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A DOLBEAULT ISOMORPHISM THEOREM IN INFINITE DIMENSIONS

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Abstract. For a large class of separable Banach spaces, we prove the real analytic Dolbeault isomorphism theorem for open subsets.

1. Introduction

Dolbeault's isomorphism theorem states that if $E \to M$ is a finite rank holomorphic vector bundle over a finite dimensional complex manifold, then its sheaf and $\overline{\partial}$-cohomology groups are canonically isomorphic for all $q \geq 0$:

$$H^q(M, E) \approx H^{0,q}_{\overline{\partial}}(M, E).$$

The case where E is the trivial bundle can be found in [D]. Our goal here is to extend this theorem to infinite dimensions. An obvious extension fails, even when M is a domain in a Banach space. Indeed, in [P], Patyi gives an example of a complex Banach space X (which even has an unconditional basis, see below) whose unit ball B has

$$H^q(B, \mathcal{O}) = 0, \qquad q \geq 1,$$

but there is a closed $f \in C^\infty_{0,1}(B)$ that is not exact, hence $H^{0,1}(B) \neq 0$. We shall, however, show that a Dolbeault–type isomorphism theorem can be proved in open sets in rather general Banach spaces — in particular in the Banach spaces Patyi considers — if the Dolbeault groups are defined in terms of real analytic forms.

Thus, let X be a complex Banach space, $\Omega \subset X$ open, $E \to \Omega$ a holomorphic Banach bundle, and $\mathcal{A}_{p,q} = \mathcal{A}^E_{p,q}$ the sheaf of real analytic (p, q)-forms on Ω, with values in E, $p, q = 0, 1, 2, \ldots$. Then the operator $\overline{\partial} : \mathcal{A}_{p,q} \to \mathcal{A}_{p,q+1}$ can be defined, much as in finite dimensions; see Section 2.

Theorem 1.1. *If X has an unconditional basis, then*

$$H^q(\Omega, E) \approx \frac{Ker\ \{\overline{\partial} : \mathcal{A}^E_{0,q}(\Omega) \to \mathcal{A}^E_{0,q+1}(\Omega)\}}{Im\ \{\overline{\partial} : \mathcal{A}^E_{0,q-1}(\Omega) \to \mathcal{A}^E_{0,q}(\Omega)\}}. \tag{1.1}$$

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The group on the left is the sheaf cohomology of germs of E-valued holomorphic q-forms over Ω; see [W]. The group on the right is a real analytic version of the Dolbeault group $H^{0,q}_{\overline{\partial}}(\Omega, E)$ mentioned above.

Not surprisingly, the theorem will be obtained by considering the sheaf $\mathcal{O}^E$ of germs of holomorphic sections of E and the complex

$$0 \to \mathcal{O}^E \to \mathcal{A}_{0,0} \xrightarrow{\overline{\partial}} \mathcal{A}_{0,1} \xrightarrow{\overline{\partial}} \dots . \tag{1.2}$$

It is known [L1, Proposition 3.2] that (1.2) is exact, unlike its $\mathcal{C}^\infty$ counterpart. Therefore the abstract de Rham Theorem (see, e.g., [W, Theorem 3.13]) would give (1.1) if we knew that the sheaves $\mathcal{A}_{0,q}$ are acyclic, i.e., $H^p(\Omega, \mathcal{A}_{0,q}) = 0$ for $p \geq 1$. This is what we are going to show, in fact in somewhat greater generality.

Theorem 1.2. *Let $X_\mathbb{R}$ be a real Banach space with unconditional basis, $\Omega \subset X_\mathbb{R}$ open, and $F \to \Omega$ a real analytic Banach bundle. Then $H^p(\Omega, F) = 0$ for $p \geq 1$.*

In [C], Cartan obtained a similar result in finite dimensions. As there, the key will be the following theorem:

Theorem 1.3. *If $X_\mathbb{R}$ is a real Banach space with unconditional basis and $X \supset X_\mathbb{R}$ its complexification, then any set $S \subset X_\mathbb{R}$ has a neighborhood basis in X consisting of pseudoconvex open sets.*

The finite dimensional case follows a similar outline. The key step was a cohomology vanishing theorem which led to the acyclicity of the resolution (1.2). In 1957, Cartan discussed the real analytic cohomology of real analytic manifolds [C]. If there is a real analytic totally real imbedding into a complex manifold and the image has a Stein neighborhood basis, then the corresponding Theorems A and B for the sheaf of germs of real analytic sections hold. As noted above, the abstract de Rham Theorem, together with acyclicity (Theorem B), imply the cohomology isomorphism theorem. In 1958, Grauert completed the picture by proving the necessary imbedding theorem as part of an investigation of the Levi Problem [G]. In Grauert's approach, a pseudoconvex neighborhood basis again plays a key role. Grauert's results do not simply carry over to infinite dimensions, since they rely on compact sets with nonempty interior. Such compact sets are not available in infinite dimensions. Still, a part of the proof of Theorem 1.3, namely the proof of Theorem 3.1, has some similarities to Grauert's method.

2. Background

Let $(X_\mathbb{R}, \| \|)$ be a real Banach space. We define the complexification $X = X_\mathbb{R} \oplus iX_\mathbb{R}$ of $X_\mathbb{R}$ as this direct sum of vector spaces with the usual complex multiplication: if $x_1, x_2 \in X_\mathbb{R}, \alpha, \beta \in \mathbb{R}$, then

$$(\alpha + i\beta)(x_1 + ix_2) = (\alpha x_1 - \beta x_2) + i(\beta x_1 + \alpha x_2).$$

Given $x_1, x_2 \in X_\mathbb{R}$ and $x = x_1 + ix_2$, we define the projections $\Re : X \to X_\mathbb{R}$ and $\Im : X \to X_\mathbb{R}$ by $\Re x = x_1$ and $\Im x = x_2$. We define the norm

$$\|x\|' = \sup_{0 \leq \theta < 2\pi} \|\Re(e^{i\theta} x)\|.$$

Since $\| \|$ agrees with $\| \|'$ on $X_\mathbb{R}$, we will write $\| \|$ for both. Both $\Re$ and $\Im$ are real linear maps of norm 1. Define conjugation, a real linear isometry conj:$X \to X$, where $\text{conj}(x) = \Re x - i\Im x$. We will also write $x \mapsto \overline{x}$ for conjugation. In general, when discussing the complexification of a Banach space, we will use a Roman capital letter with the subscript $\mathbb{R}$ to denote a real Banach space, and use the same letter without the subscript to denote its complexification. However, we will sometimes denote a real Banach space by a Roman capital letter and use the same letter with the subscript $\mathbb{C}$ to denote its complexification. For more information on complexified Banach spaces, see [S].

Again, let $X_{\mathbb{R}}, Y_{\mathbb{R}}$ be real Banach spaces. Let $\Omega \subset X_{\mathbb{R}}$ be an open set, and let $f : \Omega \to Y_{\mathbb{R}}$. Then f is said to be real analytic if there are a neighborhood $U = \operatorname{conj}(U) \subset X$ of Ω and a holomorphic function $g : U \to Y$ such that g restricts to f on Ω.

For precise definitions of manifolds, vector bundles, etc., see [L1]. It is easy to define these in the real analytic category by requiring real analytic transition functions, trivializations, etc., rather than smooth ones, just as in the finite dimensional case.

We wish to show that holomorphic functions are real analytic. But first, we will need some further background (found in [M]). Given complex Banach spaces W, Z, an open set $\Omega \subset W$, and a function $f \in \mathcal{O}(\Omega, Z)$, for each $a \in \Omega$ there are k-homogeneous polynomials $P_k : W \to Z$ such that

$$f(x) = \sum_{k=0}^{\infty} P_k(x-a) \tag{2.1}$$

in a neighborhood of a. To each k-homogeneous polynomial P_k there is associated a unique symmetric k-linear map A_k such that $P_k(x) = A_k(x, \dots, x)$. The sum (2.1) is a Taylor series, and the kth multilinear map is proportional to the iterated kth differential of f.

Define

$$\|P_k\| = \sup_{\|x\| \le 1} \|P_k(x)\|$$

and

$$\|A_k\| = \sup_{\|x_1\|, \dots, \|x_k\| \le 1} \|A_k(x_1, \dots, x_k)\|.$$

Then $\|A_k\| \le e^k \|P_k\|$ by [M, Exercise 2.G]. The Cauchy-Hadamard formula tells us that the radius of uniform convergence, R, of (2.1) is given by $R^{-1} = \limsup_k \|P_k\|^{1/k}$.

Proposition 2.1. *Let X and Y be complexified Banach spaces, $G \subset X$ open and connected, $G \cap X_{\mathbb{R}} \neq \emptyset$, and $f \in \mathcal{O}(G, Y)$. If $f|_{G \cap X_{\mathbb{R}}} = 0$, then $f = 0$.*

Proof. Choose $a \in G \cap X_{\mathbb{R}}$, and write

$$f(x) = f(a) + \sum_{k=1}^{\infty} A_k(x-a, \dots, x-a).$$

By assumption, $f(a) = 0$. But since A_k is proportional to the kth differential of f, $A_k = 0$ in all real directions. Since any $x \in X$ is a linear combination of $\Re x + i\Im x$, $A_k = 0$ by complex multilinearity. Therefore, $f = 0$. □

Now we can prove that holomorphic functions are real analytic.

Proposition 2.2. *If M and N are complex Banach manifolds, then $\mathcal{O}(M, N) \subset \mathcal{A}(M, N)$.*

Proof. Let $f \in \mathcal{O}(M,N)$. Since real analyticity is a local property, we need only consider open subsets of Banach spaces X and Y, so assume that $M \subset X$ and $N \subset Y$ are open. For $a \in M$ arbitrary, write

$$f(x) = \sum_{k=0}^{\infty} P_k(x-a) = \sum_{k=0}^{\infty} A_k(x-a,\ldots,x-a). \tag{2.2}$$

Since any complex multilinear map is also real multilinear, we can disregard the complex structure on X and Y and still write f as a sum of real multilinear maps near a. We may extend A_k to the complexification $X_{\mathbb{C}}$ of X in the natural way; if $x_1,\ldots,x_k,y_1,\ldots,y_k, \in W$, and multiplication by i refers to the complex multiplication in $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$, define the extension $\tilde{A}_k : X_{\mathbb{C}} \times \ldots \times X_{\mathbb{C}} \to Y_{\mathbb{C}}$ of A_k by the formula

$$\begin{aligned}\tilde{A}_k(x_1+iy_1,\ldots,x_k+iy_k) &= A_k(x_1,\ldots,x_k) + iA_k(x_1,\ldots,x_{k-1},y_k)\\ &\quad + iA_k(x_1,\ldots,x_{k-2},y_{k-1},x_k) + \ldots + iA_k(y_1,x_2,\ldots,x_k)\\ &\quad + i^2A_k(x_1,\ldots,x_{k-2},y_{k-1},y_k) + \ldots + i^2A_k(y_1,y_2,x_3,\ldots,x_k)\\ &\quad + \ldots + i^kA_k(y_1,\ldots,y_k).\end{aligned}$$

Observe that there are 2^k terms in this sum. To each $\tilde{A}_k$ a homogeneous polynomial $\tilde{P}_k$ can be associated. We now investigate the convergence of

$$\sum_{k=0}^{\infty} \tilde{P}_k((x+iy)-a). \tag{2.3}$$

We can estimate the terms:

$$\begin{aligned}\|\tilde{P}_k((x+iy)-a)\| &\le 2^k\|A_k\|\max(\|x-a\|,\|y\|)\\ &\le 2^ke^k\|P_k\|\max(\|x-a\|,\|y\|)^k.\end{aligned}$$

According to the Cauchy-Hadamard formula, the radius of uniform convergence R of (2.2) is given by

$$R^{-1} = \limsup_k \|P_k\|^{1/k}.$$

Thus, if we use the root test for the sum

$$\sum_{k=0}^{\infty} \|\tilde{P}_k((x+iy)-a)\|,$$

we have

$$\begin{aligned}&\limsup_k \|\tilde{P}_k((x+iy)-a)\|^{1/k}\\ &\le 2e\max(\|x-a\|,\|y\|)\limsup_k \|P_k\|^{1/k}\\ &\le 2eR^{-1}\max(\|x-a\|,\|y\|).\end{aligned}$$

Therefore (2.3) converges uniformly on a neighborhood containing the set

$$\{x+iy \in X : \max(\|x-a\|,\|y\|) < R/2e\}.$$

Thus f can be extended to a holomorphic function in an $X_{\mathbb{C}}$-neighborhood of any $a \in M$. But we would like to find a single neighborhood $G \subset X_{\mathbb{C}}$ of M to which we can extend f. We will check that the sums (2.3), which converge in a ball centered at each $a \in M$, agree on the overlaps of these balls. If so, then they define a holomorphic function on a neighborhood of M. Suppose g and g' are two extensions of f on two overlapping balls $B(a;r) = \{x \in X_{\mathbb{C}} : \|x-a\| < r\}$ and $B(a';r') = \{x \in X_{\mathbb{C}} : \|x-a'\| < r'\}$. For simplicity, we may assume $r < \operatorname{dist}(a, \partial M)$. Since $\Re(\cdot)$ has norm 1,

$$B(a;r) \cap M = \Re(B(a;r)),$$

and therefore

$$B(a;r) \cap B(a';r') \cap M = \Re(B(a;r)) \cap \Re(B(a';r')).$$

Both g and g' agree on $B(a;r) \cap B(a';r') \cap M$. By Proposition 2.1, $g - g' = 0$ on the entire convex, hence connected intersection of their domains, or $g = g'$ on the overlap, as required. □

Observe that this implies that a holomorphic vector bundle has a natural real analytic vector bundle structure. More explicitly, let M be a complex manifold, and let $E \to M$ be a holomorphic vector bundle. Then there are local holomorphic trivializations (U_α, ψ_α). By the above proposition, these holomorphic trivializations are real analytic. Furthermore, if (V_β, ϕ_β) and (W_γ, χ_γ) are atlases for M and E, respectively, then the holomorphic functions $\phi_\beta \circ \phi_\alpha^{-1}$ and $\chi_\beta \circ \chi_\alpha^{-1}$ are real analytic, and so M and E are real analytic manifolds, and E is a real analytic vector bundle over M.

Definition 2.3. Given complexified Banach spaces X and Y, and an open set $U \subset X$ such that $U = \operatorname{conj}(U)$, a function $f : U \to Y$ is called "real-type holomorphic" or, "of real type" (written $f \in \mathcal{O}_{\mathbb{R}}(U, Y)$) if f is holomorphic and commutes with conjugation, i.e., $\overline{f(x)} = f(\overline{x})$.

If this is the case, and if $x \in U \cap X_{\mathbb{R}}$, then $f(x) \in Y_{\mathbb{R}}$. Observe that the sum, direct sum, composition, etc. of two real-type holomorphic functions is again of real type. If a sequence of real-type holomorphic functions converges locally uniformly, then the limit is also a real-type holomorphic function.

A real (complex) Banach space X has a Schauder basis if there are $\{e_j\}_{j\in\mathbb{N}} \subset X$ such that any $x \in X$ can be written uniquely as a sum

$$\sum_{j=1}^{\infty} \lambda_j e_j, \qquad \lambda_j \in \mathbb{R}, \text{ respectively } \mathbb{C}. \tag{2.4}$$

If (2.4) converges unconditionally, i.e., independently of any rearrangement of terms for all $x \in X$, then $\{e_j\}$ is said to be an unconditional basis. A Schauder basis of a real Banach space $X_{\mathbb{R}}$ is also a Schauder basis of its complexification, X, and an unconditional basis of $X_{\mathbb{R}}$ is also an unconditional basis of X.

If X is a complex Banach space, $\Omega \subset X$ open, then an upper semi-continuous function $u : \Omega \to [-\infty, \infty)$ is called plurisubharmonic if for every finite dimensional subspace V, $u|V \cap \Omega$ is plurisubharmonic. A pseudoconvex set $P \subset X$ is an open set such that $-\log(\operatorname{dist}(x, \partial P))$ is plurisubharmonic in P. By convention, X itself is also pseudoconvex.

3. A pseudoconvex neighborhood basis

Theorem 1.3 is proved in two steps. The first step is the following theorem.

Theorem 3.1. *Let X be a complexified separable Banach space and let $\Omega \subset X_{\mathbb{R}}$ be open. Then there is a pseudoconvex $P \subset X$ such that $P \cap X_{\mathbb{R}} = \Omega$.*

Proof. If $\Omega = X_{\mathbb{R}}$, then $P = X$ will suffice. Otherwise, $X_{\mathbb{R}} \setminus \Omega$ is nonempty. Let $A \subset \Omega$ be a countable dense subset. For every $a \in A, b \in X_{\mathbb{R}} \setminus \Omega$, define l_{ab} to be a real linear functional on $X_{\mathbb{R}}$ of norm 1 such that $l_{ab}(a-b) = \|a-b\|$. Extend l_{ab} to a complex linear functional of real type by setting $l_{ab}x = l_{ab}(\Re x) + i l_{ab}(\Im x)$. Now let $\{a_k\}_{k=0}^{\infty}$ be an enumeration of A, and set

$$\phi_b(x) = \sum_{k=0}^{\infty} 4^{-k}(l_{a_k b}(x-b))^2 \tag{3.1}$$
$$= \sum_{k=0}^{\infty} 4^{-k}\left((l_{a_k b}x)^2 - 2(l_{a_k b}x)(l_{a_k b}b) + (l_{a_k b}b)^2\right),$$

whence for any $M > 0$, the family $\{\phi_b\}, \|b\| \le M$, is uniformly equicontinuous on bounded subsets of X. Since the series is locally uniformly convergent, ϕ_b is an entire holomorphic function. As the sum of real–type functions, ϕ_b is itself of real type. Furthermore, on Ω we can estimate ϕ_b from below uniformly in b: Let $r(x) = \operatorname{dist}(x, X_{\mathbb{R}} \setminus \Omega)$, and fix $x \in \Omega$. Choose $a_{k_0} \in A$ such that $\|x - a_{k_0}\| < r(x)/4$. Since the restriction of $l_{a_k b}$ to X is real, $(l_{a_k b}(x-b))^2 \ge 0$, so

$$\begin{aligned}\sum_{k=0}^{\infty} 4^{-k}(l_{a_k b}(x-b))^2 &\ge 4^{-k_0}(l_{a_{k_0} b}(x-b))^2\\ &\ge 4^{-k_0}(l_{a_{k_0} b}(a_{k_0}-b) + l_{a_{k_0} b}(x-a_{k_0}))^2\\ &\ge 4^{-k_0}(\|a_{k_0}-b\| - \|x-a_{k_0}\|)^2.\end{aligned}$$

The last inequality is because $l_{a_{k_0} b}(a_{k_0}-b) = \|a_{k_0}-b\|$ and because on X, $\|l_{a_{k_0} b}\| = 1$. To continue, we have

$$\begin{aligned}4^{-k_0}(\|a_{k_0}-b\| - \|x-a_{k_0}\|)^2 &> 4^{-k_0}(r(a_{k_0}) - r(x)/4)^2\\ &\ge 4^{-k_0}(3r(x)/4 - r(x)/4)^2\\ &\ge 4^{-k_0-1}r(x)^2 > 0.\end{aligned}$$

Thus, if we set

$$u(x) = \sup_{b \in X_{\mathbb{R}} \setminus \Omega} -\Re(\phi_b(x)), \qquad x \in X,$$

we have $u(x) < 0$ for $x \in \Omega$. Furthermore, $u(x) = 0$ on $X_{\mathbb{R}} \setminus \Omega$, since clearly $u \le 0$ on $X_{\mathbb{R}}$, and for any $b \in X_{\mathbb{R}} \setminus \Omega$, $\phi_b(b) = 0$.

Next we show that u is continuous. From (3.1),

$$\begin{aligned}&\liminf_{\|b\|\to\infty} \Re\phi_b(x)/\|b\|^2\\ &\ge \liminf_{\|b\|\to\infty} \left\{\Re(l_{a_0 b}(a_0-b) + l_{a_0 b}(x-a_0))^2 - \sum_{k=1}^{\infty} 4^{-k}\|x-b\|^2\right\}/\|b\|^2\\ &\ge \liminf_{\|b\|\to\infty} \|a_0-b\|^2/\|b\|^2 - 1/2 \ge 1/2,\end{aligned}$$

uniformly for x in a bounded set V. Hence, given V, $\Re\phi_b \to \infty$ as $\|b\| \to \infty$ uniformly on V, and so

$$u(x) = \sup_{b \in X_{\mathbb{R}} \setminus \Omega} -\Re\phi_b(x) = \sup_{b \in X_{\mathbb{R}} \setminus \Omega, \|b\| \le M} -\Re\phi_b(x)$$

for all $x \in V$, provided M is sufficiently large. Since $\{\phi_b : \|b\| \le M\}$ is equicontinuous on bounded subsets, it follows that u is continuous. As a continuous supremum of plurisubharmonic functions, it is also plurisubharmonic. Therefore $P = \{x \in X : u < 0\}$ is a pseudoconvex neighborhood of Ω. Since $u = 0$ on $X_{\mathbb{R}} \setminus \Omega$ and $u < 0$ on Ω, $P \cap X_{\mathbb{R}} = \Omega$. □

The second step in the proof of Theorem 1.3 is the heart of the entire matter. The goal is to show that arbitrarily "narrow" pseudoconvex neighborhoods exist. The critical tool is a theorem about real-type holomorphic domination. But first, given a Banach space X, $a \in X$, $r \in \mathbb{R}$, define

$$B(a; r) = \{x \in X : \|x - a\| < r\}$$

and $B(r) = B(0; r)$. We state a Runge-type hypothesis which we will use in the following theorem:

Hypothesis 3.2. *There is a $\mu \in (0, 1)$ such that for any Banach space $(W, \| \|_W)$, $\epsilon > 0$, and $g \in \mathcal{O}(B(1); W)$, there is an $h \in \mathcal{O}(X; W)$ that satisfies $\|g - h\|_W < \epsilon$ on $B(\mu)$.*

This allows us to state the critical theorem.

Theorem 3.3. *Let X be a complexified Banach space with a Schauder basis satisfying Hypothesis* 3.2*, let $P \subset X$ be pseudoconvex, and suppose $P = \mathrm{conj}(P)$. Let $u : P \to \mathbb{R}$ be locally Lipschitz, with $u \le 0$ on $P \cap X_{\mathbb{R}}$. Then there is a complexified Banach space Y and an $f \in \mathcal{O}_{\mathbb{R}}(P, Y)$ such that $u \le \|\Im f\|$.*

Lempert proved an analogous theorem in [L4]. This theorem was about a general (typically not complexified) Banach space with an unconditional basis X; he proved that any locally bounded function u defined on a pseudoconvex set can be dominated by the norm of a holomorphic function.

Using Theorem 3.1 and Theorem 3.3 together, we can prove the following generalized version of Theorem 1.3:

Theorem 3.4. *If $X_{\mathbb{R}}$ is a real Banach space with a Schauder basis and its complexification, $X \supset X_{\mathbb{R}}$, satisfies Hypothesis* 3.2*, then any set $S \subset X_{\mathbb{R}}$ has a neighborhood basis in X consisting of pseudoconvex open sets.*

This theorem implies Theorem 1.3 because a Banach space with an unconditional basis satisfies Hypothesis 3.2 by [L2].

Proof of Theorem 3.4. Let $G \subset X$ be an arbitrary open neighborhood of S. We construct a pseudoconvex neighborhood of S contained in G. By Theorem 3.1, there is a pseudoconvex neighborhood P of $G \cap X_{\mathbb{R}}$ such that $P \cap X_{\mathbb{R}} = G \cap X_{\mathbb{R}}$. By passing to $P \cap \mathrm{conj}(P)$, we can assume $P = \mathrm{conj}(P)$. We can also assume $G \subset P$. Define $u : P \to \mathbb{R}$ by

$$u(x) = \begin{cases} \min(1, \|\Im x\|/\mathrm{dist}(x, \partial G)), & \text{if } x \in G, \\ 1 & \text{otherwise.} \end{cases}$$

We claim that u is locally Lipschitz. In $P \setminus \overline{G}$ this is obvious, and in G it follows from the fact that $\mathrm{dist}(x, \partial G)$ is Lipschitz. If $x \in P \cap \partial G$, then since $(P \setminus G) \cap X_{\mathbb{R}} = \emptyset$, $\|\Im x\| > 0$. Therefore $\mathrm{dist}(y, \partial G) < \|\Im y\|$ for y in some neighborhood of x, and so $u = 1$ is Lipschitz there too. Theorem 3.3 implies that there is a complexified Banach space Y and an $f \in \mathcal{O}_{\mathbb{R}}(P, Y)$ such that $u \le \|\Im f\|$. Set $Q = \{\|\Im f\| < 1\}$.

Then since $\|\Im x\|$ is a convex function, Q is pseudoconvex, and $S \subset P \cap X_{\mathbb{R}} \subset Q$ since $\Im f = 0$ on $X_{\mathbb{R}}$. Furthermore, if $x \in Q$, then $\|\Im f(x)\| < 1$, so $u(x) < 1$, which implies that $x \in G$, as required. □

Theorem 3.3 remains to be proved. As in [L4], this proof will be by induction. Given an open set $P \subset X$, consider those balls $B = B(a; r)$ such that

(i) $\overline{B(a;r)} \cup \mathrm{conj}(\overline{B(a;r)}) \subset P$,
(ii) $2 \text{ diam } B \le \text{diam } P$,
(iii) $B = \mathrm{conj}(B)$ or $\overline{B} \cap \overline{\mathrm{conj}(B)} = \emptyset$.

Let $\mathfrak{B}_P$ denote the family of these balls.

Now we are ready to formulate the induction step in the form of the following proposition.

Proposition 3.5. *Let X be a complexified Banach space with Schauder basis satisfying Hypothesis 3.2. Let $P = conj(P) \subset X$ be a pseudoconvex set. If for every $B \in \mathfrak{B}_P$ there are a complexified Banach space $(V_B, \|\,\|_B)$ and an $f_B \in \mathcal{O}_{\mathbb{R}}(B \cup conj(B), V_B)$ such that $u \le \|\Im f_B\|_B$ on B, then there is a complexified Banach space $(V, \|\,\|_V)$ and an $f \in \mathcal{O}_{\mathbb{R}}(P, V)$ such that $u \le \|\Im f\|_V$ on P.*

Proposition 3.5 will be proved in Sections 4 and 5. We show below that it implies Theorem 3.3.

Proof of Theorem 3.3. Suppose not. If u cannot be dominated on P by $\|\Im f\|$ for any $f \in \mathcal{O}_{\mathbb{R}}(P, Y)$, then by Proposition 3.5 there is a $B_1 \in \mathfrak{B}_P$ such that u cannot be dominated in the same way on $B_1 \cup \mathrm{conj}(B_1)$. Replacing P with $B_1 \cup \mathrm{conj}(B_1)$, we can repeat the same argument to produce $B_{k+1} \in \mathfrak{B}_{B_k}$ such that $\overline{B}_{k+1} \subset B_k$ and u cannot be dominated on $B_{k+1} \cup \mathrm{conj}(B_{k+1})$. Since $2 \text{ diam } B_{k+1} \le \text{diam } B_k$, the B_k converge to a point $x_0 \in P$. Choose $r > 0$ such that $B = B(x_0; r) \in \mathfrak{B}_P$, and u has some Lipschitz constant $K > 0$ on $B \cup \mathrm{conj}(B)$. Since $B_k \subset B$ for some k, u cannot be dominated on $B \cup \mathrm{conj}(B)$.

Suppose first that $\overline{B} \cap X_{\mathbb{R}} = \emptyset$. Then $\Re x_0 \notin \overline{B}$, so that $\|x_0 - \Re x_0\| > r$. Set

$$f(x) = \frac{|u(x_0)| + |u(\overline{x_0})| + Kr}{\|\Im x_0\| - r} x.$$

Since $\|\Im x\| \ge \|\Im x_0\| - r$ when $x \in B$, we have $\|\Im f(x)\| \ge |u(x_0)| + Kr \ge u(x)$; and similarly $\|\Im f\| \ge u$ on $\mathrm{conj}(B)$.

In the second case, $B = \mathrm{conj}(B)$. With $f(x) = Kx$,

$$u(x) \le |u(\Re x)| + K\|x - \Re x\| \le 0 + K\|\Im x\| = \|\Im f(x)\|.$$

Thus in either case there is an $f \in \mathcal{O}(B \cup \mathrm{conj}(B), \mathbb{C})$ such that $\|\Im f\| \ge u$, contradicting our indirect assumption, and thus proving Theorem 3.3. □

4. Ball bundles

One of the tools which we will use to prove Proposition 3.5 is ball bundles over finite dimensional bases. The setup is exactly the same as in [L4]. For convenience, we include the necessary definitions and propositions here. After renorming as in [L3, Section 7], we may assume that $\{e_j\}$ is a bimonotone basis of $X_{\mathbb{R}}$, i.e., whenever $1 \le n \le N \le M \le m \le \infty$,

$$\|\sum_{j=N}^{M} \lambda_j e_j\| \le \|\sum_{j=n}^{m} \lambda_j e_j\|, \qquad \lambda_j \in \mathbb{C}.$$

This renorming respects conjugation, and the operator norms $\|\Re\|$ and $\|\Im\|$ are still both 1.

Let π_N be a projection on the first N coordinates, and $\rho_N = \mathrm{id} - \pi_N$. Fix $P \subset X$ pseudoconvex. Let $d(x) = \min\{1, \mathrm{dist}(x, X \setminus P)\}$ and, given $0 < \alpha < 1$,

$$D_N\langle\alpha\rangle = \{t \in \pi_N X : \|t\| < \alpha N, 1 < \alpha N d(t)\},$$
$$P_N\langle\alpha\rangle = \{x \in X : \pi_N x \in D_N\langle\alpha\rangle, \|\rho_N x\| < \alpha d(\pi_N x)\}.$$

These sets have the following properties (proved in [L4, Proposition 3.1]):

Proposition 4.1. *For any pseudoconvex set P and number $\alpha \in (0,1)$ the following hold:*

(a) *For each positive integer N, $P_N\langle\alpha\rangle \subset P$ is pseudoconvex.*
(b) *For fixed α, each $x \in P$ has a neighborhood that is contained in all but finitely many $P_N\langle\alpha\rangle$.*

We will use an approximation theorem for ball bundles:

Theorem 4.2. *Assume Hypothesis* 3.2 *with some $\mu \in (0,1)$. If $\gamma < 2^{-6}\mu\alpha$ and V is a complex Banach space, then any $\psi \in \mathcal{O}(P_N\langle\alpha\rangle; V)$ can be approximated by $\phi \in \mathcal{O}(P; V)$, uniformly on $P_N\langle\gamma\rangle$.*

This theorem is the same as [L3, Theorem 3.3].

Here is a proposition relating ball bundles to $\mathfrak{B}_P$.

Proposition 4.3. *Let X be a complexified Banach space, and $P = conj(P) \subset X$ pseudoconvex. For any positive integer N, and any choice of α satisfying $0 < 2^7\alpha < \mu^2 < 1$, $P_N\langle\alpha\rangle$ has a finite cover by balls $B_k = B(x_k, r_k)$ such that $B(x_k, 2r_k/\mu) \in \mathfrak{B}_P$.*

Proof. Let $A = \overline{P_N\langle\alpha\rangle} \cap \pi_N X$. Then $P_N\langle\alpha\rangle \cap \pi_N^{-1} t \subset B(t; \alpha d(t))$ for any $t \in A$. For each t there is a relatively open $U_t \subset \pi_N P$ such that $P_N\langle\alpha\rangle \cap \pi_N^{-1} U_t \subset B(t; 2\alpha d(t))$. Since $\{U_t\}_{t\in A}$ covers A, we can find a finite set $t_1, \dots, t_n \in A$ such that $\{U_{t_k}\}$ covers A. But then $\{B(t_k; 2\alpha d(t_k))\}_{k\le n}$ covers $P_N\langle\alpha\rangle$.

To construct x_k, r_k for $k = 1, \dots, n$, we shall distinguish between two cases. Fix $t_k \in A$.

Case 1: $4\alpha d(t_k)/\mu < \mathrm{dist}(t_k, X_\mathbb{R})$.

In this case, define $x_k = t_k$ and $r_k = 2\alpha d(t_k)$. We check that $B(t_k; 4\alpha d(t_k)/\mu) \in \mathfrak{B}_P$. Since $4\alpha d(t_k)/\mu < d(t_k)$, we have $\overline{B(t_k; 4\alpha d(t_k)/\mu)} \subset P$. Also,

$$2\ \mathrm{diam}(B(t_k; 4\alpha d(t_k)/\mu)) = 16\alpha d(t_k)/\mu < 2d(t_k) \le \mathrm{diam}(P).$$

Furthermore,

$$B(t_k; 4\alpha d(t_k)/\mu) \cap \mathrm{conj}(B(t_k; 4\alpha d(t_k)/\mu)) = \emptyset.$$

Therefore,

$$B(x_k; 2r_k/\mu) = B(t_k; 4\alpha d(t_k)/\mu) \in \mathfrak{B}_P.$$

Case 2: $4\alpha d(t_k)/\mu \ge \mathrm{dist}(t_k, X_\mathbb{R})$.

In this case, we will choose x_k and r_k so that $B(x_k; r_k)$ contains $B(t_k; 2\alpha d(t_k))$. Then we will show that

$$B(x_k, 2r_k/\mu) = B(x_k; d(x_k)/2) \in \mathfrak{B}_P.$$

Choose $x_k \in P \cap X_\mathbb{R}$ such that $\|x_k - t_k\| < 8\alpha d(t_k)/\mu$, and define $r_k = \mu d(x_k)$. Let $x \in B(t_k; 4\alpha d(t_k)/\mu)$. Then

$$\|x - x_k\| \le \|x - t_k\| + \|t_k - x_k\| < 16\alpha d(t_k)/\mu.$$

But

$$d(t_k) \leq d(x_k) + \|x_k - t_k\| \leq d(x_k) + 8\alpha d(t_k)/\mu,$$

so

$$d(t_k) \leq \frac{d(x_k)}{1 - 8\alpha/\mu} < 2d(x_k).$$

Therefore, $\|x - x_k\| < 2^5\alpha d(x_k)/\mu < \mu d(x_k)/4$. In other words,

$$B(t_k; 4\alpha d(t_k)/\mu) \subset B(x_k; \mu d(x_k)/4).$$

We check that $B(x_k; d(x_k)/2) \in \mathfrak{B}_P$. Since $x_k \in X_{\mathbb{R}}$,

$$B(x_k; d(x_k)/2) = \text{conj}(B(x_k; d(x_k)/2).$$

Clearly, $\overline{B(x_k; d(x_k)/2)} \subset P$. Furthermore,

$$2\text{diam}(B(x_k; d(x_k)/2)) = 2d(x_k) \leq \text{diam}(P).$$

Then $B(x_k; 2r_k/\mu) \in \mathfrak{B}_P$. □

5. The proof of Proposition 3.5

In this section, we will use the following conventions when taking direct sums of Banach spaces. Given Banach spaces $(X_j, \| \|_j), j \in J$, define the Banach space $\overline{\bigoplus}_{j \in J} X_j$ to be the set of all bounded collections $x = (x_j)$, $x_j \in X_j$, with the sup norm $\|x\| = \sup \|x_j\|_j$. Observe that the complexification of $\overline{\bigoplus}_{j \in J} X_{j\mathbb{R}}$ is $\overline{\bigoplus}_{j \in J} X_j$.

Remark 5.1. Given complexified Banach spaces X, Y, an open $G \subset X$ such that $G = \text{conj}(G)$, and a function $f \in \mathcal{O}(G, Y)$, define $f'(x) = (f(x) + \overline{f(\overline{x})})/2$ and $f''(x) = (f(x) - \overline{f(\overline{x})})/2i$. Then $f', f'' \in \mathcal{O}_{\mathbb{R}}(G, Y)$, $f' \oplus f'' \in \mathcal{O}_{\mathbb{R}}(G, Y \oplus Y)$, and

$$\|f(x)\| \leq 2\|f'(x) \oplus f''(x)\| \leq 2 \max(\|f(x)\|, \|f(\overline{x})\|).$$

Furthermore, if f is a bounded linear map, then so are f', f'' with $\|f'\|, \|f''\| \leq \|f\|$, and

$$\|f(x)\| \leq 2 \max(\|f'(x)\|, \|f''(x)\|)$$

for all $x \in G$.

Some further results are required to prove Proposition 3.5. With notation as in Section 4, if $A \subset \pi_N X \approx \mathbb{C}^N$ and $r : A \to [0, \infty)$ is continuous, define the sets

$$A(r) = \{x \in X : \pi_N x \in A, \|\rho_N x\| < r(\pi_N x)\},$$
$$A[r] = \{x \in X : \pi_N x \in A, \|\rho_N x\| \leq r(\pi_N x)\}.$$

Lemma 5.2. *Let X be a complexified Banach space with Schauder basis, $P = \text{conj}(P) \subset X$ pseudoconvex, N a positive integer, and $A_1 \subset\subset A_2 \subset\subset A_3 \subset\subset A_4 \subset (\pi_N X \cap P)$, with $A_i = \text{conj}(A_i)$ open, $i = 2, 3, 4$, and $A_1 = \text{conj}(A_1)$ compact and plurisubharmonically convex in A_4. Let $r_i : A_4 \to (0, \infty)$ be continuous, with $2r_1 < r_2 < r_3 < r_4$, $-\log r_1$ plurisubharmonic on A_4, and $r_i(x) = r_i(\overline{x})$ for $x \in A_4$. Finally, suppose that all Banach space valued holomorphic functions on $A_4(r_4)$ can be approximated by holomorphic Banach space valued functions on P, uniformly on $A_3(r_3)$. Then for any complexified Banach space V and any function $f \in \mathcal{O}_{\mathbb{R}}(X, V)$, there is a complexified Banach space W and a $g \in \mathcal{O}_{\mathbb{R}}(P, W)$ such that*

(i) *$\|g\| \leq 1$ on $A_1[r_1]$, and*
(ii) *$\|\Im g\| \geq \|\Im f\|$ on $A_3(r_3) \setminus A_2(r_2)$.*

Proof. Set $\pi = \pi_N$, $\rho = \rho_N$. First, we produce a complexified Banach space Z and a function $\phi \in \mathcal{O}_{\mathbb{R}}(P, Z)$ which has norm less than $1/8$ on $A_1[r_1]$ and greater than 2 on $A_3(r_3) \setminus A_2(r_2)$. Consider the constant function equal to 4 on X. By [L4, Lemma 4.1], there is a complex Banach space V and a function $\phi_1 \in \mathcal{O}(X, V)$ which has norm less than $1/8$ on $A_1[r_1]$ and greater than 4 on $A_3(r_3) \setminus A_2(r_2)$. Observe that the vector space V produced in [L4, Lemma 4.1] is complexified. Then by Remark 5.1, we see that $Z = V \oplus V$, $\phi(x) = \phi_1'(x) \oplus \phi_1''(x)$ will do.

Now we will need an auxiliary family of functions. Define

$$w_\lambda(z) = \lambda^2 - (z - \lambda)^2 = 2\lambda z - z^2, \qquad -1 \le \lambda \le 1.$$

Then $w_\lambda \in \mathcal{O}_{\mathbb{R}}(\mathbb{C})$. Furthermore, w_λ satisfies:

(i) $|w_\lambda(z)| \le 1/2$ whenever $|z| \le 1/8$, $-1 \le \lambda \le 1$,
(ii) $w_\lambda(z) = |z|^2$ whenever $\lambda = \Re z$, and
(iii) w_λ are uniformly bounded on bounded subsets of $\mathbb{C}$.

We are ready to define W and $g \in \mathcal{O}_{\mathbb{R}}(P, W)$. Let K be the set of all real-type linear functionals in the closed unit ball of the dual Z'. Let

$$W = \{\text{bounded maps from } K \times [-1, 1] \text{ to } \mathbb{C}\},$$

with the sup norm; it can be identified with the complexification of

$$W_{\mathbb{R}} = \{\text{bounded maps from } K \times [-1, 1] \text{ to } \mathbb{R}\}.$$

Choose q large enough so that

$$\|f(x)\| \le 2^q \qquad \text{whenever } \pi x \in A_1 \text{ and} \|\rho x\| \le 2^{-q} \max_{A_1} r_1.$$

Define a real-type holomorphic function $g : P \to W$ by

$$g(x)(k, \lambda) = (w_\lambda(k\phi(x)))^q f(\pi x + (w_\lambda(k\phi(x)))^q \rho x).$$

Note for any $x \in P$, $g(x)$ defines a bounded function from $K \times [-1, 1]$ to $\mathbb{C}$. If $x \in A_1[r_1]$, then $(w_\lambda(k\phi(x)))^q \le 2^{-q}$, and $\|f(\pi x + (w_\lambda(k\phi(x)))^q \rho x)\| \le 2^q$, so $\|g(x)\| \le 1$. On the other hand, if $x \in A_3(r_3) \setminus A_2(r_2)$, then $\|\phi(x)\| \ge 2$, so the Hahn–Banach Theorem and Remark 5.1 imply that there is a $k \in K$ such that $|k\phi(x)| = 1$. If $\lambda = \Re(k\phi(x))$, then $w_\lambda(k\phi(x)) = 1$. In this case, $g(x)(k, \lambda) = f(x)$, so $\|\Im g(x)\| \ge \|\Im f(x)\|$. Furthermore, since w_λ, k, ϕ, π, and ρ are of real type, so is g. □

Lemma 5.2 implies the following proposition, similar to [L4, Proposition 4.2]:

Proposition 5.3. *Assume Hypothesis* 3.2, *and let* $2^5\beta < \alpha < 2^{-8}\mu$. *If* $N \in \mathbb{N}$, Z *is a complexified Banach space, and* $g \in \mathcal{O}_{\mathbb{R}}(X; Z)$, *then there are a complexified Banach space* W *and* $h \in \mathcal{O}_{\mathbb{R}}(P; W)$ *such that*

(i) $\|h(x)\|_W \le 1$ *if* $x \in P_N\langle\beta\rangle$, *and*
(ii) $\|\Im h(x)\|_W \ge \|\Im g(x)\|_Z$ *if* $x \in P_{N+1}\langle\alpha\rangle \setminus P_N\langle\alpha\rangle$.

Proof. Recall that $d(x) = \min(\text{dist}(x, \partial P), 1)$. Note that since $2^5\beta < \alpha$, if $r_1 = 4\beta d, r_2 = \alpha d/4$, then $2r_1 < r_2$. Let $D_0 = \Omega \cap \pi_{N+1} X$,

$$p^N(s) = \max\left\{\frac{\pi_N s}{N}, \frac{1}{N d(s)}, \frac{\rho_N s}{d(s)}\right\}, \quad s \in D_0,$$

$$A_1 = \{s \in D_0 : p^N(s) \le 4\beta\}, \quad A_2 = \{s \in D_0 : p^N(s) < \alpha/4\},$$

$$A_i = D_{N+1}\langle\alpha_i\rangle, \quad i = 3, 4.$$

In [L4], it was shown that

$$P_N\langle\beta\rangle \subset A_1[r_1], \quad A_2(r_2) \subset P_N\langle\alpha\rangle,$$
$$A_i(r_i) = P_{N+1}\langle\alpha\rangle, \quad i = 3, 4,$$

and also that $A_1 \subset A_2$ is plurisubharmonically convex in A_4, $\overline{A_2} \subset A_3$, and $\overline{A_3} \subset A_4$. The approximation property assumed in Lemma 5.2 follows from Theorem 4.2. Therefore, Lemma 5.2 applies in this case, and yields an $h \in \mathcal{O}(P, W)$ such that $\|h\|_W \leq 1$ on $A_1[r_1] \supset P_N\langle\beta\rangle$, and $\|h\|_W \geq \|g\|_Z$ on

$$A_3(r_3) \setminus A_2(r_2) \supset P_{N+1}\langle\alpha\rangle \setminus P_N\langle\alpha\rangle.$$

□

Proposition 5.4. *Let X, V be complexified Banach spaces such that Hypothesis* 3.2 *holds for X, and let $B = B(a; r) \subset X$ satisfy $\overline{B} \cap \overline{conj(B)} = \overline{B}$ or $\emptyset$. If $U = B \cup conj(B)$ and $f \in \mathcal{O}_{\mathbb{R}}(U, V)$, then there is a complexified Banach space W and a function $g \in \mathcal{O}_{\mathbb{R}}(X, W)$ such that $\|\Im f\| \leq \|\Im g\|$ on $B(a; \mu r/2)$.*

Proof. Case 1: $B = \text{conj}(B)$. Then $a \in X_{\mathbb{R}}$.
By Hypothesis 3.2, there is an $h \in \mathcal{O}(X, V)$ such that $\|f - h\| \leq 1$ on $B(a; \mu r)$. In fact, replacing h by h' as defined in Remark 5.1, we can assume $h \in \mathcal{O}_{\mathbb{R}}(X, V)$. It follows that $f - h$ and $\Im(f - h)$ are Lipschitz on the ball $B(a; \mu r/2)$, with some Lipschitz constant M. Let $W = V \oplus X$ with the sup norm, and $g(x) = 2h(x) \oplus 2Mx$. Then for any $x \in B(a; \mu r/2)$,

$$\begin{aligned}\|\Im f(x)\| &\leq \|\Im h(x)\| + \|\Im(f(x) - h(x))\| \\ &\leq \|\Im h(x)\| + \|\Im\{(f(x) - h(x)) - (f(\Re(x)) - h(\Re(x)))\}\| \\ &\leq \|\Im h(x)\| + M\|\Im x\| \leq \|\Im g(x)\|.\end{aligned}$$

Case 2: $\overline{B} \cap \overline{\text{conj}(B)} = \emptyset$ (and therefore $\|\Im a\| - r > 0$).
By Hypothesis 3.2, there is a function $g \in \mathcal{O}(X, V \oplus X)$ such that

$$\left\| g - \left(2f \oplus \frac{2x}{\|\Im a\| - r}\right)\right\| < 1 \qquad \text{on } B(a; \mu r).$$

After replacing g with g' as in Remark 5.1, we can assume $g \in \mathcal{O}_{\mathbb{R}}(X, V \oplus \mathbb{C})$. Whenever $\|\Im f(x)\| \geq 1$, we have

$$\|\Im g(x)\| > 2\|\Im f(x)\| - 1 \geq \|\Im f(x)\|.$$

Whenever $\|\Im f(x)\| < 1$, we have

$$\|\Im g(x)\| > 2\|\Im x\|/(\|\Im a\| - r) - 1 \geq 1 > \|\Im f(x)\|$$

on $B(a; \mu r)$. □

Now we are ready to prove Proposition 3.5 (and then Theorems 3.4 and 1.3 will be fully proved).

Proof of Proposition 3.5. With μ as in Hypothesis 3.2, let $0 < 2^8\alpha < \mu^2 < 1$. By Proposition 4.3, each $P_N\langle\alpha\rangle, N = 1, 2, \ldots$, has a finite cover $\{B_k\}_{k=1}^n$, $B_k = B(x_k, r_k)$, such that $B(x_k; 2r_k/\mu) \in \mathfrak{B}_P$. For each k, there are a complexified Banach space V_k and a

$$g_k \in \mathcal{O}_{\mathbb{R}}(B(x_k; 2r_k/\mu) \cup \text{conj}(B(x_k; 2r_k/\mu)), V_k)$$

such that $u \leq \|\Im f_k\|$ on $B(x_k; 2r_k/\mu) \cup \mathrm{conj}(B(x_k; 2r_k/\mu))$. By Proposition 5.4, there are a complexified Banach space E_k and an $h_k \in \mathcal{O}_{\mathbb{R}}(X, E_k)$ such that $\|\Im h_k\| \geq \|\Im g_k\|$ on B_k. Let $W_N = \overline{\bigoplus} E_k$, and let $\phi_N \in \mathcal{O}_{\mathbb{R}}(X, W_N)$ be defined by $\phi_N(x) = (h_1(x), h_2(x), \ldots, h_n(x))$. Then $\|\Im \phi_N\| \geq u$ on $P_N\langle\alpha\rangle$. By Proposition 5.3, there is a complexified Banach space Z_N and $f_N \in \mathcal{O}_{\mathbb{R}}(P, Z_N)$ such that

(i) $\|f_N(x)\| \leq 1$ if $x \in P_N\langle\beta\rangle$, and
(ii) $\|\Im f_N(x)\| \geq \|\Im \phi_{N+1}(x)\|$ if $x \in P_{N+1}\langle\alpha\rangle \setminus P_N\langle\alpha\rangle$.

Now we can take $Z_0 = W_1$, $f_0 = \phi_1$, $Y = \overline{\bigoplus}_{N=0}^{\infty} Z_N$, and $f = (f_0, f_1, f_2, \ldots)$. By Proposition 4.1, for each $x \in P$ there is a neighborhood of x contained in some $P_N\langle\alpha\rangle$. Therefore the sequence $\{f_N\}$ is locally bounded, so $f \in \mathcal{O}_{\mathbb{R}}(P, Y)$, and $\|\Im f\| \geq u$. □

6. Acyclicity

In order to prove Theorem 1.2, we will require the following technical topological proposition, whose proof is similar to that of [C, Proposition 2].

Proposition 6.1. *Let X be a paracompact Hausdorff space, $A \subset X$ a closed subset, $\{U_i\}_{i\in I}$ a relatively open cover of A, and for each $i \in I$, $\tilde{U}_i \subset X$ a neighborhood of U_i. Let q be a positive integer. For each $I' \subset I$ of cardinality at most q, let a neighborhood $\tilde{U}_{I'} \subset \bigcap_{i\in I'} \tilde{U}_i$ of $\bigcap_{i\in I'} U_i$ be given. Then there are a neighborhood P of A, a function $\sigma : J \to I$, and an open cover $\{V_j\}_{j\in J}$ of A such that $\overline{V}_j \subset \tilde{U}_{\sigma j}$ and $\bigcap_{j\in J'} \overline{V}_j \cap P \subset \tilde{U}_{\sigma J'}$ for all $J' \subset J$ of cardinality at most q.*

Proof. By passing to a refinement, we may assume that $\{\tilde{U}_i\}$ is locally finite. Choose any locally finite open cover $\{V_j\}_{j\in J}$ of A and $\sigma : J \to I$ so that $\overline{V}_j \subset \tilde{U}_{\sigma j}$. Then for each $x \in A$ there is a neighborhood $P(x)$ of x such that

(i) $P(x) \cap V_j \neq \emptyset$ if and only if $x \in \overline{V}_j$, and
(ii) $P(x)$ is contained in all $\tilde{U}_{I'}$ containing x (of which there are finitely many).

Now define P as the union of all $P(x)$. Given $J' \subset J$ of cardinality at most q, and $y \in \bigcap_{j\in J'} \overline{V}_j \cap P$, we must show that $y \in U_{\sigma J'}$. Since $y \in P$, $y \in P(x)$ for some $x \in A$. But $P(x) \cap V_j \neq \emptyset$ for all $j \in J'$, so $x \in \bigcap_{j\in J'} \overline{V}_j \cap A \subset \tilde{U}_{\sigma J'}$. Therefore, $P(x) \subset \tilde{U}_{\sigma J'}$. □

Lemma 6.2. *Let $X_{\mathbb{R}}$ be a real Banach space, $\Omega \subset X_{\mathbb{R}}$ open, and let $F \to \Omega$ be a real analytic Banach bundle. There exist a neighborhood $W \subset X$ of Ω and a holomorphic Banach bundle $E \to W$ whose restriction to Ω is the complexification $F \otimes \mathbb{C}$ of F.*

Proof. We can assume that Ω is connected, in which case all fibers of F are isomorphic. As in the finite dimensional case, F is determined up to isomorphism by an open cover $\{U_j\}$ together with a real Banach space $Y_{\mathbb{R}}$ and real analytic transition functions

$$g_{ij} : U_i \cap U_j \to \mathrm{End}(Y_{\mathbb{R}})$$

satisfying

(i) $g_{ij}(x)g_{ji}(x) = \mathrm{id}_{Y_{\mathbb{R}}}$ for $x \in U_i \cap U_j$, and
(ii) $g_{ij}(x)g_{jk}(x)g_{ki}(x) = \mathrm{id}_{Y_{\mathbb{R}}}$ for $x \in U_i \cap U_j \cap U_k$.

It is not hard to see that the complexification of $\mathrm{End}(Y_{\mathbb{R}})$ can be naturally identified with $\mathrm{End}(Y)$. By the definition of real analytic functions, each g_{ij} can be holomorphically extended to a function $\tilde{g}_{ij}$ on a neighborhood $\tilde{U}_{ij} \subset X$ of $U_i \cap U_j$. We can take $\tilde{U}_{ij}$ so small that $\tilde{g}_{ij}(x)\tilde{g}_{ji}(x) = \mathrm{id}_Y$ on $\tilde{U}_{ij}$. Further, choose neighborhoods $\tilde{U}_{ijk}$ of $U_i \cap U_j \cap U_k$ so that $\tilde{g}_{ij}(x)\tilde{g}_{jk}(x)\tilde{g}_{ki}(x) = \mathrm{id}_Y$ for $x \in \tilde{U}_i \cap \tilde{U}_j \cap \tilde{U}_k$. If Ω is an open subset of $X_{\mathbb{R}}$, then by Proposition 6.1, after replacing $\{U_j\}$ with a refinement, we can find a neighborhood $V_j \subset X$ of each U_j such that $V_i \cap V_j \subset \tilde{U}_{ij}$. Then g_{ij} extends to $V_i \cap V_j$ holomorphically. The holomorphic extensions $\tilde{g}_{ij}$ define a holomorphic vector bundle E on $W = \bigcup V_i$, whose restriction to Ω is the complexification $F \otimes \mathbb{C}$ of F. □

We will now prove Theorem 1.2; in fact, we will prove the following more general version:

Theorem 6.3. *Let $X_{\mathbb{R}}$ be a real Banach space with a Schauder basis satisfying Hypothesis* 3.2*, let $\Omega \subset X_{\mathbb{R}}$ be open, and let $F \to \Omega$ be a real analytic Banach bundle. Then $H^p(\Omega, F) = 0$ for $p \geq 1$.*

Proof. Let $\mathfrak{U}$ be an open cover of Ω. Consider a real analytic p-cocycle $c \in C^p(\mathfrak{U}, F)$, $p \geq 1$. We wish to show that, after sufficient refinement of $\mathfrak{U}$, c becomes a coboundary. We accomplish this by complexification. It can be assumed that F is trivial over each $U \in \mathfrak{U}$. Let $E \to W$ be as in Lemma 6.2.

We extend each component of c on Ω to some holomorphic section of E over some neighborhood $\tilde{U}_{I'} \subset X$ of each $p+1$-fold intersection $\bigcap_{i \in I'} U_i$, and construct the corresponding neighborhood P of Ω and open cover $\mathfrak{V} = \{V_j\}$, as in Proposition 6.1. In view of Theorem 1.3, we can take P and each V_j to be pseudoconvex.

This enables us to apply the following theorem from [L5]:

Theorem 6.4. *Suppose X is a Banach space with a Schauder basis and Hypothesis* 3.2 *holds. If $P \subset X$ is open and pseudoconvex, $E \to P$ a locally trivial holomorphic Banach bundle, and $q \geq 1$, then $H^q(P, E) = 0$.*

This implies that we can find a holomorphic cochain $b \in C^{p-1}(\mathfrak{V}, E)$ whose coboundary is the extension of c. Taking fiberwise the real part of $b|_{\Omega}$ we obtain a $b' \in C^{p-1}(\mathfrak{U}, F)$ with $\delta b' = c$. □

Theorem 6.3 can be applied in the case where F is the Banach bundle $\mathcal{A}_{0,q}$ of real analytic $(0,q)$-forms with values in the real analytic bundle $E \to \Omega$. This yields

$$H^p(\Omega, \mathcal{A}_{0,q}) = 0, \qquad p \geq 1, \qquad q \geq 0.$$

Theorem 1.1 follows, again in greater generality:

Theorem 6.5. *If X is a complex Banach space with a Schauder basis, and satisfies Hypothesis* 3.2*, then*

$$H^q(\Omega, E) \approx \frac{\mathit{Ker}\ \{\overline{\partial} : \mathcal{A}_{0,q}(\Omega) \to \mathcal{A}_{0,q+1}(\Omega)\}}{\mathit{Im}\ \{\overline{\partial} : \mathcal{A}_{0,q-1}(\Omega) \to \mathcal{A}_{0,q}(\Omega)\}}.$$

Proof. First, recall the local solvability of the $\overline{\partial}$−equation mentioned in Section 1 and proved in [L1, Proposition 3.2]. This means that

$$0 \to \mathcal{O}^E \to \mathcal{A}_{0,0} \xrightarrow{\overline{\partial}} \mathcal{A}_{0,1} \xrightarrow{\overline{\partial}} \dots$$

is an exact sequence of sheaves of abelian groups. Second, use the above special case of the cohomology vanishing theorem. These are the two ingredients required in the hypothesis of the abstract de Rham Theorem (see, for example, [W, Theorem 3.13]). The isomorphism theorem follows at once from this. □

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