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# ON THE ASYMPTOTIC LINEARIZATION OF ACOUSTIC WAVES

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ABSTRACT. The initial value problem of a certain generalization of the nonlinear, dispersive wave equations with dissipation is rigorously studied. The solutions of the equations can be found exactly up to $O(\epsilon^2)$ in certain norms. The essential use is made of the fact that this equation is asymptotically linearizable to $O(\epsilon^2)$, i.e., the equations can be mapped to an equation which differs from a linearizable equation only in terms which are of $O(\epsilon^2)$. An application of the equations to unidirectional small amplitude acoustic waves is discussed. The general methodology used here can also be applied to other asymptotically linearizable equations.

## 1. INTRODUCTION

There exist nonlinear evolution PDEs which are *asymptotically integrable* to $O(\epsilon^2)$, i.e., they can be mapped to equations which differ from integrable equations only in terms which are of $O(\epsilon^2)$. The first example of such a map was given in [8]. An extension of the transformation used in [8] was introduced in [6], where it was also shown that the concept of *mastersymmetries* [5] provides an algorithmic approach for finding such transformations. Examples of other physically significant asymptotically integrable equations can be found in [3] and [4].

For asymptotically integrable PDEs, a natural question is if one can utilize their relation with integrable PDEs in order to obtain the long time behavior of the solution. In this paper we introduce a general formalism for answering this question, in the case that the associated integrable equation can be linearized explicitly. For concreteness, we study the following initial value problem:

(1.1)
$$u_t = u_{xx} + 2uu_x + \varepsilon \left[\alpha_1 u_{xxx} + \alpha_2 u u_{xx} + \alpha_3 u_x^2 + \left(\alpha_3 + \frac{\alpha_2}{2} - \frac{3\alpha_1}{2}\right) u^2 u_x\right],$$

(1.2)
$$u(x, 0, \epsilon) = u_0(x) + \varepsilon u_1(x),$$

where $u(x,t)$ is a real valued function of $x \in \mathbb{R},\ t \in [0, \infty)$, and $\alpha_1$, $\alpha_2$, and $\alpha_3$ are given arbitrary real constants.

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Equation (1.1) is of the form of an equation arising from a certain asymptotic expansion of the acoustic waves: Let $\rho$ denote the perturbation of the density from the equilibrium value $\rho_0$. Then $\rho$ satisfies (see [10])

$$\rho_t = \rho_{xx} + 2\rho\rho_x + \varepsilon\Big[\frac{1}{2}\rho_{xxx} + \frac{\gamma}{\gamma+1}(\rho_x^2 + \rho\rho_{xx}) + \frac{3(\gamma+\delta)}{(\gamma+1)^2}\rho^2\rho_x\Big] + 0(\varepsilon^2), \tag{1.3}$$

where

$$\gamma = \frac{P_0''\rho_0}{2a_0^2}, \qquad \delta = \frac{P_0'''\rho_0^2}{6a_0^2}, \qquad \varepsilon = -\frac{\mu}{2\rho_0 a_0 l},$$

$P_0 = P(\rho_0)$ is the equilibrium pressure, $a_0$ is the speed of sound, $l$ is a typical length, prime denotes differentiation, $\mu$ is the viscosity, and $O(\varepsilon^2)$ is the higher order terms. The equation (1.3) can be written in the form of equation (1.1) provided that $\delta = \frac{\gamma^2-4\gamma-1}{4}$.

The rigorous investigation of this initial value problem (1.1)-(1.2) has as its starting point the following formal result: Let $v(x,t)$ solve the equation

$$v_t = v_{xx} + 2vv_x + \alpha_1\varepsilon[v_{xxx} + 3vv_{xx} + 3v^2v_x + 3v_x^2]. \tag{1.4}$$

Let $\tilde{u}(x,t)$ be defined in terms of $v(x,t)$ by the explicit transformation

$$\begin{aligned} \tilde{u}(x,t) &= v(x,t) + \varepsilon P(v(x,t)), \\ P(v(x,t)) &\doteq \tfrac{3\alpha_1-\alpha_2}{2} v_x(x,t)\partial^{-1}v(x,t) + \tfrac{3\alpha_1-\alpha_3}{2} v^2(x,t), \end{aligned} \tag{1.5}$$

where $\partial^{-1}$ denotes integration with respect to $x$. Then $\tilde{u}$ solves (1.1) up to O($\epsilon^2$) terms, i.e.,

$$\begin{aligned} \tilde{u}_t &= \tilde{u}_{xx} + 2\tilde{u}\tilde{u}_x \\ &+\varepsilon\left[\alpha_1\tilde{u}_{xxx} + \alpha_2\tilde{u}\tilde{u}_{xx} + \alpha_3\tilde{u}_x^2 + \left(\alpha_3 + \tfrac{\alpha_2}{2} - \tfrac{3\alpha_1}{2}\right)\tilde{u}^2\tilde{u}_x\right] + \epsilon^2 f(x,t), \end{aligned} \tag{1.6}$$

where $f$ is a known function. Equation (1.4), which is a generalization of Burgers equation, can be linearized explicitly. Indeed, under the transformation $v = \phi_x/\phi$, equation (1.3) becomes the linear equation

$$\phi_t = \phi_{xx} + \alpha_1\varepsilon\phi_{xxx}. \tag{1.7}$$

In this paper, we will prove that the initial-value problem (1.1)-(1.2) is well posed for some $\epsilon > 0$. The $L_2$-norm and the $L_\infty$-norm of the solution $u(x,t)$ in $x$ are uniformly bounded for all $t > 0$ and decay to zero as $t \to +\infty$ in algebraic rates. Moreover, equation (1.1) is *asymptotically integrable to* $O(\epsilon^2)$, i.e., the solutions of the initial-value problem (1.1)-(1.2) are approximated up to $\epsilon^2$ by the suitable solutions of the linear equation (1.7) in $L_2$-norm and in $L_\infty$-norm in $x$ for all $t > 0$.

It is worth noting that an asymptotically integrable equation can be mapped to one of several integrable equations. For example, equation (1.1) can also be mapped to the Burgers equation

$$v_t = v_{xx} + 2vv_x. \tag{1.8}$$

Indeed, let $v(x,t)$ solve (1.8) and let $u(x,t)$ be defined in terms of $v(x,t)$ by the explicit transformation

$$u(x,t) = v(x,t) + \varepsilon[\tfrac{2\alpha_1-\alpha_3}{2}v^2(x,t) + \tfrac{3\alpha_1-\alpha_2}{2}v_x(x,t)\partial^{-1}v(x,t) - \tfrac{\alpha_1}{2}xv_t(x,t)]. \tag{1.9}$$

Then $u$ solves equation (1.1) up to $\epsilon^2$ terms [9].

The rigorous study of equation (1.1) using the transformation (1.9) instead of the transformation (1.4) was presented in [7]. It was shown in [7] that the initial-value problem (1.1) and (1.2) can be found through the solution of equation (1.8) approximately up to $O(\varepsilon^2)$ in the $L_\infty$-norm. However, for this result the initial data $u_0$ and $u_1$ have to be in a weighted $H^m(\mathbb{R})$ space, while the results obtained here are valid for a less restrictive function space.

The paper is organized as follows: In section 2, we will derive the formal result expressed by equations (1.4)-(1.6), and we will present a summary of the rigorous results. In section 3, the global well posedness of the initial value problem (1.1) and (1.2) will be derived. The derivation of the main theorem is presented in section 4.

*Notation.* The space $C^k(\mathbb{R})$, $k = 0, 1, 2, \cdots$ , denotes the space of functions defined on $\mathbb{R}$ whose first $k$ derivatives are bounded and continuous. The $L_q$-norm of a function $f$ whose $q$th-power is absolutely integrable on $\mathbb{R}$ will be denoted by $|f|_q$, for $1 \le q < \infty$. The $L_\infty$-norm of a function $f$ will be denoted by $|f|_\infty$. Let $m$ be a nonnegative integer; $W^{m,\,q}(\mathbb{R})$ will denote the Sobolev space consisting of those $L_q(\mathbb{R})$-functions whose first $m$ generalized derivatives lie in $L_q(\mathbb{R})$, with the usual norm

$$||f||_{W^{m,\,q}(\mathbb{R})} = \sum_{k=0}^{m} |f^{(k)}|_q.$$

The case $q = 2$ will be denoted by the special notation $H^m(\mathbb{R})$. The norm $H^m(\mathbb{R})$ will be denoted by $||f||_m$. The norm $||f||_{W^{m,q}(\mathbb{R})} + ||f||_{H^m(\mathbb{R})}$ will be denoted by the abbreviation $||f||_{W^{m,q}(\mathbb{R})\cap H^m(\mathbb{R})}$.

Let $X$ be a Banach space, $T$ be a positive real number and $1 \le p \le +\infty$. Then $L_p(0, T; X)$ denotes the Banach space of all measurable functions $u\colon (0, T) \to X$, such that $t \to ||u(t)||_X$ is in $L_p(0, T)$. Similarly, by $C(0, T; X)$, we denote the subspace of $L_\infty(0, T; X)$ of all continuous functions $u : [0, T] \to X$. The abbreviation $B_T^{k,l}$ will be employed for the functions: $\mathbb{R}\times\ [0, T] \to \mathbb{R}$ such that $\partial_x^i \partial_t^j u \in C(0, T; C(\mathbb{R}))$ for $0 \le i \le k$, and $0 \le j \le l$. This Banach space will be equipped with the norm

$$||u||_{B_T^{k,l}} = \sum_{0\le i\le k, 0\le j\le l} ||\partial_x^i \partial_t^j u||_{C(0,T;C(\mathbb{R}))}.$$

The space $B_T^{0,0}$ will be abbreviated by $B_T$; its norm is that of $L_\infty(\mathbb{R}\times[0, T])$. If $T = +\infty$, $L_p(\mathbb{R}^+)$, $L_p(\mathbb{R}^+; X)$ and $C(\mathbb{R}^+; X)$ will denote $L_p(0, \infty)$, $L_p(0, \infty; X)$ and $C(0, \infty; X)$, respectively. The function $\hat{f}$ will denote the Fourier transform of $f$, i.e.,

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)dx.$$

Finally, for convenience, in equation (1.1) $\alpha_3 + \frac{\alpha_2}{2} - \frac{3\alpha_1}{2}$ will be denoted by $\alpha_4$.

## 2. Formal transformations and main results

The formal result stated in the Introduction is a particular case of the following more general formal result [6], [9].

**Proposition 2.1.** *Let $\mathcal{H}(v)$ denote the ring consisting of smooth functions of $v(x,t)$ and of its $x$-derivatives. Let $\mathcal{H}_x(v)$ denote the ring consisting of the above functions and those obtained by the action of $\partial^{-1}$ and by multiplication by $x$.*

*Let $v(x,t)$ solve*

$$v_t = K(v) + \epsilon M(v), \qquad K(v),\ M(v) \in \mathcal{H}(v). \tag{2.1}$$

*Let $u(x,t)$ be defined in terms of $v(x,t)$ by*

$$u = v + \epsilon P(x,v), \quad P(x,v) \in \mathcal{H}_x(v). \tag{2.2}$$

*Then $u(x,t)$ solves*

$$u_t = K(u) + \epsilon\Big(M(u) + [P(x,u),\, K(u)]_{_L}\Big) + \epsilon^2 f(x,t), \tag{2.3}$$

*where $[\ ,\ ]_{_L}$ denote the Lie commutator*

$$[K_1(x,u),\, K_2(x,u)]_{_L} = K'_1 K_2 - K'_2 K_1. \tag{2.4}$$

*The function $f(x,t)$ has the following explicit form:*

$$\begin{aligned} f(x,t) = {} & [P(x,u),\, M(u)]_{_L} + [K(u),\, F(u)]_{_L} \\ & + P'(x,u)([P(x,u),\, K(u)]_{_L}) + K''(u)(-P(x,u)), \end{aligned} \tag{2.5}$$

*where $F(u) = P'(x,u)P(x,u)$ and primes denote Fréchet differentiation*

$$K'(x,u)Q \doteq \frac{\partial}{\partial \delta} K(x, u + \delta Q)\Big|_{\delta=0},$$

*or more explicitly*

$$K' = \frac{\partial K}{\partial u} + \frac{\partial K}{\partial u_x}\partial + \frac{\partial K}{\partial u_{xx}}\partial^2 + \cdots, \qquad \partial \doteq \partial_x.$$

*Similarly, $K''$ is denote by the $O(\delta^2)$ term,*

$$K(x, u+\delta Q) = K(x,u) + \delta K'(x,u)Q + \delta^2 K''(x,u)Q + O(\delta^3). \tag{2.6}$$ □

Equation (2.1) becomes equation (1.4) if we let

$$K(v) = v_{xx} + 2vv_x \quad \text{and} \quad M(v) = \alpha[v_{xxx} + 3(vv_{xx} + v^2 v_x + v_x^2)]. \tag{2.7}$$

Let

$$P(x,v) = P(v) = \beta v_x \partial^{-1} v + \frac{\gamma}{2} v^2. \tag{2.8}$$

By applying Proposition 2.1, a simple calculation shows that $u = v + \varepsilon P(v)$ solves

$$\begin{aligned} & u_t = u_{xx} + 2uu_x \\ & + \varepsilon[\alpha u_{xxx} + (3\alpha - 2\beta)uu_{xx} + (3\alpha-\gamma)u_x^2 + (3\alpha - \beta - \gamma)u^2 u_x] + \varepsilon^2 f(x,t). \end{aligned} \tag{2.9}$$

Renaming

$$\alpha = \alpha_1,\ \ 3\alpha - 2\beta = \alpha_2,\ \ 3\alpha - \gamma = \alpha_3,$$

equation (2.9) becomes equation (1.6) up to $\epsilon^2$ terms.

Straightforward, but tedious calculations yield the following:

$$F(u) = \beta^2 u_{xx}(\partial^{-1}u)^2 + 2\beta(\beta+\gamma)uu_x\partial^{-1}u + \beta(\frac{\gamma}{2}-\beta)u_x\partial^{-1}u^2 + \frac{\gamma^2}{2}u^3, \tag{2.10}$$

$$\begin{aligned}
[P(x,u), M(u)]_L &= -3\alpha\beta uu_{xxx} - 3\alpha(\beta+\gamma)\Big(u_xu_{xx} + 2uu_x^2\Big) \\
&\quad - \alpha(\tfrac{3\gamma}{2} + 6\beta)u^2u_{xx} - \alpha(2\beta+3\gamma)u^3u_x, \\
[K(u),\, F(u)]_L &= \tfrac{3\gamma^2+7\beta\gamma+4\beta^2}{3}u^3u_x + (5\beta\gamma+4\beta^2)u^2u_{xx} \\
&\quad + (3\gamma^2+5\beta\gamma+2\beta^2)uu_x^2 \\
&\quad - \beta(\gamma-2\beta)u_x\partial^{-1}(uu_{xx}) - 4\beta^2u_xu_{xx}(\partial^{-1}u)^2 \\
&\quad + \Big(4\beta^2uu_{xxx} + 2\beta^2u^2u_{xx} + 4\beta(\beta+\gamma)u_xu_{xx}\Big)\partial^{-1}u,
\end{aligned}$$

$$\begin{aligned}
P'(x,u)[P(x,u),K(u)]_L &= -\tfrac{3\gamma^2+4\beta\gamma+\beta^2}{3}u^3u_x - (2\beta^2+\gamma^2)uu_x^2 \\
&\quad - 2\beta\gamma u^2u_{xx} - \beta(\gamma-2\beta)u_x\partial^{-1}u_x^2 \\
&\quad - \Big(2\beta(\beta+\gamma)uu_x^2 + 2\beta^2uu_{xxx}\Big)\partial^{-1}u \\
&\quad - \Big(\beta(\beta+\gamma)u^2u_{xx} + 2\beta(\beta+\gamma)u_xu_{xx}\Big)\partial^{-1}u, \\
K''(u)(-P(x,u)) &= 2P(x,u)P(x,u)_x \\
&= 2\beta^2u_xu_{xx}(\partial^{-1}u)^2 + \gamma(\beta+\gamma)u^3u_x \\
&\quad + \Big(2\beta(\beta+\gamma)uu_x^2 + \beta\gamma u^2u_{xx}\Big)\partial^{-1}u.
\end{aligned} \tag{2.11}$$

After substituting the representation (2.11) into (2.5), we obtain the following representation for $f$:

$$\begin{aligned}
f(x,t) &= -3\alpha\beta uu_{xxx} - 3\alpha(\beta+\gamma)u_xu_{xx} + 2(\beta+\gamma)(\beta+\gamma-3\alpha)uu_x^2 \\
&+\Big(\beta(3\gamma+4\beta) - \alpha(\tfrac{3\gamma}{2}+6\beta)\Big)u^2u_{xx} + \Big((\beta+\gamma)^2 - \alpha(2\beta+3\gamma)\Big)u^3u_x \\
&-2\beta^2u_xu_{xx}(\partial^{-1}u)^2 + \Big(2\beta^2uu_{xxx} + \beta^2u^2u_{xx} + 2\beta(\beta+\gamma)u_xu_{xx}\Big)\partial^{-1}u. \quad \square
\end{aligned} \tag{2.12}$$

The following theorem will be derived.

**Theorem.** *Let $u_0(x)$ and $u_1(x)$ be in $H^2(\mathbb{R})\cap L_1(\mathbb{R})$. Then there exists a unique solution $u(x,t)$ of the initial-value problem (1.1)-(1.2). Moreover, the solution $u(x,t)$ has the asymptotic property*

$$|u(\cdot,t)|_\infty \le C_1(1+t)^{-\frac{1}{2}} \quad \textit{and} \quad |u(\cdot,t)|_2 \le C_2(1+t)^{-\frac{1}{4}} \ \text{ for all } \ t\ge 0,$$

*where the constants $C_1$ and $C_2$ depend on $||u_0||_{H^2(\mathbb{R})\cap L_1(\mathbb{R})}$ and on $||u_1||_{H^2(\mathbb{R})\cap L_1(\mathbb{R})}$.*

*Let $u_0(x)\in H^4(\mathbb{R})\cap L_1(\mathbb{R})$ and $u_1(x)\in H^3(\mathbb{R})\cap L_1(\mathbb{R})$. Define $\tilde v(x,0,\varepsilon)$ by*

$$\tilde v(x,0,\varepsilon) = u_0 + \epsilon\left[u_1 - P(u_0)\right],$$

*where $P(u)$ is defined in (1.5). Let $\tilde v(x,t,\varepsilon)$ be the solution of equation (1.4) with the initial condition $\tilde v(x,0,\varepsilon)$. Define $\tilde u(x,t,\varepsilon)$ by*

$$\tilde u(x,t,\varepsilon) = \tilde v + \varepsilon P(\tilde v).$$

*Then the solution $u(x,t,\varepsilon)$ of equation (1.1) with the initial data (1.2), satisfies*

$$|(u-\tilde u)(\cdot,t,\varepsilon)|_\infty \le C_3\varepsilon^2 \quad \textit{and} \quad |(u-\tilde u)(\cdot,t,\varepsilon)|_2 \le C_4\varepsilon^2,$$

*where the constants $C_3$ and $C_4$ depend on $||u_0||_{H^4(\mathbb{R})\cap L_1(\mathbb{R})}$ and on $||u_1||_{H^3(\mathbb{R})\cap L_1(\mathbb{R})}$.*

## 3. Some preliminary results

First, we will show that for some suitable small $\epsilon > 0$, which depends on the initial data (1.2), the initial value problem (1.1)-(1.2) is globally well-posed.

**Lemma 3.1.** *Define $\lambda_1(x,t)$ as the function whose Fourier transform is*

$$\hat{\lambda_1}(y,t) = e^{-y^2t - i\epsilon\alpha_1 y^3 t}.$$

*Then,*

$$\sup\nolimits_{t\ge 0}\int_{-\infty}^{\infty}|\lambda_1(x,t)|dx < \infty. \tag{3.1}$$

*If $\hat{\lambda_2}$ is given by*

$$\hat\lambda_2(y,t) = iye^{-y^2t - i\epsilon\alpha_1 y^3 t},$$

*then*

$$\sup\nolimits_{t\ge 0}\left\{t^{\frac{1}{2}}\int_{-\infty}^{\infty}|\lambda_2(x,t)|dx\right\} < \infty. \tag{3.2}$$

*Moreover, if $\hat{\lambda_3}$ is given by*

$$\hat\lambda_3(y,t) = y^2e^{-y^2t - i\epsilon\alpha_1 y^3 t},$$

*then*[1]

$$\sup\nolimits_{t\ge 0}\left\{\epsilon t^{\frac{1}{2}}\int_{-\infty}^{\infty}|\lambda_3(x,t)|dx\right\} < \infty. \tag{3.3}$$

*Proof.* (3.1) and (3.2) have been derived in [1] (see also [2]). In order to prove (3.3), we note that from the definition of $\lambda_3$, we have

$$\lambda_3(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y^2 e^{-y^2t - i\epsilon\alpha_1 y^3 t}e^{ixy}dy. \tag{3.4}$$

[1]Using a better decay estimate, it can be shown that for $t>0, t^{\frac{3}{2}}\int_{-\infty}^{\infty}|\lambda_3(x,t)|dx < \infty$.

By substituting $y = yt^{\frac{1}{3}}$ in (3.4) and then integrating by parts, we obtain

$$\begin{aligned}\lambda_3(x,t) &= \tfrac{1}{\sqrt{2\pi}t}\int_{-\infty}^{\infty} y^2 e^{-y^2 t^{\frac{1}{3}} - i\epsilon\alpha_1 y^3} e^{\frac{ixy}{t^{1/3}}}\, dy \\ &= \tfrac{1}{\sqrt{2\pi}t}\int_{-\infty}^{\infty} \tfrac{1}{-3i\epsilon\alpha_1} e^{\frac{ixy}{t^{1/3}} - y^2 t^{\frac{1}{3}}}\, d\big(e^{-i\epsilon\alpha_1 y^3}\big) \qquad (3.5)\\ &= \tfrac{1}{3i\sqrt{2\pi}\epsilon\alpha_1 t}\int_{-\infty}^{\infty} \big[\tfrac{ix}{t^{1/3}} - 2yt^{\frac{1}{3}}\big] e^{\frac{ixy}{t^{1/3}} - y^2 t^{\frac{1}{3}} - i\epsilon\alpha_1 y^3}\, dy.\end{aligned}$$

Hence,

$$|\lambda_3(x,t)|_1 = \tfrac{1}{3\sqrt{2\pi}\epsilon\cdot\alpha_1 t}\int_{-\infty}^{\infty}\Big|\int_{-\infty}^{\infty}\big[\tfrac{ix}{t^{\frac{1}{3}}} - 2yt^{\frac{1}{3}}\big] e^{\frac{ixy}{t^{1/3}} - y^2 t^{\frac{1}{3}} - i\epsilon\alpha_1 y^3}\, dy\Big| dx. \qquad (3.6)$$

By replacing $x/t^{\frac{1}{3}}$ by $x$ in (3.6), we find

$$\begin{aligned}|\lambda_3(x,t)|_1 &= \tfrac{1}{3\sqrt{2\pi}\epsilon\alpha_1 t^{\frac{2}{3}}}\int_{-\infty}^{\infty}\Big|\int_{-\infty}^{\infty}\big[ix - 2yt^{\frac{1}{3}}\big] e^{ixy - y^2 t^{\frac{1}{3}} - i\epsilon\alpha_1 y^3}\, dy\Big| dx \\ &\le \tfrac{C}{\epsilon\cdot t^{\frac{2}{3}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big|ix - 2yt^{\frac{1}{3}}\big|\big|e^{ixy - y^2 t^{\frac{1}{3}}}\big|\, dydx \qquad (3.7)\\ &= \tfrac{C}{\epsilon\cdot t^{\frac{2}{3}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big|ix - 2yt^{\frac{1}{3}}\big| e^{\left(-\frac{x^2}{4t^{\frac{1}{3}}} - [y - ix/(2t^{\frac{1}{3}})]^2 t^{\frac{1}{3}}\right)}\, dydx.\end{aligned}$$

Replacing in (3.7) $y$ by $y - ix/(2t^{\frac{1}{3}})$ yields

(3.8)

$$\begin{aligned}|\lambda_3(x,t)|_1 &\le \tfrac{C}{\epsilon\cdot t^{\frac{2}{3}}}\int_{-\infty}^{\infty} e^{\left(-\frac{x^2}{4t^{\frac{1}{3}}}\right)}\int_{-\infty}^{\infty}\big|ix - 2(y - ix/(2t^{\frac{1}{3}}))t^{\frac{1}{3}}\big| e^{\left(-y^2 t^{\frac{1}{3}}\right)}\, dydx \\ &\le \tfrac{C}{\epsilon\cdot t^{\frac{2}{3}}}\Big[\int_{-\infty}^{\infty}|x|\exp\big(-\tfrac{x^2}{4t^{\frac{1}{3}}}\big)dx\int_{-\infty}^{\infty}\exp\big(-y^2 t^{\frac{1}{3}}\big)dy \\ &\qquad + \int_{-\infty}^{\infty}\exp\big(-\tfrac{x^2}{4t^{\frac{1}{3}}}\big)dx\int_{-\infty}^{\infty}|y| t^{\frac{1}{3}}\exp\big(-y^2 t^{\frac{1}{3}}\big)dy\Big] \\ &\le \tfrac{C}{\epsilon\cdot t^{\frac{2}{3}}}\Big[Ct^{\frac{1}{3}}\cdot Ct^{-\frac{1}{6}} + Ct^{\frac{1}{6}}\cdot C\Big] = \tfrac{C}{\epsilon}t^{-\frac{1}{2}}.\end{aligned}$$

□

**Theorem 3.2.** *Let $u_0(x)$ and $u_1(x)$ be in $H^k(\mathbb{R})$ for $k \ge 1$. Then for $T$ sufficiently small there exists a unique solution of equation (1.1) with the initial data (1.2). Furthermore, this solution is continuous from $H^k(\mathbb{R})$ into $C(0,T;H^k(\mathbb{R}))$.*

*Proof.* We consider the nonlinear terms on the right-hand side of equation (1.1) as an external force. Formally, taking the Fourier transform of equation (1.1) with respect to the spatial variable $x$, we find

$$\hat{u}(y,t) = e^{-y^2 t - i\epsilon\alpha_1 y^3 t}\hat{u}(y,0) + \int_0^t e^{-(y^2 + i\epsilon\alpha_1 y^3)(t-\tau)}\hat{F}(y,\tau)d\tau, \qquad (3.9)$$

where $\hat{F}(y,\tau)$ is the Fourier transform of the terms

$$\begin{aligned}&2uu_x + \epsilon(\alpha_2 uu_{xx} + \alpha_3 u_x^2 + \alpha_4 u^2 u_x) \\ &= 2uu_x + \epsilon\Big\{\alpha_2\big(uu_x\big)_x + (\alpha_3 - \alpha_2)u_x^2 + \alpha_4 u^2 u_x\Big\}.\end{aligned}$$

Hence,

$$(3.10)\quad \begin{aligned} u(x,t) &= \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \lambda_1(x-y,t)(u_0(y)+\epsilon u_1(y))dy \\ &+ \tfrac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} \Big[\lambda_1(x-y,t-\tau)\Big(2uu_x+\epsilon\{(\alpha_3-\alpha_2)u_x^2+\alpha_4u^2u_x\}\Big) \\ &\qquad -\epsilon\alpha_2\lambda_2(x-y,t-\tau)uu_x\Big]dyd\tau, \end{aligned}$$

where $\lambda_1(x,t)$ and $\lambda_2(x,t)$ are defined in Lemma 3.1 and $\alpha_4 = \alpha_3 + \frac{\alpha_2}{2} - \frac{3\alpha_1}{2}$.

Define the operator $\mathbb{A}$ by

$$(3.11)\quad \begin{aligned} (\mathbb{A}u)(x,t) &= \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \lambda_1(x-y,t)(u_0(y)+\epsilon u_1(y))dy \\ &+ \tfrac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} \Big[\lambda_1(x-y,t-\tau)\Big(2uu_x+\epsilon\{(\alpha_3-\alpha_2)u_x^2+\alpha_4u^2u_x\}\Big) \\ &\qquad -\epsilon\alpha_2\lambda_2(x-y,t-\tau)uu_x\Big]dyd\tau. \end{aligned}$$

Assuming that $u_0$ and $u_1$ lie in $C^1(\mathbb{R})$, it follows that the operator $\mathbb{A}$ maps a function $u \in B_T^{1,0}$ into itself. In fact, from (3.10) one deduces that

$$(3.12)\quad \begin{aligned} (\mathbb{A}u_x)(x,t) &= \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \lambda_2(x-y,t)(u_0(y)+\epsilon u_1(y))dy \\ &+ \tfrac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} \Big[\lambda_2(x-y,t-\tau)\Big(2uu_x+\epsilon\{(\alpha_3-\alpha_2)u_x^2+\alpha_4u^2u_x\}\Big) \\ &\qquad +\epsilon\alpha_2\lambda_3(x-y,t-\tau)uu_x\Big]dyd\tau, \end{aligned}$$

where $\lambda_3(x,t)$ is defined in Lemma 3.1.

If $T$ is chosen small enough, $\mathbb{A}$ is a contraction mapping of a ball centered at the origin in $B_T^{1,0}$ into itself. Indeed, for $v$, $w \in B_T^{1,0}$ with $||v||_{B_T^{1,0}} \le R$, $||w||_{B_T^{1,0}} \le R$, applying (3.1) and (3.2) in (3.11), and (3.2) and (3.3) in (3.12) shows that

$$(3.13)\quad ||\mathbb{A}v - \mathbb{A}w||_{B_T^{1,0}} \le C(R)\sqrt{T}||v-w||_{B_T^{1,0}},$$

where the constant $C(R)$ depends on the norms of $\lambda_1$, $\lambda_2$ and $\lambda_3$ and $0 \le t \le T$. Hence the theorem is proved. □

Now we will show that for any given initial data $u_0$ and $u_1$, we can choose a small $\epsilon = \epsilon(u_0,u_1) > 0$, such that the corresponding solution of the equation (1.1) exists globally.

**Corollary 3.3.** *Let $u_0(x)$ and $u_1(x)$ be in $H^k(\mathbb{R})$ for $k \ge 1$. Then for some suitable small $\epsilon_0$, the initial value problem (1.1)-(1.2) is globally well-posed. The solution is continuous from $H^k(\mathbb{R})$ into $C(0,T;H^k(\mathbb{R}))$. Furthermore, $u_x$ and $u_{xx} \in L_2(\mathbb{R}\times\mathbb{R}^+)$. In addition,*

$$(3.14)\quad |u_x(\cdot,t)|_2 \to 0, \quad |u_{xx}(\cdot,t)|_2 \to 0, \quad \textit{and} \ \ |u(\cdot,t)|_\infty \to 0, \ \ \textit{as} \ t \to +\infty.$$

*Proof.* By Theorem 3.2, for some suitable initial data $u_0$ and $u_1$, there exists a local solution of the initial value problem (1.1)-(1.2). Suppose that there exists an increasing sequence $T_N$, $N = 1, 2, 3, \cdots$, such that

$$|u(\cdot, T_N)|_\infty > N. \tag{3.15}$$

Let $\epsilon_N$ be chosen such that

$$\mathbf{L} \cdot \epsilon_N \cdot |u(\cdot, T_N)|_\infty \leq \frac{1}{2}, \tag{3.16}$$

where $\mathbf{L} = (12|\alpha_1| + 32|\alpha_2| + 8|\alpha_3| + |\alpha_4|)$ is a constant. We multiply equation (1.1) by the combination $2u - 2u_{xx} + 8u^3$, and integrate the resulting equation over $\mathbb{R} \times [0, t]$ for $t \leq T_N$ and for all $0 < \varepsilon \leq \epsilon_N$. After integrations by parts, we find

$$\begin{aligned}
&|u(\cdot,t)|_2^2 + |u_x(\cdot,t)|_2^2 + 2|u(\cdot,t)|_4^4 \\
&\quad + 2\int_0^t [|u_x(\cdot,\tau)|_2^2 + |u_{xx}(\cdot,\tau)|_2^2 + 12|u(\cdot,\tau)u_x(\cdot,\tau)|_2^2] d\tau \\
&= ||u_0(\cdot) + \varepsilon u_1(\cdot)||_1^2 + 2|(u_0)(\cdot) + \varepsilon(u_1)(\cdot)|_4^4 - \int_0^t \int_{-\infty}^{\infty} 4uu_xu_{xx} dx d\tau \\
&\quad + 2\varepsilon \int_0^t \int_{-\infty}^{\infty} \Big((\alpha_3 - 2\alpha_2)uu_x^2 - \alpha_2 uu_{xx}^2 \\
&\qquad - (\alpha_4 + 12\alpha_1)u^2u_xu_{xx} + (4\alpha_3 - 16\alpha_2)u^3u_x^2\Big) dx d\tau.
\end{aligned} \tag{3.17}$$

Using the Cauchy-Schwarz inequality, we have

$$\int_{-\infty}^{\infty} 4|u(x,t)u_x(x,t)||u_{xx}(x,t)| dx \leq 16|u(\cdot,t)u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2$$

and

$$\int_{-\infty}^{\infty} 2\varepsilon|u(x,t)^2u_x(x,t)||u_{xx}(x,t)| dx \leq \varepsilon|u(\cdot,t)|_\infty \Big(|u(\cdot,t)u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2\Big).$$

Using these inequalities in (3.17), we find

$$\begin{aligned}
&|u(\cdot,t)|_2^2 + |u_x(\cdot,t)|_2^2 + 2|u(\cdot,t)|_4^4 \\
&\quad + \int_0^t \Big(2|u_x(\cdot,\tau)|_2^2 + |u_{xx}(\cdot,\tau)|_2^2 + 8|u(\cdot,\tau)u_x(\cdot,\tau)|_2^2\Big) d\tau \\
&\leq 8(||u_0||_1^2 + \varepsilon||u_1||_1^2) \\
&\quad + \varepsilon\mathbf{L} \int_0^t |u(\cdot,\tau)|_\infty \Big(|u_x(\cdot,\tau)|_2^2 + |u(\cdot,\tau)u_x(\cdot,\tau)|_2^2 + |u_{xx}(\cdot,\tau)|_2^2\Big) d\tau,
\end{aligned}$$

where $\mathbf{L} = (12\alpha_1 + 32|\alpha_2| + 8|\alpha_3| + |\alpha_4|)$. Because of the hypothesis (3.16), the above yields

$$\begin{aligned}
&|u(\cdot,t)|_2^2 + |u_x(\cdot,t)|_2^2 + 2|u(\cdot,t)|_4^4 \\
&\quad + \int_0^t \Big(|u_x(\cdot,\tau)|_2^2 + 7|u(\cdot,\tau)u_x(\cdot,\tau)|_2^2 + \tfrac{1}{2}|u_{xx}(\cdot,\tau)|_2^2\Big) d\tau \\
&\leq 8(||u_0||_1^2 + \epsilon_N||u_1||_1^2),
\end{aligned} \tag{3.18}$$

for $t \leq T_N$. On the other hand, by an elementary inequality and the hypothesis

(3.15) on $|u(\cdot,t)|_\infty$, we have

$$N^2 \leq |u(\cdot,t)|_\infty^2 \leq 2|u(\cdot,t)|_2|u_x(\cdot,t)|_2$$
$$\leq |u(\cdot,t)|_2^2 + |u_x(\cdot,t)|_2^2 \leq 8(||u_0||_1 + \epsilon_N||u_1||_1),$$

which is a contradiction.

Note that there is no finite limit point for the sequence $\{T_N\}_0^\infty$. Otherwise, Lemma 3.1 and the boundedness of $L_\infty$ would lead again to a contradiction. Hence, for given initial data $u_0$ and $u_1$, there exists $\epsilon_0 > 0$ and a constant $C$, which only depend on the initial data $u_0$ and $u_1$ such that

$$|u(\cdot,t)|_\infty \leq C,$$

for any $t > 0$. Therefore, for some suitable small $\epsilon_0 > 0$, the solutions of the initial value problem (1.1)-(1.2) exist globally.

From the estimate (3.18), it follows that $u_x$ and $u_{xx} \in L_2(\mathbb{R}\times\mathbb{R}^+)$. Since $|u_x(\cdot,t)|_2^2$ and $|u_{xx}(\cdot,t)|_2^2$ lie in $L_1(\mathbb{R}^+)$, and their limits as $t\to\infty$ exist, the limits must be zero. Using

$$|u(\cdot,t)|_\infty^2 \leq 2|u(\cdot,t)|_2|u_x(\cdot,t)|_2 \leq C|u_x(\cdot,t)|_2,$$

as well as the fact that the right-hand side of this inequality tends to zero as $t\to+\infty$, the result follows. □

For convenience, we will simply write $\epsilon_0$ as $\epsilon$ in the rest our statements.

**Lemma 3.4.** *Let $u_0(x)$ and $u_1(x)$ be in $H^k(\mathbb{R})$ for $k \geq 1$. If $u$ is the solution of (1.1) corresponding to initial data $u_0(x)$ and $u_1(x)$, then*

$$\lim_{t\to+\infty} t^{\frac{1}{2}}\left[|u_x(\cdot,t)|_2^2 + |u(\cdot,t)|_4^4\right] = 0 \tag{3.19}$$

*and*

$$\lim_{t\to+\infty} t^{\frac{1}{2}} \int_t^{+\infty}\left[|u_{xx}(\cdot,\tau)|_2^2 + |u(\cdot,\tau)u_x(\cdot,\tau)|_2^2\right]d\tau = 0. \tag{3.20}$$

*Proof.* We multiply (1.1) by $-2u_{xx}$ and integrate the resulting equation over $\mathbb{R}$. After integration by parts, we find

$$\frac{d}{dt}|u_x(\cdot,t)|_2^2+2|u_{xx}(\cdot,t)|_2^2=\int_{-\infty}^{\infty}\left[-4uu_xu_{xx}-2\epsilon\alpha_2uu_{xx}^2-2\epsilon\alpha_4u^2u_xu_{xx}\right]dx. \tag{3.21}$$

Similarly, we multiply (1.1) by $8u^3$ and integrate the resulting equation over $\mathbb{R}$. After integration by parts, we find

$$\begin{aligned}&2\frac{d}{dt}|u(\cdot,t)|_4^4+24|u(\cdot,t)u_x(\cdot,t)|_2^2\\&= \epsilon\int_{-\infty}^{\infty}\left[-24\alpha_1u^2u_xu_{xx}+(8\alpha_3-32\alpha_2)u^3u_x^2\right]dx.\end{aligned} \tag{3.22}$$

Adding (3.21) and (3.22), and then using Young's inequality, we derive the following inequality:

$$\begin{aligned} &\tfrac{d}{dt}\Big\{|u_x(\cdot,t)|_2^2+2|u(\cdot,t)|_4^4\Big\}+24|u(\cdot,t)u_x(\cdot,t)|_2^2+2|u_{xx}(\cdot,t)|_2^2\\ &=\textstyle\int_{-\infty}^{\infty}\Big[-4uu_xu_{xx}-2\epsilon\alpha_2uu_{xx}^2\\ &\qquad -2\epsilon(\alpha_4+12\alpha_1)u^2u_xu_{xx}+\epsilon(8\alpha_3-32\alpha_2)u^3u_x^2\Big]dx \qquad (3.23)\\ &\le\Big[\tfrac{3}{2}+2\epsilon|\alpha_2||u(\cdot,t)|_\infty\Big]|u_{xx}(\cdot,t)|_2^2\\ &+\Big[16+\Big\{8\epsilon^2(\alpha_4+12\alpha_1)^2+\epsilon(8|\alpha_3|+32|\alpha_2|)\Big\}|u(\cdot,t)|_\infty\Big]|u(\cdot,t)u_x(\cdot,t)|_2^2.\end{aligned}$$

By Corollary 3.3, there exists a positive value $T$ such that for $t\ge T$,

$$2\epsilon|\alpha_2||u(\cdot,t)|_\infty\le\frac{1}{4}\quad\text{and}\quad\Big\{8\epsilon^2(\alpha_4+12\alpha_1)^2+\epsilon(8|\alpha_3|+32|\alpha_2|)\Big\}|u(\cdot,t)|_\infty\le 2.$$

For $t\ge T$, we obtain

$$\tfrac{d(\Gamma(t))}{dt}+6|u(\cdot,t)u_x(\cdot,t)|_2^2+\tfrac{1}{4}|u_{xx}(\cdot,t)|_2^2\le 0, \tag{3.24}$$

where

$$\Gamma(t)=|u_x(\cdot,t)|_2^2+2|u(\cdot,t)|_4^4.$$

Since $|u(\cdot,t)|_2$ is bounded, $u_x$ and $u_{xx}\in L_2(\mathbb{R}\times\mathbb{R}^+)$, and

$$|u(\cdot,t)|_4^4\le|u(\cdot,t)|_\infty^2|u(\cdot,t)|_2^2\le|u_x(\cdot,t)|_2|u(\cdot,t)|_2^3,$$

it follows that $\Gamma(t)\in L_2(\mathbb{R}^+)$. Because of (3.24), it follows that

$$\textstyle\int_\tau^{+\infty}\Gamma^2(s)ds\ge\int_\tau^t\Gamma^2(s)ds\ge(t-\tau)\Gamma^2(t),$$

for $t>\tau$ and $\tau\ge T$. Hence,

$$\textstyle\int_\tau^{+\infty}\Gamma^2(s)ds\ge\limsup_{t\to+\infty}\ t\,\Gamma^2(t),$$

for $\tau$ large enough. Since $\Gamma(t)\in L_2(\mathbb{R}^+)$, the left-hand side of the above inequality can be made as small as desired by choosing $\tau$ large enough, and thus we obtain the desired result

$$\lim_{t\to+\infty}t^{\frac{1}{2}}\big(|u_x(\cdot,t)|_2^2+|u(\cdot,t)|_4^4\big)=0. \tag{3.25}$$

In order to obtain (3.20), we choose $T$ sufficiently large so that for $t\ge T$ the inequality (3.24) yields

$$\frac{d\Gamma(t)}{dt}+6|u(\cdot,t)u_x(\cdot,t)|_2^2+\frac{1}{4}|u_{xx}(\cdot,t)|_2^2\le 0. \tag{3.26}$$

Integrating this relation over the temporal interval $[t,+\infty)$ for $t\ge T$, it follows that

$$-\Gamma(t)+6\int_t^{+\infty}|u(\cdot,\tau)u_x(\cdot,\tau)|_2^2d\tau+\frac{1}{4}\int_t^{+\infty}|u_{xx}(\cdot,\tau)|_2^2d\tau\le 0,$$

and hence using equation (3.25) we find

$$\int_t^{+\infty} |u(\cdot,\tau)u_x(\cdot,\tau)|_2^2 d\tau = o(t^{-\frac{1}{2}}) \qquad \text{as } t \to +\infty,$$

and

$$\int_t^{+\infty} |u_{xx}(\cdot,\tau)|_2^2 d\tau = o(t^{-\frac{1}{2}}) \qquad \text{as } t \to +\infty.$$

The proof of the lemma is thus complete. □

**Lemma 3.5.** *Let $u_0$ and $u_1$ lie in $L_1(\mathbb{R}) \cap H^1(\mathbb{R})$. Let $u$ be the solution of equation (1.1) with the initial condition (1.2). Then, there exists a constant $C$, which depends on $u_0$ and $u_1$, such that*

$$|u(\cdot,t)|_1 \le C \quad \textit{and} \quad t^{\frac{1}{4}}|u(\cdot,t)|_2 \le C, \qquad \textit{for all } t > 0.$$

*Moreover,*

$$|u_x(\cdot,t)|_2 = o(t^{-\frac{1}{2}}), \quad \textit{as } t \to +\infty.$$

The proof of the Lemma 3.5 depends on several lemmas which will be presented shortly. Let $u$ be the solution (1.1) with the initial condition (1.2). It will be assumed throughout the remainder of this section that the hypotheses of Lemma 3.5 hold.

Let $U = \int_{-\infty}^x u(s,t)ds$ and $U = log(v)$. Then, $v$ satisfies the following nonhomogeneous diffusion equation:

$$v_t - v_{xx} = \epsilon v\Big[\alpha_1 u_{xx} + \alpha_2 u u_x + (\alpha_3 - \alpha_2)\int_{-\infty}^x u_s^2(s,t)ds + \tfrac{\alpha_4}{3}u^3\Big]. \tag{3.27}$$

Note that, for any $t \ge 0, v(x,t) = \exp(U(x,t)) = \exp(\int_{-\infty}^x u(s,t)ds)$, and thus

$$\lim_{x\to-\infty} v(x,t) = 1,$$

$$\lim_{x\to+\infty} v(x,t) = \exp\Big(\int_{-\infty}^{+\infty} u(s,t)ds\Big) \tag{3.28}$$

$$= \exp\Big(\int_{-\infty}^{+\infty} u(s,0)ds + \epsilon(\alpha_3-\alpha_2)\int_0^t\int_{-\infty}^{+\infty} u_s^2(s,\tau)dsd\tau\Big).$$

Since $|u_x(\cdot,t)|_2^2 \in L_1(\mathbb{R}^+)$, and $U$ is uniformly bounded on $\mathbb{R}\times[0,T]$ for any $T > 0$, it follows that

$$0 < \inf_{\substack{0\le t\le T,\\ x\in\mathbb{R}}} v(x,t) \le \sup_{\substack{0\le t\le T,\\ x\in\mathbb{R}}} v(x,t) < \infty. \tag{3.29}$$ □

Next, we will show that the inequalities (3.29) hold uniformly in $T$. In this respect the following lemma is useful.

**Lemma 3.6.** *Let $v(x,t)$ satisfy equation (3.27). Then*

$$0 < \inf_{x\in\mathbb{R},\ t\ge 0} v(x,t) \le \sup_{x\in\mathbb{R},\ t\ge 0} v(x,t) < \infty. \tag{3.30}$$

*Proof.* We will first prove the second inequality in (3.30), i.e.,

$$\sup_{x\in\mathbb{R},\ t\ge 0} v(x,t) < \infty.$$

If $0 \le T \le t$, the solution $v$ of (3.27) satisfies the integral equation

$$v(x,t) = \int_{-\infty}^{\infty} v(s,T)G(x-s,t-T)ds + \epsilon \int_T^t \int_{-\infty}^{\infty} G(x-s,t-\tau)\, v \Big[\alpha_1 u_{ss} + \alpha_2 u u_s + (\alpha_3-\alpha_2)\int_{-\infty}^{s} u_r^2(r,\tau)dr + \tfrac{\alpha_4}{3}u^3\Big] ds d\tau = S(x,t) + R(x,t), \tag{3.31}$$

where

$$G(x,t) = \tfrac{1}{2\sqrt{\pi t}} \exp\big(-x^2/4t\big), \qquad S(x,t) = \int_{-\infty}^{\infty} v(s,T)G(x-s,t-T)ds,$$

and

$$R(x,t) = \epsilon \int_T^t \int_{-\infty}^{\infty} G(x-s,t-\tau)\, v \Big[\alpha_1 u_{ss} + \alpha_2 u u_s + (\alpha_3-\alpha_2)\int_{-\infty}^{s} u_r^2(r,\tau)dr + \tfrac{\alpha_4}{3}u^3\Big] ds d\tau.$$

The proof of the second inequality consists of three parts, (a), (b), and (c). Part (a) involves bounding the term $\int_{-\infty}^{s} u_r^2(r,\tau)dr$ in $\mathbb{R}\times[T+\infty)$, while parts (b) and (c) involve bounding the term $\alpha_1 u_{ss} + \alpha_2 u u_s$ and $u^3$ over $\mathbb{R}\times[T+\infty)$, respectively.

**(a)** First note that

$$|G(\cdot,t-\tau)|_1 = 1.$$

Also, because of Lemma 3.4, $|u_x(\cdot,t)|_2^2 \in L_1(\mathbb{R}^+)$. Hence, the third term in the double integral of equation (3.31) can be bounded from above by

$$\begin{aligned} R_1(x,t) &= \epsilon \int_T^t \int_{-\infty}^{\infty} \Big| G(x-s,t-\tau)v(s,\tau)\int_{-\infty}^{s} u_r^2(r,\tau)dr \Big| ds d\tau \\ &\le C \sup_{T\le\tau\le t} |v(\cdot,\tau)|_\infty \int_T^t |G(\cdot,t-\tau)|_1 |u_x(\cdot,\tau)|_2^2 d\tau \\ &\le C \sup_{T\le\tau\le t} |v(\cdot,\tau)|_\infty \int_T^t |u_x(\cdot,\tau)|_2^2 d\tau = C \sup_{T\le\tau\le t} |v(\cdot,\tau)|_\infty K_1(t,T). \end{aligned}$$

Since $|u_x(\cdot,\tau)|_2^2$ is in $L_1(\mathbb{R}^+)$,

$$\lim_{T\to+\infty} \sup_{t\ge T} K_1(t,T) = 0.$$

Hence, for $T$ sufficiently large,

$$R_1(x,t) \le C K_1(t,T)\cdot \sup_{T\le\tau\le t} |v(\cdot,\tau)|_\infty \le \tfrac{1}{10} \sup_{T\le\tau\le t} |v(\cdot,\tau)|_\infty. \tag{3.32}$$

**(b)** For $T>0$ such that (3.32) is valid, the first two terms in the double integral of equation (3.31) are bounded from above by

$$\begin{aligned} R_2(x,t) &= \epsilon \int_T^t \int_{-\infty}^{\infty} \Big| G(x-s,t-\tau)v\big[\alpha_1 u_{ss} + \alpha_2 u u_s\big] \Big| ds d\tau \\ &\le C \sup_{T\le\tau\le t} |v(\cdot,\tau)|_\infty \int_T^t \int_{-\infty}^{+\infty} |G(x,t-\tau)| \Big( |u_{xx}(x,\tau)| + |u(x,\tau)u_x(x,\tau)| \Big) dx d\tau, \end{aligned}$$

where $C = max\{|\alpha_1|, |\alpha_2|\}$.Using the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} &\int_T^t \int_{-\infty}^{+\infty} |G(x,t-\tau)|\Big(|u_{xx}(x,\tau)| + |u(x,\tau)u_x(x,\tau)|\Big)dxd\tau \\ \le\quad &\int_T^t |G(\cdot,t-\tau)|_2\Big[|u_{xx}(\cdot,\tau)|_2 + |u(\cdot,\tau)u_x(\cdot,\tau)|_2\Big]d\tau. \end{aligned} \tag{3.33}$$

Using

$$|G(\cdot,t-\tau)|_2 \le C(t-\tau)^{-1/4},$$

for some constant $C$, it follows that

$$\begin{aligned} &\int_T^t |G(\cdot,t-\tau)|_2\Big[|u_{xx}(\cdot,\tau)|_2 + |u(\cdot,\tau)u_x(\cdot,\tau)|_2\Big]d\tau \\ &\le C\int_T^t \tfrac{|u_{xx}(\cdot,\tau)|_2+|u(\cdot,\tau)u_x(\cdot,\tau)|_2}{(t-\tau)^{1/4}}d\tau. \end{aligned} \tag{3.34}$$

Let $K_2$ denote

$$K_2(t,T) = \int_T^t \frac{|u_{xx}(\cdot,\tau)|_2 + |u(\cdot,\tau)u_x(\cdot,\tau)|_2}{(t-\tau)^{1/4}}d\tau.$$

For $t \in [T, 2T]$, using the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} K_2(t,T) &= \Big\{\int_T^t \tfrac{1}{(t-\tau)^{1/2}}d\tau\Big\}^{\frac12}\Big\{\int_T^t 2\Big[|u_{xx}(\cdot,\tau)|_2^2+|u(\cdot,\tau)u_x(\cdot,\tau)|_2^2\Big]d\tau\Big\}^{\frac12} \\ &\le \sqrt{2}(t-T)^{1/4}\Big\{2\int_T^{+\infty}\Big[|u_{xx}(\cdot,\tau)|_2^2 + |u(\cdot,\tau)u_x(\cdot,\tau)|_2^2\Big]d\tau\Big\}^{\frac12} \\ &\le \Big\{4T^{1/2}\int_T^{+\infty}\Big[|u_{xx}(\cdot,\tau)|_2^2 + |u(\cdot,\tau)u_x(\cdot,\tau)|_2^2\Big]d\tau\Big\}^{\frac12}, \end{aligned} \tag{3.35}$$

and the last term in (3.35) tends to zero as $T \to +\infty$, because of Lemma 3.4.

For $t > 2T$ and $\tau \in [t/2, t]$, using the inequality in (3.35) we find

$$\begin{aligned} K_2(t,T) &= \int_T^{t/2} + \int_{t/2}^t \tfrac{1}{(t-\tau)^{1/4}}\Big[|u_{xx}(\cdot,\tau)|_2 + |u(\cdot,\tau)u_x(\cdot,\tau)|_2\Big]d\tau \\ &\le \int_T^{t/2} \tfrac{1}{(t-\tau)^{1/4}}\Big[|u_{xx}(\cdot,\tau)|_2 + |u(\cdot,\tau)u_x(\cdot,\tau)|_2\Big]d\tau \\ &\quad + \Big\{4t^{1/2}\int_{t/2}^{+\infty}\Big[|u_{xx}(\cdot,\tau)|_2^2 + |u(\cdot,\tau)u_x(\cdot,\tau)|_2^2\Big]d\tau\Big\}^{\frac12}. \end{aligned} \tag{3.36}$$

Since $t > T$, the second term on the right-hand side of the last inequality tends to zero as $T \to +\infty$. The other term in (3.36) can be bounded from above by

$$\begin{aligned} &\int_T^{t/2} \tfrac{1}{(t-\tau)^{1/4}}\Big[|u_{xx}(\cdot,\tau)|_2 + |u(\cdot,\tau)u_x(\cdot,\tau)|_2\Big]d\tau \\ &\le \Big\{\int_T^{t/2} \tfrac{1}{(t-\tau)^{1/4}\tau^{3/4}}d\tau\Big\}^{\frac12}\Big\{\int_T^{t/2} \tau^{\frac34}\Big[|u_{xx}(\cdot,\tau)|_2+|u(\cdot,\tau)u_x(\cdot,\tau)|_2\Big]d\tau\Big\}^{\frac12} \\ &\le \Big\{8t^{-1/4}\int_T^t \tau^{3/4}\Big[|u_{xx}(\cdot,\tau)|_2^2 + |u(\cdot,\tau)u_x(\cdot,\tau)|_2^2\Big]d\tau\Big\}^{\frac12}. \end{aligned} \tag{3.37}$$

The inequality (3.24), derived in Lemma 3.4, yields

$$|u_{xx}(\cdot,\tau)|_2^2 + |u(\cdot,\tau)u_x(\cdot,\tau)|_2^2 \le -\frac{d}{dt}\Big[4|u_x(\cdot,\tau)|_2^2 + 8|u(\cdot,\tau)|_4^4\Big]. \tag{3.38}$$

Using (3.38), it follows that

$$
\begin{aligned}
&\int_T^t \tau^{3/4}\Big[|u_{xx}(\cdot,\tau)|_2^2+|u(\cdot,\tau)u_x(\cdot,\tau)|_2^2\Big]d\tau\\
&\le -\int_T^t \tau^{3/4}\tfrac{d}{d\tau}\Big[4|u_x(\cdot,\tau)|_2^2+8|u(\cdot,\tau)|_4^4\Big]d\tau\\
&\le T^{3/4}\Big[4|u_x(\cdot,T)|_2^2+8|u(\cdot,T)|_4^4\Big]+\tfrac{3}{4}\int_T^t \tfrac{1}{\tau^{\frac14}}\Big[4|u_x(\cdot,\tau)|_2^2+8|u(\cdot,\tau)|_4^4\Big]d\tau\\
&\le T^{3/4}\Big[4|u_x(\cdot,T)|_2^2+8|u(\cdot,T)|_4^4\Big]\\
&\qquad+\tfrac{3}{4}\Big\{\int_T^t \tfrac{d\tau}{\tau^{1/2}}\Big\}^{1/2}\Big\{C\int_T^t\Big[|u_x(\cdot,\tau)|_2^4+|u(\cdot,\tau)|_4^8\Big]d\tau\Big\}^{1/2}\\
&\le T^{3/4}\Big[4|u_x(\cdot,T)|_2^2+8|u(\cdot,T)|_4^4\Big]+Ct^{1/4}\Big\{\int_T^\infty|u_x(\cdot,\tau)|_2^2d\tau\Big\}^{1/2},
\end{aligned}
\tag{3.39}
$$

for some positive constant $C$. In the last step in (3.39), we have used the elementary inequality

$$
\begin{aligned}
|u(\cdot,T)|_4^8 &= \Big(\int_{-\infty}^{+\infty}u^4(x,t)dx\Big)^2 \le |u(\cdot,\tau)|_\infty^4|u(\cdot,\tau)|_2^4\\
&\le 4|u_x(\cdot,\tau)|_2^2|u(\cdot,\tau)|_2^6 \le C|u_x(\cdot,\tau)|_2^2.
\end{aligned}
$$

Using (3.39) in (3.37), we find that for $t>2T$,

$$
\begin{aligned}
&\int_T^{t/2}\tfrac{1}{(t-\tau)^{1/4}}\Big[|u_{xx}(\cdot,\tau)|_2+|u(\cdot,\tau)u_x(\cdot,\tau)|_2\Big]d\tau\\
&\le\Big\{CT^{1/2}\Big[|u_x(\cdot,T)|_2^2+|u(\cdot,T)|_4^4\Big]+C\Big\{\int_T^\infty|u_x(\cdot,\tau)|_2^2d\tau\Big\}^{\frac12}\Big\}^{\frac12}.
\end{aligned}
\tag{3.40}
$$

Combining (3.40) with (3.36), it follows that for $t>2T$,

$$
\begin{aligned}
K_2(t,T)&\le\Big\{CT^{1/2}\Big[|u_x(\cdot,T)|_2^2+|u(\cdot,T)|_4^4\Big]+C\Big\{\int_T^\infty|u_x(\cdot,\tau)|_2^2d\tau\Big\}^{\frac12}\Big\}^{\frac12}\\
&\qquad+\Big\{4t^{1/2}\int_{t/2}^{+\infty}\Big[|u_{xx}(\cdot,\tau)|_2^2+|u(\cdot,\tau)u_x(\cdot,\tau)|_2^2\Big]d\tau\Big\}^{\frac12}.
\end{aligned}
\tag{3.41}
$$

Using Lemma 3.4 and the fact that $|u_x(\cdot,t)|_2^2\in L_1(\mathbb{R}^+)$, (3.41) implies

$$
\lim_{T\to+\infty}\sup_{t\ge T}K_2(t,T)=0.
$$

Hence, for a suitable large $T>0$, we conclude that the first two terms in the double integral of equation (3.31) are bounded above by

$$
\begin{aligned}
R_2(x,t)&=\epsilon\int_T^t\int_{-\infty}^\infty\Big|G(x-s,t-\tau)v\big[\alpha_1u_{ss}+\alpha_2uu_s\big]\Big|dsd\tau\\
&\le CK_2(t,T)\sup_{T\le\tau\le t}|v(\cdot,\tau)|_\infty\le\tfrac15\sup_{T\le\tau\le t}|v(\cdot,\tau)|_\infty.
\end{aligned}
\tag{3.42}
$$

**(c)** Since $u(x,t)=U_x(x,t)=\frac{v_x(x,t)}{v(x,t)}$, it follows that

$$
v_x(x,t)=u(x,t)\cdot v(x,t).
$$

Replacing $vu^3$ by $v_x u^2$ and integrating by parts, the last term in the double integral of equation (3.31) becomes

$$\begin{aligned} &\epsilon \int_T^t \int_{-\infty}^{\infty} \Big( G(x-s,t-\tau) v \tfrac{\alpha_4}{3} u^3 \Big) ds d\tau \\ &= \epsilon \int_T^t \int_{-\infty}^{\infty} \Big( G(x-s,t-\tau) v_s \tfrac{\alpha_4}{3} u^2 \Big) ds d\tau \qquad (3.43) \\ &= -\tfrac{\epsilon\alpha_4}{3} \int_T^t \int_{-\infty}^{\infty} \Big( G(x-s,t-\tau) v \cdot 2uu_s + G_s(x-s,t-\tau) vu^2 \Big) ds d\tau. \end{aligned}$$

Following the arguments of part (b), it can be shown that for some suitable large $T$,

$$\begin{aligned} R_3(x,t) &= \tfrac{2\epsilon|\alpha_4|}{3} \int_T^t \int_{-\infty}^{\infty} \Big| G(x-s,t-\tau) vuu_s \Big| ds d\tau \\ &\leq CK_2(t,T) \sup_{T\leq\tau\leq t} |v(\cdot,\tau)|_\infty \leq \tfrac{1}{10} \sup_{T\leq\tau\leq t} |v(\cdot,\tau)|_\infty. \end{aligned} \qquad (3.44)$$

Applying the Cauchy-Schwarz inequality to the second term in (3.43), it follows that

$$\begin{aligned} R_4(x,t) &= \tfrac{\epsilon|\alpha_4|}{3} \int_T^t \int_{-\infty}^{\infty} \Big| G_s(x-s,t-\tau) vu^2 \Big| ds d\tau \\ &\leq \tfrac{\epsilon|\alpha_4|}{3} \sup_{T\leq\tau\leq t} \Big\{ |v(\cdot,\tau)|_\infty \Big\} \int_T^t |G_x(\cdot,t-\tau)|_2 |u(\cdot,\tau)|_4^2 d\tau. \end{aligned} \qquad (3.45)$$

A directly computation yields

$$\begin{aligned} |G_x(\cdot,t-\tau)|_2^2 &= \int_{-\infty}^{+\infty} \frac{x^2 \exp\Big(-\frac{x^2}{2(t-\tau)}\Big)}{16\pi(t-\tau)^3} dx \\ &= \tfrac{1}{4\sqrt{2}\pi(t-\tau)^{3/2}} \int_{-\infty}^{+\infty} x^2 e^{-x^2} dx = \tfrac{C}{(t-\tau)^{3/2}}. \end{aligned} \qquad (3.46)$$

For $T$ sufficiently large and for any given $\delta > 0$, by Lemma 3.4, we have

$$|u(\cdot,\tau)|_4^2 \leq \frac{\delta}{\tau^{1/4}}. \qquad (3.47)$$

The use of (3.46) and (3.47) in (3.45) shows that if $\delta$ is sufficiently small,

$$\begin{aligned} R_4(x,t) &\leq \tfrac{\epsilon|\alpha_4|}{3} \sup_{T\leq\tau\leq t} \Big\{ |v(\cdot,\tau)|_\infty \Big\} \int_T^t \tfrac{C}{(t-\tau)^{3/4}} \tfrac{\delta}{\tau^{1/4}} d\tau \\ &\leq C\delta \sup_{T\leq\tau\leq t} \Big\{ |v(\cdot,\tau)|_\infty \Big\} \int_0^1 \tfrac{d\tau}{(1-\tau)^{3/4}\tau^{1/4}} \leq \tfrac{1}{10} \sup_{T\leq\tau\leq t} \Big\{ |v(\cdot,\tau)|_\infty \Big\}. \end{aligned} \qquad (3.48)$$

Finally, note that

$$|S(\cdot,t)|_\infty \leq |G(\cdot,t)|_1 |v(\cdot,T)|_\infty = |v(\cdot,T)|_\infty. \qquad (3.49)$$

Using (3.32), (3.42), (3.44), (3.48) and (3.49), it follows that for $T$ sufficiently large, and any $t \geq T$,

$$\sup_{t\geq T} \Big\{ |v(\cdot,t)|_\infty \Big\} \leq |v(\cdot,T)|_\infty + (\tfrac{1}{10} + \tfrac{1}{5} + \tfrac{1}{10} + \tfrac{1}{10}) \sup_{t\geq T} \Big\{ |v(\cdot,t)|_\infty \Big\},$$

or equivalently,

$$\sup_{t\geq T} \Big\{ |v(\cdot,t)|_\infty \Big\} \leq 2|v(\cdot,T)|.$$

This concludes the proof of the validity of the second inequality in (3.29).

Now we prove the first inequality in (3.29), i.e.,

$$\inf_{x\in\mathbb{R},\ t\geq 0} v(x,t) > 0.$$

From equation (3.31), $v(x,t) = S(x,t) + R(x,t)$, hence

$$|v(\cdot,t) - S(\cdot,t)|_\infty = |R(\cdot,t)|_\infty. \tag{3.50}$$

By the estimates (3.32), (3.42), (3.44) and (3.48), it follows that

$$|R(\cdot,t)|_\infty \leq |R_1(\cdot,t)|_\infty + |R_2(\cdot,t)|_\infty + |R_3(\cdot,t)|_\infty + |R_4(\cdot,t)|_\infty.$$

Hence, since $v$ is uniformly bounded,

$$\begin{aligned} &\sup_{t\geq T} |v(\cdot,t) - S(\cdot,t)|_\infty \\ &\leq \sup_{t\geq T} \Big\{|v(\cdot,t)|_\infty\Big\}\Big[K_1(t,T) + 2K_2(t,T) + C\delta(T)\Big] \\ &\leq C\Big[K_1(t,T) + 2K_2(t,T) + C\delta(T)\Big]. \end{aligned}$$

Because $K_1(t,T), K_2(t,T)$, and $\delta(T) \to 0$ as $T \to +\infty$, it follows that

$$\lim_{T\to+\infty} \sup_{t\geq T} |v(\cdot,t) - S(\cdot,t)|_\infty = 0. \tag{3.51}$$

Hence,

$$\liminf_{t\to+\infty} \inf_{x\in\mathbb{R}} v(x,t) = \liminf_{t\to+\infty} \inf_{x\in\mathbb{R}} S(x,t). \tag{3.52}$$

We definite $m$ by

$$\begin{aligned} m &= \min\big\{v(-\infty,T), v(+\infty,T)\big\} \\ &= \min\Big\{1, \exp\big(\textstyle\int_{-\infty}^{\infty} u(x,0)dx + \epsilon(\alpha_3-\alpha_2)\int_0^T\int_{-\infty}^{\infty} u_x^2(x,\tau)dxd\tau\big)\Big\}. \end{aligned} \tag{3.53}$$

It is possible to choose $M$ sufficiently large, so that for $|y| > M$, $v(y,T) \geq \frac{k}{2}$. This is possible because $v(x,t)$ is uniformly bounded and $M$ is fixed. The definition of $S(x,t)$, as well as the equation $\int_{-\infty}^{+\infty} G(x,t)dx = 1$, imply

$$\begin{aligned} S(x,t) &= \textstyle\int_{-\infty}^{+\infty} G(x-s,t-T)v(s,T)ds - \int_{-\infty}^{+\infty} G(x-s,t-T)\cdot\frac{k}{2}ds + \frac{k}{2} \\ &= \tfrac{1}{\sqrt{4\pi(t-T)}}\textstyle\int_{-\infty}^{+\infty}\big(v(s,T)-\frac{k}{2}\big)\exp\Big(-\frac{(x-s)^2}{4(t-T)}\Big)ds + \frac{k}{2} \\ &\geq \tfrac{k}{2} - \tfrac{1}{\sqrt{4\pi(t-T)}}\textstyle\int_{-M}^{+M}\big|v(s,T)-\frac{k}{2}\big| \geq \frac{k}{2} - \frac{C}{(t-T)^{1/2}}. \end{aligned} \tag{3.54}$$

Hence,

$$\liminf_{t\to+\infty} \inf_{x\in\mathbb{R}} S(x,t) \geq \frac{k}{2} > 0.$$

Using (3.52), we find

$$\liminf_{t\to+\infty} \inf_{x\in\mathbb{R}} v(x,t) \geq \frac{k}{2} > 0.$$

The lemma is proved. □

Note that $\frac{v_x(x,t)}{v(x,t)} = u(x,t)$. The function $v(x,t)$ is uniformly bounded and is away from zero (see Lemma 3.6); thus, in order to prove Lemma 3.5, one only needs to prove the same results for $v_x(x,t)$.

We differentiate equation (3.27) with respect to $x$ and let $W(x,t) = v_x(x,t)$. The function $W(x,t)$ satisfies the following nonhomogeneous diffusion equation:

(3.55)
$$\begin{aligned} W_t - W_{xx} &= \epsilon \cdot \partial_x\Big(v\Big[\alpha_1 u_{xx} + \alpha_2 u u_x + (\alpha_3 - \alpha_2)\int_{-\infty}^{x} u_s^2(s,t)ds + \tfrac{\alpha_4}{3}u^3\Big]\Big) \\ &= \epsilon \cdot (\alpha_3 - \alpha_2)\Big[v u_x^2 + W\int_{-\infty}^{x} u_s^2(s,t)ds\Big] \\ &\quad + \epsilon \cdot \partial_x\Big(v\Big[\alpha_1 u_{xx} + \alpha_2 u u_x + \tfrac{\alpha_4}{3}u^3\Big]\Big). \end{aligned}$$

Replacing $vu^3$ by $v_x u^2$, we find

$$\begin{aligned} W_t - W_{xx} &= \epsilon(\alpha_3 - \alpha_2)\Big[v u_x^2 + W\int_{-\infty}^{x} u_s^2(s,t)ds\Big] \\ &\quad + \epsilon\partial_x\Big(v\Big[\alpha_1 u_{xx} + (\alpha_2 - \tfrac{2\alpha_4}{3})u u_x\Big]\Big) + \epsilon\partial_{xx}\Big(\tfrac{\alpha_4}{3}v u^2\Big) \\ &= g_1(x,t) + \partial_x\Big(g_2(x,t)\Big) + \partial_{xx}\Big(g_3(x,t)\Big). \end{aligned} \tag{3.56}$$

Let $T > 0$ be fixed, and let $W(x,t)$ be a solution of (3.56) with the initial data $W(x,T) = v_x(x,T)$. The Fourier transform of $W(x,t)$, denoted by $\hat{W}(\xi,t)$, satisfies

$$\begin{aligned} \hat{W}_t(\xi,t) + \xi^2\hat{W}(\xi,t) &= \hat{g}_1(\xi,t) + i\xi\hat{g}_2(\xi,t) - \xi^2\hat{g}_3(\xi,t), \\ \hat{W}(\xi,T) &= \hat{v}_x(\xi,T). \end{aligned} \tag{3.57}$$

Hence, for $t > T$, $\hat{W}(\xi,t)$ solves

$$\hat{W}(\xi,t) = \hat{v}_x(\xi,T)e^{\xi^2(T-t)} + \int_T^t e^{\xi^2(\tau-t)}\Big(\hat{g}_1(\xi,\tau) + i\xi\hat{g}_2(\xi,\tau) - \xi^2\hat{g}_3(\xi,\tau)\Big)d\tau.$$

Thus,

$$\begin{aligned} |W(\cdot,t)|_2 = |\hat{W}(\cdot,t)|_2 &\le \Big\{\int_{-\infty}^{+\infty}\hat{v}_x^2(\xi,T)\exp\{2\xi^2(T-t)\}d\xi\Big\}^{\frac{1}{2}} \\ &+ \int_T^t\Big\{\int_{-\infty}^{+\infty}e^{2\xi^2(\tau-t)}\Big(|\hat{g}_1(\xi,\tau)|^2 + \xi^2|\hat{g}_2(\xi,\tau)|^2 + \xi^4|\hat{g}_3(\xi,\tau)|^2\Big)d\xi\Big\}^{\frac{1}{2}}d\tau. \end{aligned} \tag{3.58}$$

We first note that $|v(\cdot,t)|_\infty$ is uniformly bounded, and also

$$|v_x(\cdot,t)|_2 = |v(\cdot,t)u(\cdot,t)|_2 \le |v(\cdot,t)|_\infty|u(\cdot,t)|_2 \le C.$$

By Parseval's theorem, the first term on the right-hand side of the inequality (3.58) is bounded from above by

$$\int_{-\infty}^{+\infty} \hat{v_x}^2(\xi,T)\exp\{2\xi^2(T-t)\}d\xi \le \int_{-\infty}^{+\infty} \hat{v_x}^2(\xi,T)d\xi = |v_x(\cdot,T)|_2^2 \le C.$$

Making the substitution $\xi = \frac{\eta}{\sqrt{t-T}}$, and applying the Dominated Convergence Theorem, we find

$$\begin{aligned}
&\lim_{t\to+\infty} t^{\frac12}\int_{-\infty}^{+\infty} \hat{v_x}^2(\xi,T)\exp\{2\xi^2(T-t)\}d\xi \\
&= \lim_{t\to+\infty} t^{\frac12}(t-T)^{-\frac12}\int_{-\infty}^{+\infty} \hat{v_x}^2\Big(\frac{\eta}{\sqrt{t-T}},T\Big)e^{-2\eta^2}d\eta \\
&= \int_{-\infty}^{+\infty} \lim_{t\to+\infty} \hat{v_x}^2\Big(\frac{\eta}{\sqrt{t-T}},T\Big)e^{-2\eta^2}d\eta \\
&= \hat{v_x}^2(0,T)\int_{-\infty}^{+\infty} e^{-2\eta^2}d\eta = (8\pi)^{-\frac12}\Big(\int_{-\infty}^{+\infty} v_x(x,T)dx\Big)^2 \\
&= \frac{1}{(8\pi)^{\frac12}}\Big(\exp\Big\{\int_{-\infty}^{+\infty} u(x,0)dx + \epsilon(\alpha_3-\alpha_2)\int_0^T\int_{-\infty}^{+\infty} u_x^2(x,\tau)dxd\tau\Big\}-1\Big)^2 \\
&= C_0(T).
\end{aligned} \tag{3.59}$$

Now we estimate the terms in the double integral in (3.58). It is expected that these terms decay faster than the term in (3.58). This fact will be proved through several lemmas. The first one is for the term $g_1(x,t)$.

**Lemma 3.7.** *Let $W(x,t) = v_x(x,t)$ satisfy equation (3.56). Then*

$$\begin{aligned}
&\int_T^t \Big\{|g_1(\xi,\tau)|^2 e^{2\xi^2(\tau-t)}d\xi\Big\}^{1/2}d\tau \\
&\le C_1(T)t^{-1/4} + C_2(T)t^{-1/8} + C_3(T)\sup_{t\ge T}|(W(\cdot,t)|_2,
\end{aligned} \tag{3.60}$$

*where $C_i(T)\to 0$ as $T\to+\infty$, for $i=1,2,$ and* 3.

*Proof.* By the definition of $g_1(x,t)$, the left-hand side of (3.60) is bounded from above by the following two terms:

$$\begin{aligned}
&\int_T^t \Big\{\int_{-\infty}^{+\infty} |\hat{g}_1(\xi,\tau)|^2\exp\{2\xi^2(\tau-t)\}d\xi\Big\}^{1/2}d\tau \\
&\le C\int_T^t\Big\{\int_{-\infty}^{+\infty}\Big[|\widehat{vu_\xi^2}(\xi,\tau)|^2 + |\widehat{W(\xi,\tau)\int_{-\infty}^{\xi}u_r^2(r,\tau)dr}|^2\Big]e^{2\xi^2(\tau-t)}d\xi\Big\}^{\frac12}d\tau.
\end{aligned} \tag{3.61}$$

For $\tau \in [T, t/2]$, by Parseval's theorem, the first term on the right-hand side of (3.61) is bounded by

$$
\begin{aligned}
&\int_T^{t/2} \left\{ \int_{-\infty}^{+\infty} |\widehat{vu^2_\xi}(\xi,\tau)|^2 e^{2\xi^2(\tau-t)} d\xi \right\}^{\frac{1}{2}} d\tau \\
&\leq \int_T^{t/2} |\widehat{vu^2_\xi}(\xi,\tau)|_\infty \left\{ \int_{-\infty}^{+\infty} e^{2\xi^2(\tau-t)} d\xi \right\}^{\frac{1}{2}} d\tau \\
&\leq C \int_T^{t/2} |v(\cdot,\tau)|_\infty |u_x(\cdot,\tau)|_2^2 \left\{ \frac{c}{\sqrt{2(t-\tau)}} \right\}^{\frac{1}{2}} d\tau \\
&\leq C t^{-1/4} \int_T^t |u_x(\cdot,\tau)|_2^2 d\tau \leq C_1(T) t^{-1/4},
\end{aligned} \tag{3.62}
$$

where $C_1(T) \to 0$ as $T \to +\infty$, since $|u_x(\cdot,\tau)|_2^2 \in L_1(\mathbb{R}^+)$.

For $\tau \in [t/2, t]$, using Parseval's theorem, the fact that $v$ is uniformly bounded, and the elementary inequality $|u_x(\cdot,\tau)|_\infty^2 \leq 2|u_x(\cdot,\tau)|_2 |u_{xx}(\cdot,\tau)|_2$, we have

$$
\begin{aligned}
&\int_{t/2}^{t} \left\{ \int_{-\infty}^{+\infty} |\widehat{vu^2_\xi}(\xi,\tau)|^2 e^{2\xi^2(\tau-t)} d\xi \right\}^{\frac{1}{2}} d\tau \\
&\leq \int_{t/2}^{t} \left\{ \int_{-\infty}^{+\infty} |\widehat{vu^2_\xi}(\xi,\tau)|^2 d\xi \right\}^{1/2} d\tau \\
&\leq C \int_{t/2}^{t} |v(\cdot,\tau) u_x^2(\cdot,\tau)|_2 d\tau \\
&\leq C \int_{t/2}^{t} |u_x(\cdot,\tau)|_2^{3/2} |u_{xx}(\cdot,\tau)|_2^{\frac{1}{2}} d\tau.
\end{aligned}
$$

Furthermore, using Hölder's inequality, and the fact that $\int_t^\infty |u_{xx}(\cdot,\tau)|_2^2 = o(t^{-1/2})$ and that $|u_x(\cdot,\tau)|_2^2 \in L_1(\mathbb{R}^+)$ because of Lemma 3.4, we find

$$
\begin{aligned}
&\int_{t/2}^{t} \left\{ \int_{-\infty}^{+\infty} |\widehat{vu^2_\xi}(\xi,\tau)|^2 e^{2\xi^2(\tau-t)} d\xi \right\}^{\frac{1}{2}} d\tau \\
&\leq C \int_{t/2}^{t} |u_x(\cdot,\tau)|_2^{\frac{3}{2}} |u_{xx}(\cdot,\tau)|_2^{\frac{1}{2}} d\tau \\
&\leq C \left\{ \int_{t/2}^{t} |u_x(\cdot,\tau)|_2^2 \right\}^{3/4} \left\{ \int_{t/2}^{t} |u_{xx}(\cdot,\tau)|_2^2 \right\}^{1/4} \\
&\leq C(T) \left\{ \int_{t/2}^{\infty} |u_{xx}(\cdot,\tau)|_2^2 \right\}^{1/4} \leq C_2(T) t^{-\frac{1}{8}},
\end{aligned} \tag{3.63}
$$

where $C_2(T) \to 0$ as $T \to +\infty$.

By using Parseval's theorem and the fact that $|u_x(\cdot,\tau)|_2^2 \in L_1(\mathbb{R}^+)$, the second term on the right-hand side of (3.61) is bounded by

$$
\begin{aligned}
&\int_T^t \Big\{ \int_{-\infty}^{+\infty} |\widehat{W(\xi,\tau)\int_{-\infty}^{\xi}} u_r^2(r,\tau)dr|^2 \exp\{2\xi^2(\tau-t)\}d\xi \Big\}^{\frac{1}{2}} d\tau \\
&\le \int_T^t \Big\{ \int_{-\infty}^{+\infty} |\widehat{W(\xi,\tau)\int_{-\infty}^{\xi}} u_r^2(r,\tau)dr|^2 d\xi \Big\}^{\frac{1}{2}} d\tau \\
&\le \int_T^t \Big\{ \int_{-\infty}^{+\infty} |W(\xi,\tau)\int_{-\infty}^{\xi} u_r^2(r,\tau)dr|^2 d\xi \Big\}^{\frac{1}{2}} d\tau \\
&\le C\int_T^t |W(\cdot,\tau)|_2 |u_x(\cdot,\tau)|_2^2 d\tau \\
&\le C\sup_{t\ge T}\Big\{|W(\cdot,t)|_2\Big\}\int_T^t |u_x(\cdot,\tau)|_2^2 d\tau \\
&\le C_3(T)\sup_{t\ge T}\Big\{|W(\cdot,t)|_2\Big\}.
\end{aligned}
\tag{3.64}
$$

Combining the results (3.62), (3.63), and (3.64) yields the result (3.60), and therefore the proof of the lemma is completed. □

Next, we will estimate the last term $g_3(x,t)$ in (3.58).

**Lemma 3.8.** *Let $W(x,t) = v_x(x,t)$ satisfy equation (3.56). Then, for any given $0 < \delta < 2$,*

$$
\int_T^t \Big\{|g_3(\xi,\tau)|^2 e^{2\xi^2(\tau-t)}d\xi\Big\}^{1/2} d\tau \le C_1(T)t^{-1/4} + C_4(T)t^{-\frac{1}{4}+\delta},
\tag{3.65}
$$

*where $C_i(T) \to 0$ as $T \to +\infty$ for $i = 1$ and 4.*

*Proof.* First, using the fact that $\max_{x>0} x^2e^{-tx} = 4/(t^2e^2)$, it follows that

$$
\xi^4 e^{2\xi^2(\tau-t)} \le \frac{1}{e^2(t-\tau)^2}.
\tag{3.66}
$$

For $\tau \in [T, t/2]$, using (3.66), Parseval's theorem, and then the fact that $|u(\cdot,\tau)|_4^4 = o(t^{-1/2})$, we find that the left-hand side of (3.65) is bounded by

$$
\begin{aligned}
&\int_T^{t/2} \Big\{ \int_{-\infty}^{+\infty} |\hat{g_3}(\xi,\tau)|^2 \xi^4 \exp\{2\xi^2(\tau-t)\}d\xi \Big\}^{\frac{1}{2}} d\tau \\
&\le \int_T^{t/2} \tfrac{C}{t-\tau}\Big\{ \int_{-\infty}^{+\infty} |\widehat{vu^2}(\xi,\tau)|^2 d\xi \Big\}^{\frac{1}{2}} d\tau \\
&\le Ct^{-1/4}\int_T^{t/2} \tfrac{1}{(t-\tau)^{3/4}}\Big\{|v(\cdot,\tau)|_\infty |u(\cdot,\tau)|_4^4\Big\}^{1/2} d\tau \\
&\le C(T)t^{-1/4}\int_T^{t/2} \tfrac{d\tau}{(t-\tau)^{3/4}\tau^{1/4}} \\
&\le C(T)t^{-1/4}\int_0^1 \tfrac{d\tau}{(1-\tau)^{3/4}\tau^{1/4}} \le C_1(T)t^{-1/4}.
\end{aligned}
\tag{3.67}
$$

Note that for any given $\delta > 0$ and $|\xi| \le 1$,

$$
|\xi|^4 e^{-2\xi^2(t-\tau)} \le |\xi|^{4-\delta} e^{-2\xi^2(t-\tau)}.
\tag{3.68}
$$

For $|\xi| \geq 1$,

$$e^{-2\xi^2(t-\tau)} \leq e^{-2(t-\tau)}. \tag{3.69}$$

Hence, using (3.68) and (3.69), we find

$$\begin{aligned} &\int_{-\infty}^{+\infty} |\widehat{vu^2}(\xi,\tau)|^2 \xi^4 \exp\{2\xi^2(\tau-t)\} d\xi \\ &\leq \int_{|\xi|\leq 1} |\widehat{vu^2}(\xi,\tau)|^2 |\xi|^{4-\delta} e^{-2\xi^2(t-\tau)} d\xi + \int_{|\xi|>1} |\widehat{vu^2}(\xi,\tau)|^2 \xi^4 e^{-2(t-\tau)} d\xi \\ &\leq \int_{-\infty}^{+\infty} |\widehat{vu^2}(\xi,\tau)|^2 |\xi|^{4-\delta} e^{-2\xi^2(t-\tau)} d\xi + \int_{-\infty}^{+\infty} |\widehat{vu^2}(\xi,\tau)|^2 \xi^4 e^{-2(t-\tau)} d\xi. \end{aligned} \tag{3.70}$$

Using the elementary inequality $\max_{x>0} x^{2-\alpha} e^{-tx} = \left(\frac{2-\alpha}{e}\right)^{2-\alpha} t^{-(2-\alpha)}$ for $\alpha < 2$, we find

$$\begin{aligned} &\int_{-\infty}^{+\infty} |\widehat{vu^2}(\xi,\tau)|^2 |\xi|^{4-\delta} e^{-2\xi^2(t-\tau)} d\xi \leq C(t-\tau)^{-(2-\alpha)} \int_{-\infty}^{+\infty} |\widehat{vu^2}(\xi,\tau)|^2 d\xi \\ &= C(t-\tau)^{-(2-\alpha)} |v(\cdot,\tau)u^2(\cdot,\tau)|_2^2 \\ &\leq C(t-\tau)^{-(2-\alpha)} |u(\cdot,\tau)|_4^4 \leq C(T)(t-\tau)^{-2+\alpha} \cdot \tau^{-\frac{1}{2}}, \end{aligned} \tag{3.71}$$

where $\alpha = \frac{\delta}{2}$, and $C(T) \to 0$ as $T \to +\infty$ because $|u(\cdot,\tau)|_4^4 = o(t^{-1/2})$. It follows from (3.71) that for $\tau \in [t/2, t]$,

$$\begin{aligned} &\int_{\frac{t}{2}}^{t} \left\{ \int_{-\infty}^{+\infty} |\widehat{vu^2}(\xi,\tau)|^2 |\xi|^{4-\delta} e^{-2\xi^2(t-\tau)} d\xi \right\}^{\frac{1}{2}} d\tau \leq C(T) \int_{\frac{t}{2}}^{t} (t-\tau)^{-1+\frac{\alpha}{2}} \tau^{-\frac{1}{4}} d\tau \\ &\leq C(T) t^{-1/4+\alpha/2} \int_{1/2}^{1} (1-\tau)^{-1+\frac{\alpha}{2}} \tau^{-\frac{\alpha}{2}} d\tau \\ &\leq C_4(T) t^{-1/4+\alpha/2} = C_4(T) t^{-\frac{1}{4}+\delta}, \end{aligned} \tag{3.72}$$

where we have renamed $\delta = \alpha/2$.

Using $v_x(x,t) = v(x,t)u(x,t)$, we find

$$\begin{aligned} |\widehat{vu^2}(\xi,\tau)|^2 \xi^4 &= |\widehat{\partial_{xx}(vu^2)}(\xi,\tau)|^2 \\ &= \left| \int_{-\infty}^{+\infty} e^{i\xi y} \left\{ vu^4 + 5vu^2 u_x + 2vu_x^2 + 2vuu_{xx} \right\}(\xi,\tau) d\xi \right|^2. \end{aligned}$$

Hence, using Parseval's theorem and the inequalities

$$|u(\cdot,\tau)|_\infty^2 \leq 2|u(\cdot,\tau)|_2 |u_x(\cdot,\tau)|_2 \quad \text{and} \quad |u_x(\cdot,\tau)|_\infty^2 \leq 2|u_x(\cdot,\tau)|_2 |u_{xx}(\cdot,\tau)|_2,$$

we find

$$\begin{aligned} &\int_{-\infty}^{+\infty} |\widehat{vu^2}(\xi,\tau)|^2 \xi^4 d\xi = |\{vu^4 + 5vu^2 u_x + 2vu_x^2 + 2vuu_{xx}\}(\cdot,\tau)|_2^2 \\ &\leq C\Big( |u^4(\cdot,\tau)|_2^2 + |u^2(\cdot,\tau)u_x(\cdot,\tau)|_2^2 + |u_x^2(\cdot,\tau)|_2^2 + |u(\cdot,\tau)u_{xx}(\cdot,\tau)|_2^2 \Big) \\ &\leq C(|u_x(\cdot,\tau)|_2^3 + |u_{xx}(\cdot,\tau)|_2^2) \leq C(T)\tau^{-3/4} + C|u_{xx}(\cdot,\tau)|_2^2. \end{aligned} \tag{3.73}$$

It follows from (3.73) that for $\tau \in [t/2, t]$,

$$\begin{aligned}
&\int_{\frac{t}{2}}^{t}\Big\{\int_{-\infty}^{+\infty}|\widehat{vu^2}(\xi,\tau)|^2\xi^4e^{-2(t-\tau)}d\xi\Big\}^{\frac{1}{2}}d\tau \\
&\le \int_{\frac{t}{2}}^{t}\Big(C(T)\tau^{-\frac{3}{8}}+C|u_{xx}(\cdot,\tau)|_2\Big)e^{-(t-\tau)}d\tau \\
&\le C(T)t^{-3/8}\int_{\frac{t}{2}}^{t}e^{-(t-\tau)}d\tau + C\Big\{\int_{\frac{t}{2}}^{t}|u_{xx}(\cdot,\tau)|_2^2d\tau\Big\}^{\frac{1}{2}}\Big\{\int_{\frac{t}{2}}^{t}e^{-2(t-\tau)}d\tau\Big\}^{\frac{1}{2}} \\
&\le C(T)\Big(t^{-3/8}+t^{-1/4}\Big) \le C_1(T)t^{-1/4},
\end{aligned} \tag{3.74}$$

where in the last inequality we have used the result in Lemma 3.4, the estimate

$$\int_{t/2}^{t}|u_{xx}(\cdot,\tau)|_2^2d\tau \le \int_{t/2}^{\infty}|u_{xx}(\cdot,\tau)|_2^2d\tau = o(t^{-1/2}),$$

as well as the fact that

$$\int_{t/2}^{t}e^{-2(t-\tau)}d\tau = \tfrac{1}{2}(1-e^{-t/2}) \le \tfrac{1}{2}.$$

Combining (3.70), (3.72) and (3.74), it follows that for $\tau \in [t/2, t]$,

$$\begin{aligned}
&\int_{\frac{t}{2}}^{t}\Big\{\int_{-\infty}^{+\infty}|v\hat{u}^2(\xi,\tau)|^2\xi^4\exp\{2\xi^2(\tau-t)\}d\xi\Big\}^{\frac{1}{2}}d\tau \\
&\le \int_{\frac{t}{2}}^{t}\Big\{\int_{-\infty}^{+\infty}|\widehat{vu^2}(\xi,\tau)|^2|\xi|^{4-\delta}e^{-2\xi^2(t-\tau)}d\xi+\int_{-\infty}^{+\infty}|\widehat{vu^2}(\xi,\tau)|^2\xi^4e^{-2(t-\tau)}d\xi\Big\}^{\frac{1}{2}}d\tau \\
&\le C_4(T)t^{-\frac{1}{4}+\delta}+C_1(T)t^{-1/4}.
\end{aligned} \tag{3.75}$$

Hence, the lemma is proved. □

Finally, we estimate the last term $g_2(x,t)$ in (3.58).

**Lemma 3.9.** *Let $W(x,t) = v_x(x,t)$ satisfy the equation (3.56). Then for any given small $\delta > 0$,*

$$\int_{T}^{t}\Big\{|g_2(\xi,\tau)|^2e^{2\xi^2(\tau-t)}d\xi\Big\}^{1/2}d\tau \le C_1(T)t^{-1/4}+C_4(T)t^{-\frac{1}{4}+\delta}, \tag{3.76}$$

*where $C_i(T) \to 0$ as $T \to +\infty$ for $i = 1$ and $4$.*

*Proof.* By considering the maximum value of the function $xe^{-tx}$ for all $x \ge 0$, we obtain the elementary inequality

$$\xi^2e^{2\xi^2(\tau-t)} \le \frac{1}{2e(t-\tau)}. \tag{3.77}$$

The use of (3.77) shows that

$$
\begin{aligned}
&\int_T^t \Big\{ \int_{-\infty}^{+\infty} |\hat{g}_2(\xi,\tau)|^2 \xi^2 \exp\{2\xi^2(\tau-t)\} d\xi \Big\}^{1/2} d\tau \\
&\le \int_T^t \frac{C}{\sqrt{t-\tau}} \Big\{ \int_{-\infty}^{+\infty} \Big( |\widehat{vu_{xx}}(\xi,\tau)|^2 + |\widehat{vuu_x}(\xi,\tau)|^2 \Big) d\xi \Big\}^{1/2} d\tau \\
&\le \int_T^t \frac{C}{\sqrt{t-\tau}} |v(\cdot,\tau)|_\infty \Big( |u_{xx}(\cdot,\tau)|_2 + |u(\cdot,\tau)u_x(\cdot,\tau)|_2 \Big) d\tau \\
&\le \int_T^{t/2} + \int_{t/2}^t \frac{C}{\sqrt{t-\tau}} \Big( |u_{xx}(\cdot,\tau)|_2 + |u(\cdot,\tau)u_x(\cdot,\tau)|_2 \Big) d\tau.
\end{aligned} \tag{3.78}
$$

For $\tau \in [T, t/2]$, in analogy with (3.35) and (3.40) of Lemma 3.6, we have

$$
\begin{aligned}
&\int_T^{t/2} \frac{1}{\sqrt{t-\tau}} \Big( |u_{xx}(\cdot,\tau)|_2 + |u(\cdot,\tau)u_x(\cdot,\tau)|_2 \Big) d\tau \\
&\le t^{-1/4} \int_T^{t/2} \frac{1}{(t-\tau)^{1/4}} \Big( |u_{xx}(\cdot,\tau)|_2 + |u(\cdot,\tau)u_x(\cdot,\tau)|_2 \Big) d\tau \\
&\le t^{-1/4} \Big\{ CT^{1/2} \Big[ |u_x(\cdot,T)|_2^2 + |u(\cdot,T)|_4^4 \Big] + C \Big\{ \int_T^\infty |u_x(\cdot,\tau)|_2^2 d\tau \Big\}^{1/2} \Big\}^{1/2} \\
&= C_1(T) t^{-1/4},
\end{aligned}
$$

where $C_1(T) \to 0$ as $T \to +\infty$ (because of (3.25) in Lemma 3.4 and the fact that $|u_x(\cdot,\tau)|_2^2 \in L_1(\mathbb{R}^+)$). Hence, it follows from (3.78) that for $\tau \in [T, t/2]$,

$$
\int_T^{t/2} \Big\{ \int_{-\infty}^{+\infty} |\hat{g}_2(\xi,\tau)|^2 \xi^2 \exp\{2\xi^2(\tau-t)\} d\xi \Big\}^{1/2} d\tau \le C_1(T) t^{-1/4}. \tag{3.79}
$$

For $\tau \in [t/2, t]$ and any given small $\delta > 0$, using Hölder's inequality we find

$$
\begin{aligned}
&\int_{t/2}^t \frac{1}{\sqrt{t-\tau}} \Big( |u_{xx}(\cdot,\tau)|_2 + |u(\cdot,\tau)u_x(\cdot,\tau)|_2 \Big) d\tau \\
&\le \Big\{ \int_{\frac{t}{2}}^t \frac{1}{(t-\tau)^{1-\beta}\tau^\beta} d\tau \Big\}^{\frac{1+2\delta}{2}} \Big\{ \int_{\frac{t}{2}}^t \tau^{\frac{2\delta}{1-2\delta}} \Big( |u_{xx}(\cdot,\tau)|_2^2 + |u(\cdot,\tau)u_x(\cdot,\tau)|_2^2 \Big) d\tau \Big\}^{\frac{1-2\delta}{2}} \\
&\le \Big\{ \int_{\frac{1}{2}}^1 \frac{1}{(1-\tau)^{1-\beta}\tau^\beta} d\tau \Big\}^{\frac{1+2\delta}{2}} \cdot t^\delta \Big\{ \int_{\frac{t}{2}}^t \Big( |u_{xx}(\cdot,\tau)|_2^2 + |u(\cdot,\tau)u_x(\cdot,\tau)|_2^2 \Big) d\tau \Big\}^{\frac{1-2\delta}{2}} \\
&\le Ct^\delta \Big\{ \int_{\frac{t}{2}}^t \Big( |u_{xx}(\cdot,\tau)|_2^2 + |u(\cdot,\tau)u_x(\cdot,\tau)|_2^2 \Big) d\tau \Big\}^{(1-2\delta)/2},
\end{aligned} \tag{3.80}
$$

where $\beta = 2\delta/(1+2\delta)$. The use of equation (3.20) of Lemma 3.4 yields

$$
\begin{aligned}
&\Big\{ \int_{t/2}^t \Big( |u_{xx}(\cdot,\tau)|_2^2 + |u(\cdot,\tau)u_x(\cdot,\tau)|_2^2 \Big) d\tau \Big\}^{(1-2\delta)/2} \\
&\le \Big\{ \int_{t/2}^\infty \Big( |u_{xx}(\cdot,\tau)|_2^2 + |u(\cdot,\tau)u_x(\cdot,\tau)|_2^2 \Big) d\tau \Big\}^{(1-2\delta)/2} \\
&\le \Big\{ C(T) t^{-1/2} \Big\}^{(1-2\delta)/2} \\
&= C_4(T) t^{-1/4+\delta/2}.
\end{aligned} \tag{3.81}
$$

It follows from (3.80) and (3.81), that for $\tau \in [t/2, t]$,

$$\int_{t/2}^{t} \Big\{ \int_{-\infty}^{+\infty} |\hat{g}_2(\xi,\tau)|^2 \xi^2 \exp\{2\xi^2(\tau - t)\} d\xi \Big\}^{1/2} d\tau \le C_4(T) t^{-1/4+\delta}, \tag{3.82}$$

where we have renamed the $\delta$ in (3.82). Hence the lemma is completed. □

*The Proof of Lemma 3.5.* Using (3.60), (3.65) and (3.76) in (3.58), we find that for $T > 0$ and sufficiently large,

$$\begin{aligned}
|W(\cdot,t)|_2 = |\hat{W}(\cdot,t)|_2 &\le \Big\{ \int_{-\infty}^{+\infty} \hat{v_x}^2(\xi,T) \exp\{2\xi^2(T-t)\} d\xi \Big\}^{\frac{1}{2}} \\
&+ \int_T^t \Big\{ \int_{-\infty}^{+\infty} e^{2\xi^2(\tau-t)} \Big( |\hat{g}_1(\xi,\tau)|^2 + \xi^2 |\hat{g}_2(\xi,\tau)|^2 + \xi^4 |\hat{g}_3(\xi,\tau)|^2 \Big) d\xi \Big\}^{\frac{1}{2}} d\tau \\
&\le C_0 t^{-\frac{1}{4}} + \int_T^t \Big\{ \int_{-\infty}^{+\infty} e^{2\xi^2(\tau-t)} \Big( |\hat{g}_1(\xi,\tau)|^2 + \xi^2 |\hat{g}_2(\xi,\tau)|^2 + \xi^4 |\hat{g}_3(\xi,\tau)|^2 \Big) d\xi \Big\}^{\frac{1}{2}} d\tau \\
&\le C_0 t^{-\frac{1}{4}} + C_1(T) t^{-\frac{1}{4}} + C_2(T) t^{-\frac{1}{8}} + C_3(T) \sup_{t \ge T} |(W(\cdot,t)|_2 + C_4(T) t^{-\frac{1}{4}+\delta}.
\end{aligned} \tag{3.83}$$

We choose $T > 0$ sufficiently large so that $C_3(T) \le 1/2$. It then follows from (3.83) that

$$|W(\cdot,t)|_2 \le C_0 t^{-1/4} + C_1(T) t^{-1/4} + C_2(T) t^{-1/8} + C_4(T) t^{-\frac{1}{4}+\delta} \tag{3.84}$$

or

$$t^{1/8} |W(\cdot,t)|_2 \le C.$$

From the earlier remark, this is equivalent to

$$t^{1/8} |u(\cdot,t)|_2 \le C. \tag{3.85}$$

Using (3.85) and (3.19), we find

$$\begin{aligned}
\int_{-\infty}^{\infty} u^4(x,t) dx &\le 2|u(\cdot,t)|_2^3 |u_x(\cdot,t)|_2 \\
&\le C(T) t^{-3/8} \cdot t^{-1/4} = C(T) t^{-5/8}.
\end{aligned} \tag{3.86}$$

Hence, $\Gamma(t) = |u_x(\cdot,t)|_2^2 + 2|u(\cdot,t)|_4^4 \in L_{\frac{8}{5}+\alpha}(\mathbb{R}^+)$ for any $\alpha > 0$. Returning to the proof of Lemma 3.4, it is easy to obtain the estimates

$$\Gamma(t) = o(t^{-\frac{5}{8}+\alpha}) \quad \text{or} \quad \lim_{t \to +\infty} t^{\frac{5}{8}-\alpha} \big( |u_x(\cdot,t)|_2^2 + |u(\cdot,t)|_4^4 \big) = 0. \tag{3.87}$$

Using the result $|u_x(\cdot,t)|_2^2 = o(t^{-\frac{5}{8}+\alpha})$ in (3.86) shows that

$$\begin{aligned}
\int_{-\infty}^{\infty} u^4(x,t) dx &\le 2|u(\cdot,t)|_2^3 |u_x(\cdot,t)|_2 \\
&\le C(T) t^{-\frac{3}{8}} \cdot t^{-\frac{5}{16}+\frac{\alpha}{2}} = C(T) t^{-\frac{11}{16}+\frac{\alpha}{2}},
\end{aligned} \tag{3.88}$$

hence

$$\Gamma(t) = |u_x(\cdot,t)|_2^2 + 2|u(\cdot,t)|_4^4 \in L_{\frac{16}{11}+\alpha}(\mathbb{R}^+),$$

for any $\alpha > 0$. Similarly, the proof of Lemma 3.4 implies

$$\Gamma(t) = o(t^{-\frac{11}{16}+\alpha}) \quad \text{or} \quad \lim_{t \to +\infty} t^{\frac{11}{16}-\alpha} \big( |u_x(\cdot,t)|_2^2 + |u(\cdot,t)|_4^4 \big) = 0. \tag{3.89}$$

Using the new result $|u_x(\cdot,t)|_2^2 = o(t^{-\frac{11}{16}+\alpha})$ in (3.86) yields

$$\begin{aligned}\int_{-\infty}^{\infty} u^4(x,t)dx &\le 2|u(\cdot,t)|_2^3|u_x(\cdot,t)|_2\\ &\le C(T)t^{-\frac{3}{8}}\cdot t^{-\frac{11}{32}+\frac{\alpha}{2}} = C(T)t^{-\frac{33}{32}+\frac{\alpha}{2}},\end{aligned}$$

hence

$$\Gamma(t) = |u_x(\cdot,t)|_2^2 + 2|u(\cdot,t)|_4^4 \in L_1(\mathbb{R}^+).$$

Returning to the proof of Lemma 3.4 once more, we deduce

$$\Gamma(t) = o(t^{-1}) \quad \text{or} \quad \lim_{t\to+\infty} t\big(|u_x(\cdot,t)|_2^2 + |u(\cdot,t)|_4^4\big) = 0. \tag{3.90}$$

Moreover,

$$\int_t^{+\infty} |u(\cdot,\tau)u_x(\cdot,\tau)|_2^2 d\tau = o(t^{-1}) \quad \text{and} \quad \int_t^{+\infty} |u_{xx}(\cdot,\tau)|_2^2 d\tau = o(t^{-1}) \text{ as } t\to+\infty.$$

With these new results, (3.63) in Lemma 3.7 is bounded above by $C_2(T)t^{-1/4}$. Hence, Lemma 3.7 is amended as follows:

$$\int_T^t \Big\{|g_1(\xi,\tau)|^2 e^{2\xi^2(\tau-t)}d\xi\Big\}^{\frac{1}{2}} d\tau \le C_1(T)t^{-\frac{1}{4}} + C_2(T)t^{-\frac{1}{4}} + C_3(T)\sup_{t\ge T}|(W(\cdot,t)|_2. \tag{3.91}$$

Using (3.90) in (3.71), the expression (3.72) in Lemma 3.8 is bounded above by $C_4(T)t^{-1/4}$. Hence, Lemma 3.8 is amended as follows:

$$\int_T^t \Big\{|g_3(\xi,\tau)|^2 e^{2\xi^2(\tau-t)}d\xi\Big\}^{1/2} d\tau \le C_1(T)t^{-1/4} + C_4(T)t^{-1/4}. \tag{3.92}$$

Finally, the expression (3.81) in Lemma 3.9 is bounded above by $C_4(T)t^{-1/4}$. Hence, Lemma 3.9 is amended as follows:

$$\int_T^t \Big\{|g_2(\xi,\tau)|^2 e^{2\xi^2(\tau-t)}d\xi\Big\}^{1/2} d\tau \le C_1(T)t^{-1/4} + C_4(T)t^{-1/4}. \tag{3.93}$$

Using (3.91), (3.92) and (3.93) in (3.58), for $T > 0$ sufficiently large such that $C_3(T) \le 1/2$, we have

$$|W(\cdot,t)|_2 \le C_0 t^{-1/4} + C_1(T)t^{-1/4} + C_2(T)t^{-1/4} + C_4(T)t^{-1/4},$$

or

$$t^{1/4}|W(\cdot,t)|_2 \le C_0 \quad \text{or} \quad t^{1/4}|u(\cdot,t)|_2 \le C,$$

for $t > T$, and the proof of Lemma 3.5 is completed. □

**Corollary 3.10.** *Let $u$ be the solution of the initial-value problem for equation (1.1) with initial data (1.2). Then there exist constants $C_1$ and $C_2$, which are independent of $t$, such that*

$$t^{\frac{3}{4}}|u_x(\cdot,t)|_2 \le C_1 \quad \text{and} \quad t^{\frac{5}{4}}|u_{xx}(\cdot,t)|_2 \le C_2,$$

*for all $t \ge 0$.*

*Proof.* First, taking the Fourier transform of equation (1.1) with respect to the spatial variable $x$, we obtain

$$\begin{aligned}\hat{u}(y,\,t) &= e^{-y^2t - i\epsilon\alpha_1 y^3 t}\hat{u}(y,0)\\ &+ \int_0^t iye^{-(y^2t+i\epsilon\alpha_1y^3t)(t-\tau)}\Big(\widehat{u^2}(y,\tau) + \epsilon\alpha_2\widehat{uu_x}(y,\tau) + \epsilon\tfrac{\alpha_4}{3}\widehat{u^3}(y,\tau)\Big)d\tau\\ &+ \int_0^t e^{-(y^2t+i\epsilon\alpha_1y^3t)(t-\tau)}\epsilon(\alpha_3-\alpha_2)\widehat{u_x^2}(y,\tau)d\tau.\end{aligned} \tag{3.94}$$

Note that by Lemma 3.5,

$$|\widehat{u^2}(\cdot,\tau)|_\infty \le C|u(\cdot,\tau)|_2^2 \le C\tau^{-1/2}. \tag{3.95}$$

Similarly, it can be shown that, for some constants $C$,

$$\begin{aligned} |\widehat{u^3}(\cdot,\tau)|_\infty &\le C|u(\cdot,\tau)|_3^3 \\ &\le C|u(\cdot,\tau)|_\infty |u(\cdot,\tau)|_2^2 \le C\tau^{-1/2} \end{aligned} \tag{3.96}$$

and

$$|\widehat{uu_x}(\cdot,\tau)|_\infty \le C|u(\cdot,\tau)|_2|u_x(\cdot,\tau)|_2 \le C\tau^{-1/2}. \tag{3.97}$$

Using (3.94), (3.95), (3.96), and the fact that $\max_{\xi>0}\xi e^{-\xi^2 t} = \frac{1}{\sqrt{2et}}$, it follows that

$$\int_0^t \left| iye^{-(y^2t+i\epsilon\alpha_1 y^3 t)(t-\tau)}\left(\widehat{u^2}(y,\tau)+\epsilon\alpha_2\widehat{uu_x}(y,\tau)+\epsilon\tfrac{\alpha_4}{3}\widehat{u^3}(y,\tau)\right)\right| d\tau$$

$$\le \int_0^t \frac{C}{\sqrt{(t-\tau)\tau}}d\tau \le C.$$

By Corollary 3.3, it can be shown that

$$\int_0^t \left| e^{-(y^2t+i\epsilon\alpha_1 y^3 t)(t-\tau)}\widehat{u_x^2}(y,\tau)\right| d\tau \le C\int_0^t |u_x(\cdot,\tau)|_2^2 d\tau \le C.$$

Hence, (3.94) yields

$$|\hat{u}(\cdot,t)|_\infty \le |\hat{u}(\cdot,0)|_\infty + C \le C_0. \tag{3.98}$$

For $T>0$ sufficiently large, using (3.24), we find

$$\begin{aligned} \frac{d}{dt}\left(t^{\frac52}|u_x(\cdot,t)|_2^2\right) &\le \frac52 t^{\frac32}\left(|u_x(\cdot,t)|_2^2 - \frac{t}{10}|u_{xx}(\cdot,t)|_2^2\right) \\ &\quad + 5t^{\frac32}\left(|u(\cdot,t)|_4^4 - \frac{6t}{5}|u(\cdot,t)u_x(\cdot,t)|_2^2\right). \end{aligned} \tag{3.99}$$

Using Parseval's theorem, we can obtain the following:

$$\begin{aligned} \frac{d}{dt}\left(t^{\frac52}|u_x(\cdot,t)|_2^2\right) &\le \frac52 t^{\frac32}\left(|u_x(\cdot,t)|_2^2 - \frac{t}{10}|u_{xx}(\cdot,t)|_2^2\right) \\ &\quad + 5t^{\frac32}\left(|u(\cdot,t)|_4^4 - \frac{6t}{5}|u(\cdot,t)u_x(\cdot,t)|_2^2\right) \\ &\le \frac52 t^{\frac32}\int_{|y|\le\sqrt{\frac{10}{t}}} y^2|\hat{u}(y,t)|^2dy + 5t^{\frac32}\int_{|y|\le\sqrt{\frac{10}{3t}}}|\widehat{u^2}(y,t)|^2dy \\ &\le \frac52 t^{\frac32}\left(\sqrt{\frac{10}{t}}\right)^3|\hat{u}(\cdot,t)|_\infty^2 + 5t^{\frac32}\sqrt{\frac{10}{3t}}|\widehat{u^2}(\cdot,t)|_\infty^2 \\ &\le C + Ct\cdot|u(\cdot,t)|_2^4 \le C; \end{aligned} \tag{3.100}$$

in the above, we have made used of the inequalities

$$|\hat{u}(\cdot,t)|_\infty \le |u(\cdot,t)|_1 \le C \quad \text{and} \quad |u(\cdot,t)|_2^2 \le \frac{C}{(1+t)^{1/2}}.$$

Hence,

$$t^{\frac{5}{2}}|u_x(\cdot,t)|_2^2 \leq C(1+t)$$

and

$$|u_x(\cdot,t)|_2^2 \leq C(1+t)^{-\frac{3}{2}} + o((1+t)^{-\frac{3}{2}}), \tag{3.101}$$

for all $t \geq 0$.

In order to derive the second result of the corollary, we differentiate equation (1.1), we multiply the resultant equation by $2u_{xxx}$ and then integrate with respect to $x$ over $\mathbb{R}$. After integration by parts, we obtain

$$\begin{aligned} &\tfrac{d}{dt}|u_{xx}(\cdot,t)|_2^2 + 2|u_{xxx}(\cdot,t)|_2^2 \\ &= -\int_{-\infty}^{\infty} 2u_{xxx}\Big(2u_x^2 + 2uu_{xx} \\ &\quad +\epsilon\big[\alpha_2(u_xu_{xx}+uu_{xxx})+\alpha_3 2u_xu_{xx}+\alpha_4(2uu_x^2+u^2u_{xx})\big]\Big)dx. \end{aligned}$$

Using the Cauchy-Schwarz inequality, as well as the elementary inequalities

$$|u(\cdot,t)|_\infty^2 \leq 2|u(\cdot,t)|_2|u_x(\cdot,t)|_2 \quad \text{and} \quad |u_x(\cdot,t)|_\infty^2 \leq 2|u_x(\cdot,t)|_2|u_{xx}(\cdot,t)|_2,$$

we find

$$\begin{aligned} &\tfrac{d}{dt}|u_{xx}(\cdot,t)|_2^2 + 2|u_{xxx}(\cdot,t)|_2^2 \\ &\leq \tfrac{1}{2}|u_{xxx}(\cdot,t)|_2^2 + C(1+|u(\cdot,t)|_\infty^2)|u_x(\cdot,t)|_\infty^2|u_x(\cdot,t)|_2^2 \\ &+C(|u(\cdot,t)|_\infty^2+|u(\cdot,t)|_\infty^4+|u_x(\cdot,t)|_\infty^2)|u_{xx}(\cdot,t)|_2^2+C|u(\cdot,t)|_\infty^2|u_{xxx}(\cdot,t)|_2^2 \\ &\leq (\tfrac{1}{2}+C|u(\cdot,t)|_2|u_x(\cdot,t)|_2)|u_{xxx}(\cdot,t)|_2^2 + C|u_x(\cdot,t)|_2^3|u_{xx}(\cdot,t)|_2 \\ &+C|u(\cdot,t)|_2|u_x(\cdot,t)|_2|u_{xx}(\cdot,t)|_2^2 + C|u_x(\cdot,t)|_2|u_{xx}(\cdot,t)|_2^3. \end{aligned} \tag{3.102}$$

We choose $T > 0$ sufficiently large such that for $t > T$,

$$C|u(\cdot,t)|_2|u_x(\cdot,t)|_2 \leq \frac{1}{2}.$$

Then for $t > T$, using (3.101) and $|u(\cdot,t)|_2 \leq Ct^{-1/4}$, we find

$$\begin{aligned} &\tfrac{d}{dt}|u_{xx}(\cdot,t)|_2^2 + |u_{xxx}(\cdot,t)|_2^2 \\ &\leq C_1|u_x(\cdot,t)|_2^3|u_{xx}(\cdot,t)|_2 + C_2|u(\cdot,t)|_2|u_x(\cdot,t)|_2|u_{xx}(\cdot,t)|_2^2 \\ &\qquad +C_3|u_x(\cdot,t)|_2|u_{xx}(\cdot,t)|_2^3 \\ &\leq C_1t^{-9/4}|u_{xx}(\cdot,t)|_2 + C_2t^{-1}|u_{xx}(\cdot,t)|_2^2 + C_3t^{-3/4}|u_{xx}(\cdot,t)|_2^3. \end{aligned} \tag{3.103}$$

Multiply (3.103) by $t^{1+\alpha}$ and then add to both sides the term $(1+\alpha)t^{\alpha}|u_{xx}(\cdot,t)|_2^2$. This yields

$$\begin{aligned}&\tfrac{d}{dt}\Big(t^{1+\alpha}|u_{xx}(\cdot,t)|_2^2\Big)\\&\le (C_1+1+\alpha)t^{\alpha}|u_{xx}(\cdot,t)|_2^2+C_2t^{\alpha-5/4}|u_{xx}(\cdot,t)|_2\\&\qquad+C_3t^{\alpha+1/4}|u_{xx}(\cdot,t)|_2^3-t^{1+\alpha}|u_{xxx}(\cdot,t)|_2^2.\end{aligned}\tag{3.104}$$

Let $\alpha=\frac{13}{8}$ in (3.104). Using Parseval's theorem and (3.98), we find

$$\begin{aligned}&\tfrac{d}{dt}\Big(t^{\frac{21}{8}}|u_{xx}(\cdot,t)|_2^2\Big)\\&\le C_1t^{\frac{13}{8}}|u_{xx}(\cdot,t)|_2^2+C_2t^{\frac{3}{8}}|u_{xx}(\cdot,t)|_2+C_3t^{\frac{15}{8}}|u_{xx}(\cdot,t)|_2^3-t^{\frac{21}{8}}|u_{xxx}(\cdot,t)|_2^2\\&\le C_0+At^{\frac{15}{8}}|u_{xx}(\cdot,t)|_2^2-t^{\frac{21}{8}}|u_{xxx}(\cdot,t)|_2^2\\&\le C_0+At^{\frac{15}{8}}\int_{|y|\le\sqrt{\frac{A}{t^{\frac{3}{4}}}}}y^4|\hat{u}(y,t)|^2dy\\&\le C_0+At^{15/8}\bigg(\sqrt{\frac{A}{t^{\frac{3}{4}}}}\bigg)^5\le C.\end{aligned}\tag{3.105}$$

It follows from (3.105) that

$$t^{\frac{21}{8}}|u_{xx}(\cdot,t)|_2^2\le Ct;\quad\text{hence}\quad |u_{xx}(\cdot,t)|_2^2\le Ct^{-\frac{13}{8}},$$

for $t>T$. With this new estimate, (3.104) is modified as

$$\begin{aligned}&\tfrac{d}{dt}\Big(t^{1+\alpha}|u_{xx}(\cdot,t)|_2^2\Big)\\&\le C_1t^{\alpha}|u_{xx}(\cdot,t)|_2^2+C_2t^{\alpha-\frac{5}{4}}|u_{xx}(\cdot,t)|_2+C_3t^{\alpha-\frac{11}{8}}|u_{xx}(\cdot,t)|_2^2-t^{1+\alpha}|u_{xxx}(\cdot,t)|_2^2.\end{aligned}$$

Choosing $\alpha=\frac{5}{2}$ and using Parseval's theorem, we find

$$\begin{aligned}&\tfrac{d}{dt}\Big(t^{\frac{7}{2}}|u_{xx}(\cdot,t)|_2^2\Big)\\&\le C_1t^{\frac{5}{2}}|u_{xx}(\cdot,t)|_2^2+C_2t^{\frac{5}{4}}|u_{xx}(\cdot,t)|_2+C_3t^{\frac{9}{8}}|u_{xx}(\cdot,t)|_2^2-t^{\frac{7}{2}}|u_{xxx}(\cdot,t)|_2^2\\&\le C_0+At^{\frac{5}{2}}|u_{xx}(\cdot,t)|_2^2-t^{\frac{7}{2}}|u_{xxx}(\cdot,t)|_2^2\\&\le C_0+At^{\frac{5}{2}}\int_{|y|\le\sqrt{\frac{A}{t}}}y^4|\hat{u}(y,t)|^2dy\le C_0+At^{\frac{5}{2}}\bigg(\sqrt{\frac{A}{t}}\bigg)^5\le C.\end{aligned}\tag{3.106}$$

Hence,

$$t^{\frac{7}{2}}|u_{xx}(\cdot,t)|_2^2\le Ct$$

and

$$|u_{xx}(\cdot,t)|_2^2\le Ct^{-\frac{5}{2}},$$

for $t>T$, thus the proof of the lemma is completed. ☐

## 4. Rigorous methodology

In this section, we will show how to select a solution of the integrable equation (1.4) in order to approximate a solution of the initial value problem (1.1)-(1.2).

Let $v(x,t)$ be the solution of equation (1.4), or

$$v_t = 2vv_x + v_{xx} + \varepsilon\alpha_1[v_{xxx} + 3(vv_{xx} + v_x^2 + v^2 v_x)], \tag{4.1}$$

with the initial condition $v(x,0)$ defined by

$$u_0(x) + \varepsilon u_1(x) = v(x,0) + \varepsilon P(v(x,0)), \tag{4.2}$$

where $P(v(x,t)) = \frac{3\alpha_1-\alpha_2}{2} v_x(x,t)\partial^{-1}v(x,t) + \frac{3\alpha_1-\alpha_3}{2} v^2(x,t)$. Let $\tilde{u}(x,t)$ be defined by

$$\tilde{u}(x,t) = v(x,t) + \varepsilon P(v(x,t)). \tag{4.3}$$

Then, from Section 2 we know that $\tilde{u}(x,t)$ will solve the equation

$$\begin{aligned}\tilde{u}_t = \tilde{u}_{xx} &+ 2\tilde{u}\tilde{u}_x \\ &+ \varepsilon\left[\alpha_1\tilde{u}_{xxx} + \alpha_2\tilde{u}\tilde{u}_{xx} + \alpha_3\tilde{u}_x^2 + \left(\alpha_3 + \tfrac{\alpha_2}{2} - \tfrac{3\alpha_1}{2}\right)\tilde{u}^2\tilde{u}_x\right] + \varepsilon^2 f(\tilde{u}),\end{aligned} \tag{4.4a}$$

with the initial data (4.2); thus

$$\tilde{u}(x,0) = u_0(x) + \varepsilon u_1(x) = v(x,0) + \varepsilon P(v(x,0)). \tag{4.4b}$$

In (4.4a), the function $f$ has the form (2.12).

**Theorem 4.1.** *Let $u_0$ and $u_1$ in (1.2) be smooth. Let $u(x,t)$ be the solution of equation (1.1) and let $\tilde{u}(x,t)$ be the solution of equation (4.4a), where both these solutions satisfy the initial condition (1.2). Then, there exists a constant C, which depends on the function $f$, such that for sufficiently small $\epsilon > 0$ and for all $t > 0$,*

$$|u(\cdot,t) - \tilde{u}(\cdot,t)|_\infty \le C(f)\varepsilon^2 \quad \text{and} \quad |u(\cdot,t) - \tilde{u}(\cdot,t)|_2 \le C(f)\varepsilon^2.$$

Before we prove Theorem 4.1, we prove some decay estimates on the solution $\tilde{u}$.

**Lemma 4.2.** *Let $w_0(x)$ be in $L_1(\mathbb{R}) \cap H^k(\mathbb{R})$ for $k = 1, 2, \cdots$. Let $w$ be the solution of the initial-value problem*

$$w_t = w_{xx} + \alpha_1\varepsilon w_{xxx}, \tag{4.5}$$

$$w(x,0) = w_0(x). \tag{4.6}$$

*Then $w$ has the following properties:*

$$\lim_{t\to+\infty} t^{i+\frac{1}{2}} \int_{-\infty}^{\infty} [\partial_x^i w(x,t)]^2 dx = \frac{1\cdot 3\cdot 5\cdots(2i-1)}{(8\pi)^{\frac{1}{2}}4^i}\left(\int_{-\infty}^{\infty} w_0(x)dx\right)^2, \tag{4.7}$$

*for $i = 0, 1, 2, \cdots, k$.*

*Proof.* See [1], and see also [2]. □

**Lemma 4.3.** *Let $w_0(x)$ be positive, belong to $L_\infty(\mathbb{R})$ and be bounded below. Thus, there exists a number $m > 0$ such that $m = \min_{x\in\mathbb{R}}\{w_0(x)\}$. Furthermore, $\frac{dw_0(x)}{dx} \in L_1(\mathbb{R}) \cap H^2(\mathbb{R})$. If $\varepsilon$ is sufficiently small, the corresponding solution $w$ of (4.5) and (4.6) satisfies*

$$0 < \inf_{x\in\mathbb{R},\ t\ge 0} w(x,t) \le \sup_{x\in\mathbb{R},\ t\ge 0} w(x,t) < \infty. \tag{4.8}$$

*Proof.* Formally taking the Fourier transform of equation (4.5) with respect to the spatial variable $x$, and taking the inverse transform after solving the relevant ordinary differential equation, we find

$$w(x,t) = \int_{-\infty}^{\infty} \lambda_1(x,t) * w_0(x)dx = \int_{-\infty}^{\infty} \lambda_1(x-\xi,t)w_0(\xi)d\xi, \tag{4.9}$$

where

$$\hat{\lambda_1}(y,t) = \exp\left(-y^2 t - i\alpha_1 \varepsilon y^3 t\right). \tag{4.10}$$

By Lemma 3.1, we have

$$\int_{-\infty}^{\infty} |\lambda_1(x,t)|dx < \infty, \tag{4.11}$$

for all $t > 0$. Using (4.11) in (4.9), we find

$$\sup_{t\geq 0} |w(x,t)|_\infty \leq |\lambda_1(\cdot,t)|_1 |w_0(\cdot)|_\infty \leq C|w_0(\cdot)|_\infty. \tag{4.12}$$

The solution of (4.5)-(4.6) can be represented as

$$w(x,t) = \int_{-\infty}^{\infty} G(x,t) * w_0(x)dx + \alpha_1\varepsilon \int_0^t \int_{-\infty}^{\infty} G(x,t-\tau) * w_{xxx}(x,\tau)dxd\tau, \tag{4.13}$$

where $G(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{|x|^2}{4t}\right)$. The function $G$ satisfies

$$\int_{-\infty}^{\infty} G(x,t)dx = 1 \quad \text{and} \quad |G(\cdot,t)|_2 \leq Ct^{-\frac{1}{4}}. \tag{4.14}$$

It follows from Lemma 4.2 and (4.14) that

$$\begin{aligned} &|\int_0^t \int_{-\infty}^{\infty} G(x-\xi,t-\tau)w_{xxx}(\xi,\tau)d\xi d\tau| \\ &\leq \int_0^t |G(\cdot,t-\tau)|_2 |w_{xxx}(\cdot,\tau)|_2 d\tau \\ &\leq C(||(w_0)_x(\cdot)||_{H^2(\mathbb{R})}) \int_0^t (t-\tau)^{-\frac{1}{4}}(1+\tau)^{-\frac{5}{4}}d\tau \\ &\leq C(||(w_0)_x(\cdot)||_{L_1(\mathbb{R})\cap H^2(\mathbb{R})}). \end{aligned} \tag{4.15}$$

Because $m = \min_{x\in\mathbb{R}}\{w_0(x)\} > 0$, using (4.15) and (4.14) in (4.13) implies that

$$w(x,t) \geq m - \alpha_1\varepsilon C||(w_0)_x(\cdot)||_{L_1(\mathbb{R})\cap H^2(\mathbb{R})} > 0, \tag{4.16}$$

for all $t > 0$ and $\varepsilon \leq \frac{m}{2\alpha_1 C(||(w_0)_x(\cdot)||_{L_1(\mathbb{R})\cap H^2(\mathbb{R})})}$. □

**Lemma 4.4.** *Let $v_0(x) \in L_1(\mathbb{R}) \cap H^3(\mathbb{R})$. Assume that $\varepsilon$ is sufficiently small so that Lemma 4.3 is valid. Then, there exists a unique solution of equation (4.1) with the initial condition $v(x,0) = v_0(x)$. This solution has the following properties:*

$$\begin{gathered} |v(\cdot,t)|_1 \leq C, \quad |v(\cdot,t)|_\infty \leq C(1+t)^{-\frac{1}{2}}, \quad |v(\cdot,t)|_2 \leq C(1+t)^{-\frac{1}{4}}, \\ |v_x(\cdot,t)|_2 \leq C(1+t)^{-\frac{3}{4}}, |v_{xx}(\cdot,t)|_2 \leq C(1+t)^{-\frac{5}{4}}, \text{ and } |v_{xxx}(\cdot,t)|_2 \leq C(1+t)^{-\frac{7}{4}}. \end{gathered} \tag{4.17}$$

*Proof.* Define $w_0(x) = e^{\int_{-\infty}^x v_0(x)dx}$. Since $v_0(x) \in L_1(\mathbb{R})$, $w_0(x)$ is bounded below. Hence, $w_0(x)$ satisfies the conditions in Lemma 4.3. Let $w(x,t)$ be the solution of (4.5) with the initial condition $w_0(x)$. By Lemma 4.3, $w(x,t)$ is positive and bounded below; thus

$$v(x,t) = \frac{w_x(x,t)}{w(x,t)}$$

is well defined. One can easily show that $v$ satisfies (4.1) with the initial condition $v(x,0) = v_0(x)$. The properties in (4.17) follow from the corresponding properties of the solution (4.5) with the initial condition $w_0(x) = v_0(x)e^{\int_{-\infty}^{x} v_0(x)dx}$ in Lemma 4.2. □

**Corollary 4.5.** *Let $v_0(x) \in L_1(\mathbb{R}) \cap H^4(\mathbb{R})$ and let $v(x,t)$ be the solution of equation (4.1) with the initial condition (4.2). Define*

$$\begin{aligned}\tilde{u}(x,t) &= v(x,t) + \epsilon\tfrac{3\alpha_1-\alpha_2}{2} v_x(x,t)\partial^{-1}v(x,t) + \epsilon\tfrac{3\alpha_1-\alpha_3}{2} v^2(x,t)\\ &= v(x,t) + \epsilon P(v(x,t)).\end{aligned}$$

*Then, $\tilde{u}(x,t)$ has the following properties:*

$$\begin{aligned}&|\tilde{u}(\cdot,t)|_1 \le C, \quad |\tilde{u}(\cdot,t)|_\infty \le C_1(1+t)^{-\frac{1}{2}} + C_2\epsilon(1+t)^{-1},\\ &|\tilde{u}(\cdot,t)|_2 \le C_1(1+t)^{-\frac{1}{4}} + C_2\epsilon(1+t)^{-\frac{3}{4}},\\ &|\tilde{u}_x(\cdot,t)|_2 \le C_1(1+t)^{-\frac{3}{4}},\\ &|\tilde{u}_{xx}(\cdot,t)|_2 \le C_1(1+t)^{-\frac{5}{4}} \quad \text{and} \quad |\tilde{u}_{xxx}(\cdot,t)|_2 \le C_1(1+t)^{-\frac{7}{4}}.\end{aligned} \tag{4.18}$$

*Furthermore, for $0 \le \xi \le 1$,*

$$(1+t)^{\frac{1}{4}+\xi}\partial^{-1}f(\tilde{u}(x,t)) \in L_2(\mathbb{R}^+; L_\infty(\mathbb{R})),$$

$$\partial^{-1}f(\tilde{u}) \in L_1(\mathbb{R}^+; L_\infty(\mathbb{R}))$$

*and*

$$f(\tilde{u}) \in L_1(\mathbb{R}^+; L_2(\mathbb{R})).$$

*Proof.* The estimates (4.18) follow directly from Lemma 4.4 and the definition $\tilde{u}(x,t) = v + \varepsilon P(v)$. Using (4.18) and the definition of $f(\tilde{u}(x,t))$ in (2.12), we find

$$\begin{aligned}&\int_0^t (1+\tau)^{\frac{1}{2}+2\xi} \sup_{x\in\mathbb{R}} |\int_{-\infty}^{x} f(\tilde{u}(y,\tau))dy|^2 d\tau \le C\int_0^t (1+\tau)^{-\frac{3}{2}} d\tau \le C(f),\\ &\int_0^t \sup_{x\in\mathbb{R}} |\int_{-\infty}^{x} f(\tilde{u}(y,\tau))dy| d\tau \le C\int_0^t (1+\tau)^{-2} d\tau \le C(f)\end{aligned}$$

and

$$\int_0^t |f(\tilde{u}(\cdot,\tau))|_2 d\tau \le C\int_0^t (1+\tau)^{-9/4} d\tau \le C(f).$$

The corollary is then proved. □

*The Proof of Theorem 4.1.* Let $u(x,t)$ be the solution of equation (1.1) with the initial condition (1.2). By using the Cole-Hopf transformation $\int_{-\infty}^{x} u(y,t)dy = \ln(z(x,t))$, $z(x,t)$ satisfies

$$z_t - z_{xx} - \alpha_1\varepsilon z_{xxx} = \varepsilon z\Big[(\alpha_2 - 3\alpha_1)uu_x + (\tfrac{\alpha_4}{3} - \alpha_1)u^3 + (\alpha_3 - \alpha_2)\int_{-\infty}^{x} u_y^2 dy\Big]. \tag{4.19}$$

Let $\tilde{u}(x,t)$ be the solution of equation (4.4a) with the initial condition (4.4b). If $\int_{-\infty}^{x} \tilde{u}(y,t)dy = \ln(\tilde{z}(x,t))$, then, $\tilde{z}(x,t)$ satisfies

$$\begin{aligned}&\tilde{z}_t - \tilde{z}_{xx} - \alpha_1\varepsilon\tilde{z}_{xxx}\\ &= \varepsilon\tilde{z}\Big[(\alpha_2 - 3\alpha_1)\tilde{u}\tilde{u}_x + (\tfrac{\alpha_4}{3} - \alpha_1)\tilde{u}^3 + \int_{-\infty}^{x}\{(\alpha_3 - \alpha_2)\tilde{u}_y^2 + \varepsilon f(\tilde{u}(y,t))\}dy\Big].\end{aligned} \tag{4.20}$$

Note that since $|u(\cdot,t)|_1$ and $|\tilde{u}(\cdot,t)|_1$ are uniformly bounded from above for all $t \geq 0$, $z(x,t)$ and $\tilde{z}(x,t)$ are well defined. Furthermore, $|z(\cdot,t)|_\infty$ and $|\tilde{z}(\cdot,t)|_\infty$ are bounded for all $t \geq 0$.

Let $Z(x,t) = z(x,t) - \tilde{z}(x,t)$, $Z_x(x,t) = z_x(x,t) - \tilde{z}_x(x,t)$, and $U(x,t) = u(x,t) - \tilde{u}(x,t)$. Then, $Z(x,t)$ satisfies

$$
\begin{aligned}
Z_t - Z_{xx} - \alpha_1 \varepsilon Z_{xxx} &= \varepsilon Z\Big[(\alpha_2 - 3\alpha_1)uu_x + (\tfrac{\alpha_4}{3} - \alpha_1)u^3 + (\alpha_3 - \alpha_2)\textstyle\int_{-\infty}^x u_y^2 dy\Big] \\
&+ \varepsilon U\Big[(\alpha_2 - 3\alpha_1)(\tilde{z}u_x - \tilde{z}\tilde{u}_x - \tilde{z}_x\tilde{u}) + (\alpha_3 - \alpha_2)\tilde{z}(u_x + \tilde{u}_x) \\
&\qquad + (\tfrac{\alpha_4}{3} - \alpha_1)\tilde{z}(u^2 + u\tilde{u} + \tilde{u}^2)\Big] \\
&+ \varepsilon(\alpha_2 - 3\alpha_1)(\tilde{z}\tilde{u}U)_x - \varepsilon\tilde{z}\textstyle\int_{-\infty}^x \{(\alpha_3 - \alpha_2)U(u_{yy} + \tilde{u}_{yy}) + \varepsilon f(\tilde{u}(y,t))\}dy \\
&= \varepsilon Z g_1(u) + \varepsilon U g_2(u, \tilde{u}, z, \tilde{z}) + \varepsilon(\alpha_2 - 3\alpha_1)(\tilde{z}\tilde{u}U)_x \\
&\quad - \varepsilon\tilde{z}\textstyle\int_{-\infty}^x \{(\alpha_3 - \alpha_2)U(u_{yy} + \tilde{u}_{yy}) + \varepsilon f(\tilde{u}(y,t))\}dy,
\end{aligned} \tag{4.21}
$$

where

$$
g_1(u) = (\alpha_2 - 3\alpha_1)uu_x + (\tfrac{\alpha_4}{3} - \alpha_1)u^3 + (\alpha_3 - \alpha_2)\textstyle\int_{-\infty}^x u_y^2 dy
$$

and

$$
\begin{aligned}
g_2(u, \tilde{u}, z, \tilde{z}) = &(\alpha_2 - 3\alpha_1)(\tilde{z}u_x - \tilde{z}\tilde{u}_x - \tilde{z}_x\tilde{u}) + (\alpha_3 - \alpha_2)\tilde{z}(u_x + \tilde{u}_x) \\
&+ (\tfrac{\alpha_4}{3} - \alpha_1)\tilde{z}(u^2 + u\tilde{u} + \tilde{u}^2).
\end{aligned}
$$

Formally, taking the Fourier transform of equation (4.21) with respect to the spatial variable $x$, and after solving the relevant ordinary differential equation and taking the inverse transform, we find

$$
\begin{aligned}
Z(x,t) = &\varepsilon \textstyle\int_0^t \int_{-\infty}^\infty \lambda_1(x - s, t - \tau)\Big[Z g_1(u) \\
&- \tilde{z}\textstyle\int_{-\infty}^s \{(\alpha_3 - \alpha_2)U(u_{yy} + \tilde{u}_{yy}) + \varepsilon f(\tilde{u}(y,t))\}dy\Big]ds d\tau \\
&+ \varepsilon \textstyle\int_0^t \int_{-\infty}^\infty \Big[\lambda_1(x - s, t - \tau)U g_2(u, \tilde{u}, z, \tilde{z}) \\
&+ \lambda_2(x - s, t - \tau)(\alpha_2 - 3\alpha_1)\tilde{z}\tilde{u}U\Big]ds d\tau.
\end{aligned} \tag{4.22}
$$

Using Lemma 3.5 and Corollary 3.10, we find

$$
\begin{aligned}
&|g_1(u(\cdot,t))|_\infty \\
&\leq C_1|u(\cdot,t)u_x(\cdot,t)|_\infty + C_2|u(\cdot,t)|_\infty^3 + C_3|u_x(\cdot,t)|_2^2 \\
&\leq C[|u(\cdot,t)|_2^{\frac{1}{2}}|u_x(\cdot,t)|_2^2|u_{xx}(\cdot,t)|_2^{\frac{1}{2}} + |u(\cdot,t)|_2^{\frac{3}{2}}|u_x(\cdot,t)|_2^{\frac{3}{2}} + |u_x(\cdot,t)|_2^2] \\
&\leq C(1+t)^{-\frac{3}{2}}.
\end{aligned} \tag{4.23}
$$

Hence, $|g_1(u(\cdot,t))|_\infty \in L_1(\mathbb{R}^+)$. Since $|\lambda_1(\cdot,t)|_1 < +\infty$, the first term in the first double integral in (4.22) is bounded from above by

$$\begin{aligned} &\varepsilon \int_0^t \int_{-\infty}^\infty |\lambda_1(x-s,t-\tau) Z g_1(u)| ds d\tau \\ &\leq \varepsilon C \sup_{0\leq\tau\leq t} \Big\{ |Z(\cdot,\tau)|_\infty \Big\} \int_0^t |g_1(u(\cdot,\tau))|_\infty |\lambda_1(\cdot,t-\tau)|_1 d\tau \\ &\leq \varepsilon C \sup_{0\leq\tau\leq t} \Big\{ |Z(\cdot,\tau)|_\infty \Big\}. \end{aligned} \tag{4.24}$$

Similarly, since $|u_{xx}(\cdot,t)|_2 \in L_1(\mathbb{R}^+)$ by Corollary 3.10 and $|\tilde{u}_{xx}(\cdot,t)|_2 \in L_1(\mathbb{R}^+)$ by Corollary 4.5, the second term in the first double integral in (4.22) is bounded from above by

$$\begin{aligned} &\varepsilon \int_0^t \int_{-\infty}^\infty |\lambda_1(x-s,t-\tau) \tilde{z} \int_{-\infty}^s \{(\alpha_3-\alpha_2) U(u_{yy}+\tilde{u}_{yy}) + \varepsilon f(\tilde{u}(y,t))\} dy | ds d\tau \\ &\leq \varepsilon C \int_0^t |\lambda_1(\cdot,t-\tau)|_1 \Big\{ |U(\cdot,\tau)|_2 [|u_{xx}(\cdot,\tau)|_2 + |\tilde{u}_{xx}(\cdot,\tau)|_2] \\ &\qquad + \sup_{x\in\mathbb{R}} | \int_{-\infty}^x \varepsilon f(\tilde{u}(y,\tau)) dy | \Big\} d\tau \\ &\leq \varepsilon C \sup_{0\leq\tau\leq t} \Big\{ |U(\cdot,\tau)|_2 \Big\} \int_0^t [|u_{xx}(\cdot,\tau)|_2 + |\tilde{u}_{xx}(\cdot,\tau)|_2] d\tau \\ &\qquad + \varepsilon^2 C \int_0^t \sup_{x\in\mathbb{R}} | \int_{-\infty}^x f(\tilde{u}(y,\tau)) dy | d\tau \\ &\leq \varepsilon C \sup_{0\leq\tau\leq t} \Big\{ |U(\cdot,\tau)|_2 \Big\} + \varepsilon^2 C \int_0^t \sup_{x\in\mathbb{R}} | \int_{-\infty}^x f(\tilde{u}(y,\tau)) dy | d\tau. \end{aligned} \tag{4.25}$$

Using the fact $\tilde{z}_x = \tilde{z}\tilde{u}$ and the elementary inequality $|u(\cdot,t)|_\infty^2 \leq 2|u(\cdot,t)|_2 |u_x(\cdot,t)|_2$, we find

$$\begin{aligned} &|g_2(u(\cdot,t),\tilde{u}(\cdot,t),z(\cdot,t),\tilde{z}(\cdot,t))|_\infty \\ &= \sup_{x\in\mathbb{R}} \Big| (\alpha_2-3\alpha_1)(\tilde{z}u_x - \tilde{z}\tilde{u}_x - \tilde{z}\tilde{u}^2) + (\alpha_3-\alpha_2)\tilde{z}(u_x+\tilde{u}_x) \\ &\qquad + (\tfrac{\alpha_4}{3}-\alpha_1)\tilde{z}(u^2+u\tilde{u}+\tilde{u}^2) \Big| \\ &\leq C \Big[ |u_x(\cdot,t)|_2^{\frac12} |u_{xx}(\cdot,t)|_2^{\frac12} + |\tilde{u}_x(\cdot,t)|_2^{\frac12} |\tilde{u}_{xx}(\cdot,t)|_2^{\frac12} \\ &\qquad + |u(\cdot,t)|_2 |u_x(\cdot,t)|_2 + |\tilde{u}(\cdot,t)|_2 |\tilde{u}_x(\cdot,t)|_2 \Big] \\ &\leq C(1+t)^{-1}. \end{aligned} \tag{4.26}$$

Using Parseval's theorem, we find

$$|\lambda_1(\cdot,t-\tau)|_2^2 = \int_{-\infty}^\infty e^{-2y^2(t-\tau)} dy = \sqrt{\frac{\pi}{2(t-\tau)}}. \tag{4.27}$$

It follows from (4.26) and (2.27) that the first term in the second double integral in (4.22) is bounded from above by

$$\begin{aligned}
&\varepsilon \int_0^t \int_{-\infty}^{\infty} |\lambda_1(x-s,t-\tau)Ug_2(u,\tilde{u},z,\tilde{z})|dsd\tau \\
&\le \varepsilon C \int_0^t |\lambda_1(\cdot,t-\tau)|_2 |U(\cdot,\tau)g_2(\cdot,\tau)|_2 d\tau \\
&\le \varepsilon C \sup_{0\le\tau\le t}\Big\{|U(\cdot,\tau)|_2\Big\} \int_0^t \tfrac{1}{(t-\tau)^{1/4}(1+\tau)} d\tau \\
&\le \varepsilon C \sup_{0\le\tau\le t}\Big\{|U(\cdot,\tau)|_2\Big\}.
\end{aligned} \tag{4.28}$$

Using Parseval's theorem, we find

$$|\lambda_2(\cdot,t-\tau)|_2^2 = \int_{-\infty}^{\infty} y^2 e^{-2y^2(t-\tau)}dy = \sqrt{\tfrac{\pi}{32(t-\tau)^3}}. \tag{4.29}$$

Finally, by using (4.29), the second term in the second double integral in (4.22) is bounded from above by

$$\begin{aligned}
&\varepsilon \int_0^t \int_{-\infty}^{\infty} |\lambda_2(x-s,t-\tau)\tilde{z}\tilde{u}U|dsd\tau \\
&\le \varepsilon C \int_0^t |\lambda_2(\cdot,t-\tau)|_2 |U(\cdot,\tau)|_2 |\tilde{u}(\cdot,\tau)|_\infty d\tau \\
&\le \varepsilon C \sup_{0\le\tau\le t}\Big\{|U(\cdot,\tau)|_2\Big\} \int_0^t \tfrac{1}{(t-\tau)^{3/4}(1+\tau)^{1/2}} d\tau \\
&\le \varepsilon C \sup_{0\le\tau\le t}\Big\{|U(\cdot,\tau)|_2\Big\}.
\end{aligned} \tag{4.30}$$

Using the estimates (4.24), (4.25), (4.28) and (4.30) in (4.22) yields that for $\varepsilon$ suitable small,

$$\begin{aligned}
&\sup_{0\le\tau\le t}\Big\{|Z(\cdot,\tau)|_\infty\Big\} \\
&\le \varepsilon C \sup_{0\le\tau\le t}\Big\{|U(\cdot,\tau)|_2\Big\} + \varepsilon^2 C \int_0^t \sup_{x\in\mathbb{R}} \Big| \int_{-\infty}^x f(\tilde{u}(y,\tau))dy\Big| d\tau.
\end{aligned} \tag{4.31}$$

We multiply equation (4.21) by $2Z_{xx}$, and then integrate the resultant equation with respect to $x$ over $\mathbb{R}$; after integration by parts, we find

$$\begin{aligned}
&|Z_x(\cdot,t)|_2^2 + 2\int_0^t |Z_{xx}(\cdot,\tau)|_2^2 d\tau \\
&= 2\varepsilon \int_0^t \int_{-\infty}^{\infty} Z_{xx} Z\Big[(\alpha_2 - 3\alpha_1)uu_x + (\tfrac{\alpha_4}{3} - \alpha_1)u^3 \\
&\qquad + Ug_2(u,\tilde{u},z,\tilde{z}) + (\alpha_2 - 3\alpha_1)(\tilde{z}\tilde{u}U)_x\Big] dxd\tau \\
&\quad -2\varepsilon \int_0^t \int_{-\infty}^{\infty} (\alpha_3 - \alpha_2)\Big[Z_x^2 \int_{-\infty}^x u_y^2 dy + Z_x Z u_x^2\Big] dxd\tau \\
&\quad +2\varepsilon \int_0^t \int_{-\infty}^{\infty} Z_x \tilde{z}_x \Big[\int_{-\infty}^x \{(\alpha_3 - \alpha_2)U(u_{yy} + \tilde{u}_{yy}) + \varepsilon f(\tilde{u}(y,\tau))\}dy\Big] dxd\tau \\
&\quad +2\varepsilon \int_0^t \int_{-\infty}^{\infty} Z_x \tilde{z}\Big[(\alpha_3 - \alpha_2)U(u_{xx} + \tilde{u}_{xx}) + \varepsilon f(\tilde{u}(x,\tau))\Big] dxd\tau.
\end{aligned} \tag{4.32}$$

By using the Cauchy-Schwarz inequality, the first double integral in (4.32) is estimated as

(4.33)
$$
\begin{aligned}
I_1(t) = \varepsilon\Big|\int_0^t\int_{-\infty}^{\infty} Z_{xx}Z\Big[(\alpha_2-3\alpha_1)uu_x+(\tfrac{\alpha_4}{3}-\alpha_1)u^3 \\
+Ug_2(u,\tilde{u},z,\tilde{z})+(\alpha_2-3\alpha_1)(\tilde{z}\tilde{u}U)_x\Big]dxd\tau\Big| \\
\le \int_0^t |Z_{xx}(\cdot,\tau)|_2^2 d\tau+\varepsilon^2 C\int_0^t\int_{-\infty}^{\infty}\Big[Z^2(u^2u_x^2+u^6) \\
+U^2g_2^2+[\tilde{z}_x^2\tilde{u}^2+\tilde{z}^2\tilde{u}_x^2]U^2+\tilde{z}^2\tilde{u}^2U_x^2\Big]dxd\tau \\
\le \int_0^t |Z_{xx}(\cdot,\tau)|_2^2 d\tau + \varepsilon^2 C\int_0^t |\tilde{z}(\cdot,\tau)\tilde{u}(\cdot,\tau)|_\infty^2 |U_x(\cdot,\tau)|_2^2 d\tau \\
+\varepsilon^2 C\sup_{0\le\tau\le t}\Big\{|Z(\cdot,\tau)|_\infty^2\Big\}\int_0^t\Big[|u(\cdot,\tau)u_x(\cdot,\tau)|_2^2+|u(\cdot,\tau)|_6^6)\Big]d\tau \\
+\varepsilon^2 C\sup_{0\le\tau\le t}\Big\{|U(\cdot,\tau)|_2^2\Big\}\int_0^t\Big[|g_2(\cdot,\tau)|_\infty^2+|\tilde{z}_x(\cdot,\tau)\tilde{u}(\cdot,\tau)|_\infty^2 \\
+|\tilde{z}(\cdot,\tau)\tilde{u}_x(\cdot,\tau)|_\infty^2\Big]d\tau.
\end{aligned}
$$

Using Lemma 3.5 and Corollary 3.10, we have

$$\int_0^t \Big[|u(\cdot,\tau)u_x(\cdot,\tau)|_2^2 + |u(\cdot,\tau)|_6^6\Big]d\tau \le C\int_0^t (1+\tau)^{-5/2}d\tau \le C.$$

Similarly, using (4.26) and Corollary 4.5, we have

$$\int_0^t \Big[|g_2(\cdot,\tau)|_\infty^2+|\tilde{z}_x(\cdot,\tau)\tilde{u}(\cdot,\tau)|_\infty^2+|\tilde{z}(\cdot,\tau)\tilde{u}_x(\cdot,\tau)|_\infty^2\Big]d\tau \le C\int_0^t (1+\tau)^{-2}d\tau \le C$$

and

$$|\tilde{z}(\cdot,\tau)\tilde{u}(\cdot,\tau)|_\infty^2 \le C(1+\tau)^{-1}.$$

Hence, using these estimates in (4.33), we find

(4.34)
$$
\begin{aligned}
I_1(t) \le \int_0^t |Z_{xx}(\cdot,\tau)|_2^2 d\tau \\
+\varepsilon^2 C\Big(\sup_{0\le\tau\le t}\Big\{|Z(\cdot,\tau)|_\infty^2\Big\}+\sup_{0\le\tau\le t}\Big\{|U(\cdot,\tau)|_2^2\Big\}+\int_0^t \tfrac{|U_x(\cdot,\tau)|_2^2}{1+\tau}d\tau\Big).
\end{aligned}
$$

By using the Cauchy-Schwarz inequality, the second double integral in (4.32) is bounded from above by

$$
\begin{aligned}
I_2(t) = \varepsilon\Big|\int_0^t\int_{-\infty}^{\infty}(\alpha_3-\alpha_2)\Big[Z_x^2\int_{-\infty}^x u_y^2 dy + Z_xZu_x^2\Big]dxd\tau\Big| \\
\le \varepsilon C\int_0^t\Big[|Z_x(\cdot,\tau)|_2^2|u_x(\cdot,\tau)|_2^2 + |Z(\cdot,\tau)|_\infty|u_x(\cdot,\tau)|_\infty|Z_x(\cdot,\tau)|_2|u_x(\cdot,\tau)|_2\Big]d\tau \\
\le \int_0^t\Big[(\delta+\varepsilon)C|Z_x(\cdot,\tau)|_2^2|u_x(\cdot,\tau)|_2^2 d\tau + \varepsilon^2\tfrac{C}{\delta}|Z(\cdot,\tau)|_\infty^2|u_x(\cdot,\tau)|_\infty^2\Big]d\tau.
\end{aligned}
$$

Using the elementary inequality $|u_x(\cdot,\tau)|_\infty^2 \le 2|u_x(\cdot,\tau)|_2|u_{xx}(\cdot,\tau)|_2$ and Corollary 3.10, we find

$$
\begin{aligned}
I_2(t) \le\ & (\delta+\varepsilon)C\int_0^t |Z_x(\cdot,\tau)|_2^2|u_x(\cdot,\tau)|_2^2 d\tau \\
& +\varepsilon^2\frac{C}{\delta}\int_0^t |Z(\cdot,\tau)|_\infty^2|u_{xx}(\cdot,\tau)|_2|u_x(\cdot,\tau)|_2 d\tau \\
\le\ & (\delta+\varepsilon)C\sup_{0\le\tau\le t}\left\{|Z_x(\cdot,\tau)|_2^2\right\}\int_0^t(1+\tau)^{-\frac{3}{2}}d\tau \\
& +\varepsilon^2\frac{C}{\delta}\sup_{0\le\tau\le t}\left\{|Z(\cdot,\tau)|_\infty^2\right\}\int_0^t(1+\tau)^{-2}d\tau \\
\le\ & (\delta+\varepsilon)C\sup_{0\le\tau\le t}\left\{|Z_x(\cdot,\tau)|_2^2\right\} \\
& +\varepsilon^2\frac{C}{\delta}\sup_{0\le\tau\le t}\left\{|Z(\cdot,\tau)|_\infty^2\right\}.
\end{aligned}
\tag{4.35}
$$

Similarly, by applying Corollary 3.10 to $u_{xx}$ and Corollary 4.5 to $\tilde{z}_x = \tilde{z}\tilde{u}$ and $\tilde{u}_{xx}$, the third integral in (4.32) is bounded from above by

$$
\begin{aligned}
I_3(t) \ =\ & \varepsilon\left|\int_0^t\int_{-\infty}^\infty Z_x\tilde{z}_x\left[\int_{-\infty}^x\{(\alpha_3-\alpha_2)U(u_{yy}+\tilde{u}_{yy})+\varepsilon f(\tilde{u}(y,\tau))\}dy\right]dxd\tau\right| \\
\le\ & \varepsilon\int_0^t |Z_x(\cdot,\tau)|_2|\tilde{z}_x(\cdot,\tau)|_2\Big[|U(\cdot,\tau)|_2(|u_{xx}(\cdot,\tau)|_2+|\tilde{u}_{xx}(\cdot,\tau)|_2) \\
& \qquad +\sup_{x\in\mathbb{R}}\left|\int_{-\infty}^x\varepsilon f(\tilde{u}(y,\tau))dy\right|\Big]d\tau \\
\le\ & \varepsilon C\int_0^t\Big[|Z_x(\cdot,\tau)|_2|U(\cdot,\tau)|_2(1+\tau)^{-3/2} \\
& \qquad +|Z_x(\cdot,\tau)|_2(1+\tau)^{-1/4}\sup_{x\in\mathbb{R}}\left|\int_{-\infty}^x\varepsilon f(\tilde{u}(y,\tau))dy\right|\Big]d\tau.
\end{aligned}
$$

For some suitable positive numbers $\delta$ and $\xi$, by using the Cauchy-Schwarz inequality we have

$$
\begin{aligned}
I_3(t) \ \le\ & \frac{\delta}{4}\int_0^t |Z_x(\cdot,\tau)|_2^2(1+\tau)^{-3/2}d\tau \\
& +\varepsilon^2\frac{C}{\delta}\int_0^t |U(\cdot,\tau)|_2^2(1+\tau)^{-3/2}d\tau \\
& +\frac{\delta}{4}\int_0^t |Z_x(\cdot,\tau)|_2^2(1+\tau)^{-(1+\xi)}d\tau \\
& +\varepsilon^4\frac{C}{\delta}\int_0^t(1+\tau)^{\frac{1}{2}+\xi}\sup_{x\in\mathbb{R}}\left|\int_{-\infty}^x f(\tilde{u}(y,\tau))dy\right|^2d\tau \\
\le\ & \delta\sup_{0\le\tau\le t}\left\{|Z_x(\cdot,\tau)|_2^2\right\}+\varepsilon^2\frac{C}{\delta}\sup_{0\le\tau\le t}\left\{|U(\cdot,\tau)|_2^2\right\} \\
& +\varepsilon^4\frac{C}{\delta}\int_0^t(1+\tau)^{\frac{1}{2}+\xi}\sup_{x\in\mathbb{R}}\left|\int_{-\infty}^x f(\tilde{u}(y,\tau))dy\right|^2d\tau.
\end{aligned}
\tag{4.36}
$$

Finally, applying Corollary 3.10 to $u_{xx}$, Lemma 4.4 to $\tilde{z}$, and Corollary 4.5 to $\tilde{u}_{xx}$, for a small $\delta > 0$, the fourth integral in (4.32) is bounded from above by

$$\begin{aligned} I_4(t) &= \varepsilon\Big|\int_0^t\int_{-\infty}^{\infty} Z_x\tilde{z}\Big[(\alpha_3-\alpha_2)U(u_{xx}+\tilde{u}_{xx})+\varepsilon f(\tilde{u}(x,\tau))\Big]dxd\tau\Big| \\ &\le \varepsilon C\int_0^t |Z_x(\cdot,\tau)|_2|\tilde{z}(\cdot,\tau)|_\infty\Big[|U(\cdot,\tau)|_\infty(|u_{xx}(\cdot,\tau)|_2 \\ &\qquad +|\tilde{u}_{xx}(\cdot,\tau)|_2)+\varepsilon|f(\tilde{u}(\cdot,\tau))|_2\Big]d\tau \\ &\le \varepsilon C\int_0^t\Big[|Z_x(\cdot,\tau)|_2|U(\cdot,\tau)|_2^{\frac{1}{2}}|U_x(\cdot,\tau)|_2^{\frac{1}{2}}(1+\tau)^{-\frac{5}{4}} \\ &\qquad +\varepsilon|Z_x(\cdot,\tau)|_2|f(\tilde{u}(\cdot,\tau))|_2\Big]d\tau \\ &\le \frac{\delta}{8}\int_0^t|Z_x(\cdot,\tau)|_2^2(1+\tau)^{-\frac{5}{4}}d\tau+\varepsilon^2\frac{C}{\delta}\int_0^t|U(\cdot,\tau)|_2|U_x(\cdot,\tau)|_2(1+\tau)^{-\frac{5}{4}}d\tau \\ &\qquad +\varepsilon^2C\sup_{0\le\tau\le t}\Big\{|Z_x(\cdot,\tau)|_2\Big\}\int_0^t|f(\tilde{u}(\cdot,\tau))|_2d\tau. \end{aligned} \tag{4.37}$$

Using the Cauchy-Schwarz inequality, we find

$$\int_0^t|U(\cdot,\tau)|_2|U_x(\cdot,\tau)|_2(1+\tau)^{-\frac{5}{4}}d\tau\le\int_0^t\Big[|U(\cdot,\tau)|_2^2(1+\tau)^{-\frac{3}{2}}+|U_x(\cdot,\tau)|_2^2(1+\tau)^{-1}\Big]d\tau.$$

For any $\delta > 0$, we have

$$\begin{aligned} &\varepsilon^2C\sup_{0\le\tau\le t}\Big\{|Z_x(\cdot,\tau)|_2\Big\}\int_0^t|f(\tilde{u}(\cdot,\tau))|_2d\tau \\ &\le\frac{\delta}{2}\sup_{0\le\tau\le t}\Big\{|Z_x(\cdot,\tau)|_2^2\Big\}+\varepsilon^4\frac{C}{\delta}\Big(\int_0^t|f(\tilde{u}(\cdot,\tau))|_2d\tau\Big)^2. \end{aligned}$$

With these estimates, (4.37) reduces to

$$\begin{aligned} I_4(t) &\le \frac{\delta}{8}\int_0^t|Z_x(\cdot,\tau)|_2^2(1+\tau)^{-\frac{5}{4}}d\tau \\ &\quad+\varepsilon^2\frac{C}{\delta}\Big(\int_0^t|U(\cdot,\tau)|_2^2(1+\tau)^{-\frac{3}{2}}d\tau+\int_0^t\frac{|U_x(\cdot,\tau)|_2^2}{1+\tau}d\tau\Big) \\ &\quad+\frac{\delta}{2}\sup_{0\le\tau\le t}\Big\{|Z_x(\cdot,\tau)|_2^2\Big\}+\varepsilon^4\frac{C}{\delta}\Big(\int_0^t|f(\tilde{u}(\cdot,\tau))|_2d\tau\Big)^2 \\ &\le\frac{\delta}{8}\sup_{0\le\tau\le t}\Big\{|Z_x(\cdot,\tau)|_2^2\Big\}\int_0^t(1+\tau)^{-\frac{5}{4}}d\tau \\ &\quad+\varepsilon^2\frac{C}{\delta}\Big(\sup_{0\le\tau\le t}|U(\cdot,\tau)|_2^2\int_0^t(1+\tau)^{-\frac{3}{2}}d\tau+\int_0^t\frac{|U_x(\cdot,\tau)|_2^2}{1+\tau}d\tau\Big) \\ &\quad+\frac{\delta}{2}\sup_{0\le\tau\le t}\Big\{|Z_x(\cdot,\tau)|_2^2\Big\}+\varepsilon^4\frac{C}{\delta}\Big(\int_0^t|f(\tilde{u}(\cdot,\tau))|_2d\tau\Big)^2 \\ &\le\delta\sup_{0\le\tau\le t}\Big\{|Z_x(\cdot,\tau)|_2^2\Big\}+\varepsilon^2C\Big(\sup_{0\le\tau\le t}\Big\{|U(\cdot,\tau)|_2^2\Big\}+\int_0^t\frac{|U_x(\cdot,\tau)|_2^2}{1+\tau}d\tau\Big) \\ &\quad+\varepsilon^4C\Big(\int_0^t|f(\tilde{u}(\cdot,\tau))|_2d\tau\Big)^2. \end{aligned} \tag{4.38}$$

Using the estimates (4.34), (4.35), (4.36) and (4.38) in (4.32), we obtain

$$\begin{aligned} &|Z_x(\cdot,t)|_2^2+\int_0^t|Z_{xx}(\cdot,\tau)|_2^2d\tau \\ &\le \varepsilon^2C_1\Big(\sup_{0\le\tau\le t}\Big\{|Z(\cdot,\tau)|_\infty^2\Big\}+\sup_{0\le\tau\le t}\Big\{|U(\cdot,\tau)|_2^2\Big\}+\int_0^t\tfrac{|U_x(\cdot,\tau)|_2^2}{1+\tau}d\tau\Big) \\ &+\varepsilon^4C\Big(\int_0^t(1+\tau)^{\frac12+\xi}\sup_{x\in\mathbb{R}}\Big|\int_{-\infty}^x f(\tilde u(y,\tau))dy\Big|^2d\tau+\big(\int_0^t|f(\tilde u(\cdot,\tau))|_2d\tau\big)^2\Big). \end{aligned} \tag{4.39}$$

On the other hand, by the definition

$$U(x,t)=u(x,t)-\tilde u(x,t)=\frac{z_x}{z}-\frac{\tilde z_x}{\tilde z}=\frac{Z_x\tilde z-Z\tilde z_x}{z\tilde z},$$

we have

$$|U(\cdot,t)|_2^2\le C\Big(|Z_x(\cdot,t)|_2^2+|Z(\cdot,t)|_\infty^2|\tilde z_x(\cdot,t)|_2^2\Big).$$

By Corollary 4.5 where $|\tilde z_x(\cdot,t)|_2^2\le C(1+t)^{-\frac12}$, we have

$$|U(\cdot,t)|_2^2\le C_2\Big(|Z_x(\cdot,t)|_2^2+|Z(\cdot,t)|_\infty^2\Big). \tag{4.40}$$

Furthermore, we note that

$$U_x(x,t)=u_x(x,t)-\tilde u_x(x,t)=\frac{\big(Z_{xx}\tilde z-Z\tilde z_{xx}\big)z\tilde z-\big(Z_x\tilde z-Z\tilde z_x\big)(z_x\tilde z+z\tilde z_x)}{(z\tilde z)^2}. \tag{4.41}$$

Using Lemma 3.5, Corollary 3.10 and Corollary 4.5 to (4.41), we find

$$\begin{aligned} &|U_x(\cdot,t)|_2^2 \\ &\le C\Big\{|Z_{xx}(\cdot,t)|_2^2+|Z_x(\cdot,t)|_2^2\big(|\tilde z_x(\cdot,t)|_2|\tilde z_{xx}(\cdot,t)|_2+|z_x(\cdot,t)|_2|z_{xx}(\cdot,t)|_2\big) \\ &\qquad+|Z(\cdot,t)|_\infty^2\big(|z_{xx}(\cdot,t)|_2^2+|\tilde z_x(\cdot,t)|_2^2\big)|\tilde z_x(\cdot,t)|_2|\tilde z_{xx}(\cdot,t)|_2 \\ &\qquad+|Z(\cdot,t)|_\infty^2\big(|z_{xx}(\cdot,t)|_2^2+|\tilde z_x(\cdot,t)|_2^2\big)|z_x(\cdot,t)|_2|z_{xx}(\cdot,t)|_2\Big\} \\ &\le C\Big\{|Z_{xx}(\cdot,t)|_2^2+|Z_x(\cdot,t)|_2^2(1+t)^{-1}+|Z(\cdot,t)|_\infty^2(1+t)^{-3/2}\Big\}. \end{aligned} \tag{4.42}$$

Multiplying (4.42) by $(1+t)^{-1}$ and then integrating the result with respect to $t$ over $[0,t]$, we have

$$\begin{aligned} \int_0^t\tfrac{|U_x(\cdot,\tau)|_2^2}{1+\tau}d\tau &\le C\Big\{\int_0^t\tfrac{|Z_{xx}(\cdot,\tau)|_2^2}{1+\tau}d\tau+\int_0^t\tfrac{|Z_x(\cdot,\tau)|_2^2}{(1+\tau)^2}d\tau+\int_0^t\tfrac{|Z(\cdot,\tau)|_\infty^2}{(1+\tau)^{5/2}}d\tau\Big\} \\ &\le C_3\Big(\int_0^t|Z_{xx}(\cdot,\tau)|_2^2d\tau+\sup_{0\le\tau\le t}\Big\{|Z_x(\cdot,\tau)|_2^2+|Z(\cdot,\tau)|_\infty^2\Big\}\Big). \end{aligned} \tag{4.43}$$

Squaring (4.31) on both sides and applying an elementary inequality, we obtain

$$\begin{aligned} \sup_{0\le\tau\le t}\Big\{|Z(\cdot,\tau)|_\infty^2\Big\}&\le\varepsilon^2C_4\sup_{0\le\tau\le t}\Big\{|U(\cdot,\tau)|_2^2\Big\} \\ &\qquad+\varepsilon^4C\Big(\int_0^t\sup_{x\in\mathbb{R}}\Big|\int_{-\infty}^x f(\tilde u(y,\tau))dy\Big|d\tau\Big)^2. \end{aligned} \tag{4.44}$$

Multiplying (4.40) by $(1+C_1+C_4)\varepsilon^2$ and (4.43) by $(1+C_1)\varepsilon^2$, respectively, then adding the results (4.39) and (4.44) together, if $\varepsilon$ is chosen sufficiently small such that

$$\varepsilon C_1 \leq \frac{1}{4}, \quad \varepsilon^2 C_2(1+C_1+C_4) \leq \frac{1}{4} \quad \text{and} \quad \varepsilon^2 C_3(1+C_1) \leq \frac{1}{4},$$

we have

$$\begin{aligned}&\sup_{0\leq\tau\leq t}\Big\{|Z_x(\cdot,\tau)|_2^2+|Z(\cdot,\tau)|_\infty^2+\varepsilon^2|U(\cdot,\tau)|_2^2\Big\}+\textstyle\int_0^t\Big(|Z_{xx}(\cdot,\tau)|_2^2+\varepsilon^2\frac{|U_x(\cdot,\tau)|_2^2}{1+\tau}\Big)d\tau\\ &\leq \varepsilon^4 C\Big[\Big(\textstyle\int_0^t \sup_{x\in\mathbb{R}}\Big|\int_{-\infty}^x f(\tilde{u}(y,\tau))dy\Big|d\tau\Big)^2\\ &\quad+\textstyle\int_0^t(1+\tau)^{\frac{1}{2}+\xi}\sup_{x\in\mathbb{R}}|\int_{-\infty}^x f(\tilde{u}(y,\tau))dy|^2d\tau+\Big(\int_0^t|f(\tilde{u}(\cdot,\tau))|_2 d\tau\Big)^2\Big].\end{aligned}$$

Applying Corollary 4.5 in above estimate yields

$$\sup_{0\leq\tau\leq t}\Big\{|Z_x(\cdot,\tau)|_2^2+|Z(\cdot,\tau)|_\infty^2\Big\}+\textstyle\int_0^t|Z_{xx}(\cdot,\tau)|_2^2d\tau \leq C(f)\varepsilon^4. \tag{4.45}$$

Using (4.45) in (4.40), we obtain

$$\sup_{0\leq\tau\leq t}|U(\cdot,\tau)|_2^2 \leq C(f)\varepsilon^4 \quad \text{or} \quad |u(\cdot,t)-\tilde{u}(\cdot,t)|_2^2 \leq C(f)\varepsilon^4, \tag{4.46}$$

for all $t>0$. Furthermore, using (4.45) in (4.43), we obtain

$$\textstyle\int_0^t \frac{|U_x(\cdot,\tau)|_2^2}{1+\tau}d\tau \leq C(f)\varepsilon^4. \tag{4.47}$$

Since $U(x,t)=u(x,t)-\tilde{u}(x,t)$, the function $U(x,t)$ satisfies

$$\begin{aligned}&U_t-U_{xx}-\alpha_1\varepsilon U_{xxx}=2Uu_x+2\tilde{u}U_x\\ &\quad+\varepsilon\Big[\alpha_2(Uu_{xx}+\tilde{u}U_{xx})+\alpha_3U_x(u_x+\tilde{u}_x)+\alpha_4[U(u+\tilde{u})\tilde{u}_x+\tilde{u}^2U_x]\Big]-\varepsilon^2 f(\tilde{u}).\end{aligned} \tag{4.48}$$

We multiply equation (4.48) by $2U_{xx}$ and then integrate the resulting equation over $\mathbb{R}\times[0,t]$. After integration by parts, we obtain

$$\begin{aligned}&|U_x(\cdot,t)|_2^2+2\textstyle\int_0^t|U_{xx}(\cdot,\tau)|_2^2d\tau\\ &=-\textstyle\int_0^t\int_{-\infty}^\infty 2U_{xx}U\Big[2u_x+\varepsilon[\alpha_2u_{xx}+\alpha_4(u+\tilde{u})\tilde{u}_x]\Big]dxd\tau\\ &\quad-\textstyle\int_0^t\int_{-\infty}^\infty 2U_{xx}U_x\Big[2\tilde{u}+\varepsilon[\alpha_3(u_x+\tilde{u}_x)+\alpha_4\tilde{u}^2]\Big]dxd\tau\\ &\quad-\textstyle\int_0^t\int_{-\infty}^\infty\Big[2\varepsilon\alpha_2U_{xx}^2\tilde{u}-2\varepsilon^2U_{xx}f(\tilde{u})\Big]dxd\tau.\end{aligned} \tag{4.49}$$

By using the Cauchy-Schwarz inequality, it follows that the first double integral in (4.49) is bounded:

$$
\begin{aligned}
I_{x1} &= \int_0^t\int_{-\infty}^{\infty} \Big|2U_{xx}U\Big[2u_x + \varepsilon[\alpha_2 u_{xx} + \alpha_4(u+\tilde{u})\tilde{u}_x]\Big]\Big| dx d\tau \\
&\le \tfrac{1}{3}\int_0^t |U_{xx}(\cdot,\tau)|_2^2 d\tau + C\int_0^t \Big[\varepsilon^2\alpha_2^2|u_{xx}(\cdot,\tau)|_2^2|U(\cdot,\tau)|_\infty^2 \\
&\qquad + |U(\cdot,\tau)|_2^2(|u_x(\cdot,\tau)|_\infty^2 + \varepsilon^2\alpha_4^2|(u+\tilde{u})\tilde{u}_x|_\infty^2)\Big] d\tau \\
&\le \tfrac{1}{3}\int_0^t |U_{xx}(\cdot,\tau)|_2^2 d\tau + \varepsilon^2 C\int_0^t |U_x(\cdot,\tau)|_2^2|u_{xx}(\cdot,\tau)|_2^2 d\tau \\
&\quad + C\sup_{0\le\tau\le t}|U(\cdot,\tau)|_2^2\int_0^t \Big[|u_x(\cdot,\tau)|_\infty^2 + |\tilde{u}_x(\cdot,\tau)|_\infty^2 + |u_{xx}(\cdot,\tau)|_2^2\Big] d\tau \\
&\le \tfrac{1}{3}\int_0^t |U_{xx}(\cdot,\tau)|_2^2 d\tau + C\varepsilon^2\int_0^t \tfrac{|U_x(\cdot,\tau)|_2^2}{1+\tau} d\tau \\
&\qquad + C\sup_{0\le\tau\le t}|U(\cdot,\tau)|_2^2\int_0^t \tfrac{1}{(1+\tau)^2} d\tau.
\end{aligned} \tag{4.50}
$$

Equations (4.46) and (4.47) imply

$$
I_{x1} \le \tfrac{1}{3}\int_0^t |U_{xx}(\cdot,\tau)|_2^2 d\tau + C(f)\varepsilon^4. \tag{4.51}
$$

In a similar way, the second double integral in (4.49) is bounded by

$$
\begin{aligned}
I_{x2} &= \int_0^t\int_{-\infty}^{\infty} \Big|2U_{xx}U_x\Big[2\tilde{u} + \varepsilon[\alpha_3(\tilde{u}_x + u_x) + \alpha_4\tilde{u}^2]\Big]\Big| dx d\tau \\
&\le \int_0^t \Big[\tfrac{1}{3}|U_{xx}(\cdot,\tau)|_2^2 + C|U_x(\cdot,\tau)|_2^2\Big(|\tilde{u}(\cdot,\tau)|_\infty^2 + |\tilde{u}_x(\cdot,\tau)|_\infty^2 + |u_x(\cdot,\tau)|_\infty^2\Big)\Big] d\tau \\
&\le \tfrac{1}{3}\int_0^t |U_{xx}(\cdot,\tau)|_2^2 d\tau + C\int_0^t \tfrac{|U_x(\cdot,\tau)|_2^2}{(1+\tau)} d\tau \\
&\le \tfrac{1}{3}\int_0^t |U_{xx}(\cdot,\tau)|_2^2 d\tau + C(f)\varepsilon^4.
\end{aligned} \tag{4.52}
$$

If $\varepsilon$ is sufficiently small, then the last double integral in (4.49) is bounded by

$$
\begin{aligned}
I_{x3} &= \int_0^t\int_{-\infty}^{\infty} \Big|2\varepsilon\alpha_2 U_{xx}^2\tilde{u} - 2\varepsilon^2 U_{xx} f(\tilde{u})\Big| dx d\tau \\
&\le \tfrac{1}{3}\int_0^t |U_{xx}(\cdot,\tau)|_2^2 d\tau + \varepsilon^4 C\int_0^t |f(\tilde{u}(\cdot,\tau))|_2^2 d\tau.
\end{aligned} \tag{4.53}
$$

Using (4.51), (4.52) and (4.53) in (4.49), we find

$$
|U_x(\cdot,t)|_2^2 + \int_0^t |U_{xx}(\cdot,\tau)|_2^2 d\tau \le C(f)\varepsilon^4. \tag{4.54}
$$

Using (4.46) and (4.54), we obtain

$$
|u(\cdot,t) - \tilde{u}(\cdot,t)|_\infty^2 = |U(\cdot,t)|_\infty^2 \le 2|U(\cdot,t)|_2|U_x(\cdot,t)|_2 \le C(f)\varepsilon^4,
$$

for all $t > 0$, and the theorem is proved. □

Now we show how to select a solution of (4.1) in order to approximate the solution of the initial value problem (1.1)-(1.2).

**Corollary 4.6.** *Let $u_0(x) \in H^4(\mathbb{R}) \cap L_1(\mathbb{R})$ and $u_1(x) \in H^3(\mathbb{R}) \cap L_1(\mathbb{R})$. Let $u(x,t)$ be the solution of (1.1) with the initial data $u_0$ and $u_1$ in (1.2). Define $\tilde{v}(x,0,\varepsilon)$ by*

$$\tilde{v}(x,0,\varepsilon) = u_0(x) + \epsilon\left[u_1(x) - P(u_0(x))\right]. \tag{4.55}$$

*Let $\tilde{v}(x,t,\varepsilon)$ be the solution (4.1) with the initial condition (4.55). Define*

$$\tilde{u}(x,t) = \tilde{v}(x,t,\varepsilon) + \varepsilon P(\tilde{v}(x,t,\varepsilon)). \tag{4.56}$$

*Then, for all $t > 0$,*

$$|u(\cdot,t) - \tilde{u}(\cdot,t)|_\infty \le C_1\varepsilon^2 \quad \text{and} \quad |u(\cdot,t) - \tilde{u}(\cdot,t)|_2 \le C_2\varepsilon^2, \tag{4.57}$$

*where $C_1$ and $C_2$ depend on $||u_0(\cdot)||_{H^4(\mathbb{R})\cap L_1(\mathbb{R})}$ and $||u_1(\cdot)||_{H^3(\mathbb{R})\cap L_1(\mathbb{R})}$.*

*Proof.* Let $v(x,t,\varepsilon)$ be the solution (4.1)-(4.2) and

$$\tilde{\tilde{u}}(x,t) = v(x,t,\varepsilon) + \varepsilon P(v(x,t,\varepsilon)).$$

By the definitions of $v(x,0,\varepsilon)$ and $\tilde{v}(x,0,\varepsilon)$, we have

$$\begin{aligned} v(x,0,\varepsilon) - \tilde{v}(x,0,\varepsilon) &= \varepsilon^2\Big(F(u_0) + \beta(u_0)_x\partial^{-1}u_1 + \beta(u_1)_x\partial^{-1}u_0 + \gamma u_0u_1\Big) \\ &= \varepsilon^2\Big(\beta^2(u_0)_{xx}(\partial^{-1}u_0)^2 + 2\beta(\beta+\gamma)u_0(u_0)_x\partial^{-1}u_0 \\ &\quad + \beta(\tfrac{\gamma}{2} - \beta)(u_0)_x\partial^{-1}(u_0)^2 + \tfrac{\gamma^2}{2}(u_0)^3 \\ &\quad + \beta(u_0)_x\partial^{-1}u_1 + \beta(u_1)_x\partial^{-1}u_0 + \gamma u_0u_1\Big) = R(u_0,u_1). \end{aligned} \tag{4.58}$$

A simple computation shows that

$$||R(u(\cdot)_0, u_1(\cdot))||_{W^{1,\infty}} \le C\varepsilon^2 \quad \text{and} \quad ||R(u(\cdot)_0, u_1(\cdot))||_1 \le C\varepsilon^2, \tag{4.59}$$

where $C$ depends on $||u_0(\cdot)||_{H^4(\mathbb{R})\cap L_1(\mathbb{R})}$ and $||u_1(\cdot)||_{H^3(\mathbb{R})\cap L_1(\mathbb{R})}$.

Since (4.1) is a linearizable equation, we can easily show that its solution depends on the initial data continuously. Thus, there exist constants $C_1$ and $C_2$, which depend on $||u_0(\cdot)||_{H^4(\mathbb{R})\cap L_1(\mathbb{R})}$ and $||u_1(\cdot)||_{H^3(\mathbb{R})\cap L_1(\mathbb{R})}$, such that

$$\begin{aligned} ||v(\cdot,t,\varepsilon) - \tilde{v}(\cdot,t,\varepsilon)||_{W^{1,\infty}(\mathbb{R})} &\le ||R(u(\cdot)_0, u_1(\cdot))||_{W^{1,\infty}(\mathbb{R})} \le C_1\varepsilon^2, \\ ||v(\cdot,t,\varepsilon) - \tilde{v}(\cdot,t,\varepsilon)||_1 &\le ||R(u(\cdot)_0, u_1(\cdot))||_1 \le C_2\varepsilon^2. \end{aligned} \tag{4.60}$$

Using the definitions (4.3) and (4.56) we find that for some constants denoted by $C_1$ and $C_2$ again,

$$|\tilde{\tilde{u}}(\cdot,t) - \tilde{u}(\cdot,t)|_\infty \le C||v(\cdot,t,\varepsilon) - \tilde{v}(\cdot,t,\varepsilon)||_{W^{1,\infty}(\mathbb{R})} \le C_1\tfrac{\varepsilon^2}{2} \tag{4.61}$$

and

$$|\tilde{\tilde{u}}(\cdot,t) - \tilde{u}(\cdot,t)|_2 \le C||v(\cdot,t,\varepsilon) - \tilde{v}(\cdot,t,\varepsilon)||_1 \le C_2\tfrac{\varepsilon^2}{2}. \tag{4.62}$$

By Theorem 4.1, there exists a constant $C_1$ which depends on the initial data $u_0$ and $u_1$, such that

$$|u(\cdot,t) - \tilde{\tilde{u}}(\cdot,t)|_\infty \le \tfrac{\varepsilon^2}{2}C_1(||u_0(\cdot)||_{H^4(\mathbb{R})\cap L_1(\mathbb{R})}, ||u_1(\cdot)||_{H^3(\mathbb{R})\cap L_1(\mathbb{R})}). \tag{4.63}$$

Hence, it follows from (4.61) and (4.63) that for all $t > 0$,

$$\begin{aligned} |u(\cdot,t) - \tilde{u}(\cdot,t)|_\infty &\le |u(\cdot,t) - \tilde{\tilde{u}}(\cdot,t)|_\infty + |\tilde{\tilde{u}}(\cdot,t) - \tilde{u}(\cdot,t)|_\infty \\ &\le \varepsilon^2C_1(||u_0(\cdot)||_{H^4(\mathbb{R})\cap L_1(\mathbb{R})}, ||u_1(\cdot)||_{H^3(\mathbb{R})\cap L_1(\mathbb{R})}). \end{aligned}$$

Similarly, applying Theorem 4.1 and (4.62), we obtain that

$$|u(\cdot,t)-\tilde{u}(\cdot,t)|_2 \le |u(\cdot,t)-\tilde{\tilde{u}}(\cdot,t)|_2+|\tilde{\tilde{u}}(\cdot,t)-\tilde{u}(\cdot,t)|_2$$

$$\le \varepsilon^2 C_2(||u_0(\cdot)||_{H^4(\mathbb{R})\cap L_1(\mathbb{R})}, ||u_1(\cdot)||_{H^3(\mathbb{R})\cap L_1(\mathbb{R})}),$$

for all $t>0$. The corollary is then proved. □

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