TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 360, Number 12, December 2008, Pages 6643–6659
S 0002-9947(08)04559-5
Article electronically published on July 22, 2008

THE GEOMETRIC GENUS OF SPLICE-QUOTIENT SINGULARITIES

TOMOHIRO OKUMA

Abstract. We prove a formula for the geometric genus of splice-quotient singularities (in the sense of Neumann and Wahl). This formula enables us to compute the invariant from the resolution graph; in fact, it reduces the computation to that for splice-quotient singularities with smaller resolution graphs. We also discuss the dimension of the first cohomology groups of certain invertible sheaves on a resolution of a splice-quotient singularity.

1. Introduction

The topology (i.e., the link or the weighted dual graph) of a normal surface singularity in general does not determine analytic invariants of the singularity. It is challenging to see what kinds of analytic invariants become topological invariants under certain conditions, and to find a formula for computing the invariants from resolution graphs. The geometric genus p_g is one of the fundamental analytic invariants of singularities. There are many studies on this invariant related to the above issue; for example, [1], [4], [13], [16], [7], [8].

Let X be a normal surface singularity whose link is a $\mathbb{Q}$-homology sphere. Then there exists the "universal abelian cover" $Y \to X$. Neumann and Wahl conjectured that if X is $\mathbb{Q}$-Gorenstein, then Y is a complete intersection of "splice type" and the geometric genus $p_g(X)$ is a topological invariant. Although this conjecture inspired research in the surface singularity theory, counterexamples are now known (see [5]). Splice type singularities are introduced by Neumann and Wahl ([14], [15], [16]). These singularities are a generalization of Brieskorn complete intersections. For a fixed resolution graph Γ, the associated splice type singularities form an equisingular family (see [15, Theorem 10.1], [18, Theorem 4.3]). Therefore, p_g of these singularities are the same and determined by Γ (this fact also follows from the result of this paper). Neumann and Wahl have developed the theory of splice type singularities and recently proved the End-Curve Theorem (Theorem 2.15), which states that if a good resolution $\widetilde{X}$ of X satisfies the "End-Curve condition" (Definition 2.13), then Y is a splice type singularity associated with the resolution graph of X (the converse is also true). In this case, X is called a splice-quotient singularity.

Received by the editors October 18, 2006 and, in revised form, March 13, 2007.

2000 *Mathematics Subject Classification.* Primary 32S25; Secondary 14B05, 14J17.

Key words and phrases. Surface singularity, geometric genus, rational homology sphere, splice type singularity, universal abelian cover.

This work was partly supported by the Grant-in-Aid for Young Scientists (B), The Ministry of Education, Culture, Sports, Science and Technology, Japan.

Suppose that X is a splice-quotient singularity with resolution graph Γ, and fix a component E_v of the exceptional set $E \subset \widetilde{X}$ such that $\delta := (E - E_v) \cdot E_v \geq 3$. We consider a filtration $\{F_n\}_{n \geq 0}$ of $\mathcal{O}_X$ associated with the prime divisor E_v, and singularities $\{X_i\}_{i=1}^{\delta}$ corresponding to connected components of the exceptional divisor $E - E_v$. Then X_i are also splice-quotient singularities following the End-Curve Theorem (Lemma 4.7). We can define an invariant c_v of the graded ring $\bigoplus_{n \geq 0} F_n/F_{n+1}$, which is sort of the constant term of the Hilbert polynomial; c_v can be computed from Γ. We prove the following.

Theorem 1.1. *If X is a splice-quotient singularity, then*

$$p_g(X) = c_v + \sum_{i=1}^{\delta} p_g(X_i).$$

By using this formula inductively, the computation of p_g is reduced to that of c_v (cf. Proposition 4.8).

We will also prove a formula for h^1 of invertible sheaves on $\widetilde{X}$ related to the eigensheaves of $\mathcal{O}_Y$ (Theorem 4.5), which implies the formula for $p_g(Y)$. In fact the above theorem is a corollary of this result.

In [9] A. Némethi and the author proved the Casson invariant conjecture of Neumann and Wahl ([13], [16, Theorem 6.3]) for splice type surface singularities by applying our formula. The conjecture can be reduced to proving an "additivity property" of the geometric genus under "splicing", while our formula relates an additivity property under "plumbing." This gap is bridged by a new method in [9, §4].

This paper is organized as follows. In Section 2, we review the basics of splice type singularities as universal abelian covers of normal surface singularities. We show that a neighborhood of a connected exceptional curve on $\widetilde{X}$ also satisfies the End-Curve condition. This will enable us to use induction on the number of "nodes" of E. In Section 3, we consider a weight filtration $\{I_n\}_{n \geq 0}$ of the local ring $\mathcal{O}_{Y,o}$ with respect to weights determined from the weighted dual graph of E; this filtration is $\mathbf{H}$-equivariant, where $\mathbf{H}$ is the Galois group of the covering $q \colon Y \to X$. Then for any $\chi \in \mathrm{Hom}(\mathbf{H}, \mathbb{C}^*)$, the Hilbert series of the χ-eigenspace $\bigoplus I_n^{\chi}/I_{n+1}^{\chi}$ of the associated graded ring is computed from Γ. (The formula will be proved in the Appendix.) By applying this fact and the Riemann-Roch formula, we compute h^1 of certain invertible sheaves related to eigensheaves of $q_*\mathcal{O}_Y$ (it also related to I_n^{χ}'s). In Section 4, we prove the main theorem. By using the vanishing theorem, the desired invariant is decomposed into one similar to c_v and invariants of "smaller" splice-quotient singularities.

Part of this paper was prepared during the author's stay at the Rényi Institute of Mathematics, Budapest, Hungary. The author would like to thank Professor András Némethi for valuable discussions and comments, and the Rényi Institute for their support and hospitality. The author would also like to thank Professor Jonathan Wahl for his helpful exposition of the End-Curve Theorem, and the referees for their careful reading and very helpful comments.

2. Universal abelian covers and splice type singularities

We introduce some notation and review some basics on the splice type singularities and the universal abelian covers of surface singularities. Although a system of splice diagram equations, which defines a splice type singularity, is associated with a weighted tree called a "splice diagram" in origin (see [15, §2] for details), we construct them in terms of "monomial cycles" on a resolution space of a surface singularity (cf. [18, §3], [15, §13]) for convenience of discussion.

Let (X, o) be a germ of a normal complex surface singularity and let $\pi\colon \widetilde{X} \to X$ be a good resolution. Then the exceptional divisor $E := \pi^{-1}(o)$ has only simple normal crossings. We denote the link of the singularity (X, o) by Σ. We may assume that X is homeomorphic to a cone over Σ. In the case where π is the minimal good resolution, the weighted dual graph of (X, o) is that of E. It is known that the weighted dual graph of (X, o) and Σ have the same information ([11]).

Assumption 2.1. We always assume that the link Σ is a rational homology sphere; it is equivalent to E being a tree of rational curves. In addition, we assume that X is not a cyclic quotient singularity. Therefore the weighted dual graph of E is not a chain (note that the universal abelian cover of a cyclic quotient singularity is nonsingular).

Let $\{E_v\}_{v\in\mathcal{V}}$ denote the set of irreducible components of E. Let

$$\mathbf{L} = \sum_{v\in\mathcal{V}} \mathbb{Z}E_v \quad \text{and} \quad \mathbf{L}_{\mathbb{Q}} = \mathbf{L}\otimes\mathbb{Q}.$$

We call an element of $\mathbf{L}$ (resp. $\mathbf{L}_{\mathbb{Q}}$) a cycle (resp. $\mathbb{Q}$-cycle). Since the intersection matrix $(E_v \cdot E_w)$ is negative definite, for any $v \in \mathcal{V}$ there exists a $\mathbb{Q}$-cycle E_v^* such that $E_v^* \cdot E_w = -\delta_{vw}$ for every $w \in \mathcal{V}$, where δ_{vw} denotes the Kronecker delta. Let

$$\mathbf{L}^* = \sum_{v\in\mathcal{V}} \mathbb{Z}E_v^* \subset \mathbf{L}_{\mathbb{Q}}.$$

We may identify $\mathbf{L}^*$ with $\mathrm{Hom}(\mathbf{L}, \mathbb{Z})$. By the assumption, $H_1(\Sigma, \mathbb{Z})$ is a finite group, and we have a natural isomorphism

$$\mathbf{H} := \mathbf{L}^*/\mathbf{L} \to H_1(\Sigma, \mathbb{Z})$$

via isomorphisms $\mathbf{L} \cong H_2(\widetilde{X}, \mathbb{Z})$ and $\mathbf{L}^* \cong H_2(\widetilde{X}, \Sigma, \mathbb{Z})$ (cf. [6, §2] or [17, §2]). Here, we identify $\mathbf{H}$ with $H_1(\Sigma, \mathbb{Z})$. The intersection pairing on $\mathbf{L}_{\mathbb{Q}}$ induces a natural pairing

$$\mathbf{H} \times \mathbf{H} \to \mathbb{Q}/\mathbb{Z}. \tag{2.1}$$

Let $\mathrm{Div}(\widetilde{X})$ denote the group of divisors on $\widetilde{X}$. We define a homomorphism

$$c_1\colon \mathrm{Div}(\widetilde{X}) \to \mathbf{L}^* \quad \text{by} \quad c_1(D) = -\sum_v (D\cdot E_v)E_v^*.$$

Clearly c_1 is surjective and $D \cdot E_v = c_1(D) \cdot E_v$ for all $v \in \mathcal{V}$.

2.1. **The universal abelian cover of X.** There uniquely exists a finite morphism $q\colon (Y, o) \to (X, o)$ of singularities that induces an unramified Galois covering $Y \setminus \{o\} \to X \setminus \{o\}$ with Galois group $\mathbf{H}$. The morphism $q\colon X \to Y$ is called the *universal abelian covering* of X.

In this subsection, we give an expression of the decomposition of $q_*\mathcal{O}_Y$ into the $\mathbf{H}$-eigensheaves, and a similar expression for the structure sheaf of a partial resolution of Y; see [17] or [6] for details. Let $\hat{\mathbf{H}} = \mathrm{Hom}(\mathbf{H}, \mathbb{C}^*)$. We define a homomorphism

$$\theta\colon \mathbf{H} \to \hat{\mathbf{H}} \quad \text{by} \quad \theta(h)(h') = \exp(2\pi\sqrt{-1}h\cdot h'),$$

where $h \cdot h' \in \mathbb{Q}/\mathbb{Z}$ is determined by (2.1). We also use the following notation:

- $\theta(h, h') := \theta(h)(h')$; then $\theta(\ ,\ )$ is symmetric.
- By abuse of notation, let θ also denote the composite $\mathbf{L}^* \to \mathbf{H} \xrightarrow{\theta} \hat{\mathbf{H}}$.
- For $D \in \mathbf{L}^*$ and $h \in \mathbf{H}$, let $\theta(h, D) := \theta(D, h) := \theta(D)(h)$.

If $\mathbb{Q}$-divisors D_1 and D_2 are $\mathbb{Q}$-linearly equivalent, i.e., $nD_1 \sim nD_2$ for some $n \in \mathbb{N}$, we write $D_1 \sim_{\mathbb{Q}} D_2$. Note that if two integral divisors on $\widetilde{X}$ are $\mathbb{Q}$-linearly equivalent, then they are also linearly equivalent because $\mathrm{Pic}(\widetilde{X})$ has no torsion by Assumption 2.1.

There exists a set $\{L_\chi\}_{\chi \in \hat{\mathbf{H}}}$ of divisors on $\widetilde{X}$ such that

(1) $\theta(c_1(L_\chi)) = \chi$,
(2) $L_\chi \sim_{\mathbb{Q}} c_1(L_\chi)$, and
(3) $[c_1(L_\chi)] = 0$, where $[D]$ denotes the integral part of a $\mathbb{Q}$-divisor D.

Clearly, such an L_χ is uniquely determined up to linear equivalence. Note that if $D \in \mathbf{L}^*$ and $\chi = \theta(D)$, then $c_1(L_\chi) - D \in \mathbf{L}$.

We define a homomorphism

$$\sigma \colon \mathbf{L}^* \to \mathrm{Div}(\widetilde{X}) \quad \text{by} \quad \sigma(D) = L_{\theta(D)} + [D].$$

Then $D \sim_{\mathbb{Q}} \sigma(D)$ for any $D \in \mathbf{L}^*$. Clearly, σ is a section of the homomorphism c_1.

Let $\mathcal{L}_\chi := \mathcal{O}_{\widetilde{X}}(-L_\chi)$.

Proposition 2.2 (cf. [17, §3.2])**.** *We have a collection $\{\mathcal{L}_\chi \otimes \mathcal{L}_{\chi'} \to \mathcal{L}_{\chi\chi'}\}$ of homomorphisms that defines an $\mathcal{O}_{\widetilde{X}}$-algebra structure of an $\mathcal{O}_{\widetilde{X}}$-module $\mathcal{A} := \bigoplus_{\chi \in \hat{\mathbf{H}}} \mathcal{L}_\chi$ such that the following are satisfied:*

(1) *The projection $\mathrm{Specan}_X\, \pi_* \mathcal{A} \to X$ coincides with $q \colon Y \to X$.*
(2) *$\widetilde{Y} := \mathrm{Specan}_{\widetilde{X}}\, \mathcal{A}$ has only cyclic quotient singularity, there is a morphism $\rho \colon \widetilde{Y} \to Y$ which is a partial resolution, and the following diagram is commutative:*

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "p"] \arrow[d, "\rho"'] & \widetilde{X} \arrow[d, "\pi"] \\ Y \arrow[r, "q"'] & X \end{tikzcd}$$

where p is the natural projection. Then, p is unramified over $\widetilde{X} \setminus E$.
(3) *The module $\mathcal{L}_\chi$ (resp. $\pi_* \mathcal{L}_\chi$) is the χ-eigensheaf of $p_* \mathcal{O}_{\widetilde{Y}}$ (resp. $q_* \mathcal{O}_Y$).*

2.2. **Splice diagram equations.** Let $\delta_v = (E - E_v) \cdot E_v$ be the number of irreducible components of E intersecting E_v. A curve E_v is called an *end* (resp. a *node*) if $\delta_v = 1$ (resp. $\delta_v \geq 3$). Let $\mathcal{E}$ (resp. $\mathcal{N}$) denote the set of indices of ends (resp. nodes). A connected component of $E - E_v$ is called a *branch* of E_v.

Definition 2.3. An element of a semigroup $\sum_{w \in \mathcal{E}} \mathbb{Z}_{\geq 0} E_w^*$, where $\mathbb{Z}_{\geq 0}$ is the set of nonnegative integers, is called a *monomial cycle*. Let $\mathbb{C}[z] := \mathbb{C}[z_w; w \in \mathcal{E}]$ be the polynomial ring in $\#\mathcal{E}$ variables. For a monomial cycle $D = \sum_{w \in \mathcal{E}} \alpha_w E_w^*$, we associate a monomial $z(D) := \prod_{w \in \mathcal{E}} z_w^{\alpha_w} \in \mathbb{C}[z]$.

Definition 2.4 (Monomial condition)**.** We say that E (or its weighted dual graph) satisfies the monomial condition if for any branch C of any node E_v, there exists a monomial cycle D such that $D - E_v^*$ is an effective integral cycle supported on C

(cf. Example 2.7). In this case, $z(D)$ is called an *admissible monomial* belonging to the branch C.

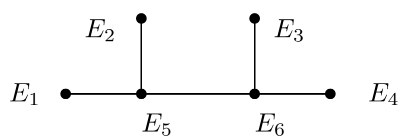
Remark 2.5. The monomial condition is equivalent to the semigroup and congruence conditions (see [15, §13]), which are required for obtaining "appropriate" splice diagram equations (cf. Theorem 2.11). Note that the original definition of admissible monomials requires only the semigroup condition ([15], [16]).

Definition 2.6. Assume that the monomial condition is satisfied. Let E_v be a node and let $C_1, \dots, C_{\delta_v}$ be the branches of E_v. Suppose $\{m_1, \dots, m_{\delta_v}\}$ is a set of admissible monomials such that m_i belongs to C_i for $i = 1, \dots, \delta_v$. Let $F = (c_{ij})$, $c_{ij} \in \mathbb{C}$, be any $((\delta_v - 2) \times \delta_v)$-matrix such that every maximal minor of it has rank $\delta_v - 2$. We define polynomials $f_1, \dots, f_{\delta_v - 2}$ by

$$\begin{pmatrix} f_1 \\ \vdots \\ f_{\delta_v - 2} \end{pmatrix} = F \begin{pmatrix} m_1 \\ \vdots \\ m_{\delta_v} \end{pmatrix}.$$

We call the set $\{f_1, \dots, f_{\delta_v - 2}\}$ a *Neumann-Wahl system* at E_v. Suppose that we have a Neumann-Wahl system $\mathcal{F}_v$ at every node E_v. Then we call the set $\mathcal{F} := \bigcup_{v \in \mathcal{N}} \mathcal{F}_v$ a *Neumann-Wahl system* associated with E. Note that $\#\mathcal{F} = \#\mathcal{E} - 2$.

Example 2.7. Suppose that $E_4^2 = -3$ and $E_w^2 = -2$ for $w \neq 4$, and the weighted dual graph of E is represented as follows:



Then the following equations show that the monomial condition is satisfied:

$$\begin{aligned} &2E_1^* = E_5^* + E_1, \quad 2E_2^* = E_5^* + E_2, \quad E_3^* + 2E_4^* = E_5^* + (E_3 + E_4 + E_6), \\ &2E_3^* = E_6^* + E_3, \quad 3E_4^* = E_6^* + E_4, \quad E_1^* + E_2^* = E_6^* + (E_1 + E_2 + E_5). \end{aligned}$$

The corresponding admissible monomials form a Neumann-Wahl system

$$z_1^2 + z_2^2 + z_3 z_4^2, \quad z_3^2 + z_4^3 + z_1 z_2.$$

Let $(a_{vw}) = -I^{-1}$, where I denotes the intersection matrix $(E_v \cdot E_w)$. Then every a_{vw} is a positive rational number and $E_v^* = \sum_{w \in \mathcal{V}} a_{vw} E_w$. We define positive integers e_v, ℓ_{vw}, and m_{vw} as follows:

$$\ell_{vw} = |\det I| a_{vw}, \quad e_v = |\det I| / \gcd \{\ell_{vw} \mid w \in \mathcal{V}\}, \quad \text{and} \quad m_{vw} = e_v a_{vw}.$$

It is easy to see that $\gcd\{m_{vw}\}_{w \in \mathcal{V}} = 1$ for every $v \in \mathcal{V}$.

Definition 2.8. For any $v \in \mathcal{V}$, we define the v-weight of the variable z_w, $w \in \mathcal{E}$, to be m_{vw}. Therefore, the v-degree of a monomial $\prod_{w \in \mathcal{E}} z_w^{\alpha_w}$ is $\sum_{w \in \mathcal{E}} \alpha_w m_{vw}$. Note that if $D = \sum_{w \in \mathcal{V}} \beta_w E_w$ is a monomial cycle, then the v-degree of $z(D)$ is equal to $e_v \beta_v = -e_v D \cdot E_v^*$.

Let $\mathbb{C}\{z\} := \mathbb{C}\{z_w; w \in \mathcal{E}\}$ be the convergent power series ring. Let $f = f_0 + f_1 \in \mathbb{C}\{z\}$, where f_0 is a nonzero quasihomogeneous polynomial with respect to the v-weight and f_1 is a series in monomials of higher v-degrees. Then we call f_0 the *v-leading form* of f, and denote it by $\mathrm{LF}_v(f)$. We also define the *v-order* of f to be the v-degree of f_0.

Definition 2.9. We consider a finite set

$$\{f_{vj_v} \mid v \in \mathcal{N},\ j_v = 1, \dots, \delta_v - 2\} \subset \mathbb{C}\{z\}.$$

If the set

$$\{\mathrm{LF}_v(f_{vj_v}) \mid v \in \mathcal{N},\ j_v = 1, \dots, \delta_v - 2\}$$

is a Neumann-Wahl system associated with E, then a system of equations

$$f_{vj_v} = 0, \quad v \in \mathcal{N}, \quad j_v = 1, \dots, \delta_v - 2,$$

is called the *splice diagram equations.* A germ of a singularity defined by the splice diagram equations in $(\mathbb{C}^{\#\mathcal{E}}, o)$ is called a *splice type singularity* or is said to be of splice type.

Theorem 2.10 (Neumann-Wahl [15, Theorem 2.6])**.** *A splice type singularity is an isolated complete intersection surface singularity.*

2.3. **H-action.** We define an action of $\mathbf{H}$ on the power series ring $\mathbb{C}\{z\}$ as follows. For any monomial cycle D and any element $h \in \mathbf{H}$, we define $h \cdot z(D) \in \mathbb{C}\{z\}$ by

$$h \cdot z(D) = \theta(h, D) z(D).$$

This extends to an action on $\mathbb{C}\{z\}$. If $\mathcal{E} = \{w_1, \dots, w_n\}$, the action corresponds to a representation

$$\mathbf{H} \to \mathrm{U}(n), \quad h \to \mathrm{D}_h,$$

where D_h denotes a diagonal matrix with i-th diagonal component $\theta(h, E^*_{w_i})$. If $\{f_{vj_v}\}$ is a Neumann-Wahl system associated with E, then for every f_{vj_v} and $h \in \mathbf{H}$, we have $h \cdot f_{vj_v} = \theta(h, E^*_v) f_{vj_v}$, i.e., f_{vj_v} is in the $\theta(E^*_v)$-eigenspace of $\mathbb{C}\{z\}$.

Theorem 2.11 (Neumann-Wahl [15, Theorem 7.2])**.** *Suppose that (Z, o) is a singularity defined by splice diagram equations*

$$f_{vj_v} = 0, \quad v \in \mathcal{N}, \quad j_v = 1, \dots, \delta_v - 2,$$

such that $h \cdot f_{vj_v} = \theta(h, E^*_v) f_{vj_v}$ for every f_{vj_v} and $h \in \mathbf{H}$. Then we have the following:

1. $\mathbf{H}$ *acts freely on $Z \backslash \{o\}$, and thus $Z' := Z/\mathbf{H}$ is a normal surface singularity.*
2. *The weighted dual graph of Z' is the same as that of X.*
3. *The quotient map $Z \to Z'$ is the universal abelian covering.*

Definition 2.12. A singularity whose universal abelian cover is of splice type (such as Z' in Theorem 2.11) is called a *splice-quotient singularity.*

Definition 2.13 (End-curve condition)**.** We say that $\widetilde{X}$ satisfies the end-curve condition if for each $w \in \mathcal{E}$ there exists an irreducible curve $H_w \subset \widetilde{X}$, not an exceptional curve, such that $H_w \cdot E = H_w \cdot E_w = 1$ and $E^*_w + H_w \sim_{\mathbb{Q}} 0$ (this implies $e_w(E^*_w + H_w) \sim 0$). In other words, the end-curve condition is equivalent to the condition that $\mathcal{O}_{\widetilde{X}}(-\sigma(E^*_w))$ has no fixed component in E for every $w \in \mathcal{E}$. In this case, a general section $s \in H^0(\mathcal{O}_{\widetilde{X}}(-\sigma(E^*_w)))$ defines a divisor $\sigma(E^*_w) + H_w$, where H_w is as above. We call such an s an *end-curve section* of E_w.

Remark 2.14. If $\widetilde{X}$ satisfies the end-curve condition, then so do the minimal good resolution and a resolution obtained by blowing up $\widetilde{X}$ at singular points of E or a point $E_w \cap H_w$, $w \in \mathcal{E}$.

The following theorem is a generalization of [16, Theorem 4.1], which was announced in [10, Theorem 6.1], and its proof will appear in [12].

Theorem 2.15 (End-Curve Theorem). *If $\widetilde{X}$ satisfies the end-curve condition, then E satisfies the monomial condition and Y is of splice type. In fact, if*

$$\psi\colon \mathbb{C}\{z_w; w \in \mathcal{E}\} \to \mathcal{O}_{Y,o}$$

is a homomorphism of $\mathbb{C}$-algebras that maps each $z_w = z(E_w^*)$ to an end-curve section of E_w, then ψ is surjective and $\operatorname{Ker}\psi$ is generated by functions $\{f_{vj_v}\}$ as in Theorem 2.11.

The following proposition implies that if X is a splice-quotient singularity, then any singularity obtained by contracting a connected exceptional curve on $\widetilde{X}$ is also a splice-quotient singularity.

Proposition 2.16. *Let $v_0 \in \mathcal{E}$, and let $\widetilde{X}' \subset \widetilde{X}$ be a sufficiently small neighborhood of the divisor $E' := E - E_{v_0}$. If $\widetilde{X}$ satisfies the end-curve condition, then so does $\widetilde{X}'$.*

Proof. Let $\mathcal{V}' = \mathcal{V} \setminus \{v_0\}$. For each $v \in \mathcal{V}'$, define a $\mathbb{Q}$-cycle $E_v^{\times} \in \sum_{w \in \mathcal{V}'} \mathbb{Q}E_w$ by the condition that $E_v^{\times} \cdot E_w = -\delta_{vw}$ for every $w \in \mathcal{V}'$. Assume that E_{v_1} intersects E_{v_0}. Since $E_{v_0}^* + H_{v_0} \sim_{\mathbb{Q}} 0$ for some curve H_{v_0} as in Definition 2.13, we have $E_{v_0}^*|_{\widetilde{X}'} \sim_{\mathbb{Q}} 0$. We can easily see that $E_{v_0}^* - a_{v_0v_0}E_{v_0} = a_{v_0v_0}E_{v_1}^{\times}$. If E_{v_1} is an end of E', then we can take $E_{v_0}|_{\widetilde{X}'}$ as a "curve H_{v_1}." Suppose $w \in \mathcal{E} \setminus \{v_0\}$. Then $E_w^* + H_w \sim_{\mathbb{Q}} 0$. Since $a_{v_0v_0}E_w^* - a_{v_0w}E_{v_0}^* = a_{v_0v_0}E_w^{\times}$, we obtain that

$$a_{v_0v_0}(E_w^{\times} + H_w|_{\widetilde{X}'}) = a_{v_0v_0}(E_w^* + H_w)|_{\widetilde{X}'} - a_{v_0w}E_{v_0}^*|_{\widetilde{X}'} \sim_{\mathbb{Q}} 0.$$

Therefore, the end-curve condition is satisfied. ☐

3. Filtration associated to a node

We use the notation of the preceding section. Assume that the end-curve condition is satisfied and that the universal abelian cover $p\colon Y \to X$ is expressed as in Theorems 2.11 and 2.15. Note that every monomial cycle D determines a homogeneous element $\psi(z(D)) \in H^0(\mathcal{L}_{\theta(D)})$ of a $\hat{\mathbf{H}}$-graded local ring $\mathcal{O}_{Y,o}$.

Throughout this section, we fix a node E_v. Let $C_1, \dots, C_{\delta_v}$ denote the branches of E_v.

Definition 3.1. For each $n \in \mathbb{Z}_{\geq 0}$, I_n denotes the ideal of $\mathcal{O}_{Y,o}$ generated by the images of the elements of $\mathbb{C}\{z\}$ having v-order $\geq n$. Let $\mathcal{G}$ denote the associated graded algebra $\bigoplus_{n \geq 0} I_n/I_{n+1}$. Let $\mathcal{G}_n = I_n/I_{n+1}$.

Theorem 3.2 (Neumann-Wahl [15, Theorem 2.6]). *Let $\{f_{wj_w}\}$ be the set of power series defining Y as in Theorem 2.11, and let I be the ideal of the polynomial ring $\mathbb{C}[z]$ generated by the v-leading forms $\{\mathrm{LF}_v(f_{wj_w})\}$. Then $\mathcal{G} \cong \mathbb{C}[z]/I$, and it is a reduced complete intersection ring.*

3.1. A geometric description of the filtration.

Let us recall the commutative diagram in Proposition 2.2. Let $F := p^{-1}(E)$. Then F is the ρ-exceptional set on $\widetilde{Y}$. Let $F_v = p^{-1}(E_v)$ (this may be not irreducible). Let $\pi'\colon \widetilde{X} \to X'$ (resp. $\rho'\colon \widetilde{Y} \to Y'$) be the morphism that contracts the divisor $E - E_v$ (resp. $F - F_v$) to normal points. Then the natural morphism $p'\colon Y' \to X'$ is finite. We have the following commutative diagram:

$$\begin{tikzcd}
\widetilde{Y} \arrow[rrr, "p"] \arrow[dr, "\rho'"] \arrow[dd, "\rho"'] & & & \widetilde{X} \arrow[dl, "\pi'"'] \arrow[dd, "\pi"] \\
& Y' \arrow[r, "p'"] \arrow[dl, "\rho_1"] & X' \arrow[dr, "\pi_1"'] & \\
Y \arrow[rrr, "q"'] & & & X
\end{tikzcd}$$

where π_1 and ρ_1 are the natural morphisms. Clearly, the exceptional sets of π_1 and ρ_1 are $E' := \pi'(E_v)$ and $F' := \rho'(F_v)$, respectively. Since $p^*E_v = e_v F_v$ (see [17, Theorem 3.4]), it follows from the definition of the v-weight that the ideal I_n is generated by the images of power series in monomials $z(D)$ satisfying $-e_v D \cdot E_v^* \geq n$ or equivalently $p^* D \geq nF_v$.

We define a positive integer a_v by

$$a_v := e_v^2 a_{vv} = e_v m_{vv}.$$

Lemma 3.3. *Let $n \in \mathbb{Z}_{\geq 0}$. Then, we have the following:*

(1) $\mathcal{O}_{\widetilde{X}}(-\sigma(E_v^*))$ *is π-generated, and* $\mathcal{O}_{\widetilde{X}}(-ne_vE_v^*)$ *is generated by functions in* $I_{na_v} \cap \mathcal{O}_{X,o}$ *near E.*

(2) $I_n = \left(\rho_*\mathcal{O}_{\widetilde{Y}}(-nF_v)\right)_o = \left(\rho_{1*}\mathcal{O}_{Y'}(-nF')\right)_o = \left(\rho_*\mathcal{O}_{\widetilde{Y}}([-n\rho'^*F'])\right)_o$, *where* $(\ )_o$ *means the stalk at $o \in Y$.*

(3) $-F'$ *is ρ_1-ample, and ρ_1 coincides with the filtered blowing up*

$$\operatorname{Projan}_Y\left(\bigoplus_{n\geq 0}\rho_{1*}\mathcal{O}_{Y'}(-nF')\right) \to Y.$$

Proof. Suppose D_1 and D_2 are monomial cycles such that each $z(D_i)$ is an admissible monomial belonging to the branch C_i. First, we note that $s_i := \psi(z(D_i)) \in H^0\left(\mathcal{O}_{\widetilde{X}}(-\sigma(D_i))\right)$ has no zero outside $\bigcup_{D_i\cdot E_w \neq 0} E_w$ (see Definition 2.13). Since $\sigma(E_v^*) \leq \sigma(D_i)$ and $\sigma(E_v^*) = \sigma(D_i)$ outside C_i for $i = 1, 2$, the sections s_1 and s_2 generate $\mathcal{O}_{\widetilde{X}}(-\sigma(E_v^*))$. We have the equality

$$\mathcal{O}_{\widetilde{X}}(-\sigma(E_v^*))^{ne_v} = \mathcal{O}_{\widetilde{X}}(-ne_vE_v^*) \subset \mathcal{O}_X$$

in the algebra $p_*\mathcal{O}_{\widetilde{Y}}$. As above, we see that sections $s_1^{ne_v}$ and $s_2^{ne_v}$ generate $\mathcal{O}_{\widetilde{X}}(-ne_vE_v^*)$. Clearly, $ne_vD_i \in \mathbf{L}$ and $-ne_vD_i \cdot E_v^* = na_v$. Hence (1) follows.

The second equality in (2) and inclusion $I_n \subset \left(\rho_*\mathcal{O}_{\widetilde{Y}}(-nF_v)\right)_o$ are clear. The last equality follows from the formula $\mathcal{O}_{Y'}(-nF') = \rho'_*\mathcal{O}_{\widetilde{Y}}([-n\rho'^*F'])$ (see [19, (2.1)]). Since $\mathcal{G}$ is reduced by Theorem 3.2, the argument of [21, (2.2)] applies to our filtration; there exist valuations $V_1, \dots, V_t$ of the quotient field of $\mathcal{O}_{Y,o}$ and rational numbers $q_1, \dots, q_t$ such that

$$I_n = \{f \in \mathcal{O}_{Y,o} \mid V_i(f) \geq nq_i \text{ for } 1 \leq i \leq t\} \quad \text{for all } n.$$

Thus it suffices to show that there exists a positive integer d such that $I_{md} = \left(\rho_{1*}\mathcal{O}_{Y'}(-mdF')\right)_o$ for all m. Let $F_v^* = \rho'^*F'$. Then we have an equality $a_vF_v^* =$

$p^*(e_vE_v^*)$ of Cartier divisors. It follows from (1) that $\mathcal{O}_{\widetilde{Y}}(-ma_vF_v^*) = I_{ma_v}\mathcal{O}_{\widetilde{Y}}$, and thus,

$$I_{ma_v} = \left(\rho_*\mathcal{O}_{\widetilde{Y}}(-ma_vF_v^*)\right)_o = \left(\rho_{1*}\mathcal{O}_{Y'}(-ma_vF')\right)_o$$

for all m. This proves (2).

Since $\mathcal{O}_{\widetilde{Y}}(-a_vF_v^*)$ is ρ-generated and trivial near $F - F_v$ but positive on F_v, the morphism ρ_1 is obtained by blowing up with respect to the ideal sheaf $\rho_*\mathcal{O}_{\widetilde{Y}}(-ma_vF_v^*)$ for some m. Hence (3) follows. □

In the next lemma, we use the a-invariant of graded rings induced by Goto and Watanabe [3, (3.1.4)] (see also [2, 3.6.13]). By Theorem 3.2 and formulas [2, 3.6.14–15], the a-invariant $a(\mathcal{G})$ of $\mathcal{G}$ is expressed as

$$\sum_{w\in\mathcal{N}}(\delta_w - 2)m_{vw} - \sum_{w\in\mathcal{E}} m_{vw} = \sum_{w\in\mathcal{V}}(\delta_w - 2)m_{vw}.$$

By the definition of the a-invariant, the n-th graded component of $H^2_{\mathcal{G}_+}(\mathcal{G})$ vanishes for $n > a(\mathcal{G})$, where $\mathcal{G}_+ = \bigoplus_{k>0}\mathcal{G}_k$.

Lemma 3.4. *$H^1(\mathcal{O}_{Y'}(-nF')) = 0$ for $n > a(\mathcal{G})$.*

Proof. We apply the arguments of Tomari-Watanabe [21, §1]. Let $\mathcal{R}$ denote the Rees algebra $\bigoplus_{n\geq 0} I_nT^n \subset \mathcal{O}_{Y,o}[T]$. We may assume that $Y = \operatorname{Spec}\mathcal{O}_{Y,o}$ and $Y' = \operatorname{Proj}\mathcal{R}$. Let $\mathcal{O}_{Y'}(n) = \widetilde{\mathcal{R}(n)}$. Since $\mathcal{O}_{Y'}(n)$ is a divisorial sheaf for every $n \geq 0$ by Proposition 1.6 of [21], it follows from Lemma 3.3 that $\mathcal{O}_{Y'}(n) \cong \mathcal{O}_{Y'}(-nF')$. On the other hand, (1.13) (ii) and (1.18) (i) of [21] imply that $H^1(\mathcal{O}_{Y'}(n)) = 0$ for $n > a(\mathcal{G})$. □

3.2. **The χ-eigenspace of the filtration.** For any module $\mathcal{M}$ with $\mathbf{H}$-action and for any $\chi \in \hat{\mathbf{H}}$, let $\mathcal{M}^\chi$ denote the χ-eigenspace of $\mathcal{M}$ (see Section 5 for the definition). The $\mathbf{H}$-action on $\mathcal{O}_{Y,o}$ induces an action on I_n. For $\chi \in \hat{\mathbf{H}}$, we have $I_n^\chi = I_n \cap \mathcal{O}_{Y,o}^\chi$ and $\mathcal{G}^\chi = I_n^\chi / I_{n+1}^\chi$.

Lemma 3.5. *For any $D \in \mathbf{L}^*$, the decomposition of the $\mathcal{O}_{\widetilde{X}}$-module $p_*\mathcal{O}_{\widetilde{Y}}(-p^*D)$ into eigensheaves is given by*

$$p_*\mathcal{O}_{\widetilde{Y}}(-p^*D) = \bigoplus_{\chi\in\hat{\mathbf{H}}} \mathcal{O}_{\widetilde{X}}(-L_\chi + [c_1(L_\chi) - D]).$$

Proof. It follows from Lemma 3.2 and the proof of Theorem 3.4 of [17]. □

We define invertible sheaves $\mathcal{L}_{\chi,n}$ and $\mathcal{L}^*_{\chi,n}$ by

$$\begin{aligned}\mathcal{L}_{\chi,n} &= \mathcal{O}_{\widetilde{X}}(-L_\chi + [c_1(L_\chi) - (n/e_v)E_v]),\\ \mathcal{L}^*_{\chi,n} &= \mathcal{O}_{\widetilde{X}}(-L_\chi + [c_1(L_\chi) - (n/m_{vv})E_v^*]).\end{aligned}$$

Lemma 3.6. *For any $\chi \in \hat{\mathbf{H}}$ and $n \in \mathbb{Z}_{\geq 0}$, we have the following:*

(1) *There exists a unique minimal effective cycle $D_{\chi,n} \in \mathbf{L}$ such that*

$$\mathcal{L}_{\chi,n}(-D_{\chi,n}) := \mathcal{L}_{\chi,n} \otimes \mathcal{O}_{\widetilde{X}}(-D_{\chi,n})$$

is π-nef.

(2) $I_n^\chi = \left(\pi_*\mathcal{L}_{\chi,n}\right)_o = \left(\pi_*\mathcal{L}^*_{\chi,n}\right)_o = \left(\pi_*\mathcal{L}_{\chi,n}(-D_{\chi,n})\right)_o.$

Proof. (1) and the equality $\pi_*\mathcal{L}_{\chi,n} = \pi_*\mathcal{L}_{\chi,n}(-D_{\chi,n})$ follows from [6, 4.2]. Note that $\rho'^*F' = m_{vv}^{-1}p^*E_v^*$. By Lemmas 3.3 and 3.5, we obtain other equalities. □

Remark 3.7. Suppose $c_1(e_v L_\chi) = \sum_{w\in\mathcal{V}} k_w E_w$. Then k_v is an integer such that $0 \le k_v < e_v$. By (2) of Lemma 3.6, we have $I_n = I_{n+1}$ if $n \not\equiv k_v \pmod{e_v}$; therefore, $\mathcal{G}^\chi = \bigoplus_{n\ge 0} \mathcal{G}^\chi_{k_v+e_v n}$.

For any $\chi \in \hat{\mathbf{H}}$, let $\mathrm{H}^\chi(t)$ denote the Hilbert series of the graded module $\mathcal{G}^\chi$, i.e.,

$$\mathrm{H}^\chi(t) = \sum_{i\ge 0} (\dim \mathcal{G}^\chi_i) t^i.$$

We define a function P^χ on $\mathbb{N}$ by

$$\mathrm{P}^\chi(n) = \sum_{i=0}^{n-1} (\dim \mathcal{G}^\chi_i).$$

Proposition 3.8 (see Section 5, Appendix)**.** *We have the following formula:*

$$\mathrm{H}^\chi(t) = \frac{1}{|\mathbf{H}|} \sum_{h\in\mathbf{H}} \chi^{-1}(h) \prod_{w\in\mathcal{V}} \left(1 - \theta(h, E^*_w) t^{m_{vw}}\right)^{\delta_w - 2}.$$

In particular, $\mathrm{H}^\chi(t)$ *and* $\mathrm{P}^\chi(n)$ *are computed from the weighted dual graph of* E.

For any cycle D and an invertible sheaf $\mathcal{L}$ on $\widetilde{X}$, $\chi(\mathcal{L} \otimes \mathcal{O}_D)$ denotes the Euler characteristic, i.e., $h^0(\mathcal{L} \otimes \mathcal{O}_D) - h^1(\mathcal{L} \otimes \mathcal{O}_D)$. By the Riemann-Roch formula,

$$\chi(\mathcal{L} \otimes \mathcal{O}_D) = -D \cdot (D + K_{\widetilde{X}})/2 + \mathcal{L} \cdot D.$$

Theorem 3.9. *For any* $\chi \in \hat{\mathbf{H}}$ *and* $n \in \mathbb{Z}_{\ge 0}$, *and any effective cycle* $D \le D_{\chi,n}$, *we have the following:*

$$\dim H^0(\mathcal{L}_\chi)/H^0(\mathcal{L}_{\chi,n}(-D)) = \mathrm{P}^\chi(n) \quad \textit{and}$$
$$h^1(\mathcal{L}_{\chi,n}(-D)) = \chi\left(\mathcal{L}_\chi \otimes \mathcal{O}_{D'}\right) - \mathrm{P}^\chi(n) + h^1(\mathcal{L}_\chi),$$

where $D' = D - [c_1(L_\chi) - (n/e_v)E_v]$. *Furthermore, these are computed from the weighted dual graph of* E *and the intersection numbers* $\mathcal{L}_\chi \cdot E_w$.

Proof. It follows from Lemma 3.6 that

$$\dim H^0(\mathcal{L}_\chi)/H^0(\mathcal{L}_{\chi,n}(-D)) = \dim I^\chi_0/I^\chi_n = \mathrm{P}^\chi(n).$$

We have $D' \ge 0$ since $[c_1(L_\chi)] = 0$. From the exact sequence

$$0 \to \mathcal{L}_{\chi,n}(-D) \to \mathcal{L}_\chi \to \mathcal{L}_\chi \otimes \mathcal{O}_{D'} \to 0,$$

we have the second formula. Finally, we have to show $h^1(\mathcal{L}_\chi)$ can be computed from the graph; however, it follows from Theorem 4.5. $\square$

4. The formulas

The situation is the same as in the preceding section. Therefore, $\widetilde{X}$ satisfies the end-curve condition, and the node E_v is again fixed. Let $\widetilde{X}_i \subset \widetilde{X}$ be a sufficiently small neighborhood of the branch C_i of E_v. Suppose that the irreducible components of C_i are indexed by a set $\mathcal{V}_i \subset \mathcal{V}$. Let $E^*_{w,i}$ denote the $\mathbb{Q}$-cycle supported on C_i satisfying $E^*_{w,i} \cdot E_{w'} = -\delta_{ww'}$ for every $w' \in \mathcal{V}_i$. We define groups $\mathbf{L}_i$, $\mathbf{L}^*_i$, and $\mathbf{H}_i$ as follows:

$$\mathbf{L}_i = \sum_{w\in\mathcal{V}_i} \mathbb{Z}E_w, \quad \mathbf{L}^*_i = \sum_{w\in\mathcal{V}_i} \mathbb{Z}E^*_{w,i}, \quad \mathbf{H}_i = \mathbf{L}^*_i/\mathbf{L}_i.$$

A map $\theta_i\colon \mathbf{H}_i \to \hat{\mathbf{H}}_i$ is defined as θ in Section 2.1; we will follow the notational convention used there. For each i, we define a map

$$\phi_i\colon \mathbf{L}^* \to \mathbf{L}_i^* \quad \text{by} \quad \sum_{w\in\mathcal{V}} \alpha_w E_w^* \mapsto \sum_{w\in\mathcal{V}_i} \alpha_w E_{w,i}^*.$$

Recall that the natural map $\{c_1(L_\chi)\}_{\chi\in\hat{\mathbf{H}}} \to \mathbf{H}$ is bijective. We define

$$\psi_i\colon \hat{\mathbf{H}} \to \hat{\mathbf{H}}_i \quad \text{by} \quad \chi \mapsto \theta_i(\phi_i(c_1(L_\chi))).$$

Remark 4.1. We have $L_\chi|_{\widetilde{X}_i} \sim_{\mathbb{Q}} \phi_i(c_1(L_\chi))$ (cf. the proof of Proposition 2.16).

There is a set $\{L_\lambda\}_{\lambda\in\hat{\mathbf{H}}_i}$ of divisors on $\widetilde{X}_i$ having properties similar to those of $\{L_\chi\}_{\chi\in\hat{\mathbf{H}}}$ (see Section 2.1). For $\chi\in\hat{\mathbf{H}}$, let $D_{\chi,i} = -[\phi_i(c_1(L_\chi))]$. Then

$$L_{\psi_i(\chi)} \sim_{\mathbb{Q}} \phi_i(c_1(L_\chi)) + D_{\chi,i}. \tag{4.1}$$

In the following, if $\mathcal{L}$ is a sheaf, then $\chi(\mathcal{L})$ denotes the Euler characteristic of $\mathcal{L}$.

Lemma 4.2. *For any $\chi\in\hat{\mathbf{H}}$ and $1\le i\le\delta_v$, we have the following:*

(1) $D_{\chi,i}\ge 0$ *and* $L_\chi|_{\widetilde{X}_i} \sim L_{\psi_i(\chi)} - D_{\chi,i}$.
(2) $H^0(\mathcal{O}_{\widetilde{X}_i}(-L_\chi)) = H^0(\mathcal{O}_{\widetilde{X}_i}(-L_{\psi_i(\chi)}))$.
(3) $h^1(\mathcal{O}_{\widetilde{X}_i}(-L_\chi)) = h^1(\mathcal{O}_{\widetilde{X}_i}(-L_{\psi_i(\chi)})) - \chi(\mathcal{O}_{D_{\chi,i}}(-L_\chi))$.

Proof. We write $L_i = L_{\psi_i(\chi)}$ and $D_i = D_{\chi,i}$. Suppose that $E_{v_1}\subset C_i$ intersects E_v, and write $c_1(L_\chi) = \beta E_v + F_1 + F_2$, where $\operatorname{Supp}(F_1)\subset C_i$ and $\operatorname{Supp}(F_2)\subset\bigcup_{j\ne i} C_j$. Then $\phi_i(c_1(L_\chi)) = -\beta E_{v_1,i}^* + F_1$. Since $[\beta E_v + F_1] = 0$, we have

$$D_i = -[-\beta E_{v_1,i}^* + F_1] \ge 0.$$

By Remark 4.1 and (4.1), we obtain (1). We have the natural inclusions

$$H^0(\mathcal{O}_{\widetilde{X}_i}(-L_i)) \subset H^0(\mathcal{O}_{\widetilde{X}_i}(-L_i + D_i)) \subset H^0(\widetilde{X}_i\setminus C_i, \mathcal{O}_{\widetilde{X}_i}(-L_i)).$$

However, $H^0(\mathcal{O}_{\widetilde{X}_i}(-L_i)) = H^0(\widetilde{X}_i\setminus C_i, \mathcal{O}_{\widetilde{X}_i}(-L_i))$ by [17, Proposition 4.3 (1)]. Therefore, (2) follows from (1). Since $D_i\ge 0$, we have the following exact sequence:

$$0\to\mathcal{O}_{\widetilde{X}_i}(-L_i)\to\mathcal{O}_{\widetilde{X}_i}(-L_i+D_i)\to\mathcal{O}_{D_i}(-L_i+D_i)\to 0.$$

Then, (3) follows from (1) and (2). □

We define Cartier divisors C, C', D, and D' as follows:

$$C := e_v E_v^*, \quad C' := \pi'_* C, \quad D := p^* C, \quad D' := \rho'_* D.$$

Then $D' = a_v F'$, where F' is the reduced exceptional divisor on X'. Since $\rho'^* D' = D$, it follows from Lemmas 3.3 and 3.6 that

$$I_{ma_v} = \left(\rho_*\mathcal{O}_{\widetilde{Y}}(-mD)\right)_o \quad \text{and} \quad I^\chi_{ma_v} = \left(\pi_*\mathcal{O}_{\widetilde{X}}(-L_\chi - mC)\right)_o.$$

Lemma 4.3. *If $m\in\mathbb{N}$ satisfies $m > a(\mathcal{G})/a_v$, then for every $\chi\in\hat{\mathbf{H}}$,*

$$H^1(\pi'_*\mathcal{O}_{\widetilde{X}}(-L_\chi - mC)) = 0.$$

Proof. By the equalities $D = \rho'^* D' = p^* C$ and Lemma 3.5,

$$p'_*\mathcal{O}_{Y'}(-mD') = \pi'_* p_*\mathcal{O}_{\widetilde{Y}}(-mD) \cong \bigoplus_{\chi\in\hat{\mathbf{H}}} \pi'_*\mathcal{O}_{\widetilde{X}}(-L_\chi - mC).$$

Since p' is finite, it follows from Lemma 3.4 that

$$H^1(p'_*\mathcal{O}_{Y'}(-mD')) = H^1(\mathcal{O}_{Y'}(-ma_vF')) = 0.$$

□

Lemma 4.4. *Assume that $m \in \mathbb{N}$ satisfies $m > a(\mathcal{G})/a_v$. For every $\chi \in \hat{\mathbf{H}}$, we have*

$$h^1(\mathcal{O}_{\widetilde{X}}(-L_\chi - mC)) = \sum_{i=1}^{\delta_v} h^1(\mathcal{O}_{\widetilde{X}_i}(-L_\chi)).$$

Proof. Let $\mathcal{L} = \mathcal{O}_{\widetilde{X}}(-L_\chi - mC)$. From the spectral sequence

$$E_2^{i,j} = H^i(R^j\pi'_*\mathcal{L}) \Rightarrow H^n(\mathcal{L}),$$

we have the following exact sequence:

$$0 \to H^1(\pi'_*\mathcal{L}) \to H^1(\mathcal{L}) \to H^0(R^1\pi'_*\mathcal{L}) \to 0.$$

However, Lemma 4.3 implies that $H^1(\mathcal{L}) \cong H^0(R^1\pi'_*\mathcal{L})$. Let $x_i \in X'$ denote the singularity obtained by contracting C_i. Then the support of $R^1\pi'_*\mathcal{L}$ is in the set $\{x_i\}_i$. Since $C = \pi'^*C'$ and C' is a Cartier divisor,

$$\left(R^1\pi'_*\mathcal{L}\right)_{x_i} \cong \left(R^1\pi'_*\mathcal{O}_{\widetilde{X}}(-L_\chi) \otimes \mathcal{O}_{X'}(-mC')\right)_{x_i} \cong \left(R^1\pi'_*\mathcal{O}_{\widetilde{X}}(-L_\chi)\right)_{x_i}.$$

This proves the lemma. □

Let K denote the canonical divisor on $\widetilde{X}$.

Theorem 4.5. *For any $\chi \in \hat{\mathbf{H}}$ and any $m \in \mathbb{N}$ greater than $a(\mathcal{G})/a_v$, we have*

$$h^1(\mathcal{L}_\chi) = \mathrm{P}^\chi(ma_v) - \frac{1}{2}(m^2a_v - me_v(K + 2L_\chi)\cdot E_v^*) + \sum_{i=1}^{\delta_v}\left(h^1(\mathcal{O}_{\widetilde{X}_i}(-L_{\psi_i(\chi)})) - \chi(\mathcal{O}_{D_{\chi,i}}(-L_\chi)\right).$$

Furthermore, $h^1(\mathcal{L}_\chi)$ is a topological invariant; in fact, it is computed from the weighted dual graph Γ of E and the intersection numbers $L_\chi \cdot E_v$.

The formula above directly induces a formula for $p_g(Y)$ because

$$p_g(Y) = h^1(\mathcal{O}_{\widetilde{Y}}) = \sum_{\chi\in\hat{\mathbf{H}}} h^1(\mathcal{L}_\chi).$$

Corollary 4.6. *$p_g(X)$ and $p_g(Y)$ can be computed from Γ. In fact, for $m > a(\mathcal{G})/a_v$, we have*

$$p_g(X) = \mathrm{P}^1(ma_v) - \frac{1}{2}(m^2a_v - me_vK\cdot E_v^*) + \sum_{i=1}^{\delta_v} p_g(X', x_i).$$

Lemma 4.7. *Every $\widetilde{X}_i$ satisfies the end-curve condition.*

Proof. It follows from Proposition 2.16. □

Proof of Theorem 4.5. From the exact sequence

$$0 \to \mathcal{O}_{\widetilde{X}}(-L_\chi - mC) \to \mathcal{O}_{\widetilde{X}}(-L_\chi) \to \mathcal{O}_{mC}(-L_\chi) \to 0$$

we obtain

$$h^1(\mathcal{L}_\chi) = \dim \frac{H^0(\mathcal{O}_{\widetilde{X}}(-L_\chi))}{H^0(\mathcal{O}_{\widetilde{X}}(-L_\chi - mC))} - \chi(\mathcal{O}_{mC}(-L_\chi)) + h^1(\mathcal{O}_{\widetilde{X}}(-L_\chi - mC)).$$

Since $C = e_v E_v^*$ and $a_v = e_v^2 a_{vv}$, using the Riemann-Roch formula we have

$$\begin{aligned}\chi(\mathcal{O}_{mC}(-L_\chi)) &= -\frac{1}{2}mC \cdot (mC + K) - mL_\chi \cdot C \\ &= \frac{1}{2}m^2 a_v - \frac{1}{2}me_v(K + 2L_\chi) \cdot E_v^*.\end{aligned}$$

On the other hand, by Lemma 3.6 we have

$$\dim \frac{H^0(\mathcal{O}_{\widetilde{X}}(-L_\chi))}{H^0(\mathcal{O}_{\widetilde{X}}(-L_\chi - mC))} = \dim I_0^\chi / I_{ma_v}^\chi = \mathrm{P}^\chi(ma_v).$$

The formula now follows from Lemmas 4.2 and 4.4.

Next, we show the second assertion by induction on $\#\mathcal{N}$. First, note that the invariant $h^1(\mathcal{L}_\chi) - \sum_{i=1}^{\delta_v} h^1(\mathcal{O}_{\widetilde{X}_i}(-L_{\psi_i(\chi)}))$ is computable from Γ. If $\#\mathcal{N} = 1$, then each C_i is a chain. Hence, every $h^1(\mathcal{O}_{\widetilde{X}_i}(-L_{\psi_i(\chi)}))$ is computable from Γ (cf. [6, 4.3]). For $\#\mathcal{N} > 1$, the invariant $h^1(\mathcal{O}_{\widetilde{X}_i}(-L_{\psi_i(\chi)}))$ is again computable from Γ by Lemma 4.7 and the hypothesis of the induction. □

In the formula of Theorem 4.5,

$$c_v^\chi := \mathrm{P}^\chi(ma_v) - \frac{1}{2}(m^2 a_v - me_v(K + 2L_\chi) \cdot E_v^*)$$

is independent of $m \gg 0$. In fact, $\mathrm{P}^\chi(mka_v)$ is a polynomial function of m for every integer $k > a(\mathcal{G})/a_v$. If $\mathrm{P}^\chi(ml)$ is also a polynomial function of m for some $l \in \mathbb{N}$, then its constant term coincides with c_v^χ. Thus, c_v^χ is an invariant of the series $\mathrm{H}^\chi(t)$.

Proposition 4.8. *If* $\mathrm{H}^\chi(t)$ *is expressed as* $p(t) + r(t)/q(t)$*, where* p*,* q*, and* r *are polynomials with* $\deg r < \deg q$*, then* $c_v^\chi = p(1)$*.*

Proof. We define a function f on $\mathbb{Z}_{\geq 0}$ by $\sum_{n\geq 0} f(n)t^n = r(t)/q(t)$. For a positive integer l, let $P_l(m) = \sum_{n=0}^{ml-1} f(n)$. Then it is sufficient to prove that $P_l(m)$ is a polynomial function with $P_l(0) = 0$ for some $l > \deg p$. We may assume that q and r have no common root. By Proposition 3.8, there exists a positive integer $N > \deg p$ such that $\alpha^N = 1$ for every root α of $q(t)$. Therefore it follows from [20, Corollary 1.5] that there exist polynomials $f_0, \dots, f_{N-1}$ such that $f(n) = f_i(n)$ if $n \equiv i \pmod N$. Then $P_N(m) = \sum_{i=0}^{N-1}\sum_{n=0}^{m-1} f_i(nN + i)$. Since $f_i(nN + i)$ is a polynomial function of n for each i, $\sum_{n=0}^{m-1} f_i(nN + i)$ is a polynomial function of m whose constant term is zero. This proves the assertion. □

Example 4.9. Let us consider a weighted dual graph Γ represented as in Figure 1. We can show that this graph satisfies the monomial condition. Indeed, the set of the following functions, z_i being the variable corresponding to the end w_i, is a Neumann-Wahl system associated with Γ:

$$z_1 + z_3^2 + z_4 z_5 \text{ (at } v_0), \quad z_1^3 + z_2^4 + z_3^{30} \text{ (at } v_1), \quad z_1 z_2^7 z_3 + z_4^3 + z_5^3 \text{ (at } v_2).$$

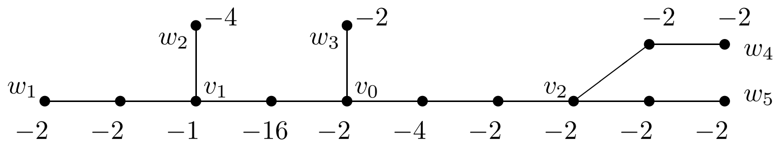


FIGURE 1

(This is an example of a splice type singularity of embedding dimension less than $\#\mathcal{E}$.) Now, we suppose that X is a splice-quotient singularity with weighted dual graph Γ. Then, X is numerical Gorenstein (i.e., $c_1(K_{\widetilde{X}}) \in \mathbf{L}$) and $p_a(Z) = 1 + Z \cdot (Z + K_{\widetilde{X}})/2 = 4$, where Z is Artin's fundamental cycle. We can show that X is actually a Gorenstein singularity, i.e., $K_{\widetilde{X}} \sim c_1(K_{\widetilde{X}})$ holds. Since any splice-quotient singularity is $\mathbb{Q}$-Gorenstein, $K_{\widetilde{X}} \sim_{\mathbb{Q}} c_1(K_{\widetilde{X}})$. On the other hand, the condition $H^1(\mathcal{O}_E) = 0$ implies that $\operatorname{Pic}(\widetilde{X})$ has no torsion. Therefore $K_{\widetilde{X}} \sim c_1(K_{\widetilde{X}})$.

Since $|\mathbf{H}| = 36$, the universal abelian covering of X is not trivial. In the following we compute the geometric genus of X by applying our formula (to do this we will need a computer algebra system).

Recall that the group $\mathbf{H}$ is generated by $\{E_w^*\}_{w\in\mathcal{V}}$ and the relations on the generators are given by the intersection matrix $I = (E_v \cdot E_w)$. By taking the Hermite normal form of I, we have a simplified expression of $\mathbf{H}$. For example, $\mathbf{H}$ has a set of generators $\{E_{w_2}^*, E_{w_3}^*, E_{w_4}^*\}$ with relations $2E_{w_2}^*, 6E_{w_3}, E_{w_2}^* + 3E_{w_3}^* + 3E_{w_4}^* \in \mathbf{L}$. Let $H_{v_0}(t)$ denote the Hilbert series associated with the node v_0. Let $\Lambda = \left\{\lambda \in (\mathbb{Z}_{\geq 0})^3 \mid \lambda_1 < 2, \lambda_2 < 6, \lambda_3 < 3\right\}$ and $E_\lambda^* = \sum \lambda_i E_{w_{i+1}}^*$ for $\lambda \in \Lambda$. Then,

$$
\begin{aligned}
H_{v_0}(t) &= \frac{1}{36} \sum_{\lambda\in\Lambda} \prod_{w\in\mathcal{V}} \left(1 - \exp(2\pi\sqrt{-1}E_\lambda^* \cdot E_w^*)t^{m_{v_0 w}}\right)^{\delta_w - 2} \\
&= \frac{t^{24} - t^{21} + t^{18} - t^{15} + 3t^{12} - t^9 + t^6 - t^3 + 1}{t^{15} - t^{12} - t^3 + 1} \\
&= \frac{4t^{12} - 2t^9 + 2t^6 - 2t^3 + 1}{t^{15} - t^{12} - t^3 + 1} + t^9 + t^3.
\end{aligned}
$$

By Proposition 4.8, we have $c_{v_0} = 2$. Next, let Γ_i denote the branch of v_0 containing v_i and let $H_{\Gamma_i, v_i}(t)$ denote the Hilbert series associated with the node v_i on Γ_i $(i = 1, 2)$. Then, we have

$$
\begin{aligned}
H_{\Gamma_1,v_1}(t) &= \frac{t^{36} - t^{33} + t^{24} - t^{18} + t^{12} - t^3 + 1}{t^{19} - t^{16} - t^3 + 1} \\
&= \frac{t^{12} + t^8 + t^4 - t^3 - t^2 - t + 1}{t^{19} - t^{16} - t^3 + 1} + t^{17} + t^5 + t^2 + t, \\
H_{\Gamma_2,v_2}(t) &= \frac{t^{24} + 1}{t^{20} - t^{14} - t^6 + 1} = \frac{t^{18} + t^{10} - t^4 + 1}{t^{20} - t^{14} - t^6 + 1} + t^4.
\end{aligned}
$$

Therefore $c_{\Gamma_1,v_1} = 4$ and $c_{\Gamma_2,v_2} = 1$ (the second also follows from the fact that Γ_2 corresponds to a minimally elliptic singularity). Now, we conclude that

$$
p_g(X) = c_{v_0} + c_{\Gamma_1,v_1} + c_{\Gamma_2,v_2} = 7.
$$

5. Appendix: Molien's formula

Let $S = \mathbb{C}[z_1, \dots, z_n]$ be the polynomial ring in n variables. Suppose that the grading on S is given by $\deg z_j = w_j \in \mathbb{N}$. Let $S_i \subset S$ denote the i-th graded component. Let G be a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$ consisting of diagonal matrices. Then G acts linearly on S in the usual manner and every graded component S_i is invariant under the action. We denote the action by $(g, f) \mapsto g \cdot f \in S$ for $(g, f) \in G \times S$. For any character $\chi \in \hat{G} = \mathrm{Hom}(G, \mathbb{C}^*)$, the χ-eigenspace S^χ is the space

$$\{f \in S \mid g \cdot f = \chi(g) f \text{ for all } g \in G\}.$$

Then S^χ is also graded. Let $S_i^\chi = S^\chi \cap S_i$. We denote by $\mathcal{H}^\chi(t)$ the Hilbert series of S^χ, i.e.,

$$\mathcal{H}^\chi(t) = \sum_{i \geq 0} (\dim S_i^\chi) t^i.$$

Then as in the proof of [2, Theorem 6.4.8], we have the following

Theorem 5.1 (Molien's formula)**.** *If g_j denotes the j-th diagonal component of g,*

$$\mathcal{H}^\chi(t) = \frac{1}{|G|} \sum_{g \in G} \frac{\chi(g)^{-1}}{\prod_{j=1}^n (1 - g_j t^{w_j})}.$$

Next, we consider a complete intersection case. Let $f_1, \dots, f_m \in S$ be a regular sequence of homogeneous elements with $\deg(f_i) = d_i$, and $I := (f_1, \dots, f_m) \subset S$. Assume that there exist $\chi_1, \dots, \chi_m \in \hat{G}$ such that $g \cdot f_i = \chi_i(g) f_i$ for all $g \in G$. Then S/I is a graded $\mathbb{C}$-algebra with a natural G-action, and every graded component $(S/I)_i$ is invariant under this action. Let $H^\chi(t)$ denote the Hilbert series of the χ-eigenspace $(S/I)^\chi$.

Theorem 5.2. *We have the formula*

$$H^\chi(t) = \frac{1}{|G|} \sum_{g \in G} \chi^{-1}(g) \frac{\prod_{i=1}^m (1 - \chi_i(g) t^{d_i})}{\prod_{j=1}^n (1 - g_j t^{w_j})}.$$

Let us outline the proof. We consider the Koszul complex of the sequence $f_1, \dots, f_m$. We use the following notation:

- Let $\Gamma_p = \{(i_1, \dots, i_p) \mid i_j \in \mathbb{N},\ 1 \leq i_1 < \cdots < i_p \leq m\}$.
- For $\gamma = (i_1, \dots, i_p) \in \Gamma_p$, we set $d_\gamma := \sum_{j=1}^p d_{i_j}$, $\chi_\gamma := \prod_{j=1}^p \chi_{i_j}$ and $\gamma \setminus i_j := (i_1, \dots, \hat{i_j}, \dots, i_p)$, where $\hat{i_j}$ means that i_j is to be omitted.
- Let K_p denote a graded S-module with free basis v_γ, $\gamma \in \Gamma_p$, defined as

$$K_p = \bigoplus_{\gamma \in \Gamma_p} S(-d_\gamma) v_\gamma, \qquad K_0 = S.$$

Then an S-linear map $d : K_p \to K_{p-1}$ is defined by

$$d(v_\gamma) = \sum_{j=1}^p (-1)^{j+1} f_{i_j} v_{\gamma \setminus i_j}, \quad d(v_{(i_1)}) = f_{i_1}.$$

The Koszul complex $K_\bullet$ is a graded complex with differentials of degree 0, and the following is exact (see 1.6.14 and 1.6.15 of [2]):

$$0 \to K_m \xrightarrow{d} \cdots \xrightarrow{d} K_1 \xrightarrow{d} K_0 \to S/I \to 0,$$

where $K_0 \to S/I$ is the canonical surjection. Let $K_p^\chi \subset K_p$ be a submodule defined by

$$K_p^\chi = \bigoplus_{\gamma \in \Gamma_p} S(-d_\gamma)^{\chi\chi_\gamma^{-1}} v_\gamma, \quad K_0^\chi = S^\chi.$$

After some computation we obtain

Lemma 5.3. *We have $d(K_p^\chi) \subset K_{p-1}^\chi$. Moreover, the following complex is exact:*

$$0 \to K_m^\chi \xrightarrow{d} \cdots \xrightarrow{d} K_1^\chi \xrightarrow{d} K_0^\chi \to (S/I)^\chi \to 0.$$

If we denote the Hilbert series of a graded module $\mathcal{M}$ by $H_{\mathcal{M}}(t)$, then Lemma 5.3 immediately implies

$$H^\chi(t) = \sum_{p=0}^{m} (-1)^p H_{K_p^\chi}(t).$$

Obviously,

$$H_{K_p^\chi}(t) = \sum_{\gamma \in \Gamma_p} H_{S(-d_\gamma)^{\chi\chi_\gamma^{-1}}}(t), \quad H_{S(-d_\gamma)^{\chi\chi_\gamma^{-1}}}(t) = t^{d_\gamma} \mathcal{H}^{\chi\chi_\gamma^{-1}}(t).$$

By applying Theorem 5.1, we obtain the desired formula

$$H^\chi(t) = \frac{1}{|G|} \sum_{g \in G} \frac{\chi^{-1}(g)}{\prod_{j=1}^{n} (1 - g_j t^{w_j})} \sum_{p=0}^{m} \sum_{\gamma \in \Gamma_p} (-1)^p \chi_\gamma(g) t^{d_\gamma}.$$

Proof of Proposition 3.8. We use the notation of Theorem 3.2. For $w \in \mathcal{E}$ (resp. $w \in \mathcal{N}$), the v-degree of z_w (resp. $\mathrm{LF}_v(f_{wj_w})$) is m_{vw}. For $h \in \mathbf{H}$, $h \cdot f_{wj_w} = \theta(h, E_w^*) f_{wj_w}$; the same holds for z_w, $w \in \mathcal{E}$ (see Section 2.3). Note that g_j is determined by $g \cdot z_j = g_j z_j$. Now, recall that j_w moves from 1 to $\delta_w - 2$ if $w \in \mathcal{N}$ and $\delta_w - 2 = -1$ if $w \in \mathcal{E}$. Thus, it follows from Theorem 5.2 that

$$\mathrm{H}^\chi(t) = \frac{1}{|\mathbf{H}|} \sum_{h \in \mathbf{H}} \chi^{-1}(h) \prod_{w \in \mathcal{V}} (1 - \theta(h, E_w^*) t^{m_{vw}})^{\delta_w - 2}.$$

References

[1] M. Artin, *On isolated rational singularities of surfaces*, Amer. J. Math. **88** (1966), 129–136. MR0199191 (33:7340)

[2] W. Bruns and J. Herzog, *Cohen-Macaulay rings* (*Revised edition*), Cambridge Stud. Adv. Math., vol. 39, Cambridge Univ. Press, Cambridge, 1998.

[3] S. Goto and K.-i. Watanabe, *On graded rings. I*, J. Math. Soc. Japan **30** (1978), no. 2, 179–213. MR494707 (81m:13021)

[4] H. Laufer, *On minimally elliptic singularities*, Amer. J. Math. **99** (1977), 1257–1295. MR0568898 (58:27961)

[5] I. Luengo-Velasco, A. Melle-Hernández, and A. Némethi, *Links and analytic invariants of superisolated singularities*, J. Algebraic Geom. **14** (2005), no. 3, 543–565. MR2129010 (2005m:32057)

[6] A. Némethi, *Line bundles associated with normal surface singularities*, arXiv:math.AG/0310084.

[7] ______, *"Weakly" elliptic Gorenstein singularities of surfaces*, Invent. Math. **137** (1999), no. 1, 145–167. MR1703331 (2000e:32037)

[8] A. Némethi and L. I. Nicolaescu, *Seiberg-Witten invariants and surface singularities*, Geom. Topol. **6** (2002), 269–328. MR1914570 (2003i:14048)

[9] A. Némethi and T. Okuma, *On the Casson Invariant Conjecture of Neumann–Wahl*, arXiv:math.AG/0610465, to appear in J. Algebraic Geom.

[10] W. D. Neumann, *Graph 3-manifolds, splice diagrams, singularities*, Singularity theory, World Sci. Publ., Hackensack, NJ, 2007, pp. 787–817. MR2342940

[11] ______, *A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves*, Trans. Amer. Math. Soc. **268** (1981), no. 2, 299–344. MR632532 (84a:32015)

[12] W. D. Neumann and J. Wahl, *The end curve theorem for normal complex surface singularities*, arXiv:0804.4644v1.

[13] ______, *Casson invariant of links of singularities*, Comment. Math. Helv. **65** (1990), 58–78. MR1036128 (91c:57022)

[14] ______, *Universal abelian covers of surface singularities*, Trends in singularities, Trends Math., Birkhäuser, Basel, 2002, pp. 181–190. MR1900786 (2003c:32028)

[15] ______, *Complete intersection singularities of splice type as universal abelian covers*, Geom. Topol. **9** (2005), 699–755. MR2140991 (2006i:32037)

[16] ______, *Complex surface singularities with integral homology sphere links*, Geom. Topol. **9** (2005), 757–811. MR2140992 (2006b:32042)

[17] T. Okuma, *Universal abelian covers of rational surface singularities*, J. London Math. Soc. (2) **70** (2004), 307–324. MR2078895 (2005e:14006)

[18] ______, *Universal abelian covers of certain surface singularities*, Math. Ann. **334** (2006), 753–773. MR2209255 (2007f:32036)

[19] F. Sakai, *Weil divisors on normal surfaces*, Duke Math. J. **51** (1984), no. 4, 877–887. MR771385 (86m:14025)

[20] R. Stanley, *Combinatorics and commutative algebra*, second ed., Progress in Mathematics, vol. 41, Birkhäuser Boston, Inc., Boston, MA, 1996. MR1453579 (98h:05001)

[21] M. Tomari and K.-i. Watanabe, *Filtered rings, filtered blowing-ups and normal two-dimensional singularities with "star-shaped" resolution*, Publ. Res. Inst. Math. Sci. **25** (1989), 681–740. MR1031224 (91a:14010)

Department of Education, Yamagata University, Yamagata 990-8560, Japan
E-mail address: okuma@e.yamagata-u.ac.jp