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THE ZERO SET OF SEMI-INVARIANTS FOR EXTENDED DYNKIN QUIVERS

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ABSTRACT. We show that the set of common zeros $\mathcal{Z}_{Q,\mathbf{d}}$ of all semi-invariants vanishing at 0 on the variety $\mathrm{rep}(Q, \mathbf{d})$ of all representations with dimension vector $\mathbf{d}$ of an extended Dynkin quiver Q under the group $\mathrm{GL}(\mathbf{d})$ is a complete intersection if $\mathbf{d}$ is "big enough". In case $\mathrm{rep}(Q, \mathbf{d})$ does not contain an open $\mathrm{GL}(\mathbf{d})$-orbit, which is the case not considered so far, the number of irreducible components of $\mathcal{Z}_{Q,\mathbf{d}}$ grows with $\mathbf{d}$, except if Q is an oriented cycle.

1. INTRODUCTION AND MAIN RESULT

1.1. Let k be an algebraically closed field, and let $Q = (Q_0, Q_1, t, h)$ be a quiver with n vertices and a finite set Q_1 of arrows $\alpha : t\alpha \to h\alpha$, where $t\alpha$ and $h\alpha$ denote the tail and the head of α, respectively.

A representation of Q over k is a collection $(X(i);\, i \in Q_0)$ of finite dimensional k-vector spaces together with a collection $(X(\alpha) : X(t\alpha) \to X(h\alpha);\, \alpha \in Q_1)$ of k-linear maps. A morphism $f : X \to Y$ between two representations is a collection $(f(i) : X(i) \to Y(i))$ of k-linear maps such that

$$f(h\alpha) \circ X(\alpha) = Y(\alpha) \circ f(t\alpha) \qquad \text{for all } \alpha \in Q_1.$$

By $\sigma(X)$ we denote the number of pairwise non-isomorphic indecomposable direct summands occurring in a decomposition of X into indecomposables. According to the theorem of Krull-Schmidt, $\sigma(X)$ is well-defined. The dimension vector of a representation X of Q is the vector

$$\mathbf{dim}\, X = (\dim X(i))_{i\in Q_0} \in \mathbb{N}^{Q_0}.$$

We denote the category of representations of Q by $\mathrm{rep}(Q)$, and for any vector $\mathbf{d} = (d_i)_{i\in Q_0} \in \mathbb{N}^{Q_0}$,

$$\mathrm{rep}(Q, \mathbf{d}) = \prod_{\alpha\in Q_1} \mathrm{Mat}(d_{h\alpha} \times d_{t\alpha}, k)$$

is the vector space of representations X of Q with $X(i) = k^{d_i}$, $i \in Q_0$. The group

$$\mathrm{GL}(\mathbf{d}) = \prod_{i\in Q_0} \mathrm{GL}(d_i, k)$$

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acts on $\mathrm{rep}(Q,\mathbf{d})$ by

$$((g_i)_{i\in Q_0}\star X)(\alpha)=g_{h\alpha}\circ X(\alpha)\circ g_{t\alpha}^{-1}.$$

Note that the $\mathrm{GL}(\mathbf{d})$-orbit of X consists of the representations Y in $\mathrm{rep}(Q,\mathbf{d})$ which are isomorphic to X.

A regular function $f\in k[\mathrm{rep}(Q,\mathbf{d})]$ is called a semi-invariant if, for any $g\in \mathrm{GL}(\mathbf{d})$, $g\star f=\chi(g)f$ for some group homomorphism $\chi:\mathrm{GL}(\mathbf{d})\to k^*$ which is a regular function on $\mathrm{GL}(\mathbf{d})$, the so-called weight of f. Note that the k-algebra generated by all semi-invariants is just the algebra $k[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{SL}(\mathbf{d})}$ of polynomial functions which are invariant under the product

$$\mathrm{SL}(\mathbf{d})=\prod_{i\in Q_0}\mathrm{SL}(d_i,k).$$

Indeed, the algebra $k[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{SL}(\mathbf{d})}$ is the direct sum of the spaces of $\mathrm{GL}(\mathbf{d})$-semi-invariants of weight χ, where χ ranges over all characters of $\mathrm{GL}(\mathbf{d})$.

In case $\mathbf{d}$ is a sincere prehomogeneous dimension vector, i.e. if $d_i>0$ for all i and if the orbit $\mathrm{GL}(\mathbf{d})\star T$ of some representation T is open and dense, the algebra $k[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{SL}(\mathbf{d})}$ is a polynomial algebra in $n-\sigma(T)$ generators. In fact, the generators correspond bijectively to the simple objects in the perpendicular category $T^{\perp}$, the full subcategory of $\mathrm{rep}(Q)$ whose objects Y satisfy $\mathrm{Hom}_Q(T,Y)=\mathrm{Ext}^1_Q(T,Y)=0$ [12]. We showed in [8] that the variety of common zeros $\mathcal{Z}_{Q,\mathbf{d}}$ of all non-constant semi-invariants is a complete intersection if each of the pairwise non-isomorphic indecomposable direct summands T_i of

$$T=\bigoplus_{i=1}^{\sigma(T)}T_i^{\lambda_i}$$

arises with a sufficiently large multiplicity λ_i. Choosing λ_i larger still, we obtain that $\mathcal{Z}_{Q,\mathbf{d}}$ is irreducible. By [9], $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection or irreducible, for $\lambda_i\geq 3$ or $\lambda_i\geq 4$, respectively, if Q is a tame quiver, i.e. a disjoint union of Dynkin quivers and extended Dynkin quivers. Chang and Weyman, the first to consider this question, showed in [2] that $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection for any $\mathbf{d}$ if Q is a Dynkin quiver of type $\mathbb{A}_n$.

The interest in knowing when $k[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{SL}(\mathbf{d})}$ is non-singular (or equivalently is a polynomial ring), and when $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection comes from the following fact: Assume a reductive algebraic group G acts regularly and linearly on a finite dimensional $\mathbb{C}$-vector space V. The action of G on V is called

(i) coregular if the algebraic quotient [5] $V//G$ has no singularities,
(ii) cofree if $\mathbb{C}[V]$ is a free module over the invariant ring $\mathbb{C}[V]^G$,
(iii) equidimensional if the dimension of $V//G$ equals the codimension in V of the set of common zeros of all G-invariants which vanish at 0.

G. Schwarz proved in [16] that an action is cofree if and only if it is coregular and equidimensional. He classified all coregular and cofree representations of connected simple algebraic groups ([14], [15]). In [7], P. Littelmann classified all coregular and cofree irreducible representations of semisimple groups.

We have recalled above that in case $\mathbf{d}$ is a prehomogeneous dimension vector the action of $\mathrm{SL}(\mathbf{d})$ on $\mathrm{rep}(Q,\mathbf{d})$ is always coregular, and it is equidimensional if $\mathbf{d}$ is "big enough".

Unfortunately, most dimension vectors fail to be prehomogeneous, except for Dynkin quivers, in which case they all are. The algebra $k[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{SL}(\mathbf{d})}$ for an arbitrary pair $(Q,\mathbf{d})$ is not known. If Q is an extended Dynkin quiver, however, Skowroński and Weyman have obtained in [17] a complete description of $k[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{SL}(\mathbf{d})}$ by generators and relations for an arbitrary dimension vector $\mathbf{d}$. It turns out that in most cases the action of $\mathrm{SL}(\mathbf{d})$ on $\mathrm{rep}(Q,\mathbf{d})$ is coregular (compare 1.3).

Our purpose in the present paper is to study $\mathcal{Z}_{Q,\mathbf{d}}$ for an extended Dynkin quiver and an arbitrary dimension vector. We find that, as in the prehomogeneous case, the action of $\mathrm{SL}(\mathbf{d})$ on $\mathrm{rep}(Q,\mathbf{d})$ is equidimensional if $\mathbf{d}$ is "big enough". But $\mathcal{Z}_{Q,\mathbf{d}}$ does not become irreducible with growing $\mathbf{d}$. In fact, except for the oriented cycle, the number of its irreducible components increases with $\mathbf{d}$.

1.2. From now on until the end of Section 4, we assume that Q does not contain oriented cycles. We will usually not repeat this assumption. We will treat the oriented cycle separately in Section 5.

We need to recall a few facts and definitions, mostly from [11], before we can state our results. For a quiver Q, the Euler bilinear form $\langle -,-\rangle : \mathbb{Z}^{Q_0}\times\mathbb{Z}^{Q_0}\to\mathbb{Z}$ is defined by $\langle \mathbf{x},\mathbf{y}\rangle = \sum_{i\in Q_0} x_i y_i - \sum_{\alpha\in Q_1} x_{t\alpha} y_{h\alpha}$. The associated quadratic form $q_Q : \mathbb{Z}^{Q_0}\to\mathbb{Z}$, given by $q_Q(\mathbf{x}) = \langle \mathbf{x},\mathbf{x}\rangle$, is the Euler quadratic form. If X and Y are representations of Q, we have

$$\langle \mathbf{dim}\, X, \mathbf{dim}\, Y\rangle = [X,Y] - {}^1[X,Y],$$

where we set $[X,Y] = \dim_k \mathrm{Hom}_Q(X,Y)$ and ${}^1[X,Y] = \dim_k \mathrm{Ext}^1_Q(X,Y)$.

Assume that Q is an extended Dynkin quiver. Then the Euler quadratic form is positive semi-definite, and its radical is $\mathbb{Z}\mathbf{h}$ for a unique vector $\mathbf{h}\in\mathbb{N}^{Q_0}$. The defect is the linear form $\partial : \mathbb{Z}^{Q_0}\to\mathbb{Z}$ given by $\partial(\mathbf{x}) = \langle \mathbf{h},\mathbf{x}\rangle = -\langle \mathbf{x},\mathbf{h}\rangle$. A representation X is called preprojective, regular, or preinjective, if $\partial(\mathbf{dim}\, U) < 0$, $= 0$, or > 0, respectively, for every indecomposable direct summand U of X. Any representation X has a unique decomposition $X = X_{\mathcal{P}}\oplus X_{\mathcal{R}}\oplus X_{\mathcal{I}}$, where $X_{\mathcal{P}}$, $X_{\mathcal{R}}$, $X_{\mathcal{I}}$ are preprojective, regular, and preinjective, respectively.

The regular representations form an exact abelian subcategory $\mathcal{R}$ of $\mathrm{rep}(Q)$. The category $\mathcal{R}$ decomposes into a $\mathbb{P}^1(k)$-family $\coprod_{\mu\in\mathbb{P}^1(k)}\mathcal{R}_\mu$ of uniserial categories. For each μ, the category $\mathcal{R}_\mu$ contains a finite number r_μ of simple objects; their dimension vectors add up to $\mathbf{h}$. The set $\mathcal{E} = \{\mu\in\mathbb{P}^1(k);\ r_\mu > 1\}$ has at most three elements. For $\mu\notin\mathcal{E}$, we denote the unique simple representations of $\mathcal{R}_\mu$ by H_μ. A simple regular representation with dimension vector $\mathbf{h}$ is called homogeneous. We have $\sum(r_\mu - 1) = \#Q_0 - 2 = n-2$.

If $\mathbf{d}$ is not a prehomogeneous dimension vector, then representations in $\mathrm{rep}(Q,\mathbf{d})$ are necessarily generically regular. In fact, $\mathrm{rep}(Q,\mathbf{d})$ contains an open subset consisting of representations $H_{\mu_1}\oplus\cdots\oplus H_{\mu_p}\oplus V$, where $p\geq 1$ and $\mu_1,\ldots,\mu_p\notin\mathcal{E}$ are pairwise different, any indecomposable direct summand of V belongs to $\mathcal{R}_\mu$ for some $\mu\in\mathcal{E}$, and $\mathbf{dim}\, V = \mathbf{e}$ is prehomogeneous. We call the decomposition $\mathbf{d} = p\cdot\mathbf{h} + \mathbf{e}$ the canonical decomposition of $\mathbf{d}$.

1.3. We are now ready to state our results.

As a consequence of the theorem of Skowroński and Weyman [17, Thm.21], we know that the algebra $k[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{SL}(\mathbf{d})}$ is a polynomial ring if Q is an extended Dynkin quiver, $\mathbf{d} = p\cdot\mathbf{h}+\mathbf{e}$ is not prehomogeneous and $p\geq 2$. If Q is of type

$\widetilde{\mathbb{A}}_{n-1}$, this is true even if $p = 1$. Moreover, if the algebra $k[\operatorname{rep}(Q, \mathbf{d})]^{\operatorname{SL}(\mathbf{d})}$ is not a polynomial algebra, the categorical quotient $\operatorname{rep}(Q, \mathbf{d})/\!/ \operatorname{SL}(\mathbf{d})$ is singular. Indeed, its coordinate ring is of the form

$$k[X_1, \dots, X_a, Y_1, \dots, Y_b, Z_1, \dots, Z_c]/(f),$$

for some polynomial $f = X_1 \cdots X_{a'} + Y_1 \cdots Y_{b'} + Z_1 \cdots Z_{c'}$ with $a' \leq a$, $b' \leq b$, $c' \leq c$, and $a', b', c' \geq 2$, and f has a singularity at 0.

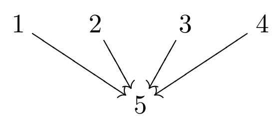
Theorem 1. *Let Q be an extended Dynkin quiver different from the oriented cycle, $\mathbf{d} \in \mathbb{N}^{Q_0}$ a non-prehomogeneous dimension vector with canonical decomposition $\mathbf{d} = p \cdot \mathbf{h} + \mathbf{e}$, and $V = \bigoplus_{i=1}^{\sigma(V)} V_i^{\lambda_i}$ a representation in the open orbit of $\operatorname{rep}(Q, \mathbf{e})$ with V_i indecomposable and pairwise non-isomorphic. Assume that either Q is of type $\widetilde{\mathbb{A}}_{n-1}$ or else $p \geq 3$ and $\lambda_i \geq 3$ for all i. Then we have:*

(i) *The action of $\operatorname{SL}(\mathbf{d})$ on $\operatorname{rep}(Q, \mathbf{d})$ is equidimensional.*
(ii) *Each irreducible component of $\mathcal{Z}_{Q,\mathbf{d}}$ is the closure of a $\operatorname{GL}(\mathbf{d})$-orbit.*
(iii) *The number of irreducible components of $\mathcal{Z}_{Q,\mathbf{d}}$ is at least $p - 2$.*

Remark 1.1. Note that the behavior of the number of irreducible components of $\mathcal{Z}_{Q,\mathbf{d}}$ is quite different from what happens for $\mathbf{d}$ prehomogeneous, since in that case $\mathcal{Z}_{Q,N\cdot\mathbf{d}}$ is irreducible for N large.

Remark 1.2. If Q is of type $\widetilde{\mathbb{D}}_{n-1}$, it can be shown that the assumption on λ_i may be dropped if $p \geq 4$. We do not know if such a tradeoff is possible if Q is of type $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, or $\widetilde{\mathbb{E}}_8$.

Remark 1.3. Our arguments do not extend to the case $\mathbf{d} = 2 \cdot \mathbf{h}$. Indeed, for

$Q =$ 

, $\mathbf{h} =$ 1 1 1 1 / 2

the algebra $k[\operatorname{rep}(Q, \mathbf{d})]^{\operatorname{SL}(2\cdot\mathbf{h})}$ is a polynomial ring in 6 variables by [17, Thm. 17], but $\mathcal{Z}_{Q,\mathbf{d}}$ contains the representation $X = X_1 \oplus X_2$, where X_1 is the one-dimensional representation supported at the vertex 5 and the orbit of the representation X_2 is open in $\operatorname{rep}(Q, 2 \cdot \mathbf{h} - \mathbf{dim}\, X_1)$, and

$$\operatorname{codim} \overline{\operatorname{GL}(2 \cdot \mathbf{h}) \star X} = {}^1[X, X] = {}^1[X_2, X_1] = 5.$$

The paper is organized as follows: In Section 2 we generalize Schofield's result relating semi-invariants to objects in some perpendicular category [12] to non-prehomogeneous dimension vectors. In Section 3 we describe these generalized perpendicular categories, and we prove our main result in Section 4. The last section is devoted to the oriented cycle. As in that case there exist non-constant $\operatorname{GL}(\mathbf{d})$-invariants on $\operatorname{rep}(Q, \mathbf{d})$, and we need to modify the description of $\mathcal{Z}_{Q,\mathbf{d}}$ slightly. We obtain the following corollary.

Corollary 1.4. *If Q is an $\widetilde{\mathbb{A}}_{n-1}$-quiver, then $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection for any dimension vector $\mathbf{d}$.*

2. Semi-invariants and perpendicular categories

2.1. Let Q be a quiver, $\mathbf{d}, \mathbf{e} \in \mathbb{N}^{Q_0}$, and let X, Y be representations of Q with $\mathbf{dim}\, X = \mathbf{d}$ and $\mathbf{dim}\, Y = \mathbf{e}$. Consider the linear map

$$\mathcal{F}_{X,Y} : \bigoplus_{i \in Q_0} \mathrm{Hom}_k(k^{d_i}, k^{e_i}) \to \bigoplus_{\alpha \in Q_1} \mathrm{Hom}_k(k^{d_{t\alpha}}, k^{e_{h\alpha}})$$

which sends $(g_i; i \in Q_0)$ to $(h_\alpha; \alpha \in Q_1)$ with $h_\alpha = g_{h\alpha} \circ X(\alpha) - Y(\alpha) \circ g_{t\alpha}$. Note that $\mathrm{Ker}\, \mathcal{F}_{X,Y} = \mathrm{Hom}_Q(X,Y)$ and $\mathrm{Coker}\, \mathcal{F}_{X,Y} = \mathrm{Ext}^1_Q(X,Y)$, which implies that

$$\langle \mathbf{d}, \mathbf{e} \rangle = [X,Y] - {}^1[X,Y].$$

If we assume that $\langle \mathbf{d}, \mathbf{e} \rangle = 0$, the linear map $\mathcal{F}_{X,Y}$ will be represented by a square matrix $H_{X,Y}$ (with respect to some bases), and the determinant $\det H_{X,Y}$ is a $\mathrm{GL}(\mathbf{d}) \times \mathrm{GL}(\mathbf{e})$ semi-invariant on $\mathrm{rep}(Q,\mathbf{d}) \times \mathrm{rep}(Q,\mathbf{e})$ by [12]. We denote by $f_Y \in k[\mathrm{rep}(Q,\mathbf{d})]$ the semi-invariant associated to a representation Y. Note that, for a short exact sequence

$$0 \to Y' \to Y \to Y'' \to 0$$

with $\langle \mathbf{d}, \mathbf{dim}\, Y' \rangle = \langle \mathbf{d}, \mathbf{dim}\, Y \rangle = \langle \mathbf{d}, \mathbf{dim}\, Y'' \rangle = 0$, we have that f_Y is a non-zero multiple of $f_{Y'} \cdot f_{Y''}$ [3].

The semi-invariant f_Y does not vanish identically on $\mathrm{rep}(Q,\mathbf{d})$ if and only if there exists some $T \in \mathrm{rep}(Q,\mathbf{d})$ with $[T,Y] = {}^1[T,Y] = 0$. We define the perpendicular category $\mathbf{d}^\perp$ to be the full subcategory of $\mathrm{rep}(Q)$ whose objects are the representations Y of Q for which there is a $T \in \mathrm{rep}(Q,\mathbf{d})$ with $[T,Y] = {}^1[T,Y] = 0$. Note that in this case $[X,Y] = {}^1[X,Y] = 0$ for X in some dense open subset of $\mathrm{rep}(Q,\mathbf{d})$, as $[-,Y] = 0$ and ${}^1[-,Y] = 0$ are open conditions. In case $\mathbf{d}$ is prehomogeneous, $\mathbf{d}^\perp$ is just the category $T^\perp$ introduced by Schofield in [12], where $T \in \mathrm{rep}(Q,\mathbf{d})$ lies in the open orbit.

The following result from [3], which in characteristic zero also follows from [13], relates $\mathbf{d}^\perp$ to semi-invariants.

Proposition 2.1. *If Q does not contain oriented cycles and $\mathbf{d} \in \mathbb{N}^{Q_0}$, the functions f_Y, $Y \in \mathbf{d}^\perp$, span the space $k[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{SL}(\mathbf{d})}$.*

2.2. We start with the following lemma.

Lemma 2.2. *$\mathbf{d}^\perp$ is an exact abelian subcategory of $\mathrm{rep}(Q)$.*

Proof. Obviously $\mathbf{d}^\perp$ is closed under taking direct summands. Let Y' and Y'' belong to $\mathbf{d}^\perp$. Then there is a $T \in \mathrm{rep}(Q,\mathbf{d})$ for which

$$[T,Y'] = {}^1[T,Y'] = [T,Y''] = {}^1[T,Y''] = 0.$$

Let $f : Y' \to Y''$ be a homomorphism. Then we get two induced long exact sequences

$$\begin{aligned} 0 &\to \mathrm{Hom}_Q(T,\mathrm{Ker}(f)) \to \mathrm{Hom}_Q(T,Y') \to \mathrm{Hom}_Q(T,\mathrm{Im}(f)) \to \\ &\to \mathrm{Ext}^1_Q(T,\mathrm{Ker}(f)) \to \mathrm{Ext}^1_Q(T,Y') \to \mathrm{Ext}^1_Q(T,\mathrm{Im}(f)) \to 0, \\ 0 &\to \mathrm{Hom}_Q(T,\mathrm{Im}(f)) \to \mathrm{Hom}_Q(T,Y'') \to \mathrm{Hom}_Q(T,\mathrm{Coker}(f)) \to \\ &\to \mathrm{Ext}^1_Q(T,\mathrm{Im}(f)) \to \mathrm{Ext}^1_Q(T,Y'') \to \mathrm{Ext}^1_Q(T,\mathrm{Coker}(f)) \to 0. \end{aligned}$$

This implies that

$$\begin{aligned}[T, \mathrm{Ker}(f)] &= [T, \mathrm{Im}(f)] = {}^1[T, \mathrm{Ker}(f)] = {}^1[T, \mathrm{Im}(f)] \\ &= [T, \mathrm{Coker}(f)] = {}^1[T, \mathrm{Coker}(f)] = 0.\end{aligned}$$

Hence the subcategory $\mathbf{d}^\perp$ is closed under kernels, images and cokernels. If Y is an extension

$$0 \to Y' \to Y \to Y'' \to 0$$

of Y'' by Y', considering the long exact sequence obtained from mapping T to this short exact sequence yields $Y \in \mathbf{d}^\perp$. □

If $\mathbf{d}$ is prehomogeneous, $\mathbf{d}^\perp$ is equivalent to the category of representations of some quiver by [12]. For arbitrary $\mathbf{d}$, $\mathbf{d}^\perp$ may have infinitely many simple objects, however. We will compute $\mathbf{d}^\perp$ in case Q is an extended Dynkin quiver and $\mathbf{d}$ is not prehomogeneous in Section 3.

2.3. For further reference, we collect a few properties of $\mathbf{d}^\perp$.

Proposition 2.3. *Let $\mathbf{d} = \mathbf{d}' + \mathbf{d}''$ with $\mathbf{d}, \mathbf{d}', \mathbf{d}'' \in \mathbb{N}^{Q_0}$, and suppose that generically a representation $T \in \mathrm{rep}(Q, \mathbf{d})$ has a subrepresentation $T' \in \mathrm{rep}(Q, \mathbf{d}')$. Then*

$$\mathbf{d}^\perp \cap (\mathbf{d}')^\perp = (\mathbf{d}'')^\perp \cap (\mathbf{d}')^\perp = (\mathbf{d}'')^\perp \cap \mathbf{d}^\perp.$$

Proof. We only prove the first equality. Suppose $Y \in (\mathbf{d}'')^\perp \cap (\mathbf{d}')^\perp$, and choose $T' \in \mathrm{rep}(Q, \mathbf{d}')$ with $[T', Y] = {}^1[T', Y] = 0$ and $T'' \in \mathrm{rep}(Q, \mathbf{d}'')$ with $[T'', Y] = {}^1[T'', Y] = 0$. Then obviously $[T' \oplus T'', Y] = {}^1[T' \oplus T'', Y] = 0$, which implies $Y \in \mathbf{d}^\perp$. Conversely, if $Y \in \mathbf{d}^\perp \cap (\mathbf{d}')^\perp$, choose $T \in \mathrm{rep}(Q, \mathbf{d})$ with $[T, Y] = {}^1[T, Y] = 0$ and having a subrepresentation $T' \subseteq T$ with $\mathbf{dim}\, T' = \mathbf{d}'$. Note that ${}^1[T, Y] = 0$ implies ${}^1[T', Y] = 0$ as the map $\mathrm{Ext}^1_Q(T, Y) \to \mathrm{Ext}^1_Q(T', Y)$ is surjective. But as $Y \in (\mathbf{d}')^\perp$ we have

$$\langle \mathbf{d}', \mathbf{dim}\, Y \rangle = [T', Y] - {}^1[T', Y] = 0$$

and thus $[T', Y] = 0$. Applying the functor $\mathrm{Hom}_Q(-, Y)$ we find $[T/T', Y] = {}^1[T/T', Y] = 0$ and thus $Y \in (\mathbf{d}')^\perp \cap (\mathbf{d}'')^\perp$. □

Corollary 2.4. *Let z be a sink of Q, and denote by $\boldsymbol{\varepsilon}_z \in \mathbb{N}^{Q_0}$ the vector given by $\boldsymbol{\varepsilon}_z(y) = \delta_{z,y}$. For $\mathbf{d} \in \mathbb{N}^{Q_0}$, set $\overline{\mathbf{d}} = \mathbf{d} - d_z \cdot \boldsymbol{\varepsilon}_z$. Then*

$$\{Y \in \mathbf{d}^\perp;\ Y(z) = 0\} = \{Y \in (\overline{\mathbf{d}})^\perp;\ Y(z) = 0\}.$$

Proof. We may assume that $d_z > 0$. Apply Proposition 2.3 for $\mathbf{d}' = d_z \cdot \boldsymbol{\varepsilon}_z$, and note that

$$(\mathbf{d}')^\perp = (\boldsymbol{\varepsilon}_z)^\perp = \{Y \in \mathrm{rep}(Q);\ Y(z) = 0\},$$

as the one dimensional representation E_z supported at z is simple projective, and $\mathrm{Hom}_Q(E_z, Y)$ is isomorphic to $Y(z)$. □

Proposition 2.5. *Let $\mathbf{d} = \mathbf{d}' + \mathbf{d}''$, and suppose that any T in a dense open set $\mathcal{U} \subseteq \mathrm{rep}(Q, \mathbf{d})$ decomposes as $T = T' \oplus T''$ for some $T' \in \mathrm{rep}(Q, \mathbf{d}')$ and some $T'' \in \mathrm{rep}(Q, \mathbf{d}'')$. Then $\mathbf{d}^\perp = (\mathbf{d}')^\perp \cap (\mathbf{d}'')^\perp$.*

Proof. The inclusion $(\mathbf{d}')^\perp \cap (\mathbf{d}'')^\perp \subseteq \mathbf{d}^\perp$ follows from Proposition 2.3. Conversely, let $Y \in \mathbf{d}^\perp$, and choose $T \in \mathcal{U}$ with $[T, Y] = {}^1[T, Y] = 0$. Then clearly

$$[T', Y] = {}^1[T', Y] = [T'', Y] = {}^1[T'', Y] = 0$$

if $T = T' \oplus T''$ is a decomposition with $\mathbf{dim}\, T' = \mathbf{d}'$ and $\mathbf{dim}\, T'' = \mathbf{d}''$. □

2.4. Finally, we wish to study the behavior of $\mathbf{d}^{\perp}$ under reflection functors. Let z be a sink of Q, and let $\alpha_j : y_j \to z$, $j = 1, \dots, s$ be the arrows of Q with head z. Define a new quiver Q', obtained from Q by deleting z and $\alpha_1, \dots, \alpha_s$ and by adding a new vertex z' as well as arrows $\beta_j : z' \to y_j$, $j = 1, \dots, s$. Let E_z and $E'_{z'}$ be the one-dimensional representations of Q and Q', supported at z and z', respectively. Note that E_z is simple projective in $\mathrm{rep}(Q)$ and $E'_{z'}$ is simple injective in $\mathrm{rep}(Q')$.

We consider the reflection functor

$$\mathcal{F} : \mathrm{rep}(Q) \to \mathrm{rep}(Q')$$

associated with z. Recall that

$$(\mathcal{F}X)(i) = \begin{cases} X(i) & i \neq z', \\ \mathrm{Ker}\left(\bigoplus X(y_j) \xrightarrow{[X(\alpha_1), \dots, X(\alpha_s)]} X(z)\right) & i = z', \end{cases}$$

and that

$$(\mathcal{F}X)(\beta_l) : (\mathcal{F}X)(z') \to (\mathcal{F}X)(y_l) = X(y_l)$$

is the inclusion of $(\mathcal{F}X)(z')$ into $\bigoplus_{j=1}^{s} X(y_j)$ followed by the projection to $X(y_l)$ (see [1], [4]). The functor $\mathcal{F}$ restricts to an equivalence

$$\mathcal{F} : (\mathrm{rep}(Q))' \to (\mathrm{rep}(Q'))'$$

from the full subcategory $(\mathrm{rep}(Q))'$ of $\mathrm{rep}(Q)$ whose objects do not contain E_z as a direct summand, or equivalently have no non-trivial morphisms to E_z, to the full subcategory $(\mathrm{rep}(Q'))'$ of $\mathrm{rep}(Q')$ whose objects do not contain $E'_{z'}$ as a direct summand.

Suppose $\mathbf{d} \in \mathbb{N}^{Q_0}$ satisfies $d_z < \sum_{h\alpha = z} d_{t\alpha}$, and define $\mathbf{d}' \in \mathbb{Z}^{Q'_0}$ by

$$d'_x = \begin{cases} d_x, & x \neq z', \\ \sum_{h\alpha = z} d_{t\alpha} - d_z, & x = z'. \end{cases}$$

Note that $d'_{z'} > 0$. For $T \in (\mathrm{rep}(Q))'$ with $\mathbf{dim}\, T = \mathbf{d}$, we have $\mathcal{F}T \in (\mathrm{rep}(Q'))'$ with $\mathbf{dim}\, \mathcal{F}T = \mathbf{d}'$.

Proposition 2.6. *Let Q be a quiver with a sink z, $\mathbf{d} \in \mathbb{N}^{Q_0}$ with $d_z < \sum_{h\alpha = z} d_{t\alpha}$, and let Q', $\mathbf{d}'$ be defined as above. Then $\mathcal{F}(\mathbf{d}^{\perp_Q}) = (\mathbf{d}')^{\perp_{Q'}}$.*

Proof. We will prove that $\mathcal{F}Y \in (\mathbf{d}')^{\perp_{Q'}}$ for $Y \in \mathbf{d}^{\perp_Q}$; the other inclusion is obtained from using the reflection functor $\mathcal{F}^{-1} : (\mathrm{rep}(Q'))' \to (\mathrm{rep}(Q))'$. Choose $T \in \mathrm{rep}(Q, \mathbf{d})$ such that $[T, Y] = {}^1[T, Y] = 0$ and such that E_z is not a direct summand of T. This is possible as generically $[T, E_z] = 0$ on $\mathrm{rep}(Q, \mathbf{d})$. Note that E_z is not a direct summand of Y either as

$$\langle \mathbf{d}, \boldsymbol{\varepsilon}_z \rangle = -d'_{z'} = -{}^1[T, E_z] < 0.$$

But then we have $T, Y \in (\mathrm{rep}(Q))'$, and we know

$$[\mathcal{F}T, \mathcal{F}Y] = [T, Y] = 0 \qquad \text{and} \qquad {}^1[\mathcal{F}T, \mathcal{F}Y] = {}^1[T, Y] = 0.$$

So $\mathcal{F}Y$ belongs to $(\mathbf{d}')^{\perp_{Q'}}$. □

3. Extended Dynkin quivers

3.1. Throughout this section Q is an extended Dynkin quiver. Remember that by $\mathcal{E}$ we denote the set $\mathcal{E} = \{\mu \in \mathbb{P}^1(k);\ r_\mu > 1\}$, where r_μ is the number of simple objects in the category R_μ. We need to recall two more results from [11].

Lemma 3.1. *Let $X_\mathcal{P}$, $X_\mathcal{I}$, X_μ be a non-zero preprojective, preinjective and regular representation in $\mathcal{R}_\mu$, respectively, $\mu \in \mathbb{P}^1(k)$. Then we have*

(i) $[X_\mu, X_\mathcal{P}] = {}^1[X_\mathcal{P}, X_\mu] = 0$ *for all* μ,
(ii) $[X_\mathcal{I}, X_\mu] = {}^1[X_\mu, X_\mathcal{I}] = 0$ *for all* μ,
(iii) $[X_\mathcal{I}, X_\mathcal{P}] = {}^1[X_\mathcal{P}, X_\mathcal{I}] = 0$,
(iv) $[X_\mu, X_\nu] = {}^1[X_\mu, X_\nu] = 0$ *for* $\mu \neq \nu$,
(v) $[X_\mathcal{P}, X_\mu] > 0$ *and* $[X_\mu, X_\mathcal{I}] > 0$ *if* $\mu \notin \mathcal{E}$.

3.2. For $r \geq 1$ we denote by C_r the oriented cycle with r vertices:

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{r-1}} r, \qquad r \xrightarrow{\alpha_r} 1$$

We call a representation X of C_r nilpotent if there is a positive integer N_X such that $X(\pi) = 0$ for any path π of length greater than or equal to N_X.

Lemma 3.2. *For $\mu \in \mathbb{P}^1(k)$, the category $\mathcal{R}_\mu$ is equivalent to the category of nilpotent representations of the oriented cycle C_{r_μ}.*

Fix a non-prehomogeneous dimension vector $\mathbf{d}$ with canonical decomposition $\mathbf{d} = p \cdot \mathbf{h} + \mathbf{e}$, $p \geq 1$. We choose $V \in \operatorname{rep}(Q, \mathbf{e})$ such that the $\mathrm{GL}(\mathbf{e})$-orbit of V is open, and we decompose $V = \bigoplus_{\mu \in \mathcal{E}} V_\mu$, $V_\mu \in \mathcal{R}_\mu$. With this notation we have the following results.

Proposition 3.3. *Let $\mathbf{e}_\mu = \mathbf{dim}\, V_\mu$ for $\mu \in \mathcal{E}$. Then:*

(i) $\mathbf{h}^\perp = \coprod_{\mu \in \mathbb{P}^1(k)} \mathcal{R}_\mu$.
(ii) $\mathbf{d}^\perp = \mathbf{h}^\perp \cap \mathbf{e}^\perp$.
(iii) *An indecomposable representation $Y \in \mathcal{R}_\mu$ belongs to $\mathbf{e}^\perp$ if and only if either $\mu \notin \mathcal{E}$ or, for $\mu \in \mathcal{E}$, $Y \in (\mathbf{e}_\mu)^\perp$.*

Proof. *(i)* and *(iii)* follow directly from Lemma 3.1, and *(ii)* is a consequence of Proposition 2.5. □

Our next goal is to describe $(\mathbf{e}_\mu)^\perp$ for $\mu \in \mathcal{E}$. Fix $r \geq 1$, and set $C = C_r$. By $\mathcal{N}$ we denote the full subcategory of $\operatorname{rep}(C)$ whose objects are the nilpotent representations. Note that $\mathcal{N}$ is an exact abelian subcategory of $\operatorname{rep}(C)$. Let T be a representation in $\mathcal{N}$ having a dense open orbit in $\operatorname{rep}(C, \mathbf{d})$, where $\mathbf{d} = \mathbf{dim}\, T$. Up to renumbering the vertices of C, we may suppose that $d_r \leq d_i$ for any vertex i of C. Then the composition $T(\alpha_{r-1}) \circ \ldots \circ T(\alpha_1) \circ T(\alpha_r)$ is generically an automorphism. If d_r were positive, then T could not be nilpotent. So we see that $d_r = 0$.

An indecomposable representation Y in $\mathcal{N}$ is uniquely determined by its socle, which is simple and thus corresponds to a vertex i of C, and its dimension l. Let ω be the path in C of length l stopping at i; it is the shortest path stopping at i with $Y(\omega) = 0$. Note that in this way we obtain a bijection from the set of indecomposable representations in $\mathcal{N}$, up to isomorphism, to the set of all paths of positive length in C. If ω is such a path, we let Y_ω be the corresponding indecomposable.

The following lemma is not difficult; its proof is left to the reader. By W^* we denote the dual of the vector space W.

Lemma 3.4. *Let ω be a path of positive length in C. Then we have for any representation X of C:*

$$\operatorname{Hom}_C(X, Y_\omega) \simeq (\operatorname{Coker} X(\omega))^* \qquad \text{and} \qquad \operatorname{Ext}^1_C(X, Y_\omega) \simeq (\operatorname{Ker} X(\omega))^* .$$

We obtain the following consequence.

Corollary 3.5. *Let ω be a path of positive length in C, let T be a representation in $\mathcal{N}$ with ${}^1[T,T] = 0$, and set $\mathbf{d} = \mathbf{dim}\, T$. Then we have:*

(i) *$Y_\omega \in \mathbf{d}^\perp$ if and only if $d_x \geq d_{h\omega} = d_{t\omega}$ for all vertices of ω,*
(ii) *Y_ω is a simple object in $\mathbf{d}^\perp$ if and only if $d_x > d_{h\omega} = d_{t\omega}$ for all inner vertices of ω.*

Here we set $t\omega = t\beta_1$ and $h\omega = h\beta_t$ for $\omega = \beta_t \cdots \beta_1$, and we call x a vertex of ω if x is a tail or a head of some β_i. An inner vertex of ω is a vertex of the form $t\beta_i$, $i > 1$.

Proof. (i) From Lemma 3.4 we see that $Y_\omega \in \mathbf{d}^\perp$ if and only if $T(\omega)$ is an isomorphism and therefore $d_{h\omega} = d_{t\omega}$. As $T(\omega)$ factors through $T(x)$ for any vertex x of ω, we find $d_x \geq d_{t\omega} = d_{h\omega}$. Conversely, this condition implies that $T(\omega)$, which is a composition of generic maps, one for each arrow β_i, is an isomorphism.

(ii) Assume Y_ω is simple and there is an inner vertex x with $d_{t\omega} = d_x = d_{h\omega}$. Denote by ω' the subpath of ω from x to $h\omega$. By *(i)*, $Y_{\omega'}$ belongs to $\mathbf{d}^\perp$, and clearly $Y_{\omega'}$ is a proper subrepresentation of Y_ω. Conversely, a representation Y_ω with $d_x > d_{h\omega} = d_{t\omega}$ for all inner vertices cannot have any proper subrepresentation in $\mathbf{d}^\perp$, again by *(i)*. □

Proposition 3.6. *Let T be a representation in $\mathcal{N}$ with ${}^1[T,T] = 0$, and set $\mathbf{d} = \mathbf{dim}\, T$. Then $\mathbf{d}^{\perp_\mathcal{N}}$ is an abelian category with $\#(C)_0 - \sigma(T) = r - \sigma(T)$ simple objects.*

Proof. Let $\widehat{C}$ be the quiver obtained from $C = C_r$ by deleting the arrow α_r; it is an $\mathbb{A}_r$-quiver. Recall that T in fact lies in $\operatorname{rep}(\widehat{C}, \mathbf{d})$ as $d_r = 0$. Our strategy is to show that $\mathbf{d}^{\perp_\mathcal{N}}$ and $\mathbf{d}^{\perp_{\widehat{C}}}$ have the same simple objects. Then our result follows from Schofield's result [12, Thm. 2.5], as $\widehat{C}$ is a quiver of finite representation type and thus $\mathbf{d}$ is prehomogeneous when viewed as a dimension vector for $\widehat{C}$.

For $Y \in \operatorname{rep}(\widehat{C})$, we have

$$\operatorname{Hom}_C(T, Y) = \operatorname{Hom}_{\widehat{C}}(T, Y) \qquad \text{and} \qquad \langle \mathbf{d}, \mathbf{dim}\, Y\rangle_C = \langle \mathbf{d}, \mathbf{dim}\, Y\rangle_{\widehat{C}},$$

as $d_r = 0$. We conclude that $\mathbf{d}^{\perp_{\widehat{C}}} = \mathbf{d}^{\perp_\mathcal{N}} \cap \operatorname{rep}(\widehat{C})$. Let Y_ω be a simple object of $\mathbf{d}^{\perp_\mathcal{N}}$. By Corollary 3.5, the vertex r cannot be an inner vertex of ω, as $d_r = 0$. Then $Y_\omega(\alpha_r) = 0$ by the definition of Y_ω, and hence $Y_\omega \in \operatorname{rep}(\widehat{C})$. □

Proposition 3.7. *Let Q be an extended Dynkin quiver and $\mathbf{d} \in \mathbb{N}^{Q_0}$ a non-prehomogeneous dimension vector with canonical decomposition $\mathbf{d} = p\cdot\mathbf{h} + \mathbf{e}$, $p \geq 1$. If either $p \geq 2$ or Q is an $\widetilde{\mathbb{A}}_{n-1}$-quiver, the algebra $k[\operatorname{rep}(Q, \mathbf{d})]^{\mathrm{SL}(\mathbf{d})}$ is a polynomial ring in $n + p - 1 - \sigma(V)$ variables, where $V \in \operatorname{rep}(Q, \mathbf{e})$ has an open orbit.*

Proof. The main theorem of Skowroński and Weyman in [17] says that, under our assumptions, $k[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{SL}(\mathbf{d})}$ is the quotient of a polynomial ring $k[c_0,\ldots,c_p,f_Y]$ by an ideal generated by $\#\mathcal{E}$ relations, each allowing for the cancellation of one of the c_i's from the list of generators, where Y ranges over the simple non-homogeneous objects in $\mathbf{d}^\perp$. Indeed, by Lemma 3.2 and Corollary 3.5, the simple objects in $\mathcal{R}_\mu$ correspond bijectively to the "admissible arcs" of [17]. The number of simple objects of $\mathbf{d}^\perp$ which belong to $\mathcal{R}_\mu$ is $r_\mu-\sigma(V_\mu)$, where $V=\bigoplus_{\mu\in\mathcal{E}}V_\mu$, by Proposition 3.6. So the number of simple non-homogeneous objects in $\mathbf{d}^\perp$ equals

$$\sum_{\mu\in\mathcal{E}}(r_\mu-\sigma(V_\mu))=\sum_{\mu\in\mathcal{E}}r_\mu-\sigma(V)=n-2+\#\mathcal{E}-\sigma(V)$$

as $\sum_{\mu\in\mathcal{E}}(r_\mu-1)=n-2$. Taking into account the $\#\mathcal{E}$ relations, we conclude that $k[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{SL}(\mathbf{d})}$ is a polynomial ring on

$$(p+1)+(n-2+\#\mathcal{E}-\sigma(V))-\#\mathcal{E}=p+n-1-\sigma(V)$$

generators. □

4. Proof of the theorem

4.1. We recall the notation and assumptions for our theorem and keep them fixed throughout this section: Q is an extended Dynkin quiver with $\#Q_0=n$, not an oriented cycle, $\mathbf{d}\in\mathbb{N}^{Q_0}$ is a non-prehomogeneous dimension vector with canonical decomposition $\mathbf{d}=p\cdot\mathbf{h}+\mathbf{e}$, $V=\bigoplus_{i=1}^{\sigma(V)}V_i^{\lambda_i}$ is a representation in the open orbit of $\mathrm{rep}(Q,\mathbf{e})$ with V_i indecomposable and pairwise non-isomorphic. We assume that either Q is of type $\widetilde{\mathbb{A}}_{n-1}$ or else $p\geq 3$ and $\lambda_i\geq 3$ for $i=1,\ldots,\sigma(V)$. Note that by Proposition 2.1, the variety $\mathcal{Z}_{Q,\mathbf{d}}$ of common zeros of all non-constant semi-invariants has the following description:

$$\begin{aligned}\mathcal{Z}_{Q,\mathbf{d}}&=\{X\in\mathrm{rep}(Q,\mathbf{d});\ [X,Y]\neq 0\text{ for all }Y\in\mathbf{d}^\perp,Y\neq 0\}\\&=\{X\in\mathrm{rep}(Q,\mathbf{d});\ {}^1[X,Y]\neq 0\text{ for all }Y\in\mathbf{d}^\perp,Y\neq 0\}.\end{aligned}$$

The next result is an immediate consequence of Lemma 3.1 and Proposition 3.3.

Proposition 4.1. *Any representation X in $\mathcal{Z}_{Q,\mathbf{d}}$ has a nonzero preprojective (and a preinjective) direct summand.*

The following corollary implies part (ii) of our theorem.

Corollary 4.2. *Any irreducible component $\mathcal{C}$ of $\mathcal{Z}_{Q,\mathbf{d}}$ is the closure of some orbit $\mathrm{GL}(\mathbf{d})\star X$.*

We call $X\in\mathcal{Z}_{Q,\mathbf{d}}$ generic if $\overline{\mathrm{GL}(\mathbf{d})\star X}$ is an irreducible component of $\mathcal{Z}_{Q,\mathbf{d}}$.

Proof. Otherwise, $\mathcal{C}$ contains an infinite number of distinct orbits of maximal dimension, none of which belong to any other irreducible component of $\mathcal{Z}_{Q,\mathbf{d}}$. Some must be given by representations having a direct summand from $\mathcal{R}_\mu$, $\mu\notin\mathcal{E}$. If $X=X_1\oplus X_2\oplus X_3$ is one of them with $X_1\neq 0$ preprojective and $X_2\in\mathcal{R}_\mu$, $\mu\notin\mathcal{E}$, there exists a non-split extension

$$0\to X_1\to\widetilde{X}_1\to X_2\to 0,$$

by Lemma 3.1. Note that $\widetilde{X}_1$ still contains a non-zero preprojective summand, as

$$\partial(\mathbf{dim}\,\widetilde{X}_1)=\partial(\mathbf{dim}\,X_1)+\partial(\mathbf{dim}\,X_2)<0.$$

Therefore $[\widetilde{X}_1, H_\nu] \neq 0$ for any $\nu \notin \mathcal{E}$. For $Y \in \mathbf{d}^\perp$ simple non-homogeneous, we have ${}^1[X_2, Y] = 0$ by Lemma 3.1. Mapping the short exact sequence above to Y, we conclude that

$$[\widetilde{X}_1 \oplus X_3, Y] = [X_1 \oplus X_2 \oplus X_3, Y].$$

Hence $\widetilde{X} = \widetilde{X}_1 \oplus X_3$ still belongs to $\mathcal{Z}_{Q,\mathbf{d}}$, and even to $\mathcal{C}$, as X lies in the closure of $\mathrm{GL}(\mathbf{d}) \star \widetilde{X}$. This contradicts the maximality of the dimension of $\mathrm{GL}(\mathbf{d}) \star X$. □

We fix a sink z in Q, and we define Q' as in Section 2.4. Note that generically a representation T in $\mathrm{rep}(Q, \mathbf{d})$ is regular and hence does not contain the simple projective E_z as a direct summand and thus $[T, E_z] = 0$. But E_z does not belong to $\mathbf{d}^\perp$ either. We conclude that

$$\langle \mathbf{d}, \boldsymbol{\varepsilon}_z \rangle = d_z - \sum_{j=1}^{s} d_{y_j} = [T, E_z] - {}^1[T, E_z] < 0,$$

and therefore we may apply Proposition 2.6. The same arguments yield that either $e_z = 0$ or else $e_z < \sum_{j=1}^{s} e_{y_j}$. In either case we have $e'_{z'} = \sum_{j=1}^{s} e_{y_j} - e_z \geq 0$.

Proposition 4.3. *If $X \in \mathcal{Z}_{Q,\mathbf{d}}$ does not contain the simple projective E_z as a direct summand, we have that X is generic in $\mathcal{Z}_{Q,\mathbf{d}}$ if and only if $\mathcal{F}X$ is generic in $\mathcal{Z}_{Q',\mathbf{d}'}$. Moreover,*

$$\mathrm{codim}_{\mathrm{rep}(Q,\mathbf{d})} \overline{\mathrm{GL}(\mathbf{d}) \star X} = \mathrm{codim}_{\mathrm{rep}(Q',\mathbf{d}')} \overline{\mathrm{GL}(\mathbf{d}') \star \mathcal{F}X}.$$

Proof. We know that $X \in (\mathrm{rep}(Q))'$ and $\mathcal{F}X \in (\mathrm{rep}(Q'))'$. The sets

$$\mathrm{rep}(Q, \mathbf{d})' = \mathrm{rep}(Q, \mathbf{d}) \cap (\mathrm{rep}(Q))', \quad \mathrm{rep}(Q', \mathbf{d}')' = \mathrm{rep}(Q', \mathbf{d}') \cap (\mathrm{rep}(Q'))'$$

are open as $[-, E_z] = 0$ and $[E'_{z'}, -] = 0$ are open conditions. Moreover, they are related by a fiber bundle construction [6]. In particular, there is a bijection compatible with $\mathcal{F}$ between the $\mathrm{GL}(\mathbf{d})$-orbits in $\mathrm{rep}(Q, \mathbf{d})'$ and the $\mathrm{GL}(\mathbf{d}')$-orbits in $\mathrm{rep}(Q', \mathbf{d}')'$, preserving their codimensions, closures and inclusions between closures. Hence the claim follows from Proposition 2.6. □

4.2. By $\overline{Q}$ we denote the full subquiver of Q with vertex set $Q_0 \setminus \{z\}$, and we set $\overline{\mathbf{d}} = \mathbf{d}|_{\overline{Q}_0}$. For $X \in \mathrm{rep}(Q)$, we denote by $\overline{X} \in \mathrm{rep}(\overline{Q})$ the restriction of X to $\overline{Q}$. We set

$$\mathcal{Z}''_{Q,\mathbf{d}} = \{X \in \mathcal{Z}_{Q,\mathbf{d}};\ E_z \text{ is a direct summand of } X\}.$$

As a generic $X \in \mathcal{Z}_{Q,\mathbf{d}}$ contains a non-zero preprojective direct summand and as any indecomposable preprojective representation becomes simple projective under a suitable series of reflection functors, part (i) of our theorem will follow if we show that

$$\mathrm{codim}_{\mathrm{rep}(Q,\mathbf{d})} \mathcal{Z}''_{Q,\mathbf{d}} = n + p - 1 - \sigma(V).$$

Proposition 4.4. *The map*

$$\mathrm{rep}(Q, \mathbf{d}) \to \mathrm{rep}(\overline{Q}, \overline{\mathbf{d}}) \times \mathrm{Mat}(d_z \times (\sum_{j=1}^{s} d_{y_j}), k)$$

sending X to $\big(\overline{X}, (X(\alpha_1) \cdots X(\alpha_s))\big)$ restricts to an isomorphism

$$\mathcal{Z}''_{Q,\mathbf{d}} \simeq \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} \times \mathcal{M},$$

where $\mathcal{M} = \{A \in \mathrm{Mat}(d_z \times (\sum_{j=1}^{s} d_{y_j}), k);\ \mathrm{rank}\, A < d_z\}$.

Proof. Indeed, $X \in \mathrm{rep}(Q, \mathbf{d})$ belongs to $\mathcal{Z}''_{Q,\mathbf{d}}$ if and only if $X \simeq X' \oplus E_z$ and $[X', Y] \neq 0$ for all non-zero $Y \in \mathbf{d}^\perp$ with $Y(z) = 0$. By Corollary 2.4, these are exactly the objects of $(\overline{\mathbf{d}})^\perp$ in $\mathrm{rep}(\overline{Q}, \overline{\mathbf{d}})$, extended by 0 to Q. But if $Y(z) = 0$, we have $[X', Y] = [\overline{X}, Y]$. The result follows. □

Let $\overline{H}$ be a representation in the open orbit of $\mathrm{rep}(\overline{Q}, \overline{\mathbf{h}})$, and note that the image $\overline{H}_\mu$ of the simple homogeneous representation H_μ is isomorphic to $\overline{H}$ for any $\mu \notin \mathcal{E}$. We will need the following lemma.

Lemma 4.5. (i) $\sigma(\overline{H}) \leq 3h_z - 2 = 3(h_z - 1) + 1$.
(ii) *If either Q is an $\widetilde{\mathbb{A}}_{n-1}$-quiver or else $V = \bigoplus V_i^{\lambda_i}$ with $\lambda_i \geq 3$ for all i, then* $\sigma(\overline{V}) \leq \sigma(V) + e'_{z'}$.

Proof. *(i)* Clearly $\overline{H}$ has at most as many pairwise non-isomorphic direct summands as $\sum_{j=1}^s \dim_k H_\mu(y_j)$, $\mu \notin \mathcal{E}$, which implies that

$$\sigma(\overline{H}) \leq \sum_{j=1}^{s} h_{y_j} = 2h_z.$$

The last equality follows from

$$0 = \langle \boldsymbol{\varepsilon}_z, \mathbf{h} \rangle + \langle \mathbf{h}, \boldsymbol{\varepsilon}_z \rangle = h_z + h_z - \sum_{j=1}^{s} h_{y_j}.$$

This yields our claim except in the case $h_z = 1$. But in that case, $\overline{Q}$ is a Dynkin quiver, and we have

$$[\overline{H}, \overline{H}] = \langle \overline{\mathbf{h}}, \overline{\mathbf{h}} \rangle_{\overline{Q}} = \langle \overline{\mathbf{h}}, \overline{\mathbf{h}} \rangle_Q = \langle \mathbf{h} - \boldsymbol{\varepsilon}_z, \mathbf{h} - \boldsymbol{\varepsilon}_z \rangle = \langle \boldsymbol{\varepsilon}_z, \boldsymbol{\varepsilon}_z \rangle = 1.$$

Recall that ${}^1[\overline{H}, \overline{H}] = 0$ as the orbit of $\overline{H}$ is open. In particular, $\overline{H}$ is indecomposable and $\sigma(\overline{H}) = 1$.

(ii) We restrict V to the support of $\mathbf{e}$, which is a tame quiver K for which $\mathbf{e}$ is a sincere dimension vector. If $e_z = 0$, i.e. if z is not a vertex of K, we have $\overline{K} = K$, $\overline{V} = V$ and $e'_{z'} \geq 0$. Otherwise, z is a sink of K, and we can apply our results from [9]: $\mathcal{Z}_{K,\mathbf{e}}$ is a complete intersection of codimension $\#K_0 - \sigma(V)$, and $\mathcal{Z}_{\overline{K},\overline{\mathbf{e}}}$ is a complete intersection of codimension $\#K_0 - 1 - \sigma(\overline{V})$. Note that either every connected component of K is a quiver of $\mathbb{A}$-type or else the multiplicity λ_i of every indecomposable $V_i|_K$ arising in $V|_K$ is at least 3. Set

$$\mathcal{Z}''_{K,\mathbf{e}} = \{X \in \mathcal{Z}_{K,\mathbf{e}};\ [X, E_z] \neq 0\}$$

and remember that, as in Proposition 4.4, $\mathcal{Z}''_{K,\mathbf{e}} \xrightarrow{\sim} \mathcal{Z}_{\overline{K},\overline{\mathbf{e}}} \times \mathcal{M}'$ with

$$\mathcal{M}' = \{B \in \mathrm{Mat}(e_z \times \sum_{j=1}^{s} e_{y_j}, k);\ \mathrm{rank}(B) < e_z\}.$$

We conclude that

$$\begin{aligned} \#K_0 - \sigma(V) = \mathrm{codim}_{\mathrm{rep}(K,\mathbf{e})} \mathcal{Z}_{K,\mathbf{e}} &\leq \mathrm{codim}_{\mathrm{rep}(K,\mathbf{e})} \mathcal{Z}''_{K,\mathbf{e}} \\ &= \#K_0 - 1 - \sigma(\overline{V}) + e'_{z'} + 1, \end{aligned}$$

as $\mathrm{codim}\, \mathcal{M}' = \sum_{j=1}^s e_{y_j} - e_z + 1 = e'_{z'} + 1$. The result follows. □

We are now ready to finish the proof of part *(i)* of our theorem. We need to show that

$$\operatorname{codim}_{\operatorname{rep}(Q,\mathbf{d})} \mathcal{Z}''_{Q,\mathbf{d}} \geq n+p-1-\sigma(V).$$

Note that $\overline{Q}$ is a disjoint union of Dynkin quivers; in fact it is an $\mathbb{A}_{n-1}$-quiver if Q is an $\widetilde{\mathbb{A}}_{n-1}$-quiver. In the remaining cases, the multiplicity of any indecomposable direct summand in the representation $\overline{H}^p \oplus \overline{V} \in \operatorname{rep}(\overline{Q}, \overline{\mathbf{d}})$, whose $\operatorname{GL}(\overline{\mathbf{d}})$-orbit is open, is at least 3. Therefore we know from [9] that

$$\operatorname{codim} \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} = n-1-\sigma(\overline{V} \oplus \overline{H}^p).$$

We compute, using Proposition 4.4, the preceding lemma, and [9] for $(\overline{Q}, \overline{\mathbf{d}})$:

$$\begin{aligned}
&\operatorname{codim}_{\operatorname{rep}(Q,\mathbf{d})} \mathcal{Z}''_{Q,\mathbf{d}} - (n+p-1-\sigma(V))\\
&\quad = \operatorname{codim}_{\operatorname{rep}(\overline{Q},\overline{\mathbf{d}})} \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} + (d'_{z'}+1) - (n+p-1-\sigma(V))\\
&\quad = (n-1-\sigma(\overline{V} \oplus \overline{H}^p)) + (ph_z + e'_{z'} + 1) - (n+p-1-\sigma(V))\\
&\quad = (\sigma(\overline{V}) + \sigma(\overline{H}) - \sigma(\overline{V} \oplus \overline{H})) + (p(h_z-1) + 1 - \sigma(\overline{H}))\\
&\qquad + (\sigma(V) + e'_{z'} - \sigma(\overline{V})) \geq 0.
\end{aligned}$$

In the last sum, each summand is non-negative. This is obvious for the first one, and it follows from the preceding lemma for the second and the third. Note that either $p \geq 3$ or else Q is an $\widetilde{\mathbb{A}}_{n-1}$-quiver and $h_z = 1$.

4.3. Finally, we prove part *(iii)* of the theorem.

Lemma 4.6. *For $\mathbf{d} = p\mathbf{h} + \mathbf{e}$, $p \geq 3$, $\mathcal{Z}_{Q,\mathbf{d}}$ contains a generic representation $X = X_{\mathcal{P}} \oplus X_{\mathcal{R}} \oplus X_{\mathcal{I}}$ with $X_{\mathcal{P}}$ preprojective, $X_{\mathcal{R}}$ regular, and $X_{\mathcal{I}}$ preinjective such that the preprojective part $X_{\mathcal{P}}$ has defect $\partial(X_{\mathcal{P}}) = -1$.*

Note that in particular $X_{\mathcal{P}}$ is indecomposable.

Proof. We assume there is a sink z with $h_z = 1$. If no such sink exists, there is a source y with $h_y = 1$, and applying the dual arguments we find that $\partial(X_{\mathcal{I}}) = 1$, which implies $\partial(X_{\mathcal{P}}) = -1$.

We start from a representation W with dimension vector $p\mathbf{h}$ which is a direct sum of all simple regular non-homogeneous representations and some simple homogeneous representations (if necessary). Obviously there is an exact sequence in $\operatorname{rep}(Q)$ of the form

$$0 \to E_z \to W \to W' \to 0$$

for some W'. Then no indecomposable direct summand of W' is preprojective, by Lemma 3.1. Let $X' = E_z \oplus W'$. Thus the defect of the preprojective part E_z of X' equals $\langle \mathbf{h}, \boldsymbol{\varepsilon}_z \rangle = -h_z = -1$. Observe that $[X', Y] \neq 0$ for any regular repesentation $Y \neq 0$. Indeed, if Y is simple homogeneous, then $[X', Y] \geq [E_z, Y] > 0$, and if Y is simple non-homogeneous, then $[X', Y] \geq [W, Y] > 0$. Thus X' lies in $\mathcal{Z}_{Q,\mathbf{d}}$.

X' belongs to $\overline{\operatorname{GL}(\mathbf{d}) \star X}$ for some generic X, which we decompose as $X = X_{\mathcal{P}} \oplus X_{\mathcal{R}} \oplus X_{\mathcal{I}}$ with $X_{\mathcal{P}}$ preprojective, $X_{\mathcal{R}}$ regular, and $X_{\mathcal{I}}$ preinjective. By Proposition 4.1, we know that $X_{\mathcal{P}} \neq 0$ and thus $\partial(X_{\mathcal{P}}) \leq -1$. Choose $\nu \in \mathbb{P}^1(k) \setminus \mathcal{E}$ in such a way that H_ν is not a direct summand of X nor of X', and remember that $[X, H_\nu] \leq [X', H_\nu]$. Using Lemma 3.1 we compute:

$$\begin{aligned}
1 &\leq -\partial(X_{\mathcal{P}}) = \langle \operatorname{\mathbf{dim}} X_{\mathcal{P}}, \mathbf{h} \rangle = [X_{\mathcal{P}}, H_\nu] = [X, H_\nu]\\
&\leq [X', H_\nu] = -\partial(X'_{\mathcal{P}}) = 1.
\end{aligned}$$

□

For each natural number r, the vector $\mathbf{dim}\, X_{\mathcal{P}} + r\mathbf{h}$ is the dimension vector of some indecomposable representation $X_{\mathcal{P}}[r]$, which is still preprojective with defect -1 and thus has an open orbit in $\mathrm{rep}(Q, \mathbf{dim}\, X_{\mathcal{P}} + r\mathbf{h})$. For $s \in \mathbb{N}$ we let $X_{\mathcal{I}}[s]$ be an indecomposable representation with $\mathbf{dim}\, X_{\mathcal{I}}[s] = \mathbf{dim}\, X_{\mathcal{I}} + s\mathbf{h}$. Set $X[r,s] = X_{\mathcal{P}}[r] \oplus X_{\mathcal{R}} \oplus X_{\mathcal{I}}[s]$. The following result implies part *(iii)* of our theorem:

Proposition 4.7. *The representations $X[r,s]$ are pairwise non-isomorphic and generic in $\mathcal{Z}_{Q,\mathbf{d}+(r+s)\mathbf{h}}$.*

Proof. Choose $Y \in (\mathbf{d} + (r+s)\mathbf{h})^{\perp} = \mathbf{d}^{\perp}$, and remember that Y is regular and has defect 0. Therefore we have

$$[X_{\mathcal{P}}[r], Y] = \langle \mathbf{dim}\, X_{\mathcal{P}}[r], \mathbf{dim}\, Y\rangle = \langle \mathbf{dim}\, X_{\mathcal{P}}, \mathbf{dim}\, Y\rangle = [X_{\mathcal{P}}, Y],$$

and thus

$$[X[r,s], Y] = [X_{\mathcal{P}}, Y] + [X_{\mathcal{R}}, Y] = [X, Y] > 0$$

and $X[r,s]$ belongs to $\mathcal{Z}_{Q,\mathbf{d}+(r+s)\mathbf{h}}$.

In order to show that $X[r,s]$ is generic, it is enough to prove

$$^1[X[r,s], X[r,s]] = n + p + r + s - 1 - \sigma(V),$$

which follows if

$$^1[X[r,s], X[r,s]] = {}^1[X, X] + r + s.$$

As above we have

$$^1[X_{\mathcal{R}}, X_{\mathcal{P}}[r]] = {}^1[X_{\mathcal{R}}, X_{\mathcal{P}}] \qquad \text{and} \qquad {}^1[X_{\mathcal{I}}[s], X_{\mathcal{R}}] = {}^1[X_{\mathcal{I}}, X_{\mathcal{R}}].$$

We compute

$$\begin{aligned} ^1[X_{\mathcal{I}}[s], X_{\mathcal{P}}[r]] &= -\langle \mathbf{dim}\, X_{\mathcal{I}}[s], \mathbf{dim}\, X_{\mathcal{P}}[r]\rangle \\ &= -\langle \mathbf{dim}\, X_{\mathcal{I}} + s\mathbf{h}, \mathbf{dim}\, X_{\mathcal{P}} + r\mathbf{h}\rangle \\ &= {}^1[X_{\mathcal{I}}, X_{\mathcal{P}}] - s\partial(\mathbf{dim}\, X_{\mathcal{P}}) + r\partial(\mathbf{dim}\, X_{\mathcal{I}}) \\ &= {}^1[X_{\mathcal{I}}, X_{\mathcal{P}}] + s + r. \end{aligned}$$

By Lemma 3.1 these are the only terms that do not vanish. □

5. The oriented cycle

5.1. In this section we wish to generalize our results to the only extended Dynkin quiver not considered so far, the oriented cycle, i.e. the quiver Q with $Q_0 = \{1, 2, \ldots, n\}$ and $Q_1 = \{\alpha_i : i \to (i+1);\ i \in Q_0\}$; we view the elements of Q_0 as representatives of $\mathbb{Z}/n\mathbb{Z}$. The category $\mathrm{rep}(Q)$ of finite dimensional representations of Q decomposes into a family $\coprod_{\mu\in k} \mathcal{R}_\mu$ of uniserial categories $\mathcal{R}_\mu$ parametrized by $\mu \in k$. For $\mu \neq 0$, $\mathcal{R}_\mu$ contains a unique simple representation H_μ with $\mathbf{dim}\, H_\mu = \mathbf{h}$, where $h_i = 1$ for $i \in Q_0$. For $\mu = 0$, $\mathcal{R}_0$ consists of all nilpotent representations (compare Section 3.2). If $n \geq 2$, its simple objects are just the one-dimensional representations of Q.

We recall the description of the semi-invariants for $\mathrm{rep}(Q, \mathbf{d})$ from [10] and [17]. For $\mathbf{d} \in \mathbb{N}^{Q_0}$, we may assume that $d_1 = p \leq d_i$, $i \in Q_0$, up to renumbering the vertices of Q. For $X \in \mathrm{rep}(Q, \mathbf{d})$, the coefficients $c_1(X), \ldots, c_p(X)$ of the characteristic polynomial

$$\det(T - X(\alpha_n) \cdot \ldots \cdot X(\alpha_1)) = T^p + c_1(X)T^{p-1} + \cdots + c_p(X)$$

are clearly invariant under $\mathrm{GL}(\mathbf{d})$. For two integers $i < j \leq i+n$, the path $\alpha_{j-1}\cdots\alpha_i$ is called an admissible arc $A = [i,j]$ if $d_i = d_j < d_m$ for all m with $i < m < j$. For any admissible arc $A = [i,j]$ the determinant

$$f_A(X) = \det\left(X(\alpha_{j-1})\cdot\ldots\cdot X(\alpha_i)\right)$$

is a semi-invariant. We call an admissible arc $B = [i,j]$ minimal if $d_i = d_j = p$.

Proposition 5.1. *Let Q be the oriented cycle, and let $\mathbf{d} \in \mathbb{N}^{Q_0}$ with $d_1 = p \leq d_i$ for all $i \in Q_0$.*

(i) *The algebra of semi-invariants is the polynomial algebra*

$$k[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{SL}(\mathbf{d})} = k[c_1,\ldots,c_p;\{f_A\}]/(\prod f_B - c_p),$$

where A ranges over all admissible and B over all minimal admissible arcs of Q.

(ii) *The algebra $k[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{GL}(\mathbf{d})}$ of $\mathrm{GL}(\mathbf{d})$-invariants is a polynomial ring in $c_1,\ldots,c_p$.*

Proof. Both statements are essentially contained in [17]: *(i)* is stated explicitly, and *(ii)* is the fact that an invariant is a semi-invariant with trivial weight. □

Theorem 2. *Let Q be the oriented cycle, $\mathbf{d} \in \mathbb{N}^{Q_0}$ with $d_1 = p \leq d_i$ for all $i \in Q_0$.*

(i) *The set $\mathcal{Z}_{Q,\mathbf{d}}$ of common zeros of $c_1,\ldots,c_p;\{f_A\}$ is a complete intersection, where A ranges over all admissible arcs of Q.*

(ii) *The set $\mathcal{N}_{\mathbf{d}}$ of nilpotent representations in $\mathrm{rep}(Q,\mathbf{d})$ is the set of common zeros of $c_1,\ldots,c_p$; it is a complete intersection.*

As nilpotent representations do not depend on parameters, any irreducible component of $\mathcal{Z}_{Q,\mathbf{d}}$ or $\mathcal{N}_{\mathbf{d}}$ is the closure of an orbit. The number of irreducible components of $\mathcal{Z}_{Q,\mathbf{d}}$ can be shown to be bounded for the oriented cycle. Note that as a consequence of our two theorems, we obtain that $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection for any $\widetilde{\mathbb{A}}_{n-1}$-quiver.

5.2. Our strategy is to compare $\mathrm{rep}(Q,\mathbf{d})$ with $\mathrm{rep}(\widetilde{Q},\widetilde{\mathbf{d}})$, where

$$\widetilde{Q} = \underbrace{1 \xrightarrow{\alpha_1} 2 \dashrightarrow \cdots \dashrightarrow n \xrightarrow{\alpha_n} 0}_{\beta\colon 1 \to 0}$$

and

$$\widetilde{d}_i = \begin{cases} d_i & 1 \leq i \leq n, \\ p = d_1 & i = 0. \end{cases}$$

For $Y \in \mathrm{rep}(\widetilde{Q},\widetilde{\mathbf{d}})$ the coefficients $\widetilde{c}_0(Y),\ldots,\widetilde{c}_p(Y)$ of the polynomial

$$\det(Y(\beta)T - Y(\alpha_n)\cdot\ldots\cdot Y(\alpha_1)) = \widetilde{c}_0(Y)T^p + \widetilde{c}_1(Y)T^{p-1} + \cdots + \widetilde{c}_p(Y)$$

are semi-invariants; note that $\widetilde{c}_0(Y) = \det(Y(\beta))$. For every admissible arc $A = [i,j]$ of Q as defined in Section 5.1, there is a semi-invariant $\widetilde{f}_A$ given by

$$\widetilde{f}_A(Y) = \det(Y(\alpha_{j-1})\cdot\ldots\cdot Y(\alpha_i)).$$

From [17] we know that

$$k[\mathrm{rep}(\widetilde{Q},\widetilde{\mathbf{d}})]^{\mathrm{SL}(\widetilde{\mathbf{d}})} = k[\widetilde{c}_1,\ldots,\widetilde{c}_p;\{\widetilde{f}_A\}]/(\prod \widetilde{f}_B - \widetilde{c}_p),$$

where A ranges over all admissible and B over all minimal admissible arcs for Q.

We leave the proof of the following lemma to the reader.

Lemma 5.2. *The map*

$$\Phi :\ \mathrm{GL}(p) \times \mathrm{rep}(Q, \mathbf{d}) \to \mathrm{rep}(\widetilde{Q}, \widetilde{\mathbf{d}})$$

given by

$$\Phi(s, X)(\gamma) = \begin{cases} X(\alpha_i) & \gamma = \alpha_i,\ i < n, \\ s \cdot X(\alpha_n) & \gamma = \alpha_n, \\ s & \gamma = \beta \end{cases}$$

is a $\mathrm{GL}(\widetilde{\mathbf{d}})$*-equivariant open immersion onto the set*

$$\mathrm{rep}(\widetilde{Q}, \widetilde{\mathbf{d}})' = \left\{ Y \in \mathrm{rep}(\widetilde{Q}, \widetilde{\mathbf{d}});\ \det(Y(\beta)) \neq 0 \right\},$$

where $\mathrm{GL}(\widetilde{\mathbf{d}})$ *acts on* $\mathrm{GL}(p) \times \mathrm{rep}(Q, \mathbf{d})$ *by*

$$h \star (s, X) = (h_0 s h_1^{-1}, \overline{h} \star X)$$

and $\overline{h}_i = h_i$ *for* $i \in Q_0$.

Note that by definition,

$$\Phi^*(\widetilde{c}_i)(s, X) = \begin{cases} \det(s) \cdot c_i(X) & i = 1, \dots, p, \\ \det(s) & i = 0; \end{cases}$$

$$\Phi^*(\widetilde{f}_A)(s, X) = \begin{cases} \det(s) \cdot f_A(X) & \text{if } \alpha_n \text{ belongs to } A, \\ f_A(X) & \text{otherwise.} \end{cases}$$

We conclude that the zero set $\mathcal{V}(c_1, \dots, c_p, \{f_A\}) = \mathcal{Z}_{Q,\mathbf{d}} \subseteq \mathrm{rep}(Q, \mathbf{d})$ has the same codimension as the zero set $\mathcal{V}(\widetilde{c}_1, \dots, \widetilde{c}_p, \{\widetilde{f}_A\}) = \mathcal{Y}_{\widetilde{Q},\widetilde{\mathbf{d}}} \subseteq \mathrm{rep}(\widetilde{Q}, \widetilde{\mathbf{d}})$. As $\mathcal{Z}_{\widetilde{Q},\widetilde{\mathbf{d}}} = \mathcal{V}(\widetilde{c}_0) \cap \mathcal{Y}_{\widetilde{Q},\widetilde{\mathbf{d}}}$ is a complete intersection by Theorem 1, $\mathcal{Y}_{\widetilde{Q},\widetilde{\mathbf{d}}}$ is as well. This proves part *(i)*.

As for part *(ii)*, we need to study the set $\mathcal{V}(c_1, \dots, c_p)$ of common zeros of $c_1, \dots, c_{p-1}$, $c_p = \prod f_B$, where B ranges over all minimal admissible arcs. As

$$\mathcal{V}(c_1, \dots, c_p) = \bigcup_B \mathcal{V}(c_1, \dots, c_{p-1}, f_B),$$

this is a complete intersection as well. Clearly, a representation X is nilpotent if and only if the characteristic polynomial of $X(\alpha_n) \cdot \ldots \cdot X(\alpha_1)$ is T^p.

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