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# DIRICHLET REGULARITY OF SUBANALYTIC DOMAINS

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ABSTRACT. Let $\Omega$ be a bounded and subanalytic domain in $\mathbb{R}^n$, $n \geq 2$. We show that the set of boundary points of $\Omega$ which are regular with respect to the Dirichlet problem is again subanalytic. Moreover, we give sharp upper bounds for the dimension of the set of irregular boundary points. This enables us to decide whether the domain has a classical Green function. In dimensions 2 and 3, this is the case, given some mild and necessary conditions on the topology of the domain.

## INTRODUCTION

**Subanalytic geometry.** The category of subanalytic sets and maps is an outstanding and very important framework to study sets and maps which have singularities but which still show a "tame" behaviour (see, for example, Bierstone-Milman [2], Denef-Van den Dries [6], Hironaka [10] and Łojasiewicz [16], [17]). A subanalytic set is locally a projection of a bounded semianalytic set, both of which are defined below.

*Semianalytic sets.* A set $A \subset \mathbb{R}^n$, $n \geq 1$, is called semianalytic if the following holds:

For each $x_0 \in \mathbb{R}^n$ there are open neighbourhoods $U, V$ of $x_0$ with $\overline{U} \subset V$ and there are real analytic functions $f_i, g_{i_1}, \ldots, g_{i_k}$ on $V$, $1 \leq i \leq \ell$, such that

$$A \cap U = \bigcup_{1 \leq i \leq \ell} \{x \in U \mid f_i(x) = 0,\ g_{i_1}(x) > 0, \ldots, g_{i_k}(x) > 0\}.$$

*Subanalytic sets.* A set $B \subset \mathbb{R}^n$, $n \geq 1$, is called subanalytic if the following holds:

For each $x_0 \in \mathbb{R}^n$ there is an open neighbourhood $U$ of $x_0$, some $m \geq n$ and some bounded semianalytic set $A \subset \mathbb{R}^m$ such that $B \cap U = \pi_n(A)$, where

$$\pi_n \colon \mathbb{R}^m \longrightarrow \mathbb{R}^n, (x_1, \ldots, x_m) \longmapsto (x_1, \ldots, x_n)$$

is the projection on the first $n$ coordinates. A map is subanalytic if its graph is a subanalytic set.

What do we mean by "tame" behaviour? The connected components of a subanalytic set are again subanalytic and locally finite. Hence, a bounded subanalytic set has finitely many components, each of which is subanalytic. Subanalytic sets can be subanalytically stratified and they show fine metric properties (see, for example, Kurdyka [12], Łojasiewicz [16], [17] and Parusinski [18]). These properties allow a good description of measure quantities such as volume or of certain classes of

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functions in the subanalytic case (see Comte [5], Kurdyka-Raby [13] and Kurdyka-Xiao [14]).

A point of a subanalytic set is called regular if the set is an analytic manifold at this point; otherwise it is called singular. As a consequence of subanalytic stratification, the set of singular points of a subanalytic set is again subanalytic and has lower dimension.

There is another concept of regularity (for boundary points of a given domain) originating from the theory of partial differential equations. This notion of regularity is defined via the behaviour of the solutions to the Laplace equation with Dirichlet boundary condition, the so-called *Dirichlet problem.* Although we leave the subanalytic category by solving these partial differential equations, we are nevertheless able to capture the notion of Dirichlet regularity in the context of subanalytic geometry. We introduce Dirichlet regularity, starting with the

**Dirichlet problem.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain (a domain is an open and connected set) and let $f \in C(\partial\Omega)$; i.e. $f$ is a continuous function on the boundary of $\Omega$. Then the *Dirichlet problem* for $f$ is as follows:

Is there a function $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ solving the boundary value problem

$$\begin{aligned} \Delta u = 0 \quad &\text{in} \quad \Omega, \\ u = f \quad &\text{on} \quad \partial\Omega? \end{aligned}$$

Thereby $\Delta := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ is the Laplace operator. A function fulfilling the first equation (the PDE given by the elliptic linear differential operator $\Delta$) is called harmonic on $\Omega$. The class of functions harmonic on $\Omega$ is denoted $\mathcal{H}(\Omega)$. If the answer to the above question is yes, the solution $u$ is unique by the maximum principle for harmonic functions. It is called the *classical Dirichlet solution* for the boundary function $f$.

The Dirichlet problem has many connections to and applications in physics. For example, we can think of $f$ as a given temperature distribution on the boundary of $\Omega$. Then $u$ is the temperature distribution on $\Omega$ in equilibrium.

The domain $\Omega$ is called *Dirichlet regular* if the classical Dirichlet solution exists for every continuous boundary function. For example, the open ball $B_1(0) \subset \mathbb{R}^n$, $n \geq 2$, is regular and the classical Dirichlet solution for a given continuous function on the boundary can be found by the Poisson integral. On the contrary, the punctured open ball $\dot{B}_1(0) \subset \mathbb{R}^n$, $n \geq 2$ is not regular (see Armitage-Gardiner [1, Example 6.1.1]). For example, the function taking the value 0 on the sphere and the value 1 in the center has no classical Dirichlet solution. However, in this case one can give a *generalized Dirichlet solution* using the Perron-Wiener-Brelot method (see Perron [20], Wiener [22], [23] and Brelot [3]):

Let $f \in C(\partial\Omega)$. Then

$$H_f := \inf\{u \in \mathcal{H}^*(\Omega) \mid \varliminf_{\substack{y \to x \\ y \in \Omega}} u(y) \geq f(x) \quad \text{for all} \quad x \in \partial\Omega\}$$

is harmonic on $\Omega$ (see, for example, [1, Chapter 6]). Thereby $\mathcal{H}^*(\Omega)$ is the space of *superharmonic functions* on $\Omega$ (see [1, Chapter 3] for their definition and properties). The generalized Dirichlet solution coincides with the classical one whenever the latter exists (see [1, Remark 6.2.7]).

There is a local version of regularity: A boundary point $x \in \partial\,\Omega$ is called *Dirichlet regular* if

$$\lim_{\substack{y\to x\\ y\in\Omega}} H_f(y) = f(x) \quad \text{for all} \quad f \in C(\partial\,\Omega).$$

Otherwise it is called *irregular*. (From now on we often omit the word Dirichlet. In what follows we always mean Dirichlet regular when we write regular.)

Hence a domain is regular if and only if each of its boundary points is regular. The notion of Dirichlet regularity is closely connected to the *Green function* of the given domain. This is an important function for the theory of the Laplace operator and it codes the geometry of the domain. It is defined as follows. Let $y \in \Omega$. Then $G_y := K_y - H_{K_y|\partial\Omega}$, where

$$K_y(x) := \begin{cases} -\log|x-y| & \text{if} \quad n = 2, \\ \frac{1}{|x-y|^{n-2}} & \text{if} \quad n \geq 3 \end{cases}$$

for $x \in \Omega$ is called the *generalized Green function* of $\Omega$ with pole $y$ (and $K_y$ is called the *Poisson kernel* with pole $y$). The regularity of a boundary point can be checked by the generalized Green function (see [1, Theorem 6.8.3]):

$$x \in \partial\Omega \quad \text{is regular} \iff \lim_{\substack{w\to x\\ w\in\Omega}} G_y(w) = 0 \quad \text{for any resp. all} \quad y \in \Omega.$$

Hence a domain is regular if and only if it has a classical Green function, i.e. if and only if the generalized Green function is continuously extendable to the boundary by 0.

**Main results.** We investigate the set of Dirichlet regular boundary points of a bounded subanalytic domain. In [14], Kurdyka and Xiao showed that each subanalytic domain which is a bounded cell (see Van den Dries [7, Chapter 3] for the definition of a cell) is regular. This observation is based on the so-called cone condition (see [1, Theorem 6.6.15]). In general, however, subanalytic domains are not regular, an example is given by the punctured open ball; instead we establish the following:

**Theorem A.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded and subanalytic domain in $\mathbb{R}^n$, $n \geq 2$. Then the set of regular boundary points of $\Omega$ is subanalytic.*

We give sharp estimates for the dimension of the set of irregular boundary points.

We call a boundary point of a subanalytic domain in $\mathbb{R}^n$ admissible if the local dimension of the complement of the domain at this point is at least $n-1$. Non-admissible boundary points turn out to be irregular. The domain is called admissible if all its boundary points are admissible. We obtain:

**Theorem B.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded and subanalytic domain which is admissible. Then the set of irregular boundary points of $\Omega$ has dimension less than or equal to $n-4$. This upper bound is sharp.*

As a corollary we get the following result on the existence of classical Green functions, generalizing [14, 3.1] for subanalytic domains in $\mathbb{R}^2$ or $\mathbb{R}^3$:

**Corollary.** *Let $\Omega$ be a bounded and subanalytic domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $\Omega$ has a classical Green function iff $\Omega$ is admissible.*

To obtain these theorems we use Wiener's criterion for the regularity of a boundary point (see, for example, Wermer [21, Chapter 19] or the preliminary section). This criterion is based on the concept of capacity and is of an analytic nature. By using the "tame" behaviour of subanalytic sets we are able to translate it in a geometric criterion in the subanalytic case.

*Remark.* A generalization of subanalytic geometry is given by the so-called o-minimal structures on $\mathbb{R}$ (see [7] for their definition and basic properties). Bounded subanalytic sets are precisely the sets which are definable in the polynomially bounded o-minimal structure $\mathbb{R}_{\mathrm{an}}$ and bounded. Mutatis mutandis the above results hold in any polynomially bounded o-minimal structure on $\mathbb{R}$; the proofs go through. Theorem B and the above Corollary do not hold in arbitrary o-minimal structures if $n \geq 3$ as the Lebesgue spine shows (see [1, Remark 6.6.17]).

**Overview.** In a preliminary section we introduce a certain stratification of subanalytic sets, which we use throughout the paper. We mention Wiener's criterion and prove results which allow us to estimate capacities. This enables us to give a geometric criterion for a boundary point of a subanalytic domain to be regular. In Section 1 we do this for reachable boundary points (a boundary point of a subanalytic domain in $\mathbb{R}^n$ is called reachable if the complement of the domain has local dimension $n$ at this point). In Section 2 we generalize this result to admissible boundary points. As a consequence we can handle all boundary points. This allows us to prove Theorem A and Theorem B in Section 3. In addition we consider subanalytic families of bounded domains.

## 0. Preliminaries

0.1. **Stratification of subanalytic sets.** We use the following stratification of a bounded and subanalytic set, which we call good stratification:

**Good Stratification (see** [12, Remark 5.1 and Theorem A]**).** Let $A \subset \mathbb{R}^n$ be a bounded, subanalytic set. Then there is a finite partition $\mathcal{T}$ of $A$ in good strata, i.e. in subanalytic connected $C^1$-manifolds with the following properties:

(i) Let $S, T \in \mathcal{T}$ with $S \cap (\overline{T} \setminus T) \neq \emptyset$. Then $S \subset (\overline{T} \setminus T)$ and $\dim S < \dim T$.

(ii) For every $T \in \mathcal{T}$, after a suitable orthogonal coordinate transformation, there is a subanalytic $C^1$-function $f\colon U \to \mathbb{R}^{n-\dim T}$ with $U \subset \mathbb{R}^{\dim T}$ a domain such that $T = \operatorname{graph} f$ and such that there are constants $L > 0$ and $C > 0$ with $|f(x) - f(y)| \leq L|x - y|$ and $|Df(x)| \leq C$ for all $x, y \in U$. Thereby $Df(x)$ denotes the Jacobian of $f$ at a point $x \in U$. We write $f(T)$, $U(T)$, $L(T)$ and $C(T)$ for these data.

Moreover, given a finite family $\mathcal{A}$ of subanalytic sets in $\mathbb{R}^n$ we can choose $\mathcal{T}$ to be compatible with $\mathcal{A}$; i.e. the following holds. Let $A' \in \mathcal{A}$ and $T \in \mathcal{T}$ with $A' \cap T \neq \emptyset$. Then $T \subset A'$.

0.2. **Some potential theory.** We mention the following useful fact, which we use throughout the paper and which is easily derived from the fact that the class of harmonic functions is closed under composition with translations and orthogonal coordinate transformations.

Let $G \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain, let $x \in \partial G$ and let $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ be a translation or an orthogonal transformation. Then $x$ is a regular boundary point of $\partial G$ iff $\rho(x)$ is a regular boundary point of $\partial\rho(G)$.

Let $n \geq 3$. We use the criterion of Wiener to show regularity or irregularity of boundary points.

**Wiener's criterion (see** [21, Chapter 19] **and** [1, Theorem 7.7.2]**).** Let $G \subset \mathbb{R}^n$, $n \geq 3$, be an arbitrary bounded domain and let $x \in \partial G$. Then the following holds:

$$x \quad \text{is regular} \iff \int\limits_0^1 \frac{c((\mathbb{R}^n \setminus G) \cap \overline{B}_r(x))}{r^{n-1}}\, dr = \infty.$$

Here

$$c(E) := \sup\{\mu(E) \mid \mu \in \mathcal{M}_c^+(E),\ U^\mu \leq 1\}$$

is the capacity of a Borel set $E \subset \mathbb{R}^n$, $n \geq 3$, $\mathcal{M}_c^+(E)$ is the set of positive Borel measures with compact support contained in $E$ and

$$U^\mu \colon \mathbb{R}^n \longrightarrow [0, \infty],\ x \longmapsto \int\limits_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-2}}$$

is the Newton potential of some given $\mu \in \mathcal{M}_c^+(E)$. Note that capacity is invariant under translation and orthogonal coordinate transformations (see [1, Chapter 4] and [11] for more information on capacity).

We will use the following estimates of capacity in the subanalytic context.

**Lemma 0.2.1.** *Let $G \subset \mathbb{R}^n$, $n \geq 3$, be an arbitrary bounded domain. Assume that $\mu \in \mathcal{M}_c^+(G)$ has the following properties:*

(i) $\mu(G) = 1$,

(ii) $\operatorname{supp} \mu$ *is connected,*

(iii) $U^\mu \equiv +\infty$ *on* $\operatorname{supp} \mu$.

*Then* $c(\overline{G}) \geq \frac{1}{\max\limits_{x \in \partial G} U^\mu(x)}$.

*Proof.* A Newton potential $U^\mu$ is superharmonic on $\mathbb{R}^n$ and harmonic on $\mathbb{R}^n \setminus \operatorname{supp} \mu$ (see [1, Definition 4.2.1 and Theorem 4.2.3]). Hence it is lower semicontinuous on $\mathbb{R}^n$ (see [1, Definition 3.1.2]) and continuous on $\mathbb{R}^n \setminus \operatorname{supp} \mu$. Since $U^\mu \equiv +\infty$ on $\operatorname{supp} \mu$ we obtain that $U^\mu$ is continuous. Therefore $U^\mu$ takes a maximum $M \in \mathbb{R}_{>0}$ on $\partial G \subset \mathbb{R}^n \setminus \operatorname{supp} \mu$. We set $V := \{x \in \mathbb{R}^n \mid U^\mu(x) > M\}$. Then $V$ turns out to be connected:

Let $V_1, V_2$ be open subsets with $V_1 \dot\cup V_2 = V$. Since $\operatorname{supp} \mu$ is contained in $V$ and connected we may assume that $\operatorname{supp} \mu \subset V_1$. Then $\overline{V_2} \subset \mathbb{R}^n \setminus \operatorname{supp} \mu$. Assume that $V_2 \neq \emptyset$. Let $M' := \max\limits_{x \in \overline{V_2}} U^\mu(x) \in \mathbb{R}_{>0}$. Then $M' > M$. Due to the maximum principle for harmonic functions (see [1, Theorem 1.2.4]), there is some $x \in \partial V_2$ with $U^\mu(x) = M'$. But then $M' \leq M$ since $x \notin V$, which is a contradiction.

Since $V = (G \cap V) \dot\cup ((\mathbb{R}^n \setminus \overline{G}) \cap V)$ and $\operatorname{supp} \mu \subset G \cap V$ we obtain $V \subset G$ (and hence $\overline{V} \subset \overline{G}$) by the connectedness of $V$. We define

$$u \colon \mathbb{R}^n \longrightarrow \mathbb{R},\ x \longmapsto \begin{cases} U^\mu(x) & \quad x \notin \overline{V}, \\ & \text{if} \\ M & \quad x \in \overline{V}. \end{cases}$$

Then $u \in \mathcal{H}^*(\mathbb{R}^n) \cap \mathcal{H}(\mathbb{R}^n \setminus \overline{V})$ (see [1, Corollary 3.2.4]). Since

$$\lim_{x \to \infty} u(x) = \lim_{x \to \infty} U^\mu(x) = 0,$$

we see with [21, Theorem 18.1] that $u$ is a potential. Thus there is some $\tilde{\mu} \in \mathcal{M}_c^+(\overline{V})$ with $u = U^{\tilde{\mu}}$. We have

$$\begin{aligned} 1 &= \mu(\text{supp }\mu) = \lim_{x\to\infty} |x|^{n-2} U^{\mu}(x) \\ &= \lim_{x\to\infty} |x|^{n-2} u(x) = \lim_{x\to\infty} |x|^{n-2} U^{\tilde{\mu}}(x) = \tilde{\mu}(\overline{V}). \end{aligned}$$

Hence $\tilde{\mu}$ is the equilibrium measure for $\overline{V}$ (see [21, Theorem 9.1]), and we deduce (see [21, Theorem 7.1c]) that

$$c(\overline{G}) \geq c(\overline{V}) = \frac{1}{\max\limits_{x \in \partial G} U^{\mu}(x)}. \qquad \square$$

**Lemma 0.2.2.** *Let $G \subset \mathbb{R}^n$, $n \geq 3$, be an arbitrary bounded domain. Assume that $\mu \in \mathcal{M}_c^+(\overline{G})$ has the following properties:*

(i) $\mu(\overline{G}) = 1$,
(ii) $U^{\mu}|_{\text{supp }\mu} \equiv +\infty$.

*Then* $c(\overline{G}) \leq \frac{1}{\min\limits_{x\in\overline{G}} U^{\mu}(x)}$.

*Proof.* As in the previous proof we see that $U^{\mu}$ is continuous. So $U^{\mu}$ takes a minimum $m \in \mathbb{R}_{>0}$ on $\overline{G}$. We set $K := \{x \in \mathbb{R}^n \mid U^{\mu}(x) \geq m\}$. Then $K$ is compact with $\overline{G} \subset K$. We consider the function

$$u\colon \mathbb{R}^n \longrightarrow \mathbb{R},\ x \longmapsto \begin{cases} U^{\mu}(x) & \quad x \notin K, \\ & \text{if} \\ m & \quad x \in K. \end{cases}$$

Using a similar argument as in the previous lemma, $u$ turns out to be the equilibrium potential to $K$. So we get (see [21, Theorem 7.1c]) that

$$c(\overline{G}) \leq c(K) = \frac{1}{\min\limits_{x \in \overline{G}} U^{\mu}(x)}.$$

$\square$

## 1. Regularity of reachable boundary points of bounded subanalytic domains

We introduce the notion of reachable boundary points of a bounded and subanalytic domain. In dimension $n = 2$ or $n = 3$, every reachable boundary point is regular. In dimension $n \geq 4$, this is not necessarily the case. We give a necessary and sufficient geometric condition.

**General assumption.** Let $\Omega$ be a bounded and subanalytic domain in $\mathbb{R}^n$, $n \geq 2$.

**Definition 1.1.** A boundary point $x \in \partial\Omega$ is called reachable if one of the following equivalent conditions is fulfilled:

i) $x \notin \overset{\circ}{\overline{\Omega}}$,
ii) $x \in \overline{(\mathbb{R}^n \setminus \overline{\Omega})}$,
iii) $\dim_x(\mathbb{R}^n \setminus \Omega) = n$.

Thereby $\dim_x A$ denotes the local dimension of a subanalytic set $A \subset \mathbb{R}^n$ at some point $x \in \overline{A}$.

We set $\Omega_{\text{re}} := \Omega \cup \{x \in \partial\Omega \mid x \text{ is not reachable}\}$. We call $\Omega$ reachable if $\partial\Omega$ has only reachable boundary points, i.e. if $\Omega = \Omega_{\text{re}}$ or equivalently if $\Omega = \overset{\circ}{\overline{\Omega}}$.

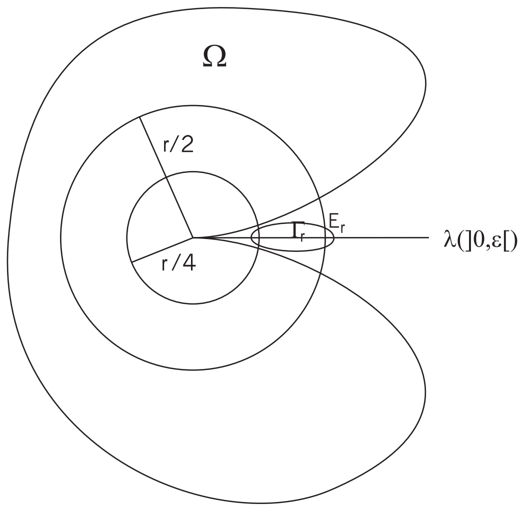


FIGURE 1. Two-dimensional illustration to the proof of Theorem 1.3

*Remark* 1.2. Condition iii) is equivalent to Conditions i) and ii) by stratification. The set $\Omega_{\mathrm{re}}$ is a bounded and subanalytic domain which is reachable. Moreover, $\partial\Omega_{\mathrm{re}} \subset \partial\Omega$ and a reachable boundary point of $\Omega$ is a boundary point of $\Omega_{\mathrm{re}}$.

**Theorem 1.3.** *Let $\Omega \subset \mathbb{R}^2$ or $\Omega \subset \mathbb{R}^3$. Then every reachable boundary point of $\Omega$ is regular.*

*Proof.* The case $n = 2$ is an immediate consequence of the classical lemma of Lebesgue (see Helms [9, Theorem 8.26]), which holds for arbitrary domains, and the curve selection lemma in subanalytic geometry (see [17, p. 1589]).

Let $n = 3$ and let $x$ be a reachable boundary point of $\Omega$. We may assume that $x = 0$. By the curve selection lemma and good stratification, there is, after a suitable orthogonal coordinate transformation, a subanalytic $C^1$-curve $\lambda = (\lambda_1, \lambda_2, \lambda_3)\colon ]0, \varepsilon[\to \mathbb{R}^3$, $\varepsilon > 0$, with the following properties:

(i) $\lambda(]0, \varepsilon[) \subset \mathbb{R}^3 \setminus \overline{\Omega}$,
(ii) $\lim\limits_{t\to 0} \lambda(t) = 0$,
(iii) $\lambda_1(t) = t$ for all $t \in ]0, \varepsilon[$,
(iv) there is some $C > 0$ with $|\lambda'(t)| \leq C$ for all $t \in ]0, \varepsilon[$,
(v) $t \mapsto |\lambda(t)|$ is strongly increasing on $]0, \varepsilon[$.

By Łojasiewicz's inequality (see [2, Theorem 6.4]) there are constants $0 < d < 1$ and $\sigma > 1$ such that $\mathrm{dist}(x, \overline{\Omega}) \geq d|x|^\sigma$ for all $x \in \lambda(]0, \varepsilon[)$. For $r \in ]0, \varepsilon[$ we set $t_r := |\lambda|^{-1}\left(\frac{r}{2}\right), \Gamma_r := \lambda(]t_{\frac{r}{2}}, t_r[)$ and $E_r := \{x \in \mathbb{R}^3 \mid \mathrm{dist}(x, \Gamma_r) < d\left(\frac{r}{4}\right)^\sigma\}$. By construction the domain $E_r$ is contained in $(\mathbb{R}^3 \setminus \overline{\Omega}) \cap B_r(0)$ (see Figure 1).

We define the measure $\mu_r$ on the manifold $\Gamma_r$ by

$$\int\limits_{\Gamma_r} f\, d\mu_r := \frac{1}{t_r - t_{\frac{r}{2}}} \int\limits_{t_{\frac{r}{2}}}^{t_r} f(\lambda(t))dt \quad \text{for all} \quad f \in C_c(\Gamma_r).$$

We extend $\mu_r$ by 0 to $\mathbb{R}^3$. The measure $\mu_r \in \mathcal{M}_c^+(E_r)$ fulfills the conditions of Lemma 0.2.1:

(i) $\mu_r(E_r) = 1$,
(ii) $\operatorname{supp} \mu_r = \overline{\Gamma_r} = \lambda([t_{\frac{r}{2}}, t_r])$ is connected,
(iii) $U^{\mu_r} \equiv +\infty$ on $\operatorname{supp} \mu_r$.

To use Lemma 0.2.1 we need to estimate the potential $U^{\mu_r}$ on $\partial E_r$. Let $x \in \partial E_r$. For $t \in ]t_{\frac{r}{2}}, t_r[$ we have

$$\max_{1\le j\le 3} |x_j - \lambda_j(t)| \ \ge \frac{1}{\sqrt{3}}\ d\left(\frac{r}{4}\right)^\sigma .$$

With $D_r := \frac{1}{\sqrt{3}}\ d\left(\frac{r}{4}\right)^\sigma$ we set

$$\begin{aligned}
I_r^1(x) &:= \left\{ t \in ]t_{\frac{r}{2}}, t_r[ \quad | \quad |x_1 - \lambda_1(t)| \ge D_r \right\},\\
I_r^2(x) &:= \left\{ t \in ]t_{\frac{r}{2}}, t_r[ \quad | \quad |x_2 - \lambda_2(t)| \ge D_r \right\} \setminus I_r^1(x),\\
I_r^3(x) &:= \left\{ t \in ]t_{\frac{r}{2}}, t_r[ \quad | \quad |x_3 - \lambda_3(t)| \ge D_r \right\} \setminus (I_r^1(x) \cup I_r^2(x)).
\end{aligned}$$

We can estimate $U^{\mu_r}(x)$ as follows:

$$\begin{aligned}
U^{\mu_r}(x) &= \int_{\Gamma_r} \tfrac{d\mu_r(y)}{|x-y|} = \tfrac{1}{t_r - t_{\frac{r}{2}}} \int_{t_{\frac{r}{2}}}^{t_r} \tfrac{dt}{|x-\lambda(t)|}\\
&= \tfrac{1}{t_r - t_{\frac{r}{2}}} \sum_{j=1}^{3} \int_{I_r^j(x)} \tfrac{dt}{|x-\lambda(t)|}\\
&\le \tfrac{1}{t_r - t_{\frac{r}{2}}} \left( \int_{I_r^1} \tfrac{dt}{|x_1 - t|} + 2 \int_{t_{\frac{r}{2}}}^{t_r} \tfrac{dt}{\sqrt{(x_1-t)^2 + D_r^2}} \right).
\end{aligned}$$

By evaluating these integrals and using the definition of $I_r^1(x)$, we find some $c > 0$ such that

$$U^{\mu_r}(x) \le \frac{c}{t_r - t_{\frac{r}{2}}} |\log D_r|$$

for $x \in \partial E_r$ and all sufficiently small $r > 0$. Hence we obtain some $c' > 0$ such that

$$U^{\mu_r}(x) \le \frac{c'\sigma}{t_r - t_{\frac{r}{2}}} |\log r|$$

for all $x \in \partial E_r$ and $0 < r \le r_0$ with some small $r_0 < 1$. Since $t_r - t_{\frac{r}{2}} \ge \frac{r}{4C}$ we get with Lemma 0.2.1 and $\varepsilon := (4Cc'\sigma)^{-1}$ that

$$\begin{aligned}
c((\mathbb{R}^3 \setminus \Omega) \cap \overline{B}_r(0)) &\ge c(\overline{E_r})\\
&\ge \left( \max_{x \in \partial E_r} U^{\mu_r}(x) \right)^{-1}\\
&\ge \varepsilon \frac{r}{|\log r|} \quad \text{for all} \quad r \le r_0.
\end{aligned}$$

Hence

$$\int_0^1 \frac{c((\mathbb{R}^3 \setminus \Omega) \cap \overline{B}_r(0))}{r^2} dr \ \ge\ \varepsilon \int_0^{r_0} \frac{dr}{r|\log r|} = \infty,$$

and as a consequence of Wiener's criterion, the boundary point is regular. $\square$

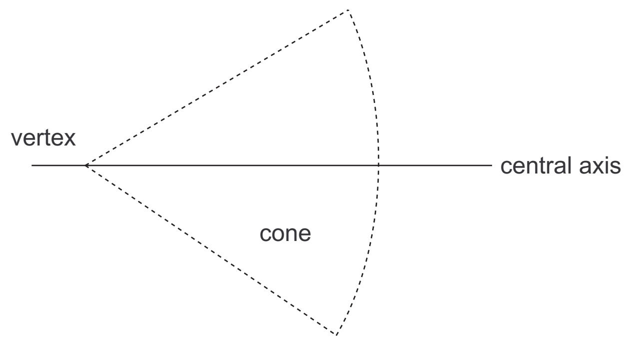


FIGURE 2. Cone in $\mathbb{R}^2$

A reachable boundary point of a bounded and subanalytic domain in $\mathbb{R}^n$, $n \geq 4$, is not necessarily regular:

**Example 1.4** (see [1, Remark 6.6.17])**.** Let $n \geq 4$. We consider the subanalytic (in fact semialgebraic) domains

$$G_\sigma := \{x \in \mathbb{R}^n \mid \sqrt{x_2^2 + \cdots + x_n^2} > x_1^\sigma \quad \text{for } x_1 \geq 0\} \cap B_1(0)$$

with $\sigma \in \mathbb{Q}$, $\sigma > 0$. Then 0 is a reachable boundary point of $G_\sigma$. It is regular iff $\sigma \leq 1$.

This special example can be proven with well-known techniques, using ellipsoids. With the results of Section 0.2 we obtain a criterion for arbitrary subanalytic domains. We shall need the notion of a cone.

**Definition 1.5.** A cone $K \subset \mathbb{R}^n$ with vertex $x \in \mathbb{R}^n$ and central vector $v \in \mathbb{R}^n \setminus \{0\}$ is a set

$$K := \left\{ y \in \mathbb{R}^n \setminus \{x\} \mid \frac{\langle y - x, v \rangle}{|y - x|\,|v|} > \alpha \right\} \cap B_r(x)$$

with some $\alpha > 0$ and some $r > 0$. We call $x + \mathbb{R}v$ the central axis of $K$.

*Remark* 1.6. We stress that a cone is properly contained in one of the open half-spaces defined by the hyperplane which contains the vertex and whose normal vector is the central vector of the cone.

**Theorem 1.7.** *Let $\Omega$ be a bounded and subanalytic domain in $\mathbb{R}^n$, $n \geq 4$. Let $x \in \partial\Omega$ be a reachable boundary point. Then the following are equivalent:*

(i) *$x$ is a regular boundary point of $\Omega_{\mathrm{re}}$.*

(ii) *There is a cone $K$ with vertex $x$, and an affine subspace $E$ through $x$ of codimension 2, which contains the central axis of $K$, such that the projection of $(\mathbb{R}^n \setminus \overline{\Omega_{\mathrm{re}}}) \cap K$ onto $E$ contains a cone in $E$ with vertex $x$.*

*Remark* 1.8. a) If $x$ is a regular boundary point of $\Omega_{\mathrm{re}}$, then it is also a regular boundary point of $\Omega$.

b) It is crucial in Theorem 1.7 ii) that the affine subspace contains the central axis of the cone as one can see by Example 1.4.

We shall need some preparation for the proof of Theorem 1.7 regarding cones and integrals.

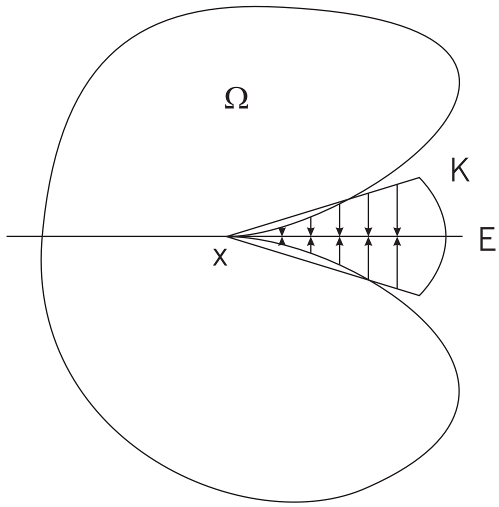


FIGURE 3. Two-dimensional illustration of Theorem 1.7

*Remark* 1.9. a) Let $A \subset \mathbb{R}^n$ be a subanalytic set and let $x \in \overline{A}$. By the proof of Proposition 2.1 in [13] the set $A$ contains a cone with vertex $x \in \overline{A}$ if and only if the volume density of $A$ at $x$, defined by

$$\theta(A,x) := \lim_{r\to 0} \frac{\mathrm{Vol}_n(A \cap B_r(x))}{r^n},$$

is greater than 0 (thereby $\mathrm{Vol}_k$ denotes the $k$-dimensional Hausdorff measure in $\mathbb{R}^n$).

b) As a consequence of a) we obtain the following. Let $A \subset \mathbb{R}^n$ be a subanalytic set and let $x \in \overline{A}$. Assume that $A$ contains a cone with vertex $x$. Let $\mathcal{A}$ be a finite partition of $A$ into subanalytic sets. Then some $A' \in \mathcal{A}$ contains a cone with vertex $x$.

**Definition 1.10.** Let $A \subset \mathbb{R}^n$ be a subanalytic set, let $x \in \overline{A}$ and let $\sigma > 0$. We set $\sigma(A,x) := \{y \in \mathbb{R}^n \mid \mathrm{dist}(y,A) \le |y-x|^\sigma\}$.

**Proposition 1.11.** *Let $A \subset \mathbb{R}^n$ be a subanalytic set and let $0 \in \overline{A}$. Assume that $A$ contains no cone with vertex $0$. Then there is a finite family $\mathcal{T}$ of good strata and there is some $\sigma > 1$ such that the following holds:*

i) *$\dim T < n$ for all $T \in \mathcal{T}$ and every $U(T)$ (compare with the definition of good stratification in Section 0.1) contains a cone with vertex $0$ $(\in \mathbb{R}^{\dim T})$.*
ii) *$A \cap B_r(0) \subset \bigcup_{T\in\mathcal{T}} \sigma(T,0) \cap B_r(0)$ for all sufficiently small $r > 0$.*

*Proof.* a) We show first that there is some subanalytic set $B \subset \mathbb{R}^n$ with $\dim B < n$ and some $\widehat{\sigma} > 1$ such that $A \cap B_r(0) \subset \widehat{\sigma}(B,0) \cap B_r(0)$ for all sufficiently small $r > 0$. By [13, Proposition 2.1] we see that $\theta(A,0) = \mathrm{Vol}_n(C_0(A) \cap \overline{B}_1(0))$, where

$$C_0(A) := \{x \in \mathbb{R}^n \mid \forall\, \varepsilon > 0 \quad \exists\, y \in A \quad \exists\, \lambda \ge 0 \quad \text{such that } |y| < \varepsilon \quad \text{and} \quad |\lambda y - x| < \varepsilon\}$$

is the tangent cone of $A$ at 0 (see [13, Definition 1.1]). By Remark 1.9 we get that $\theta(A,0) = 0$. Since $C_0(A)$ is subanalytic (see [13, Lemma 1.2]) we obtain

$\dim C_0(A) < n$. We consider the function

$$D\colon ]0,1[\longrightarrow \mathbb{R}_{\geq 0},\ t \longmapsto \sup_{y\in A\cap B_t(0)} \operatorname{dist}(y,(C_0(A))).$$

Since $D$ is subanalytic we find some $\sigma' \in \mathbb{Q}_{>0}$ with $\lim_{t\to 0} D(t)/t^{\sigma'} \in \mathbb{R}_{>0}$ (see, for example, [2, Lemma 5.3]). By the definition of the tangent cone we get that $\sigma' > 1$. Hence $B := C_0(A)$ and any $\widehat{\sigma}$ with $1 < \widehat{\sigma} < \sigma'$ fulfill the requirement. b) We apply a good stratification $\mathcal{T}$ to $B$. Let $T \in \mathcal{T}$. If $U(T)$ contains a cone with vertex 0, we keep it. Otherwise we apply step a) to $U(T)$. Since the dimension is lowered in this procedure and since in dimension 1 cones and open intervals are the same we get the claim. $\square$

**Corollary 1.12.** *Let $A \subset \mathbb{R}^n$ be a subanalytic set and let $x \in \overline{A}$. The following holds:*

(i) *$A$ contains a cone with vertex $x$ iff $\mathrm{Vol}_n(A \cap \overline{B}_r(x)) \geq C\, r^n$ for all sufficiently small $r > 0$ and some $C > 0$.*
(ii) *$A$ contains no cone with vertex $x$ iff $\mathrm{Vol}_n(A \cap \overline{B}_r(x)) \leq C\, r^{n-1} r^{\sigma}$ for all sufficiently small $r > 0$, some $\sigma > 1$ and some $C > 0$.*

*Proof.* (i) is clear by Remark 1.9.

(ii) The direction from the right to the left is a consequence of (i). We show the other direction. We may assume that $x = 0$. We use the notion of Proposition 1.11. Let $T \in \mathcal{T}$. By elementary integration we find some $\sigma'$ with $1 < \sigma' \leq \sigma$ and some $C > 0$ such that $\mathrm{Vol}_n(\sigma(T,0) \cap B_r(0)) \leq C\, r^{n-1} r^{\sigma'}$ for all sufficiently small $r > 0$. This proves the claim. $\square$

We shall consider later the following integrals:

**Definition 1.13.** a) Let $n \geq 3$ and let $a, b > 0$. We set

$$I_n(a,b) := \int_0^b \cdots \int_0^b \frac{dx_1 \ldots dx_{n-2}}{(x_1^2 + \cdots + x_{n-2}^2 + a^2)^{\frac{n-2}{2}}}.$$

b) Let $n \geq 4$, let $1 \leq \ell \leq n-3$ and let $a > 0$. We set

$$J_{n,\ell}(a) := \int_0^1 \cdots \int_0^1 \frac{dx_1 \ldots dx_\ell}{(x_1^2 + \cdots + x_\ell^2 + a^2)^{\frac{n-2}{2}}}.$$

**Lemma 1.14.** (i) *There are $C_n > 0$ such that $I_n(a,b) \leq C_n |\log a|$ for all suffciently small $a, b > 0$.*
(ii) *There are $C_{n,\ell} > 0$ such that $J_{n,\ell}(a) \geq C_{n,\ell} \left(\frac{1}{n}\right)^{n-2-\ell}$ for all sufficiently small $a > 0$.*

*Proof.* We use the following integral formulas (see Gröbner-Hofreiter [8, pp. 15, 49]): let $k \in \mathbb{N}$ and let $c > 0$. Then

$$\begin{aligned}\int \frac{ds}{(s^2+c^2)^k} &= s\sum_{j=1}^{k-1} \frac{(2k-3;-2;j-1)}{(k-1;-1;j)} \frac{1}{(2c^2)^j (s^2+c^2)^{k-1}} \\ &\quad + \frac{(1;2;k-1)}{(k-1)!} \frac{1}{(2c^2)^{k-1}} \int \frac{ds}{s^2+c^2} + C,\end{aligned} \tag{1}$$

$$\int \frac{ds}{(s^2+c^2)^{k+\frac{1}{2}}} = s\sum_{j=1}^{k-1} \frac{(2k-2;-2;j)}{(2k-1;-2;j+1)} \frac{1}{c^{2j+2}(s^2+c^2)^{k-j-\frac{1}{2}}} + C. \tag{2}$$

Thereby $(m;d;j) := \prod_{i=0}^{j-1}(m+id)$ for $m, d \in \mathbb{R}$ and $j \in \mathbb{N}_0$. Note that all terms are positive in (1) and (2), and call this condition $(*)$.

(i): To prove (i) we show the following claims. Let $k \in \mathbb{N}$.

$\alpha$) There is some $D_k > 0$ such that

$$\int_0^t \frac{ds}{(s^2+c^2)^k} \leq D_k \frac{1}{(c^2)^{k-\frac{1}{2}}} \quad \text{for all} \quad t > 0.$$

$\beta$) There is some $\widehat{D}_k > 0$ such that

$$\int_0^t \frac{ds}{(s^2+c^2)^{k+\frac{1}{2}}} \leq \widehat{D}_k \frac{1}{c^{2k}} \quad \text{for all} \quad t > 0.$$

We have for $j = 1, \ldots, k-1$ that $(c^2)^j(s^2+c^2)^{k-j} \geq (c^2)^{k-1}(s^2+c^2) \geq (c^2)^{k-1}2sc = 2s(c^2)^{k-\frac{1}{2}}$. With $\int_0^t \frac{ds}{s^2+c^2} = \frac{1}{c}\arctan\frac{t}{c}$ we get by (1) and $(*)$ that $\alpha$) holds.

We have for $j = 1, \ldots, k-1$ that $c^{2j+2}(s^2+c^2)^{k-j-\frac{1}{2}} \geq c^{2k}(s^2+c^2)^{\frac{1}{2}} \geq c^{2k}s$. With (2) and $(*)$ we obtain $\beta$).

We show (i) by induction on $n \geq 3$:

$n = 3$: By [8, p. 45] we have $I_3(a,b) = \int_0^b \frac{dx}{\sqrt{x^2+a^2}} = \log(b + \sqrt{b^2+a^2}) - \log a$ and the claim follows.

$n \to n+1$: Case 1: $n+1$ is even. We obtain by $\alpha$) with $s := x_{n-1}$ and $c^2 := x_1^2 + \cdots + x_{n-2}^2 + a^2$ that $I_{n+1}(a,b) \leq D_{\frac{n-1}{2}} I_n(a,b)$ and get the claim by the induction hypothesis.

Case 2: $n+1$ is odd. We obtain by $\beta$) that $I_{n+1}(a,b) \leq \widehat{D}_{\frac{n-2}{2}} I_n(a,b)$ and get the claim by the induction hypothesis.

(ii): To prove (ii) we show the following claims. Let $k, m \in \mathbb{N}$.

$\gamma$) There is some $D_{k,m} > 0$ such that

$$\int_0^1 \frac{ds}{(s^2+c^2)^k} \geq D_{k,m} \frac{1}{(c^2)^{k-\frac{1}{2}}} \quad \text{for all} \quad c^2 \leq m.$$

$\delta$) There is some $\widehat{D}_{k,m} > 0$ such that

$$\int_0^1 \frac{ds}{(s^2+c^2)^{k+\frac{1}{2}}} \geq \widehat{D}_{k,m} \frac{1}{(c^2)^k} \quad \text{for all} \quad c^2 \leq m.$$

Using $(*)$, claims $\gamma$) and $\delta$) follow by considering only the summand next to the constant $C$ in (1), resp. (2). The proof of (ii) can be done by induction in the same way as the proof of (i). □

*Proof of Theorem* 1.7. We replace $\Omega_{\mathrm{re}}$ by $\Omega$; i.e. we assume that $\Omega$ is reachable. We may also assume that $x = 0$.

(ii) $\Rightarrow$ (i):

We show first the following:

*Claim.* There is a good stratum $\widehat{\Gamma}$ with $\dim \widehat{\Gamma} = n-2$ and $\widehat{\Gamma} \subset \mathbb{R}^n \setminus \overline{\Omega}$ such that $U(\widehat{\Gamma})$ contains a cone $\widehat{K}$ with vertex in 0.

*Proof of the claim.* We may assume that $E = \{x \in \mathbb{R}^n \mid x_{n-1} = x_n = 0\}$ and that the central axis of $K$ is given by the first unit vector. Let $U := (\mathbb{R}^n \setminus \overline{\Omega}) \cap K$ and $\pi \colon \mathbb{R}^n \to \mathbb{R}^{n-2}$ be the projection onto $E$. By assumption $\pi(U)$ contains a cone with vertex 0. By [7, p. 94] there is a subanalytic map $\varphi \colon \pi(U) \to \mathbb{R}^n$ with graph $\varphi \subset U$. Applying a stratification to graph $\varphi$ we find with Remark 1.9 b) a cone $K' \subset E$ with vertex 0 such that $\psi := \varphi|_{K'}$ is a $C^1$-function. Since $\Lambda :=$ graph $\psi \subset K$ we find by the definition of a cone (see Remark 1.6) some $c > 0$ such that $\pi(\Lambda \cap B_r(0)) \supset K' \cap B_{cr}(0)$ for all suffi¡ciently small $r > 0$. Hence there is some $C > 0$ such that $\mathrm{Vol}_{n-2}(\Lambda \cap B_r(0)) \geq C\, r^{n-2}$ for all sufficiently small $r > 0$ $(**)$. Let $\mathcal{T}$ be a good stratification of $\Lambda$. Then $\dim T \leq n-2$ for all $T \in \mathcal{T}$. Let $\mathcal{T}' := \{T \in \mathcal{T} \mid \dim T = n-2\}$. Then $\mathcal{T}' \neq \emptyset$ since $\dim \Lambda = n-2$. Assume that no $U(T)$, $T \in \mathcal{T}'$, contains a cone with vertex 0. By Corollary 1.11 we find some $\sigma > 1$ and some $D > 0$ such that $\mathrm{Vol}_{n-2}(U(T) \cap B_r(0)) \leq D\, r^{n-3} r^{\sigma}$ for all sufficiently small $r > 0$ and all $T \in \mathcal{T}'$. Since the Jacobians of $f(T)$ are bounded (see Section 0.1) we find some $\widehat{D} > 0$ such that $\mathrm{Vol}_{n-2}(T \cap B_r(0)) \leq \widehat{D}\, r^{n-3} r^{\sigma}$ for all sufficiently small $r > 0$ and all $T \in \mathcal{T}'$. But this contradicts $(**)$ and the claim is proven.

Let $\widehat{\Gamma}$ be a good stratum as in the claim. We have $U(\widehat{\Gamma}) \subset \mathbb{R}^{n-2}$ in a coordinate system suitable for $T$ (compare with Section 0.1). With $\pi \colon \mathbb{R}^n \to \mathbb{R}^{n-2}$ we denote again the projection on the first $n-2$ coordinates. We can choose the cone $\widehat{K}$ such that $\overline{\mathrm{graph}\, g} \cap \overline{\Omega} = \{0\}$, where $g := f(\widehat{\Gamma})|_{\widehat{K}}$. By Lojasiewicz's inequality (see [2, Theorem 6.4]) there are constants $\sigma > 1$ and $0 < d < 1$ such that $\mathrm{dist}(x, \overline{\Omega}) \geq d\, |x|^{\sigma}$ for all $x \in \Gamma :=$ graph $g$. By the Lipschitz condition on $g$ (see Section 0.1) we find some constant $0 < c < 1$ such that $\pi(\Gamma \cap B_r(0)) \supset \widehat{K} \cap B_{cr}(0)$ for all sufficiently small $r > 0$. We set

$$\widehat{K}_r := \{x \in \widehat{K} \mid c\,\tfrac{r}{4} < |x| < c\,\tfrac{r}{2}\}, \quad \Gamma_r := \mathrm{graph}\, g|_{\widehat{K}_r} \quad \text{and}$$
$$E_r := \{x \in \mathbb{R}^n \mid \mathrm{dist}(x, \Gamma_r) < d\left(\tfrac{r}{4}\right)^{\sigma}\}.$$

The domain $E_r$ is contained in $(\mathbb{R}^n \setminus \overline{\Omega}) \cap B_r(0)$. We define the measure $\mu_r$ on the manifold $\Gamma_r$ by

$$\int_{\Gamma_r} f\, d\mu_r = \frac{1}{\mathrm{Vol}_{n-2}(\widehat{K}_r)} \int_{\widehat{K}_r} f(\widehat{g}(x'))dx' \quad \text{for all} \quad f \in C_c(\Gamma_r),$$

where $\widehat{g}\colon \widehat{K} \to \mathbb{R}^n$, $x' \mapsto (x', g(x'))$. We extend $\mu_r$ by 0 to $\mathbb{R}^n$. The measure $\mu_r \in \mathcal{M}_c^+(E_r)$ fulfills the conditions of Lemma 0.2.1:

(i) $\mu_r(E_r) = 1$,
(ii) $\operatorname{supp} \mu_r = \overline{\Gamma_r}$ is connected,
(iii) $U^{\mu_r} \equiv +\infty$ on $\operatorname{supp} \mu_r$.

We estimate the potential $U^{\mu_r}$ on $\partial E_r$. Let $x = (x', x_{n-1}, x_n) \in \partial E_r$. With $D_r := \frac{1}{\sqrt{n}} d\left(\frac{r}{4}\right)^\sigma$ we set

$$\begin{aligned}
U_r^1(x) &:= \left\{ y' \in \widehat{K}_r \quad \middle| \quad |x_1 - \widehat{g}_1(y')| \geq D_r \right\},\\
U_r^2(x) &:= \left\{ y' \in \widehat{K}_r \quad \middle| \quad |x_2 - \widehat{g}_2(y')| \geq D_r \right\} \setminus U_r^1(x),\\
&\;\;\vdots\\
U_r^n(x) &:= \left\{ y' \in \widehat{K}_r \quad \middle| \quad |x_n - \widehat{g}_n(y')| \geq D_r \right\} \setminus \bigcup_{1 \leq j \leq n-1} U_r^j(x).
\end{aligned}$$

We get

$$\begin{aligned}
U^{\mu_r}(x) &= \int_{\Gamma_r} \frac{d\mu_r(y)}{|x-y|^{n-2}} = \frac{1}{\mathrm{Vol}_{n-2}(\widehat{K}_r)} \int_{\widehat{K}_r} \frac{dy'}{|x-\widehat{g}(y')|^{n-2}}\\
&= \frac{1}{\mathrm{Vol}_{n-2}(\widehat{K}_r)} \sum_{j=1}^{n} \int_{U_r^j(x)} \frac{dy'}{|x-\widehat{g}(y')|^{n-2}}\\
&\leq \frac{1}{\mathrm{Vol}_{n-2}(\widehat{K}_r)} \left( \sum_{j=1}^{n-2} \int_{U_r^j(x)} \frac{dy'}{|x'-y'|^{n-2}} \right.\\
&\qquad\qquad \left. + \int_{U_r^{n-1}(x)} \frac{dy'}{(|x'-y'|^2+D_r^2)^{n-2}} + \int_{U_r^n(x)} \frac{dy'}{(|x'-y'|^2+D_r^2)^{n-2}} \right)\\
&\leq \frac{\widehat{D}}{\mathrm{Vol}_{n-2}(\widehat{K}_r)} \left( (n-2) \int_{[0,r]^{n-2}} \frac{dx_1 \ldots dx_{n-2}}{((x_1+D_r)^2+x_2^2+\cdots+x_{n-2}^2)^{\frac{n-2}{2}}} \right.\\
&\qquad\qquad \left. +2 \int_{[0,r]^{n-2}} \frac{dx_1 \ldots dx_{n-2}}{(x_1^2+x_2^2+\cdots+x_{n-2}^2+D_r^2)^{\frac{n-2}{2}}} \right)\\
&\overset{(1.12)}{\leq} \frac{\widehat{D}}{\mathrm{Vol}_{n-2}(\widehat{K}_r)}\, 2n\, I_n(D_r) \overset{(1.13)}{\leq} \frac{\widehat{D}}{\mathrm{Vol}_{n-2}(\widehat{K}_r)}\, 2n\, C_n\, |\log D_r|
\end{aligned}$$

for all sufficiently small $r > 0$ and some $\widehat{D} > 0$. Since there is some $D' > 0$ such that $\mathrm{Vol}_{n-2}(\widehat{K}_r) \geq D'\, r^{n-2}$ for all sufficiently small $r > 0$, we find some small $r_0$ with $0 < r_0 < 1$ and some $D > 0$ such that

$$U^{\mu_r}(x) \leq D\, \frac{|\log r|}{r^{n-2}}$$

for all $0 < r \leq r_0$. Hence we get by Lemma 0.2.1,

$$\begin{aligned} c((\mathbb{R}^n \setminus \Omega) \cap \overline{B_r}(0)) &\geq c(\overline{E_r}) \\ &\geq \left( \max_{x \in \partial E_r} U^{\mu_r}(x) \right)^{-1} \\ &\geq \frac{1}{D} \frac{r^{n-2}}{|\log r|} \end{aligned}$$

for all $r \leq r_0$. We conclude that

$$\int_0^1 \frac{c((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_r(0))}{r^{n-1}} dr \;\geq\; \frac{1}{D} \int_0^{r_0} \frac{dr}{r|\log r|} = \infty,$$

which means that the boundary point is regular according to Wiener's criterion.

(i) $\Longrightarrow$ (ii):

Assume that (ii) fails. Certainly, $\mathbb{R}^n \setminus \overline{\Omega}$ contains no cone with vertex 0. Let $\mathcal{T}$ be a finite family of good strata as in Proposition 1.11. Let $T \in \mathcal{T}$.

*Claim* 1. $\dim T \leq n-3$.

*Proof of Claim* 1. Let $k := \dim T$. In a suitable coordinate system we have that $U(T) \subset \mathbb{R}^k \subset \mathbb{R}^n$. According to Proposition 1.11 there is some cone $\widehat{K} \subset \mathbb{R}^k$ with vertex 0 such that $\widehat{K} \subset U(T)$. Let $K \subset \mathbb{R}^n$ be a cone with vertex 0 with the same central axis as $\widehat{K}$, which contains $\widehat{K}$ and the set

$$\{x = (x', x'') \in \mathbb{R}^n \mid x' \in \widehat{K},\ |x''| \leq L(T)|x'|\}.$$

Let $g := f(T)|_{\widehat{K}}$ and $\Gamma :=$ graph $g$. Then $\Gamma \subset K$ and $\Gamma \cap B_r(0) \subset \mathbb{R}^n \setminus \overline{\Omega}$ for all sufficiently small $r > 0$ by Proposition 1.11 ii). Hence the projection of $(\mathbb{R}^n \setminus \overline{\Omega}) \cap K$ onto $\mathbb{R}^k$ contains $\widehat{K} \cap B_r(0)$ for some small $r > 0$. Since (ii) fails by assumption we get that $k \leq n-3$. This proves Claim 1.

Let $\sigma > 1$ be as in Proposition 1.11. Then $(\mathbb{R}^n \setminus \Omega) \cap B_r(0) \subset \bigcup_{T \in \mathcal{T}} \sigma(T, 0) \cap B_r(0)$ for all sufficiently small $r > 0$ (+). We choose $\widehat{\sigma}$ with $1 < \widehat{\sigma} < \sigma$. Let $T \in \mathcal{T}$.

*Claim* 2. There is some constant $C = C(T)$ such that $c(\sigma(T,0) \cap \overline{B}_r(0)) \leq C\, r^{n-3} r^{\widehat{\sigma}}$ for all sufficiently small $r > 0$.

Assume that Claim 2 holds. Using (+) and the subadditivity of capacity (see [1, Corollary 5.4.5]) we find then some $r_0 > 0$ and some $C > 0$ such that $c((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_r(0)) \leq C r^{n-3} r^{\widehat{\sigma}}$ for all $0 < r \leq r_0$. Hence

$$\int_0^{r_0} \frac{c((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_r(0))}{r^{n-1}} dr \leq C \int_0^{r_0} r^{\widehat{\sigma}-2} dr < \infty.$$

By Wiener's criterion we get that 0 is an irregular boundary point of $\Omega$, contradicting (i). So (i) $\Longrightarrow$ (ii) is proven if Claim 2 holds.

*Proof of Claim* 2. Let $k := \dim T$. In a suitable coordinate system we have $U(T) \subset \mathbb{R}^k$. By [19, Theorem 1] we can extend $f(T)$ to a Lipschitz function $h\colon \mathbb{R}^k \to \mathbb{R}^{n-k}$ with Lipschitz constant $L := \sqrt{n-k}L(T)$. For $r > 0$ we set $\Gamma_r :=$ graph $h|_{B_r(0)}$

(with $B_r(0) \subset \mathbb{R}^k$) and $E_r := \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, \Gamma_r) \le r^{\widehat{\sigma}}\}$. Then $\sigma(T,0) \cap \overline{B}_r(0) \subset E_r$ for all sufficiently small $r > 0$ (++). We define the measure $\mu_r$ on $\Gamma_r$ by

$$\int_{\Gamma_r} f\, d\mu_r = \frac{1}{\operatorname{Vol}_k(B_r(0))} \int_{B_r(0)\subset\mathbb{R}^k} f(\widehat{h}(x'))dx' \quad \text{for all} \quad f \in C_c(\Gamma_r)$$

where $\widehat{h}\colon \mathbb{R}^k \to \mathbb{R}^n$, $x' \mapsto (x', h(x'))$. We extend $\mu_r$ by 0 to $\mathbb{R}^n$. The measure $\mu_r \in \mathcal{M}_c^+(\overline{E_r})$ fulfills the conditions of Lemma 0.2.2:

(i) $\mu_r(E_r) = 1$,
(ii) $U^{\mu_r}|_{\operatorname{supp}\, \mu_r} \equiv +\infty$.

We want to estimate the potential $U^{\mu_r}$ on $\overline{E_r}$. Let $x \in \overline{E_r}$. Then there is some $x' \in \overline{B}_r(0) \subset \mathbb{R}^k$ with $\operatorname{dist}(x, \Gamma_r) = |x - \widehat{h}(x')| \le r^{\widehat{\sigma}}$. We get

$$\begin{aligned}
U^{\mu_r}(x) &= \int_{\Gamma_r} \tfrac{d\mu_r}{|x-y|^{n-2}} \\
&= \tfrac{1}{\operatorname{Vol}_k(B_r(0))} \int_{B_r(0)} \tfrac{dy'}{|x-\widehat{h}(y')|^{n-2}} \\
&\ge \tfrac{1}{\operatorname{Vol}_k(B_r(0))} \int_{B_r(0)} \tfrac{dy'}{(r^{2\widehat{\sigma}}+|\widehat{h}(x')-\widehat{h}(y')|^2))^{\frac{n-2}{2}}} \\
&\ge \tfrac{C_1}{\operatorname{Vol}_k(B_r(0))} \int_{B_r(0)} \tfrac{dy'}{\left(\left(\frac{r^{\widehat{\sigma}}}{1+L}\right)^2+|x'-y'|^2\right)^{\frac{n-2}{2}}} \\
&\ge \tfrac{C_2}{\operatorname{Vol}_k(B_r(0))} \int_{[0,C_3 r]^k} \tfrac{dx_1\ldots dx_k}{\left(\left(\frac{r^{\widehat{\sigma}}}{1+L}\right)^2+x_1^2+\cdots+x_k^2\right)^{\frac{n-2}{2}}}
\end{aligned}$$

with constants $C_1, C_2, C_3 > 0$ dependent on $k$ and $L$ but independent from $r$ and $x$. After a suitable substitution of variables we find by Lemma 1.14 constants $D_1, D_2, D_3 > 0$ with the same property as the above constants, such that

$$\begin{aligned}
U^{\mu_r}(x) &\ge \frac{D_1}{\operatorname{Vol}_k(B_r(0))}\ \frac{1}{r^{n-2-k}}\ J_{n,k}\left(D_2 \frac{r^{\widehat{\sigma}}}{r}\right) \\
&\ge D_3 \frac{1}{r^k}\ \frac{1}{(r^{\widehat{\sigma}})^{n-2-k}}.
\end{aligned}$$

By Lemma 0.2.2 we obtain with $D := D_3^{-1}$,

$$c(\overline{E_r}) \le \left(\min_{x \in \overline{E_r}} U^{\mu_r(x)}\right)^{-1} \le D\, r^k (r^{\widehat{\sigma}})^{n-2-k}.$$

We get Claim 2 by (++) and Claim 1. □

## 2. Regularity of arbitrary boundary points of bounded subanalytic domains

We answered the question of regularity of reachable boundary points inside the subanalytic category. We generalize these results to admissible boundary points. As a consequence we can handle all boundary points by general facts from potential theory. As in Section 1 we make the following assumption.

**General assumption.** $\Omega$ denotes a bounded and subanalytic domain in $\mathbb{R}^n$, $n \geq 2$.

**Definition 2.1.** A boundary point $x \in \partial\,\Omega$ is called admissible if $\dim_x(\mathbb{R}^n \setminus \Omega) \geq n-1$. We set

$$\Omega_{\mathrm{ad}} := \Omega \cup \{x \in \partial\,\Omega \mid x \text{ is not admissible}\}.$$

We call $\Omega$ admissible if $\partial\,\Omega$ has only admissible boundary points, i.e. if $\Omega = \Omega_{\mathrm{ad}}$.

*Remark* 2.2. The set $\Omega_{\mathrm{ad}}$ is a bounded and subanalytic domain which is admissible. Moreover, $\partial\Omega_{\mathrm{ad}} \subset \partial\Omega$ and an admissible boundary point of $\Omega$ is a boundary point of $\Omega_{\mathrm{ad}}$. In the case $n = 2$ a boundary point is admissible iff it is not an isolated boundary point.

**Proposition 2.3.** *Let $x \in \partial\,\Omega$. Then $x$ is a regular boundary point of $\Omega$ iff $x$ is a regular boundary point of $\Omega_{\mathrm{ad}}$.*

*Proof.* "$\Leftarrow$": Let $x \in \partial\Omega_{\mathrm{ad}}$ be a regular boundary point of $\Omega_{\mathrm{ad}}$. By Remark 2.2 we get that $x \in \partial\Omega$. Since $\Omega \subset \Omega_{\mathrm{ad}}$, we obtain that $x$ is a regular boundary point of $\Omega$ (compare with [1, p. 180]).

"$\Rightarrow$": Let $x \in \partial\Omega$ be a regular boundary point of $\Omega$. We show first that $x$ is an admissible boundary point of $\Omega$. Assume that $\dim_x(\mathbb{R}^n \setminus \Omega) \leq n-2$. In the case $n = 2$, the boundary point $x$ would be isolated and hence irregular (compare with [1, Example 6.6.1]). In the case $n \geq 3$, there would be some $R > 0$ such that $(\mathbb{R}^n \setminus \Omega) \cap B_R(x)$ is a finite union of subanalytic $C^1$-manifolds of dimension smaller than or equal to $n-2$. But then $c((\mathbb{R}^n \setminus G) \cap B_R(x)) = 0$ (see [1, pp. 123, 141]), which means, according to Wiener's criterion, that $x$ is irregular. So $x$ is admissible and hence $x \in \partial\Omega_{\mathrm{ad}}$ by Remark 2.2. Finally we have to show that $x$ is a regular boundary point of $\Omega_{\mathrm{ad}}$. In the case $n = 2$ there is a subanalytic $C^1$-manifold of dimension one or two in the complement of $\Omega_{\mathrm{ad}}$ with $x$ in its closure. By the curve selection lemma (see [17, p. 1589]) and the lemma of Lebesgue (see [9, Theorem 8.26]) we see that $x$ is a regular boundary point of $\Omega_{\mathrm{ad}}$. In the case $n \geq 3$, there is some $R > 0$ such that

$$(\mathbb{R}^n \setminus \Omega_{\mathrm{ad}}) \cap B_R(x) = ((\mathbb{R}^n \setminus \Omega) \cap B_R(x)) \setminus \Gamma$$

with $\Gamma$ subanalytic and $\dim \Gamma \leq n-2$. Using that $c(\Gamma) = 0$ we can see by the subadditivity of capacity (see [1, Corollary 5.4.5]) and Wiener's criterion that $x$ is a regular boundary point of $\Omega_{\mathrm{ad}}$. $\square$

In the case $n = 2, 3$ the following holds:

**Theorem 2.4.** *Let $\Omega$ be a bounded and subanalytic domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. Let $x \in \partial\Omega$. Then $x$ is a regular boundary point of $\Omega$ iff $x$ is admissible.*

*Proof.* If $x$ is a regular boundary point of $\Omega$, then $x$ is an admissible boundary point of $\Omega$ by Proposition 2.3 and Remark 2.2. Let $x$ be an admissible boundary point of $\Omega$. We have to show that $x$ is regular. In the case $n = 2$, we get by Remark 2.2 that $x$ is not an isolated boundary point and the claim follows with the curve selection lemma (see [17, p. 1589]) and the lemma of Lebesgue (see [9, Theorem 8.26]). In the case $n = 3$, the claim can be shown with Theorem 1.3 in the same way as Theorem 2.6 (ii) $\Rightarrow$ (i) below. We omit the proof. $\square$

**Corollary 2.5.** *Let $\Omega$ be a bounded and subanalytic domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $\Omega$ is regular iff $\Omega$ is admissible.*

In the case $n \geq 4$, we can generalize Theorem 1.6 to admissible boundary points. We reduce the problem to the case of reachable boundary points and use the fact that the capacity of a compact set coincides with the capacity of its boundary.

**Theorem 2.6.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 4$, be a bounded and subanalytic domain. Let $x \in \partial\Omega$. Then the following are equivalent:*

(i) *$x$ is a regular boundary point of $\Omega$.*
(ii) *There is a cone $K$ with vertex $x$ and an affine subspace $E$ through $x$ of codimension 2, which contains the central axis of $K$, such that the projection of $(\mathbb{R}^n \setminus \Omega_{\mathrm{ad}}) \cap K$ onto $E$ contains a cone in $E$ with vertex $x$.*

*Proof.* By Proposition 2.3 we may assume that $\Omega = \Omega_{\mathrm{ad}}$. We may also assume that $x = 0$.

(ii) $\Longrightarrow$ (i):

We choose a good stratification $\mathcal{T}$ of $(\mathbb{R}^n \setminus \Omega) \cap K$. Let $\pi\colon \mathbb{R}^n \to E$ be the projection onto $E$. By assumption $\pi((\mathbb{R}^n \setminus \Omega) \cap K)$ contains a cone with vertex 0. Since $\pi((\mathbb{R}^n \setminus \Omega) \cap K) = \bigcup_{T \in \mathcal{T}} \pi(T)$ there is by Remark 1.8 b) some $S \in \mathcal{T}$ such that $\pi(S)$ contains a cone with vertex 0. According to the definition of an admissible boundary point there is some $T \in \mathcal{T}$ with $\dim T \geq n-1$ and $S \subset \overline{T}$. Since $\overline{\pi(T)} \supset \pi(\overline{T})$ we see that $\pi(T)$ contains a cone with vertex 0. If $\dim T = n$ we can apply Theorem 1.6 (ii) $\Rightarrow$ (i) and we are done. Let $\dim T = n-1$. We may assume (after shrinking $U(T)$ if necessary) that $\overline{T} \setminus \{0\} \subset (\mathbb{R}^n \setminus \Omega) \cap K$. We show that

$$\int_0^1 \frac{c(\overline{T} \cap \overline{B}_r(0))}{r^{n-1}} dr = \infty \tag{1}$$

and we are done by Wiener's criterion. Working in a suitable coordinate system we have $T = \text{graph } f(T)$ with $f(T)\colon U(T) \subset \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ (compare with Section 0.1). Let $\mathcal{K}$ be a finite covering of $U(T)$ by cones in $\mathbb{R}^{n-1}$ each with vertex 0.

Since $T = \bigcup_{\widehat{K} \in \mathcal{K}} \text{graph} f(T)|_{\widehat{K} \cap U(T)}$ we find by Remark 1.8 b) some $\widehat{K} \in \mathcal{K}$ such that $\pi(\text{graph} f(T)|_{\widehat{K} \cap U(T)})$ contains a cone in $E$ with vertex 0. Let $U := \widehat{K} \cap U(T)$, $f := f(T)|_U$ and $\Gamma := \text{graph } f$. Note that $f$ can be extended to a Lipschitz function $\overline{f}$ on $\overline{U}$ with the same Lipschitz constant as $f$. Let $\sigma > 1$. We set

$$F := \{x = (x', x_n) \in \mathbb{R}^n \mid x' \in \overline{U},\ |x_n - \overline{f}(x')| \leq (\text{dist}(x', \partial U))^\sigma\}.$$

We have $\overline{\Gamma} \subset F$. We consider the subanalytic domain $V := B_1(0) \setminus F$. The regular boundary point $0 \in \partial V$ fulfills condition (ii) of Theorem 1.7; hence it is regular. With Wiener's criterion we get

$$\int_0^1 \frac{c(F \cap \overline{B}_r(0))}{r^{n-1}} dr = \infty. \tag{2}$$

We want to compare the capacity of $F \cap \overline{B}_r(0)$ and of $\overline{\Gamma} \cap \overline{B}_r(0)$. By the definition of $U$ and $F$ and by the Lipschitz condition on $\overline{f}$ we can find a cone $K' \subset \mathbb{R}^n$ with vertex 0 with the same central axis as $\widehat{K}$ and which contains $F$. Hence there is some constant $c$ with $0 < c < 1$ such that for all sufficiently small $r > 0$ we can

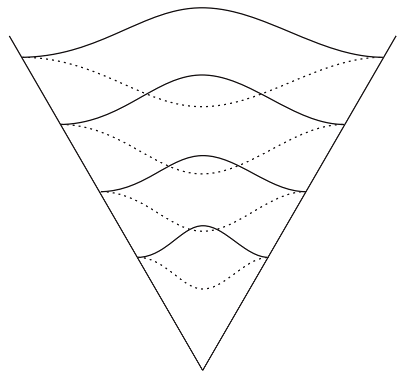

FIGURE 4. $F$ in dimension 3

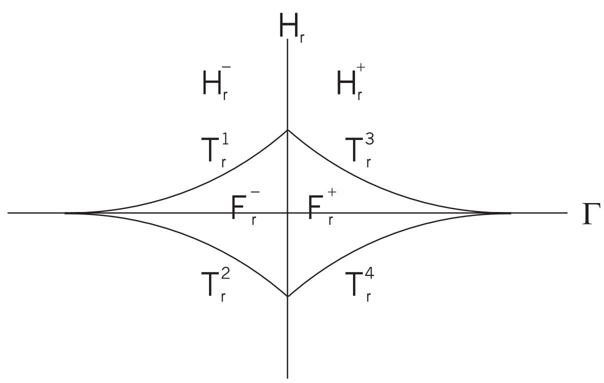


FIGURE 5. Two-dimensional illustration

find a hyperplane

$$H_r := \{x \in \mathbb{R}^n \mid x_1 v_1 + \cdots + x_n v_n = \alpha_r\}$$

(with $v := (v_1, \ldots, v_n)$ the central vector of $K'$ and some $\alpha_r > 0$), such that

$$F \cap \overline{B}_{cr}(0) \subset F \cap H_r^- \subset F \cap \overline{B}_r(0),$$

where $H_r^- := \{x \in \mathbb{R}^n \mid x_1 v_1 + \cdots + x_n v_n \le \alpha_r\}$. Let $F_r^- := F \cap H_r^-$ and let $F_r^+$ be the reflection of $F_r^-$ at $H_r$. We set $F_r := F_r^- \cup F_r^+$. The boundary $\partial F_r$ of $F_r$ is the union of the following sets:

$$\begin{aligned}
T_r^1 &:= \{x = (x', x_n) \in \partial F_r \mid x \in F_r^-,\ x_n \ge \overline{f}(x')\},\\
T_r^2 &:= \{x = (x', x_n) \in \partial F_r \mid x \in F_r^-,\ x_n \le \overline{f}(x')\},\\
T_r^3 &:= \text{the reflection of} \quad T_r^1 \quad \text{at} \quad H_r,\\
T_r^4 &:= \text{the reflection of} \quad T_r^2 \quad \text{at} \quad H_r.
\end{aligned}$$

We consider the map

$$\varphi\colon F \longrightarrow \overline{\Gamma},\ (x', x_n) \longmapsto (x', \overline{f}(x')).$$

Then $\varphi(T_r^1) \subset \overline{\Gamma} \cap \overline{B}_{dr}(0)$ with some constant $d \geq 1$ (independent from $r$). Let $\mu \in \mathcal{M}_c^+(T_r^1)$ such that $U^\mu \leq 1$ on $T_r^1$. We define $\tilde{\mu} \in \mathcal{M}_c^+(\varphi(T_r^1))$ by

$$\int_{\varphi(T_r^1)} h\, d\tilde{\mu} = \int_{T_r^1} h \circ \varphi\, d\mu \quad \text{for all} \quad h \in C_c(\varphi(T_r^1)).$$

By construction there is some $L_1 > 0$ with $|\varphi(x) - \varphi(y)| \geq L_1 |x-y|$ for all $x, y \in T_r^1$ and all sufficiently small $r > 0$. Thus $U^{\tilde{\mu}} \leq \frac{1}{L_1^{n-2}}$ on $\varphi(T_r^1)$ by the definition of the Newton potential. Hence according to the definition of capacity (see Section 0.2), we have that $c(\varphi(T_r^1)) \geq L_1^{n-2} c(T_r^1)$. The same argument gives some $L_2 > 0$ with $c(\varphi(T_r^2)) \geq L_2^{n-2} c(T_r^2)$. Using the fact that $c(E) = c(\partial E)$ for a compact set $E$ (see [1, Lemma 5.4.2]) and the subadditivity of capacity (see [1, Corollary 5.4.5]) we conclude, with $L := \min\{L_1, L_2\}$, and after enlarging $d$ if necessary, that

$$\begin{aligned} c(\overline{\Gamma} \cap \overline{B}_{dr}(0)) \quad & \geq \max\{c(\varphi(T_r^1), \varphi(T_r^2)\} \\ & \geq L^{n-2} \ \max\{c(T_r^1), c(T_r^2)\} \\ & \geq \tfrac{L^{n-2}}{4}\ c(\partial F_r) \\ & = \tfrac{L^{n-2}}{4}\ c(F_r) \\ & \geq \tfrac{L^{n-2}}{8}\ c(F \cap H_r^-) \\ & \geq \tfrac{L^{n-2}}{8}\ c(F \cap \overline{B}_{cr}(0)) \end{aligned}$$

for all sufficiently small $r > 0$. Rescaling (1) and (2) we see that (1) holds.

(i) $\Longrightarrow$ (ii)

Using Proposition 1.11 the proof proceeds similarly to the proof of Theorem 1.7, (i) $\Rightarrow$ (ii). □

## 3. Proofs of Theorems A and B

With the results we have obtained on reachable and admissible boundary points we can prove the theorems stated in the introduction.

*Proof of Theorem* A. By Proposition 2.3 and Remark 2.2 we may assume that $\Omega$ is admissible. In the cases $n = 2$ and $n = 3$, the claim follows from Corollary 2.5. Let $n \geq 4$. Condition (ii) of Theorem 2.6 is a subanalytic condition, even a semialgebraic one (compare with [4, chapter 2]). This gives us that the set of regular boundary points of $\Omega$ is subanalytic. □

*Proof of Theorem* B. We replace $\Omega_{\text{ad}}$ by $\Omega$; i.e. we assume that $\Omega$ is admissible.

a) We show that the set of irregular boundary points of $\Omega$ has then dimension less than or equal to $n-4$. In the case $n = 2$ or $n = 3$, this holds by Corollary 2.5. Thus let $n \geq 4$. Let $r > 0$ with $\overline{\Omega} \subset B_r(0)$. We choose a good stratification $\mathcal{T}$ of $B_r(0)$ which is compatible with $\Omega$ and $\partial\Omega$. Let $T \in \mathcal{T}$ with $T \subset \partial\Omega$ and $\dim T =: k \geq n-3$.

*Case* 1. $k = n-2$ or $k = n-1$. Let $x \in T$. We show that $x$ is a regular boundary point of $\Omega$. We may assume that $x = 0$. We work in a coordinate system suitable for $T$. Since $U(T) \subset \mathbb{R}^k$ is open and since $0 \in U(T)$ we can choose a cone $K' \subset \mathbb{R}^k$ with vertex 0 which is contained in $U(T)$. By the Lipschitz condition on $f(T)$ we find a cone $K \subset \mathbb{R}^n$ with the same central axis as $K'$ such that graph $f(T)|_{K'} \subset K$. Then condition (ii) of Theorem 2.6 is fulfilled with this cone $K$.

*Case* 2. $k = n-3$. Since $\Omega$ is admissible we find some $S_0 \in \mathcal{T}$ with $S_0 \subset \mathbb{R}^n \setminus \Omega$, $T \subset \overline{S_0} \setminus S_0$ and $\dim S_0 \geq n-1$. Applying Whitney's wing lemma (see [16, p. 101] and [15, Lemma 1.7]) we find a finite family $\mathcal{S}$ of good strata of dimension $n-2$ contained in $\mathbb{R}^n \setminus \Omega$ such that $T \subset \bigcup_{S \in \mathcal{S}} \overline{S} \setminus S$. Let $S \in \mathcal{S}$, let $A := (\overline{S} \setminus S) \cap \partial\Omega$ and let $B$ be the set of irregular boundary points of $\Omega$ contained in $A$. We show that $\dim B \leq n-4$ and we are done. We work in a coordinate system suitable for $S$. Let $U := U(S) \subset \mathbb{R}^{n-2}$ and $f := f(S)$. Note that $f$ can be extended to a Lipschitz function $\overline{f} \colon \overline{U} \to \mathbb{R}^2$ and that $\overline{S} = \operatorname{graph} \overline{f}$.

*Claim.* Let $C$ be the set of all $x' \in \overline{U}$ such that $\overline{U}$ contains a cone in $\mathbb{R}^{n-2}$ with vertex $x'$. Then $\dim(\overline{U} \setminus C) \leq n-4$.

Proof of the claim: Obviously $U \subset C$. By stratification the boundary $\partial U$ is a $C^1$-manifold of pure dimension $n-3$ outside a subanalytic set $D$ of dimension at most $n-4$. Therefore $\overline{U} \setminus C \subset D$ and the claim is proven.

By the Lipschitz condition on $\overline{f}$ we see that every $(x', \overline{f}(x'))$ with $x' \in C$ fulfills condition (ii) of Theorem 2.6 and we are done.

b) It follows from the example below that the upper bounds are sharp. □

**Example 3.1.** Let $n \geq 4$ and let

$$E := \{x \in \mathbb{R}^n \mid x_{n-3} = \cdots = x_n = 0\}.$$

We consider

$$\Omega := \{x = (x', x'') \in \mathbb{R}^{n-3} \times \mathbb{R}^3 \mid |x''| > x_{n-3}^2 \quad \text{for} \quad x_{n-3} \geq 0\} \cap B_1(0).$$

By Theorem 1.7 we see that $E \cap \overline{B}_{\frac{1}{2}}(0)$ is contained in the set of irregular boundary points of the reachable subanalytic domain $\Omega$. □

**Corollary 3.2.** *Let $\Omega$ be a subanalytic and bounded domain. The set of irregular boundary points of $\Omega$ has dimension less than or equal to $n-2$. This upper bound is sharp.*

*Proof.* By Proposition 2.3 we know that the set of irregular boundary points of $\Omega$ is the union of the set of the irregular boundary points of $\Omega_{\mathrm{ad}}$ and the set of non-admissible boundary points of $\Omega$. The first set has, according to Theorem B, dimension less than or equal to $n-4$, the second one less than or equal to $n-2$. □

Recall the connection between the existence of classical Green functions and Dirichlet regularity explained in the introduction. Using this we obtain the corollary stated in the introduction as a reformulation of Corollary 2.4. The developed geometric description of Dirichlet regularity leads to a parametric version:

**Definition 3.3.** Let $S \subset \mathbb{R}^n \times \mathbb{R}^m$ be a set. For $a \in \mathbb{R}^m$, let $S_a := \{x \in \mathbb{R}^n \mid (x, a) \in S\}$.

**Theorem 3.4.** *Let $S \subset \mathbb{R}^n \times \mathbb{R}^m$ be a subanalytic set such that $S_a \subset \mathbb{R}^n$ is a bounded domain for each $a \in \mathbb{R}^m$. Then the set*

$$\{a \in \mathbb{R}^m \mid S_a \text{ is regular}\} = \{a \in \mathbb{R}^m \mid S_a \text{ has a classical Green function}\}$$

*is subanalytic.*

*Proof.* This follows from Proposition 2.3, Theorem 2.4 and Theorem 2.6. □

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