 1 \le i \le n.$$

Employing the Hoffmann-Jørgensen inequality (see, for instance, inequality (6.6) in [13]), we can conclude that

$$\mathbb{P}\left\{\max_{1\le k\le n}\|\sum_{i=1}^k \Delta_i\| \ge 4\epsilon y\right\} \le \left(\mathbb{P}\left\{\max_{1\le k\le n}\|\sum_{i=1}^k \Delta_i\| \ge \epsilon y\right\}\right)^2, \tag{3.7}$$

which in turn is

$$\le \left(\sum_{i=1}^n \mathbb{P}\{\Delta_i \ne 0\}\right)^2 \le \left(\sum_{i=1}^n \mathbb{P}\{\|Z_i\| \ge \rho\epsilon y\}\right)^2.$$

Using Markov's inequality and recalling the definition of ρ we see that this last term is bounded above by

$$(2D_{\epsilon,\eta})^{2s}\beta^2(y)(\log(1/\beta(y)))^{2s} \leq K_s(2D_{\epsilon,\eta})^{2s}\beta(y),$$

where $K_s > 0$ is a constant so that $(\log a)^{2s} \leq K_s a, a \geq 1$. We can conclude that

$$\mathbb{P}\left\{\max_{1\leq k\leq n} \|\sum_{i=1}^{k} \Delta_i\| \geq 4\epsilon y\right\} \leq C'\beta(y), \tag{3.8}$$

where $C' = K_s(2D_{\epsilon,\eta})^{2s}$.

Next set $\Delta_i' := Z_i I\{\|Z_i\| > \epsilon y\}, 1 \leq i \leq n$. Then we have once more by Markov's inequality

$$\mathbb{P}\left\{\max_{1\leq k\leq n} \|\sum_{i=1}^{k} \Delta_i'\| \neq 0\right\} \leq \epsilon^{-s}\beta(y). \tag{3.9}$$

Combining inequalities (3.6), (3.8) and (3.9), we see that if $\beta(y) < 1$ we have

$$\begin{aligned} &\mathbb{P}\left\{\max_{1\leq k\leq n} \|\sum_{i=1}^{k}(Z_i - \mathbb{E}Y_i)\| \geq (1+\eta)\mathbb{E}\|\sum_{i=1}^{n} Y_i'\| + (1+4\epsilon)y\right\} \\ \leq\;& \exp\left(-\frac{y^2}{(2+\epsilon)\Lambda_n}\right) + C''\beta(y), \end{aligned}$$

where $C'' = 1 + C' + \epsilon^{-s}$. A simple application of the triangular inequality gives

$$\mathbb{P}\left\{\max_{1\leq k\leq n} \|\sum_{i=1}^{k} Z_i\| \geq b_n' + (1+4\epsilon)y\right\} \leq \exp\left(-\frac{y^2}{(2+\epsilon)\Lambda_n}\right) + C''\beta(y), \tag{3.10}$$

where

$$\begin{aligned} b_n' &= (1+\eta)\mathbb{E}\|\sum_{i=1}^{n} Y_i'\| + \max_{1\leq k\leq n} \|\sum_{i=1}^{k} \mathbb{E}Y_i\| \\ &\leq (1+\eta)\mathbb{E}\|\sum_{i=1}^{n} Y_i\| + 3\max_{1\leq k\leq n} \|\sum_{i=1}^{k} \mathbb{E}Y_i\|. \end{aligned}$$

Further note that

$$\begin{aligned} \mathbb{E}\|\sum_{i=1}^{n} Y_i\| &\leq \mathbb{E}\|\sum_{i=1}^{n} Z_i\| + \mathbb{E}\|\sum_{i=1}^{n} Z_i I\{\|Z_i\| \geq \rho\epsilon y\}\| \\ &\leq \mathbb{E}\|\sum_{i=1}^{n} Z_i\| + \sum_{i=1}^{n} \mathbb{E}\|Z_i\| I\{\|Z_i\| \geq \rho\epsilon y\} =: \mathbb{E}\|\sum_{i=1}^{n} Z_i\| + \delta_n. \end{aligned}$$

As we have $\mathbb{E}Z_i = 0, 1 \leq i \leq n$, it also follows that $\max_{1\leq k\leq n} \|\sum_{i=1}^{k} \mathbb{E}Y_i\| \leq \delta_n$ and consequently,

$$b_n' \leq (1+\eta)\mathbb{E}\|\sum_{i=1}^{n} Z_i\| + 5\delta_n. \tag{3.11}$$

Furthermore, we have,

$$\delta_n \leq y\beta(y)/\{\rho\epsilon\}^{s-1} \leq \epsilon y,$$

provided that $\beta(y) \leq \epsilon^s \rho^{s-1}$.

It is easily checked that if $\rho < 1$, we have $\beta(y)/(\epsilon^s \rho^{s-1}) \le \beta(y)/(\epsilon^s \rho^s) \le (C''\beta(y))^{1/2}$. (We are assuming that $\beta(y) \le 1$.) Consequently, $\delta_n \le \epsilon y$ whenever $C''\beta(y) \le 1$ and $\rho < 1$. This is also true if $\rho = 1$, as we have $C'' \ge \epsilon^{-s}$.

We thus can conclude if $\beta(y) \le 1/C'' < 1$:

$$\mathbb{P}\left\{\max_{1\le k\le n} \|\sum_{i=1}^{k} Z_i\| \ge (1+\eta)\mathbb{E}\|\sum_{i=1}^{n} Z_i\| + (1+9\epsilon)y\right\} \le \exp\left(-\frac{y^2}{(2+\epsilon)\Lambda_n}\right) + C''\beta(y). \tag{3.12}$$

The above inequality is of course trivial if $\beta(y) > 1/C''$, and consequently (3.12) holds for all $y > 0$. Setting $y = t/(1+9\epsilon)$ and choosing ϵ in (3.12) so small that $(2+\epsilon)(1+9\epsilon)^2 \le 2+\delta$, we obtain the assertion. □

4. A general result on almost sure convergence

Let c_n be a sequence of real numbers satisfying the following two conditions:

$$c_n/\sqrt{n} \nearrow \infty \tag{4.1}$$

and

$$\forall\, \epsilon > 0\, \exists\, m_\epsilon \ge 1 : c_n/c_m \le (1+\epsilon)(n/m), m_\epsilon \le m < n. \tag{4.2}$$

Note that condition (4.2) is satisfied for any sequence a_n considered in Section 2.

Let H be defined as in Section 1, that is,

$$H(t) = \sup_{f\in B_1^*} \mathbb{E} f^2(X) I\{\|X\| \le t\}, t \ge 0.$$

Set

$$\alpha_0 = \sup\left\{\alpha \ge 0 : \sum_{n=1}^{\infty} n^{-1} \exp\left(-\frac{\alpha^2 c_n^2}{2nH(c_n)}\right) = \infty\right\}.$$

In general α_0 can be any number in $[0,\infty]$. If we are assuming that $\mathbb{E}f^2(X) < \infty, f \in B^*$ and we choose $c_n = \sqrt{2nLLn}$, it follows that $\alpha_0^2 = \sigma^2 = \sup_{f\in B_1^*} \mathbb{E}f^2(X)$.

Our main result in this section is the following generalization of (1.7).

Theorem 4.1. *Let $X, X_1, X_2, \ldots$ be i.i.d. mean zero random variables taking values in a separable Banach space B. Assume that*

$$\sum_{n=1}^{\infty} \mathbb{P}\{\|X\| \ge c_n\} < \infty, \tag{4.3}$$

where c_n is a sequence of positive real numbers satisfying conditions (4.1) *and* (4.2).

Then we have with probability one,

$$\alpha_0 \vee \beta_0 \le \limsup_{n\to\infty} \|S_n\|/c_n \le \alpha_0 + \beta_0, \tag{4.4}$$

where $\beta_0 = \limsup_{n\to\infty} \mathbb{E}\|S_n\|/c_n$.

The following lemma which is more or less known shows that β_0 is finite whenever $\{S_n/c_n; n \ge 1\}$ is bounded in probability and that $\beta_0 = 0$ if $S_n/c_n \xrightarrow{\mathbb{P}} 0$. So in the latter case we see that the lim sup in (4.4) is equal to α_0.

Lemma 4.1. *Let $X, X_1, X_2, \ldots$ be i.i.d. B-valued random variables with mean zero and let $S_n = \sum_{i=1}^n X_i, n \geq 1$. Let $\{c_n\}$ be a sequence of positive real numbers satisfying conditions* (4.1) *and* (4.2). *Under assumption* (4.3) *we have:*

(a) $\{S_n/c_n; n \geq 1\}$ *is bounded in probability* $\Longleftrightarrow \limsup_{n\to\infty} \mathbb{E}\|S_n\|/c_n < \infty$.

(b) $S_n/c_n \xrightarrow{\mathbb{P}} 0 \Longleftrightarrow \mathbb{E}\|S_n\|/c_n \to 0$.

Proof. We only need to prove the implications "$\Rightarrow$", and by a standard symmetrization argument it is enough to do that for symmetric random variables. We have for any $\epsilon > 0$,

$$\mathbb{E}\|S_n\| \leq \mathbb{E}\left\|\sum_{i=1}^n X_i I\{\|X_i\| \leq \epsilon c_n\}\right\| + n\mathbb{E}\|X\| I\{\|X\| > \epsilon c_n\}. \tag{4.5}$$

The last term is of order $o(c_n)$ under assumption (4.3) (see Lemma 1, [5]). Using the trivial inequality

$$\mathbb{P}\left\{\|\sum_{i=1}^n X_i I\{\|X_i\| \leq \epsilon c_n\}\| \geq x\right\} \leq \mathbb{P}\{\|S_n\| \geq x\} + n\mathbb{P}\{\|X\| \geq \epsilon c_n\},$$

in conjunction with Proposition 6.8 in [13], we find that if $\{S_n/c_n\}$ is bounded in probability, the first term in (4.5) is $\leq C(\epsilon)c_n$, where $C(\epsilon) < \infty$. Consequently, we have in this case $\mathbb{E}\|S_n\|/c_n < \infty$. Assuming $S_n/c_n \xrightarrow{\mathbb{P}} 0$, one can choose $C(\epsilon)$ so that $C(\epsilon) \to 0$ as $\epsilon \to 0$. Since we can make ϵ arbitrarily small, it follows that $\mathbb{E}\|S_n\|/c_n \to 0$ if $S_n/c_n \xrightarrow{\mathbb{P}} 0$. ☐

If B is a type 2 Banach space, assumption (4.3) implies that $\mathbb{E}\|S_n\| = o(c_n)$. (See Lemma 6, [3]. The proof given there also works under the present conditions on $\{c_n\}$.) Therefore we have in any type 2 space $\beta_0 = 0$, and the lim sup in (4.4) is equal to α_0. Recalling that finite-dimensional spaces are type 2 spaces, we see that this result extends Theorem 3 in [5]. Also note that the conditions on $\{c_n\}$ are general enough so that one can infer Theorem 3 in [3] from the present Theorem 4.1 as well (without using randomization and Sudakov minoration).

We now turn to the proof of Theorem 4.1. We assume throughout that condition (4.3) is satisfied. Using essentially the same argument as in Lemma 3 of [4], one can infer from the definition of α_0 that whenever $n_j \nearrow \infty$ is a subsequence satisfying for large enough j,

$$1 < a_1 < n_{j+1}/n_j \leq a_2 < \infty, \tag{4.6}$$

we have

$$\sum_{j=1}^\infty \exp\left(-\frac{\alpha^2 c_{n_j}^2}{2n_j H(c_{n_j})}\right) \begin{cases} = \infty & \text{if } \alpha < \alpha_0, \\ < \infty & \text{if } \alpha > \alpha_0. \end{cases} \tag{4.7}$$

4.1. **The upper bound part.** W.l.o.g. we can and do assume in this part that $\alpha_0 + \beta_0 < \infty$.

We first note that due to (4.1) and assumption (4.3) we have for any subsequence $\{n_j\}$ satisfying (4.6),

$$\sum_{j=1}^\infty n_j \mathbb{E}\|X\|^3 I\{\|X\| \leq c_{n_j}\}/c_{n_j}^3 < \infty. \tag{4.8}$$

(See, for instance, Lemma 7.1, [16].)

Moreover, we have as $n \to \infty$,

$$n\mathbb{E}\|X\|I\{\|X\| \geq c_n\} \leq \sum_{i=1}^{n} \mathbb{E}\|X\|I\{\|X\| \geq c_i\} = o(c_n). \tag{4.9}$$

This last fact follows as in the proof of Lemma 10 in [3] replacing γ_n by c_n.

Set $X'_n = X_n I\{\|X_n\| \leq c_n\}, n \geq 1$, and denote the sum of the first n of these variables by $S'_n, n \geq 1$. We obviously have

$$\sum_{n=1}^{\infty} \mathbb{P}\{X_n \neq X'_n\} < \infty$$

so that with probability one, $X_n = X'_n$ eventually. Due to relation (4.9) we have $\mathbb{E}S'_n = o(c_n)$, and consequently it is enough to show that

$$\limsup_{n\to\infty} \|S'_n - \mathbb{E}S'_n\|/c_n \leq \alpha_0 + \beta_0 \text{ a.s.} \tag{4.10}$$

This follows via Borel-Cantelli once we have proven for any $0 < \delta < 1$

$$\sum_{j=1}^{\infty} \mathbb{P}\left\{\max_{1\leq n\leq n_{j+1}} \|S'_n - \mathbb{E}S'_n\| \geq \{\alpha_0 + \delta + \beta_0(1+\delta)\}(1+2\delta)c_{n_j}\right\} < \infty, \tag{4.11}$$

where $n_j \sim \rho^j$ for a suitable $\rho > 1$.

In order to apply Theorem 3.1 we need an upper bound for $b_n := \mathbb{E}\|S'_n - \mathbb{E}S'_n\|$. Using essentially the same argument as in the proof of Theorem 3.1 we find that this quantity is less than or equal to

$$\mathbb{E}\|S_n\| + 2\sum_{i=1}^{n} \mathbb{E}\|X\|I\{\|X\| \geq c_i\}.$$

On account of fact (4.9) and condition (4.2) we have for large enough j,

$$b_{n_{j+1}} \leq (1+\delta)\beta_0 c_{n_{j+1}} \leq (1+2\delta)\beta_0 c_{n_j},$$

provided we have chosen $\rho < (1+2\delta)/(1+\delta)$.

From Theorem 3.1 (where we set $\eta = \delta$) and the c_r-inequality, it now follows for large j that

$$\begin{aligned}
&\mathbb{P}\left\{\max_{1\leq n\leq n_{j+1}} \|S'_n - \mathbb{E}S'_n\| \geq \{\alpha_0 + \delta + \beta_0(1+\delta)\}(1+2\delta)c_{n_j}\right\} \\
\leq\quad &\exp\left(-\frac{(\alpha_0+\delta)^2(1+2\delta)^2 c_{n_j}^2}{(2+\delta)n_{j+1}H(c_{n_{j+1}})}\right) + \frac{8Cn_{j+1}\mathbb{E}\|X\|^3 I\{\|X\| \leq c_{n_{j+1}}\}}{(\alpha_0+\delta)^3(1+2\delta)^3 c_{n_j}^3} \\
\leq\quad &\exp\left(-\frac{(\alpha_0+\delta)^2 c_{n_{j+1}}^2}{2n_{j+1}H(c_{n_{j+1}})}\right) + \frac{8Cn_{j+1}\mathbb{E}\|X\|^3 I\{\|X\| \leq c_{n_{j+1}}\}}{(\alpha_0+\delta)^3(1+\delta)^3 c_{n_{j+1}}^3}.
\end{aligned}$$

Recalling relations (4.7) and (4.8) it is easy now to see that (4.11) holds and the proof of the upper bound is complete.

4.2. **The first lower bound.** In this part of the proof we assume that $\alpha_0 > 0$ (possibly infinite), and we shall show that we have with probability one

$$\limsup_{n\to\infty} \|S_n\|/c_n \geq \alpha, 0 < \alpha < \alpha_0. \tag{4.12}$$

Given $0 < \alpha < \alpha_0$, we can further assume

$$\limsup_{n\to\infty} \mathbb{P}\{\|S_n\| \geq \alpha c_n\} \leq 1/2. \tag{4.13}$$

Otherwise, we would have

$$\mathbb{P}\{\limsup_{n\to\infty} \|S_n\|/c_n \geq \alpha\} \geq \limsup_{n\to\infty} \mathbb{P}\{\|S_n\| \geq \alpha c_n\} > 1/2,$$

which implies (4.12) via the 0-1 law of Hewitt-Savage.

We first prove that under the assumptions (4.3) and (4.13) we have for any sequence $\{n_j\}$ satisfying condition (4.6),

$$\sum_{j=1}^{\infty} \mathbb{P}\{\|S_{n_j}\| \geq \alpha c_{n_j}\} = \infty. \tag{4.14}$$

To that end we choose for any j a functional $f_j \in B_1^*$ so that

$$\mathbb{E} f_j^2(X) I\{\|X\| \leq c_{n_j}\} \geq (1-\epsilon) H(c_{n_j}),$$

where $0 < \epsilon < 1$ will be specified later on.

Set for $j, k \geq 1$,

$$\begin{aligned} \xi_{j,k} &:= f_j(X_k) I\{\|X_k\| \leq c_{n_j}\}, \\ \xi'_{j,k} &:= \xi_{j,k} - \mathbb{E}\xi_{j,k}. \end{aligned}$$

Then it is easy to see that

$$\mathbb{P}\{\|S_{n_j}\| \geq \alpha c_{n_j}\} \geq \mathbb{P}\left\{\sum_{k=1}^{n_j} \xi_{j,k} \geq \alpha c_{n_j}\right\} - n_j \mathbb{P}\{\|X\| \geq c_{n_j}\}. \tag{4.15}$$

From assumption (4.3) it immediately follows that

$$\sum_{j=1}^{\infty} n_j \mathbb{P}\{\|X\| \geq c_{n_j}\} < \infty.$$

Moreover, we have $|\mathbb{E}\xi_{j,k}| \leq \mathbb{E}\|X\| I\{\|X\| \geq c_{n_j}\}$, which in view of fact (4.9) is of order $o(c_{n_j}/n_j)$. Consequently, in order to prove (4.14) it is enough to show that for a suitable $0 < \epsilon < 1$,

$$\sum_{j=1}^{\infty} \mathbb{P}\left\{\sum_{k=1}^{n_j} \xi'_{j,k} \geq (1+\epsilon)\alpha c_{n_j}\right\} = \infty. \tag{4.16}$$

To estimate these probabilities we employ a non-uniform bound on the rate of convergence in the central limit theorem (see, e.g., Theorem 5.17 on page 168 in [15]). We can conclude that

$$\begin{aligned} &\mathbb{P}\left\{\sum_{k=1}^{n_j} \xi'_{j,k} \geq (1+\epsilon)\alpha c_{n_j}\right\} \\ &\geq \mathbb{P}\{\sigma_j \zeta \geq (1+\epsilon)\alpha c_{n_j}/\sqrt{n_j}\} - A\alpha^{-3}(1+\epsilon)^{-3} n_j \mathbb{E}|\xi'_{j,1}|^3 c_{n_j}^{-3}, \end{aligned} \tag{4.17}$$

where ζ is a standard normal variable, $\sigma_j^2 = \mathrm{Var}(\xi_{j,1})$ and A is an absolute constant.

Noting that $\mathbb{E}|\xi'_{j,1}|^3 \le 8\mathbb{E}|\xi_{j,1}|^3 \le 8\mathbb{E}\|X\|^3 I\{\|X\| \le c_{n_j}\}$, we can infer from fact (4.8) that

$$\sum_{j=1}^{\infty} n_j \mathbb{E}|\xi'_{j,1}|^3 c_{n_j}^{-3} < \infty.$$

Therefore, relation (4.16) and consequently (4.14) follow if we can show that

$$\sum_{j=1}^{\infty} \mathbb{P}\{\sigma_j \zeta \ge (1+\epsilon)\alpha c_{n_j}/\sqrt{n_j}\} = \infty. \tag{4.18}$$

Let $\mathbb{N}_0 = \{j \ge 1 : H(c_{n_j}) \le c_{n_j}^2/n_j^2\}$. Then it is easily checked that for any $\eta > 0$,

$$\sum_{j \in \mathbb{N}_0} \exp\left(-\frac{\eta c_{n_j}^2}{2n_j H(c_{n_j})}\right) < \infty. \tag{4.19}$$

Furthermore, we have for large $j \notin \mathbb{N}_0$,

$$\begin{aligned} \sigma_j^2 &= \mathbb{E}f_j^2(X)I\{\|X\| \le c_{n_j}\} - (\mathbb{E}f_j(X)I\{\|X\| \le c_{n_j}\})^2 \\ &= \mathbb{E}f_j^2(X)I\{\|X\| \le c_{n_j}\} - (\mathbb{E}f_j(X)I\{\|X\| > c_{n_j}\})^2 \\ &\ge (1-\epsilon)H(c_{n_j}) - (\mathbb{E}\|X\|I\{\|X\| \ge c_{n_j}\})^2 \\ &\ge (1-2\epsilon)H(c_{n_j}). \end{aligned}$$

Here we have used that for large j, $\mathbb{E}\|X\|I\{\|X\| \ge c_{n_j}\} \le \sqrt{\epsilon}c_{n_j}/n_j$ (see fact (4.9)).

Employing a standard lower bound for the tail probabilites of normal random variables, we can conclude that for large $j \notin \mathbb{N}_0$,

$$\mathbb{P}\{\sigma_j \zeta \ge (1+\epsilon)\alpha c_{n_j}/\sqrt{n_j}\} \ge \exp\left(-\frac{(1+\epsilon)^2\alpha^2 c_{n_j}^2}{2n_j(1-3\epsilon)H(c_{n_j})}\right).$$

Choosing ϵ so small that $\alpha(1+\epsilon)/\sqrt{1-3\epsilon}) < \alpha_0$ we obtain (4.18) from relations (4.7) and (4.19). This implies relation (4.14).

We are now ready to finish the proof of (4.12) by a standard argument.

Set $m_k = \sum_{j=1}^k n_j, n \ge 1$, where $n_j = [(1+\delta^{-2})^j]$ with $0 < \delta < 1/2$.

Note that we then have $n_{j+1}/n_j \ge \delta^{-2}$ and consequently by (4.1),

$$c_{n_{j+1}} \ge \delta^{-1} c_{n_j}, j \ge 1. \tag{4.20}$$

Likewise it follows that

$$m_k \le n_k \left(\sum_{i=0}^{k-1} \delta^{2i}\right) \le n_k/(1-\delta^2).$$

If k is large enough we can conclude from (4.2) that

$$c_{m_k}/c_{n_k} \le (1+\delta)m_k/n_k \le (1-\delta)^{-1}. \tag{4.21}$$

Define for $k \ge 1$,

$$\begin{aligned} F_k &:= \left\{\|S_{m_k} - S_{m_{k-1}}\| \ge \alpha c_{n_k}\right\}, \\ G_k &:= \left\{\|S_{m_{k-1}}\| \le 2\alpha\delta c_{n_k}\right\}. \end{aligned}$$

Note that due to relations (4.20) and (4.21) we have for large k,

$$\mathbb{P}(G_k) \ge \mathbb{P}\left\{\|S_{m_{k-1}}\| \le 2\alpha c_{n_{k-1}}\right\} \ge \mathbb{P}\left\{\|S_{m_{k-1}}\| \le 2(1-\delta)\alpha c_{m_{k-1}}\right\}.$$

Thus (recall (4.13)) $\mathbb{P}(G_k) \geq 1/2$ for large k. In view of (4.14) we have $\sum_{k=1}^{\infty} \mathbb{P}(F_k) = \infty$. The events F_k and G_k are independent. Thus we can conclude via Lemma 3.4 in [15] that

$$\mathbb{P}(F_k \cap G_k \text{ infinitely often}) = 1.$$

We clearly have

$$F_k \cap G_k \subset \{\|S_{m_k}\| \geq \alpha(1-2\delta)c_{n_k}\},$$

which is due to relation (4.21)

$$\subset \{\|S_{m_k}\| \geq \alpha(1-2\delta)(1-\delta)c_{m_k}\}$$

provided that k is large enough.

It follows that with probability one,

$$\limsup_{k\to\infty} \|S_{m_k}\|/c_{m_k} \geq \alpha(1-2\delta)(1-\delta).$$

Since we can choose δ arbitrarily small, this implies statement (4.12).

4.3. **The second lower bound.** We now assume that $\alpha_0 + \beta_0 < \infty$. If $\alpha_0 = \infty$ the lower bound follows from Section 4.2. Likewise, if $\beta_0 = \infty$ we can obtain it from Lemma 4.1.

We use essentially the same argument as in Theorem 7 of [1]. There is a small complication: we cannot show for all sequences $\{c_n\}$ satisfying the above conditions that $\mathbb{E}[\sup_n \|S_n\|/c_n] < \infty$. Therefore, we prove this first for $S'_n = \sum_{i=1}^n X'_i, n \geq 1$, where the random variables X'_n are defined as in Section 4.1, that is,

$$X'_n = X_n I\{\|X_n\| \leq c_n\}, n \geq 1.$$

From the upper bound part (see Section 4.1) it follows that we have with probability one,

$$\limsup_{n\to\infty} \|S'_n\|/c_n \leq \alpha_0 + \beta_0 < \infty. \tag{4.22}$$

Since $\sup_n \|X'_n\|/c_n \leq 1$ we obtain from Corollary 6.12 in [13] that

$$\mathbb{E}\left[\sup_n \|S'_n\|/c_n\right] < \infty. \tag{4.23}$$

Using the fact that with probability one, $\limsup_{n\to\infty} \|S_n\|/c_n$ is constant, Fatou's lemma implies that with probability one,

$$\limsup_{n\to\infty} \|S_n\|/c_n = \limsup_{n\to\infty} \|S'_n\|/c_n \geq \limsup_{n\to\infty} \mathbb{E}\|S'_n\|/c_n. \tag{4.24}$$

In view of (4.9) we have as $n \to \infty$

$$|\mathbb{E}\|S'_n\| - \mathbb{E}\|S_n\|| \leq \sum_{i=1}^n \mathbb{E}\|X\| I\{\|X\| \geq c_i\} = o(c_n),$$

and we find that with probability one,

$$\limsup_{n\to\infty} \|S_n\|/c_n \geq \beta_0.$$

This completes the proof of Theorem 4.1.

5. Proof of Theorem 2.3

We only prove Theorem 2.3. The proofs of the corollaries are exactly as in the real-valued case, and they are omitted. Also Theorems 2.1 and 2.2 follow directly from Theorem 4.1. As for Theorem 2.3, we first note that in view of Lemma 4.1(b) we have $\beta_0 = 0$ in Theorem 4.1. Therefore we only need to show

$$\alpha_0 \leq \lambda \tag{5.1}$$

and

$$\alpha_0 \geq (1-q)^{1/2}\lambda, \ h \in \mathcal{H}_q. \tag{5.2}$$

By regular variation of the function Ψ^{-1} we can infer from (2.9) that

$$\limsup_{n\to\infty} (LLn)H(a_n/LLn)/h(n) = \lambda^2/2.$$

The proof of (5.2) then goes exactly as in the real-valued case (see the proof of (4.13) on page 1619, [5]), and thus it can be omitted as well. In order to prove the corresponding upper bound in the real-valued case, we used another result, namely Theorem 4 in our previous paper [5]. It is possible to extend this result to Banach space valued random variables, but there is also a more direct argument for deriving the upper bound (5.1) which we shall give below.

Proof of (5.1). If $\lambda = \infty$ the upper bound is trivial. Thus we can assume that $\lambda \in [0, \infty)$. We have to show for any $\alpha > \lambda$,

$$\sum_{n=1}^{\infty} n^{-1} \exp\left(-\frac{\alpha^2 h(n)}{2H(a_n)}\right) < \infty. \tag{5.3}$$

Set $\delta = (\alpha - \lambda)/3$. Then we clearly have for large enough n,

$$H(a_n/LLn) \leq \frac{(\lambda+\delta)^2}{2}\frac{h(n)}{LLn}.$$

Setting $\mathbb{N}_0 = \{n : H(a_n) - H(a_n/LLn) \leq \delta\lambda h(n)/LLn\}$ we get for large $n \in \mathbb{N}_0$,

$$H(a_n) \leq \frac{(\lambda+2\delta)^2}{2}\frac{h(n)}{LLn},$$

and consequently

$$\sum_{n\in\mathbb{N}_0} n^{-1} \exp\left(-\frac{\alpha^2 h(n)}{2H(a_n)}\right) < \infty. \tag{5.4}$$

Further note that we trivially have

$$\sum_{n=1}^{\infty} \frac{H(a_n) - H(a_n/LLn)}{a_n^2 LLn} \leq \sum_{n=1}^{\infty} \frac{\mathbb{E}\|X\|^3 I\{\|X\| \leq a_n\}}{a_n^3} < \infty.$$

The latter series is finite because we are assuming $\mathbb{E}\Psi^{-1}(\|X\|) < \infty$. (See, for instance, Lemma 5(a), [3].) It follows that

$$\sum_{n\notin\mathbb{N}_0} \frac{1}{n(LLn)^2} < \infty. \tag{5.5}$$

Condition (2.9) implies that for large enough n, and $0 < \epsilon < 1$,

$$H(a_n) \le (\lambda^2+1)\frac{a_n^2 LLa_n}{\Psi^{-1}(a_n LLa_n)} \le C_\epsilon \frac{a_n^2 LLa_n}{\Psi^{-1}(a_n)(LLa_n)^{2-\epsilon}} \le C'_\epsilon \frac{h(n)}{(LLn)^{1-\epsilon}},$$

where we have used the fact that Ψ^{-1} is regularly varying at infinity with index 2. (C_ϵ and C'_ϵ are positive constants.)

We can now infer from (5.5) that

$$\sum_{n \notin \mathbb{N}_0} n^{-1} \exp\left(-\frac{\alpha^2 h(n)}{2H(a_n)}\right) \le \sum_{n \notin \mathbb{N}_0} n^{-1} \exp(-\alpha^2 (2C'_\epsilon)^{-1} (LLn)^{1-\epsilon}) < \infty. \tag{5.6}$$

Combining (5.4) and (5.6) we see that the series in (5.3) is finite, and our proof of (5.1) is complete. □

Acknowledgements

The authors would like to thank J. Kuelbs and D. Mason for their support and interest in this work. Thanks are also due to to the referee who made a number of useful suggestions in helping us improve the presentation.

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