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RANDOM GAPS

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Abstract. It is proved that there exists an (ω_1, ω_1) Souslin gap in the Boolean algebra $(L^0(\nu)/\operatorname{Fin}, \subseteq^*_{\mathrm{ae}})$ for every nonseparable measure ν. Thus a Souslin, also known as destructible, (ω_1, ω_1) gap in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ can always be constructed from uncountably many random reals. We explain how to obtain the corresponding conclusion from the hypothesis that Lebesgue measure can be extended to all subsets of the real line (RVM).

1. Introduction

A *pregap* in a Boolean algebra $(\mathcal{B}, \leq)$ is an orthogonal pair (A, B) of subsets of $\mathcal{B}$, i.e.

(i) $a \wedge b = 0$ for all $a \in A$ and $b \in B$,

and it is a *gap* if additionally there is no element c of $\mathcal{B}$ such that

(ii) $a < c$ for all $a \in A$, and $b < -c$ for all $c \in B$.

Such an element c is said to *interpolate* the pregap. A *linear pregap* is a pregap (A, B) where both A and B are linearly ordered by $\leq$, and for a pair of linear order types (φ, ψ), a (φ, ψ) *pregap* in a Boolean algebra $(\mathcal{B}, \leq)$ is a linear pregap (A, B) where $\operatorname{otp}(A, \leq) = \varphi$ and $\operatorname{otp}(B, \leq) = \psi$. Thus (A, B) is a (φ, ψ) *gap* if it is a (φ, ψ) pregap for which no element of $\mathcal{B}$ can be used to extend (A, B) to a $(\varphi + 1, \psi)$ pregap or a $(\varphi, \psi + 1)$ pregap.

We let $\mathbb{N}$ denote the set $\{0, 1, 2, \dots\}$ of nonnegative integers, and $\mathcal{P}(\mathbb{N})$ the power set of $\mathbb{N}$ quasi-ordered by $a \subseteq^* b$ if $a \setminus b$ is finite. $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ denotes the equivalence classes of $\mathcal{P}(\mathbb{N})$ modulo the equivalence relation of finite set difference, with the induced partial ordering $[a] \subseteq^* [b]$ if $a \subseteq^* b$. One can find results on gaps in the Boolean algebra $(\mathcal{P}(\mathbb{N})/\operatorname{Fin}, \subseteq^*)$ dating back to the second half of the 19th century, including a basic result of du Bois-Reymond [2] appearing in 1873, and Hadamard's Theorem [12] that there are no (ω, ω) gaps in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. Indeed, one of the major achievements in early Set Theory was Hausdorff's construction [14] of an (ω_1, ω_1) gap in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ (he actually first constructed in his 1909 paper [13] an (ω_1, ω_1)

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gap in a different structure, that has a simple translation to a gap in $\mathcal{P}(\mathbb{N}) / \mathrm{Fin}$). See [25] for the history of gaps.

1.1. **Destructibility.** While being a pregap in $\mathcal{P}(\mathbb{N}) / \mathrm{Fin}$ is absolute between any "reasonable" models (e.g., transitive models of some large enough fragment of ZF), the property of being a gap is not. For example, if (A, B) is an (ω_1, ω_1) gap in $\mathcal{P}(\mathbb{N}) / \mathrm{Fin}$ and $\mathcal{Q}$ is a poset which collapses $\aleph_1$, then by Hadamard's theorem, which generalizes to any limit ordinal of countable cofinality, forcing with $\mathcal{Q}$ must introduce an element of $\mathcal{P}(\mathbb{N}) / \mathrm{Fin}$ which interpolates (A, B) and thus renders it a nongap. Avoiding this particular example, an (ω_1, ω_1) pregap (A, B) in $\mathcal{P}(\mathbb{N}) / \mathrm{Fin}$ is called *destructible* if there is an $\aleph_1$ preserving poset which forces that (A, B) is not a gap.

The statement "all (ω_1, ω_1) gaps in $\mathcal{P}(\mathbb{N}) / \mathrm{Fin}$ are indestructible" is in fact a Ramsey-theoretic statement which is closely analogous to Souslin's Hypothesis. This becomes clear when one considers the characterization of destructibility in Theorem 1.3 below. We credit Theorems 1.1 and 1.3 to Kunen [22] and Woodin [33], respectively. The elegant Ramsey-theoretic presentation is due to Todorčević [30]. See, e.g., [25], [30] and [29] for proofs.

When working with pregaps in $\mathcal{P}(\mathbb{N}) / \mathrm{Fin}$ one often works with representatives (i.e. subsets of $\mathbb{N}$) of the equivalence classes. With every pair of families $a_i, b_i \subseteq \mathbb{N}$ indexed by $i \in I$ we associate a partition $[I]^2 = K_0 \cup K_1$ via

$$\{i, j\} \in K_0 \quad \text{iff} \quad (a_i \cap b_j) \cup (a_j \cap b_i) = \emptyset. \tag{1}$$

In the case where (the equivalence classes of) $(a_i, b_i : i \in I)$ is a pregap in $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$, we may assume (in order to avoid trivialities and thereby obtain more concise results) that the representatives have been chosen so that

$$a_i \cap b_i = \emptyset \quad \text{for all } i \in I. \tag{$\bigstar$}$$

The following theorem characterizes (ω_1, ω_1) gaps in $\mathcal{P}(\mathbb{N}) / \mathrm{Fin}$.

Theorem 1.1 (Kunen)**.** *For every (ω_1, ω_1) pregap in $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ with representatives chosen satisfying condition $\bigstar$ the following are equivalent.*
- (a) *$(a_\alpha, b_\alpha : \alpha < \omega_1)$ is a gap.*
- (b) *There is no uncountable 0-homogeneous subset of ω_1.*
- (c) *The poset of all finite 1-homogeneous subsets of ω_1 has the ccc.*

Remark 1.2. Only the case $I = \omega_1$ was considered because in general the theorem is false, although for (κ, κ) pregaps with κ regular and uncountable, it does generalize by replacing "uncountable" with "cardinality κ" and "ccc" with "κ-cc".

By switching the colors 0 and 1 one obtains a characterization of destructibility.

Theorem 1.3 (Woodin)**.** *For every (ω_1, ω_1) pregap in $\mathcal{P}(\mathbb{N}) / \mathrm{Fin}$ with representatives satisfying $\bigstar$ the following are equivalent.*
- (a) *$(a_\alpha, b_\alpha : \alpha < \omega_1)$ is destructible.*
- (b) *There is no uncountable 1-homogeneous subset of ω_1.*
- (c) *The poset of all finite 0-homogeneous subsets of ω_1 has the ccc.*

Considering condition (b) of Theorem 1.1, condition (c) of Theorem 1.3 says that there is a poset with the ccc forcing that $(a_\alpha, b_\alpha : \alpha < \omega_1)$ is not a gap. A Souslin tree itself has the ccc and forces an ω_1-branch through itself, which is analogous to forcing an interpolation of a gap, and condition (b) of Theorem 1.3 can be viewed in

Table 1. Souslin's Hypothesis

	Measurable algebra		
Axiom	**trivial**	**separable**	**nonseparable**
$\diamondsuit$	False	False	False
CH	Undecided	Undecided	Undecided
ZFC	Undecided	Undecided	Undecided
$(*)$	True	True	True
$\mathrm{MA}_{\aleph_1}$	True	True	True

Table 2. All gaps are indestructible

	Measurable algebra		
Axiom	**trivial**	**separable**	**nonseparable**
$\diamondsuit$	False	False	False
CH	Undecided	False	False
ZFC	Undecided	Undecided	False
$(*)$	True	Undecided	False
$\mathrm{MA}_{\aleph_1}$	True	True	False

analogy with the property that Souslin trees have no uncountable antichains. With this in mind, we provisionally refer to destructible and *Souslin* (ω_1, ω_1) pregaps interchangeably, until further definitions are made in Section 1.4.

Remark 1.4. It is a theorem of Kunen [22] (see also e.g. Scheepers [25]) that for cardinals κ and λ, every (κ, λ) pregap in $\mathcal{P}(\mathbb{N}) / \mathrm{Fin}$ with either $\kappa \neq \omega_1$ or $\lambda \neq \omega_1$ can be interpolated by a ccc poset. E.g. when $\kappa = \lambda \neq \omega_1$, the corresponding condition Theorem 1.3(c) (i.e. replace "ω_1" with "κ") holds for any such pregap.

The analogy goes further when one considers the influence of four additional set theoretic axioms: the existence of a diamond sequence ($\diamondsuit$); the Continuum Hypothesis (CH); the principle $(*)$ for ideals of countable subsets of ω_1, a consequence of PFA which is consistent with CH and entails many combinatorial consequences of PFA (see [1,31]); and Martin's Axiom for $\aleph_1$ many dense subsets ($\mathrm{MA}_{\aleph_1}$). Jensen proved that $\diamondsuit$ implies the existence of a Souslin tree, while Todorčević proved in Dow's paper [5] (he took credit for this in a private communication) that $\diamondsuit$ implies the existence of a destructible (ω_1, ω_1) gap. It is a theorem of Jensen ([4]) that SH is consistent with CH, while Abraham and Todorčević [1] proved that $(*)$ implies both SH and "all (ω_1, ω_1) gaps are indestructible" (and in particular "all (ω_1, ω_1) gaps are indestructible" is consistent with CH). It is a theorem of Solovay and Tennenbaum [28] that $\mathrm{MA}_{\aleph_1}$ implies SH, while it a theorem of Kunen [22] that $\mathrm{MA}_{\aleph_1}$ implies that there are no Souslin (ω_1, ω_1) gaps in $\mathcal{P}(\mathbb{N}) / \mathrm{Fin}$. These results are summarized in the **trivial** column of Tables 1 and 2.

The analogy can be carried still further by considering the influence of Cohen forcing. It is a theorem of Shelah [27] that Cohen forcing (i.e. adding one Cohen real) always produces a Souslin tree, while it is a theorem of Todorčević (see [29]) that Cohen forcing produces a Souslin gap.

This naturally leads us at once to consider the influence of the other fundamental forcing notion, random forcing: a separable measurable algebra (adding one random real), or more generally forcing with an arbitrary measurable algebra (possibly nonseparable, adding uncountably many random reals).

Let us recall here that a Boolean algebra $\mathcal{B}$ is *measurable* if there is a function $\mu : \mathcal{B} \to [0, \infty)$ such that $(\mathcal{B}, \mu)$ is a measure algebra. The measure μ has a naturally associated metric d_μ on $\mathcal{B}$, where the distance between a and b is $d_\mu(a, b) = \mu(a \triangle b)$. The measure algebra is *(non)separable* if the metric topology on $\mathcal{B}$ is (non)separable. Note that given a measurable algebra $\mathcal{B}$, separability is independent of the choice of measure μ; this follows from Maharam's Theorem, which is discussed below.

The influence on Souslin's Hypothesis of forcing with some measurable algebra over a model satisfying any of the above axioms is known and summarized in Table 1. It is a theorem of Laver [24] that under $\mathrm{MA}_{\aleph_1}$, forcing with any measurable algebra preserves SH, and it is a theorem of the author [20] that assuming $(*)$, forcing with any measurable algebra preserves SH. Note that the other rows follow from these results because Souslin trees are preserved by forcing notions satisfying property K (i.e. Knaster's chain condition), and in particular by any measurable algebra.

It is a result of the author [17] that under $\mathrm{MA}_{\aleph_1}$, all gaps are indestructible in any forcing extension by a separable measurable algebra. In [19] the author proves that under CH, adding one random real produces a Souslin gap (the CH row and **separable** column of Table 2); and this is the first place where the analogy breaks down, proving that: *Souslin's Hypothesis is consistent with the existence of a Souslin* (ω_1, ω_1) *gap in* $\mathcal{P}(\mathbb{N}) / \mathrm{Fin}$. The following relatively old question of Woodin has received the attention of a number of authors.

Question 1 (Woodin)**.** Does $\mathrm{MA}_{\aleph_1}$ imply that all (ω_1, ω_1) gaps in $\mathcal{P}(\mathbb{N}) / \mathrm{Fin}$ are indestructible in any forcing extension by a measurable algebra?

The main result of this paper, the "False" in the ZFC row of the **nonseparable** column of Table 2, answers Question 1 negatively and it completes the picture by allowing us to fill in the remainder of Table 2:

Theorem 1. *Let* $\mathcal{R}$ *be a nonseparable measurable algebra. Then some condition in* $\mathcal{R}^+$ *forces that there exists a Souslin* (ω_1, ω_1) *gap in* $\mathcal{P}(\mathbb{N}) / \mathrm{Fin}$.

1.2. **Terminology.** We have been calling $(\mathcal{B}, \leq)$ a Boolean algebra to indicate that $(\mathcal{B}, \leq)$ is a partial ordering with minimum and maximum elements 0 and 1, resp., such that every two elements $x, y \in \mathcal{B}$ have both an infimum $x \wedge y$ and a supremum $x \vee y$ and each element also has a complement $-x = \bigvee\{z \in \mathcal{B} : x \wedge z = 0\}$.[1] This agrees with the usual definition of a Boolean algebra as a structure of the form $(\mathcal{B}, \wedge, \vee, -, 0, 1)$, in that $(\mathcal{B}, \leq)$, with $x \leq y$ defined by $x - y = 0$, has the above properties iff this structure satisfies the axioms of a Boolean algebra.

Recall that a *measure space* is a triple (X, Σ, ν) where Σ is a σ-algebra of subsets of X consisting of the ν*-measurable sets*, $\nu : \Sigma \to [0, \infty]$ is a function with $\nu(\emptyset) = 0$ and

$$\nu\left(\bigcup_{n=0}^{\infty} E_n\right) = \sum_{n=0}^{\infty} \nu(E_n) \tag{2}$$

whenever $(E_n : n \in \mathbb{N})$ is a sequence of pairwise disjoint ν-measurable sets. A *measure algebra* is a pair $(\mathcal{B}, \mu)$ where $\mathcal{B}$ is a σ-complete Boolean algebra and

[1] We use the notation $x - y$ to abbreviate $x \wedge (-y)$.

$\mu : \mathcal{B} \to [0, \infty]$ is a function with $\mu(0) = 0$ and $\mu\big(\bigvee_{n=0}^{\infty} a_n\big) = \sum_{n=0}^{\infty} \mu(a_n)$ whenever $a_m \wedge a_n = 0$ for all $m \neq n$. A *probability space* is a measure space (X, Σ, ν) with $\nu(X) = 1$; similarly, a *probability algebra* is a measure algebra $(\mathcal{R}, \mu)$ with $\mu(1) = 1$. A *σ-finite* measure space is a measure space with ν-measurable sets $(E_n : n \in \mathbb{N})$ such that $X = \bigcup_{n=0}^{\infty} E_n$ and $\nu(E_n) < \infty$ for all n; a σ-finite measure algebra is defined analogously.

For a measure space (X, ν) we write $\mathcal{N}_\nu$ for the ideal of all $A \subseteq X$ with $\nu(A) = 0$. These sets are called *null* (or *negligible*). The *measure algebra of a measure space* (X, Σ, ν) is the quotient $\Sigma \,/\, \mathcal{N}_\nu$ with the well-defined measure $\mu([E]) = \nu(E)$.

We recall some further definitions we shall need. The *additivity* of a measure algebra $\operatorname{add}(\mu)$ is the smallest cardinal κ for which there is a pairwise incompatible family $\mathcal{A} \subseteq \mathcal{R}$ of cardinality κ such that the equality

$$\mu\left(\bigvee \mathcal{A}\right) = \bigvee_{a \in \mathcal{A}} \mu(a) \tag{3}$$

fails. Thus $\operatorname{add}(\mu) \geq \aleph_1$. A similar definition can be made for a measure space. Note that the additivity of the ideal $\mathcal{N}_\nu$ satisfies $\operatorname{add}(\mathcal{N}_\nu) = \operatorname{add}(\nu)$. For some cardinal κ, we say that a measure ν is *κ-additive* to indicate that $\operatorname{add}(\nu) \geq \kappa$. A measurable set $E \subseteq X$ is called an *atom* if it is not null and every $F \subseteq E$ with $\mu(F) < \mu(E)$ is null; similarly, an *atom* can be defined for a nonzero element of a measure algebra.

1.3. **Applications.**

1.3.1. *Souslin trees versus gaps.* An immediate consequence is a simple construction[2] of a model satisfying SH which also has a destructible gap:

Corollary 1.5. *Adding uncountably many random reals to any model of* ZFC *satisfying* $\mathrm{MA}_{\aleph_1}$ *gives a model of "there are no Souslin trees" and "there is a Souslin* (ω_1, ω_1) *gap in* $(\mathcal{P}(\mathbb{N}) \,/\, \mathrm{Fin}, \subseteq^*)$*".*

1.3.2. *Consequences of* RVM. Another corollary is that the classical hypothesis that the Lebesgue measure can be extended to all subsets of the real line $\mathbb{R}$, implies the existence of a destructible (ω_1, ω_1) gap in $\mathcal{P}(\mathbb{N}) \,/\, \mathrm{Fin}$. This is an immediate consequence of Theorem 1 and known absoluteness results for forcing extensions by a large enough measure algebra, from a real-valued measurable cardinal. We thankfully acknowledge Stevo Todorčević for suggesting Corollary 1.9 (in May 2002).

This absoluteness is in fact an axiomatization of random forcing. We state a theorem (Theorem 1.7) to this effect, without proof, some variation of which is folklore. It says that if a statement of reasonable complexity is forced to hold in the extension by a large enough measurable algebra, then this statement is a consequence of the axiom that the Lebesgue measure extends to all subsets of the real line. This axiom is often called RVM.

We will not give the proof here, or even a full explanation of the terminology. For a complete proof, and also a variation of Theorem 1.7, with other aspects of the mathematical background explained, we refer the reader to the supplement to this paper [21]; and for further reading we also suggest the paper [3] by Caicedo. Recall that an *atomlessly measurable cardinal* κ is an uncountable cardinal carrying an

[2]The point is that it is relatively difficult to construct a model of $(*)$ and CH, and this consistency result was originally proved by adding one random real to such a model.

atomless κ-additive probability measure with domain $\mathcal{P}(\kappa)$. It is a classical theorem of Ulam [32] that RVM is equivalent to the existence of an atomlessly measurable cardinal κ, and that κ is larger than the least weakly inaccessible cardinal but $\kappa \leq 2^{\aleph_0}$.

Notation 1.6. For a set X, we write $(\mathcal{R}_{(X)}, \mu_{(X)})$ for the measure algebra of the measure space $\{0,1\}^X$ with its Haar probability measure.

A word on Maharam's Theorem is also in order here. It states roughly that every σ-finite measure algebra (cf. §1.2), up to an isomorphism, has a simple decomposition into measure algebras of the form $(\mathcal{R}_{(\theta)}, \mu_{(\theta)})$, where θ is some cardinal. A measurable algebra (cf. §1.1) is *homogeneous* in the forcing sense iff it is isomorphic to $\mathcal{R}_{(\theta)}$ for some cardinal θ. The *Maharam type* of such a measurable algebra is the cardinal θ. By Maharam's Theorem, every σ-finite measurable algebra $\mathcal{R}$ has a dense set of $z \in \mathcal{R}^+$ such that $\mathcal{R}_z = \{x \in \mathcal{R} : x \leq z\}$ is homogeneous. Note that a homogeneous measurable algebra is nonseparable iff its Maharam type is at least $\aleph_1$. For more information on Maharam's Theorem, see also e.g. [8, 18].

Theorem 1.7. *Suppose κ is an atomlessly measurable cardinal. If $\varphi(x,y)$ is a Π_1 formula and $a \in H_\kappa$ is a parameter such that every homogeneous measurable algebra of large enough Maharam type forces the statement*

$$\exists x \in H_{\check{\kappa}^+} \ \varphi(x, \check{a}), \tag{4}$$

then $V \models \ulcorner \exists x\, \varphi(x,a) \urcorner$.

Remark 1.8. There is an important theorem of Gitik and Shelah [10] (see also [7]) giving a lower bound on the Maharam type of the measure algebras associated with atomlessly measurable cardinals: *Let X be a set. If $\nu : \mathcal{P}(X) \to [0,1]$ is an atomless probability measure, then the measure algebra $\mathcal{P}(X) / \mathcal{N}_\nu$ has Maharam type at least* $\min\{\operatorname{add}(\nu)^{+\omega}, 2^{\operatorname{add}(\nu)}\}$. (Moreover they proved in [11] that the Maharam type is $2^{2^{\aleph_0}}$ in the case $X = 2^{\aleph_0}$ and $\operatorname{add}(\nu) = 2^{\aleph_0}$.) The "large enough Maharam type" in Theorem 1.7 is the Maharam type of $\mathcal{P}(\kappa) / \mathcal{N}_\nu$, and thus is at least $\min\{\kappa^{+\omega}, 2^\kappa\}$.

Corollary 1.9 (ZFC + RVM). *There exists a Souslin (ω_1, ω_1) gap in $\mathcal{P}(\mathbb{N}) / \mathrm{Fin}$.*

Proof. Using Woodin's Theorem (Theorem 1.3), it is routine to obtain a Π_1 formula $\varphi(x,y,a)$ so that $\varphi(x,y,\omega_1)$ holds iff (x,y) forms a Souslin (ω_1,ω_1) gap in $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$. By the assumption that the Lebesgue measure on the real line can be extended to a measure whose domain is all of $\mathcal{P}(\mathbb{R})$, there exists an atomlessly measurable cardinal κ. Since (ω_1,ω_1) gaps are objects of $H_{\aleph_2}$, by Theorem 1, every homogeneous measure algebra of Maharam type at least $\aleph_1$ forces $\ulcorner \exists x, y \in H_{\check{\kappa}^+} \ \varphi(x,y,\check{\omega}_1) \urcorner$. Therefore, from Theorem 1.7 we conclude that there exists (x,y) satisfying $\varphi(x,y,\omega_1)$, completing the proof. □

1.4. **Souslin gaps.** If (A,B) is a gap, not necessarily linear, where the cardinalities of A and B are both at most $\aleph_1$, then forcing with a poset which collapses $\aleph_1$ will still interpolate the gap. Hence, it makes sense to extend the definition of *destructible* to include this situation.

Broadening our scope to include arbitrary pregaps in $\mathcal{P}(\mathbb{N}) / \mathrm{Fin}$, Theorem 1.3 is no longer true. In condition (c), the poset of finite 0-homogeneous subsets will force an interpolation of an uncountable subgap, but if the gap is nonlinear it will not necessarily interpolate the whole gap. We would like to isolate this condition (c), as the Souslin property.

Definition 1.10. A pregap (A, B) in $\mathcal{P}(\mathbb{N}) / \mathrm{Fin}$ is called *Souslin* if every uncountable family $\mathcal{F}$ of finite subsets of $A \times B$ (i.e. finite subpregaps of (A, B)) has an $\aleph_1$ preserving forcing extension where there exists $d \in \mathcal{P}(\mathbb{N}) / \mathrm{Fin}$ such that

$$\{F \in \mathcal{F} : d \text{ interpolates } F\} \text{ is uncountable.} \tag{5}$$

Note we need only consider families $\mathcal{F}$ of size $\aleph_1$, and that by a Δ-system argument we can assume $\mathcal{F}$ is pairwise disjoint.

This Souslin property of gaps can be characterized in terms of representative subsets of $\mathbb{N}$. For this it will be convenient to reconsider the requirement $\bigstar$. Suppose $(a_i, b_i : i \in I)$ is an indexing of two sequences of members of $\mathcal{P}(\mathbb{N})$. Instead of partitioning the subsets of cardinality two $[I]^2$ as in (1), we can define partitions of the pairs $I^2 = L_0^k \cup L_1^k$ and generalize by adding a parameter $k \in \mathbb{N}$, as in

$$(i, j) \in L_0^k \quad \text{iff} \quad (a_i \cap b_j \setminus k) \cup (a_j \cap b_i \setminus k) = \emptyset, \tag{6}$$

and we write L_0 and L_1 for L_0^0 and L_1^0, respectively. Then the condition $\bigstar$ is equivalent to "$L_0(i, i)$ for all $i \in I$", and assuming this is satisfied, a subset of I is K_0-homogeneous iff it is L_0-homogeneous. Notice that the quantification in the next theorem is significantly different than in Theorems 1.1 and 1.3.

Theorem 1.11. *Let (A, B) be a pregap in $\mathcal{P}(\mathbb{N}) / \mathrm{Fin}$. Then the following are equivalent.*

(a) *(A, B) is Souslin.*
(b) *For every choice $(a_i, b_i : i \in I)$ of representatives the associated poset of all finite L_0-homogeneous subsets of I has the ccc.*
(c) *There exists an indexing $(a_i, b_i : i \in I)$ of representatives such that for every k the associated poset of all finite L_0^k-homogeneous subsets of I has the ccc.*

Proof. (a)→(b): Suppose that (A, B) is Souslin and $(a_i, b_i : i \in I)$ is a choice of representatives. Let $J_\alpha \in \mathrm{Fin}_I$ $(\alpha < \omega_1)$ be a family of 0-homogeneous finite subsets of I. By going to an uncountable subset we can assume that they are all the same size, say $J_\alpha = \{i_0^\alpha, \dots, i_{n-1}^\alpha\}$ for all α. Now go to an $\aleph_1$ preserving forcing extension with $d \subseteq \mathbb{N}$ such that

$$X = \big\{\alpha < \omega_1 : d \text{ interpolates } \{(a_i, b_i) : i \in J_\alpha\}\big\} \text{ is uncountable.} \tag{7}$$

By going to an uncountable subset of X we can assume there is a $k \in \mathbb{N}$ such that $a_i \setminus k \subseteq d$ and $b_i \setminus k \subseteq d^{\mathsf{c}}$ for all $i \in J_\alpha$ for all $\alpha \in X$. Finally, we can pick $\alpha \neq \beta$ in X such that both $\{(a_i, b_i) : i \in J_\alpha\}$ and $\{(a_i, b_i) : i \in J_\beta\}$ have the same trace on k. Then $(J_\alpha \cup J_\beta)^2 \subseteq K_0$ because for all $l, m = 0, \dots, n-1$,

$$\begin{aligned} a_{i_l^\alpha} \cap b_{i_m^\beta} &= (a_{i_l^\alpha} \cap b_{i_m^\beta} \cap k) \cup (a_{i_l^\alpha} \cap b_{i_m^\beta} \setminus k) \\ &= a_{i_l^\alpha} \cap b_{i_m^\beta} \cap k \cup \emptyset \\ &= a_{i_l^\alpha} \cap b_{i_m^\alpha} \cap k \\ &= \emptyset, \end{aligned} \tag{8}$$

as $a_{i_l^\alpha} \cap b_{i_m^\alpha} = \emptyset$ for all l, m by homogeneity.

(b)→(c): Take *any* indexing $(a_i, b_i : i \in I)$ of representatives. Suppose $\mathcal{F}$ is an uncountable family of finite L_0^k-homogeneous subsets of I for some fixed $k \in \mathbb{N}$. Applying condition (b) to the indexing $(a_i \setminus k, b_i \setminus k : i \in I)$ of representatives, since each $F \in \mathcal{F}$ is 0-homogeneous for the associated partition, it follows that there exist $F \neq G$ in $\mathcal{F}$ such that $(F \cup G)^2 \subseteq L_0^k$.

(c)→(a): Suppose that $(a_i, b_i : i \in I)$ is an indexing of representatives satisfying the hypothesis of (c). For each $F \in \mathcal{F}$ choose a finite $J_F \subseteq I$ such that $F \subseteq \{a_i : i \in J_F\} \times \{b_i : i \in J_F\}$, and then find $k_F \in \mathbb{N}$ large enough so that

$$J_F^2 \subseteq L_0^{k_F}. \tag{9}$$

Pick $k \in \mathbb{N}$ large enough so that $\mathcal{F}_0 = \{F \in \mathcal{F} : k_F = k\}$ is uncountable. It follows from condition (c) that the poset of all finite $\mathcal{H} \subseteq \mathcal{F}_0$ such that $\bigcup_{F \in \mathcal{H}} J_F$ is L_0^k-homogeneous has the ccc, and therefore it can have only countably many atoms and thus forces an uncountable subset $\mathcal{G} \subseteq \mathcal{F}_0$, each member of which is interpolated by $\bigcup_{F \in \mathcal{G}} \bigcup_{i \in J_F} a_i$. □

Pregaps have been called *Luzin* in the literature (e.g. [6]) when there exists an uncountable K_1-homogeneous subset of I for some indexing of representatives satisfying ★. By Theorem 1.11 (a)→(b), every Souslin pregap in $\mathcal{P}(\mathbb{N}) / \mathrm{Fin}$ is non-Luzin, and thus for pregaps of size at most $\aleph_1$,

$$\text{destructible} \rightarrow \text{Souslin} \rightarrow \text{non-Luzin}. \tag{10}$$

In the realm of linear gaps, Souslin gaps are precisely the non-Luzin ones, and for linear gaps of size $\aleph_1$ all three notions coincide by Theorem 1.3. It may be worth revisiting this concept in a separate article.

1.5. **Gaps in** $(L^0(\nu)/\mathrm{Fin}, \subseteq^*_{\mathrm{ae}})$**.** Let $(\mathcal{R}, \mu)$ be the measure algebra of some σ-finite measure space (X, Σ, ν). In considering $\mathcal{R}$-names for gaps in $\mathcal{P}(\mathbb{N}) / \mathrm{Fin}$, we are led to consider gaps in another Boolean algebra. Recall that a *random variable on X with codomain S*, where S is some topological space, is an almost everywhere defined function f, measurable for the completion of μ, taking values in S; in other words, f is defined on a conegligible subset of X and there is a conegligible $C \subseteq X$ such that for every open $U \subseteq S$, $f^{-1}[U] \cap C \in \Sigma$. The family of all equivalence classes of random variables on X with codomain $\mathbb{R}$ over the equivalence relation $f(x) = g(x)$ almost everywhere, is the standard space $L^0(\nu)$ from functional analysis. We are interested here in taking $\mathcal{P}(\mathbb{N})$ to be our reals; hence, we define $L^0(\nu) / \mathrm{Fin}$ to be the family of all equivalence classes of random variables on X with codomain $\mathcal{P}(\mathbb{N})$ modulo the relation that $f(x) \triangle g(x)$ is finite almost everywhere, and order $L^0(\nu) / \mathrm{Fin}$ by defining $[f] \subseteq^*_{\text{æ}} [g]$ if $f(x) \subseteq^* g(x)$ almost everywhere. This clearly defines a Boolean algebra. By a *random gap* we mean a (pre)gap in a Boolean algebra of this form.

We have seen that Maharam's Theorem says that for purposes of forcing, we can restrict our attention to measure algebras of the form $(\mathcal{R}_{(\theta)}, \mu_{(\theta)})$ for some cardinal θ. It is well known (e.g. [26]) that there is a direct correspondence between $\mathcal{R}_{(\theta)}$-names $\dot{a}$ for a subset of $\mathbb{N}$ and random variables f on the underlying measure space $\{0,1\}^\theta$ with codomain $\mathcal{P}(\mathbb{N})$, via

$$\left\| f(\dot{r}) = \dot{a} \right\| = 1, \tag{11}$$

where $\dot{r}$ is a name for the generic object in $\{0,1\}^\theta$. Of course f must be interpreted correctly in the forcing extension to make sense of (11), but the point is that every measurable function $f : \{0,1\}^\theta \to \mathcal{P}(\mathbb{N})$ is equal almost everywhere to a

Baire function[3] $g : \{0,1\}^\theta \to \mathcal{P}(\mathbb{N})$, and every such Baire function can be coded by a countable sequence of ordinals so that every suitable model containing this sequence has a correct interpretation of g as a Baire function from $\{0,1\}^\theta$ into $\mathcal{P}(\mathbb{N})$. Henceforth, we will dot random variables (e.g. $\dot{a}$) to emphasize this correspondence.

For two of these random variables $\dot{a}$ and $\dot{b}$, clearly $\|\dot{a}(\dot{r}) \subseteq^* \dot{b}(\dot{r})\| = 1$ iff $[\dot{a}] \subseteq^*_{æ} [\dot{b}]$ in $L^0(\mu_{(\theta)})$ / Fin. Thus a pregap in $(L^0(\mu_{(\theta)})\,/\,\mathrm{Fin}, \subseteq^*_{æ})$ is the same thing as an $\mathcal{R}_{(\theta)}$-name for a pregap in $(\mathcal{P}(\mathbb{N})\,/\,\mathrm{Fin}, \subseteq^*)$, and a pregap in $L^0(\mu_{(\theta)})$ / Fin is a gap iff when viewed as an $\mathcal{R}_{(\theta)}$-name for a pregap it is a gap in $\mathcal{P}(\mathbb{N})$ / Fin with positive probability. We also would like to extend the correspondence to the notion of Souslin gaps, keeping in mind Theorem 1.11.

Definition 1.12. A pregap (A, B) in $L^0(\nu)$ / Fin is *Souslin* if there is an enumeration $(\dot{a}_\alpha, \dot{b}_\alpha : \alpha < \omega_1)$ of representatives, so that for every sequence $(E_\xi, \Gamma_\xi : \xi < \omega_1)$ and every $k \in \mathbb{N}$, where $E_\xi \in \Sigma$ has positive finite measure, $\Gamma_\xi \subseteq \omega_1$ is finite, and

$$\mathrm{Pr}\left[\bigcup_{\alpha,\beta\in\Gamma_\xi} \dot{a}_\alpha \cap \dot{b}_\beta \setminus k = \emptyset \,\middle|\, E_\xi\right] = 1 \quad \text{for all } \xi \tag{12}$$

(equivalently, $\bigcup_{\alpha,\beta\in\Gamma_\xi} \dot{a}_\alpha(x) \cap \dot{b}_\beta(x) \setminus k = \emptyset$ almost everywhere in E_ξ), there exist $\xi \neq \eta$ such that $\nu(E_\xi \cap E_\eta) \neq 0$ and

$$\mathrm{Pr}\left[\bigcup_{\alpha\in\Gamma_\xi} \bigcup_{\beta\in\Gamma_\eta} (\dot{a}_\alpha \cap \dot{b}_\beta \setminus k) \cup (\dot{a}_\beta \cap \dot{b}_\alpha \setminus k) = \emptyset \,\middle|\, E_\xi \cap E_\eta\right] \neq 0. \tag{13}$$

Note that in the case where $\nu = \mu_{(\theta)}$, when identifying random variables $\dot{a}$ on $\{0,1\}^\theta$ with codomain $\mathcal{P}(\mathbb{N})$ with the $\mathcal{R}_{(\theta)}$-name $\dot{a}(\dot{r})$ for a subset of $\mathbb{N}$, then replacing each E_ξ with $x_\xi \in \mathcal{R}^+$, condition (12) becomes

$$x_\xi \wedge \bigvee_{\alpha,\beta\in\Gamma_\xi} \|\dot{a}_\alpha \cap \dot{b}_\beta \setminus k \neq \emptyset\| = 0 \quad \text{for all } \xi, \tag{14}$$

and condition (13) becomes, with $x_\xi \wedge x_\eta \neq 0$,

$$x_\xi \wedge x_\eta - \bigvee_{\alpha\in\Gamma_\xi} \bigvee_{\beta\in\Gamma_\eta} \|(\dot{a}_\alpha \cap \dot{b}_\beta \setminus k) \cup (\dot{a}_\beta \cap \dot{b}_\alpha \setminus k) \neq \emptyset\| \neq 0. \tag{15}$$

Thus, by Theorem 1.11, a pregap in $L^0(\mu_{(\theta)})$ is Souslin iff it is forced to be a Souslin pregap with probability one when viewed as an $\mathcal{R}_{(\theta)}$-name.

For a measure space (X, Σ, ν), ν is called nonseparable if the measure algebra of the measure space is nonseparable. Thus we obtain the following equivalent formulation of Theorem 1.

Theorem 2. *There exists an (ω_1, ω_1) Souslin gap in $(L^0(\nu)\,/\,\mathrm{Fin}, \subseteq^*_{æ})$ whenever ν is a nonseparable σ-finite measure.*

Let us note that gaps in $L^0(\nu)$ / Fin are a generalization of gaps in $\mathcal{P}(\mathbb{N})$ / Fin, because if (X, Σ, ν) is a trivial measure space with one element of measure one, then $(L^0(\nu)\,/\,\mathrm{Fin}, \subseteq^*_{æ})$ and $(\mathcal{P}(\mathbb{N})\,/\,\mathrm{Fin}, \subseteq^*)$ are isomorphic. Finally, we notice that the existence of an (ω_1, ω_1) gap in $L^0(\nu)$ / Fin for any σ-finite measure ν, follows

[3] We mean Baire measurable function, where the Baire sets are members of the smallest σ-algebra for which every continuous function is measurable. This is the σ-algebra generated by the clopen sets in the case of a Cantor cube $\{0,1\}^X$.

from Hausdorff's Theorem on the existence of these gaps in $\mathcal{P}(\mathbb{N}) \,/\, \mathrm{Fin}$ applied in the appropriate random forcing extension.

1.5.1. *Destructible gaps in* $L^0(\nu)/\,\mathrm{Fin}$. The fact that there is no gap (A, B) in $\mathcal{P}(\mathbb{N}) \,/\, \mathrm{Fin}$ with both A and B countable generalizes to gaps in $L^0(\nu) \,/\, \mathrm{Fin}$ for any σ-finite measure ν. Therefore, an (ω_1, ω_1) gap in $L^0(\nu) \,/\, \mathrm{Fin}$ is interpolated by collapsing $\aleph_1$. Thus we can define a destructible (ω_1, ω_1) gap in $L^0(\nu) \,/\, \mathrm{Fin}$ as one which can be interpolated by a forcing which preserves $\aleph_1$. Note that, in $L^0(\nu) \,/\, \mathrm{Fin}$, destructible (ω_1, ω_1) gaps and Souslin (ω_1, ω_1) gaps are two different things! Indeed in proving Theorem 2 we shall construct a Souslin gap which is indestructible. On the other hand, in [19], an (ω_1, ω_1) gap in $L^0(\nu) \,/\, \mathrm{Fin}$ with ν separable, is constructed with the aid of the Continuum Hypothesis, and this gap is both destructible and Souslin; moreover, this gap can be destroyed by a poset with the property K, which cannot happen to a gap in $\mathcal{P}(\mathbb{N}) \,/\, \mathrm{Fin}$. It is not hard to prove that every destructible (ω_1, ω_1) gap in $L^0(\nu) \,/\, \mathrm{Fin}$ must be Souslin.

2. Measure-theoretic characterizations

There are some simple measure-theoretic characterizations, or more precisely necessary and/or sufficient conditions, of when an $\mathcal{R}$-name for a pregap names a Souslin gap and related phenomena. This is our explanation for the complex behavior observed in the interactions between gaps in $\mathcal{P}(\mathbb{N}) \,/\, \mathrm{Fin}$ and random forcing. For example, in [19] it is shown that for a separable probability algebra $(\mathcal{R}, \nu)$, an $\mathcal{R}$-name $(\dot{a}_\alpha, \dot{b}_\alpha : \alpha < \omega_1)$ for a pregap in $\mathcal{P}(\mathbb{N}) \,/\, \mathrm{Fin}$ is forced to be Souslin with positive probability if

$$\mu\big(\big\|\dot{a}_\alpha \cap \dot{b}_\beta \neq \emptyset\big\|\big) < \frac{1}{2} \quad \text{for all } \alpha, \beta < \omega_1 \tag{16}$$

(or less than some other constant below 1). The statement (16) is far simpler than the definition of destructibility or its Ramsey-theoretic characterization in Theorem 1.3. Note that unlike Theorem 1.3, the characterization in equation (16) is by no means equivalent to destructibility; in the case of the trivial measure algebra it implies that the pregap is a nongap.

We have also obtained in [16] a simple necessary condition for an $\mathcal{R}$-name for a pregap to satisfy the *Hausdorff property*, namely that the strengthening of indestructibility to $\{\alpha < \beta : a_\alpha \cap b_\beta \subseteq k\}$ is finite for all $k \in \mathbb{N}$, for all $\beta < \omega_1$, which was satisfied by Hausdorff's original construction of an (ω_1, ω_1) gap. It states that if the equation (32) holds for some $h \in c_0$ (i.e. $\lim_{n\to\infty} h(n) = 0$), then we do not have a name for a gap satisfying Hausdorff's condition. In fact, in [16] an $\mathcal{R}$-name (with $\mathcal{R}$ nonseparable) for an indestructible gap not satisfying Hausdorff's condition is constructed by satisfying (32) for an $h \in \ell^1$ (i.e. $\sum_{n=0}^{\infty} h(n) < \infty$). Therefore, the characterization of the Souslin property in Theorem 3 is not valid with the hypothesis (32) alone.

We were unable to find a sufficient condition for the Souslin property with an arbitrary measurable algebra purely by putting constraints on the measures of various events. This lead us to probabilistic considerations in our goal to obtain a characterization of Souslin gaps in $L^0(\nu) \,/\, \mathrm{Fin}$ (this is achieved in Theorem 3 below).

Recall that a family $\{\mathcal{S}_i : i \in I\}$ of subsets of some measure algebra $(\mathcal{R}, \mu)$ is *stochastically independent* if $\mu(x \wedge y) = \mu(x)\mu(y)$ for all $x \in \mathcal{S}_i$ and $y \in \mathcal{S}_j$, for all $i \neq j$ in I. Two elements $x, y \in \mathcal{R}$ are called stochastically independent when $\big\{\{x\}, \{y\}\big\}$ is a stochastically independent family.

The following basic result of probability theory is not used for the characterization of a Souslin gap, but is needed later on in the actual construction of an (ω_1, ω_1) Souslin gap in $L^0(\nu) \,/\, \mathrm{Fin}$ (a proof can be found in [21]).

Lemma 2.1. *Let $\big\{\{x_i, y_i\} : i \in I\big\}$ be a stochastically independent family such that $x_i \wedge y_i = 0$ for all $i \in I$. Then for all $A, B \subseteq I$, $\mu\big(\bigvee_{i\in A} x_i \wedge \bigvee_{i \in B} y_i\big) \leq \mu\big(\bigvee_{i \in A} x_i\big)\mu\big(\bigvee_{i\in B} y_i\big)$ with equality iff either $A \cap B = \emptyset$ or $x_i = 1$ or $y_i = 1$ for some $i \in A \cup B$. In other words, $\bigvee_{i\in A} x_i$ is* unfavourable *for $\bigvee_{i\in B} y_i$.*

Recall that for $A \subseteq X$, a subset S of the Cantor cube $\{0,1\}^X$ is called *determined by coordinates in A* or *A-determined* if for all $y, z \in \{0,1\}^X$,

$$y \restriction A = z \restriction A \qquad \text{implies} \qquad y \in S \quad \text{iff} \quad z \in S. \tag{17}$$

Also, $x \in \mathcal{R}_{(X)}$ is *determined by coordinates in A* or *A-determined* if x is a member of the natural identification of $\mathcal{R}_{(A)}$ with a subalgebra of $\mathcal{R}_{(X)}$ (cf. Notation 1.6). Thus $x \in \mathcal{R}_{(X)}$ is A-determined iff it has an A-determined representative $S \subseteq \{0,1\}^X$. The set of *coordinates determining* x is the minimum $A \subseteq X$ which determines x. Such an A always exists; indeed it is the set of all $\xi \in X$ such that

$$\pi_{\langle \xi, 0\rangle}(x \wedge \langle \xi, 0\rangle) \neq \pi_{\langle \xi, 1\rangle}(x \wedge \langle \xi, 1\rangle), \tag{18}$$

where $\pi_{\langle \xi, i\rangle} : (\mathcal{R}_{(X)})_{\langle \xi, i\rangle} \to \mathcal{R}_{(X \setminus \{\xi\})}$ is the natural isomorphism as in Corollary 2.3, and $\langle \xi, i\rangle$ denotes the element of the measure algebra represented by $\{z \in \{0,1\}^X : z(\xi) = i\}$. Recall that every member of $\mathcal{R}_{(X)}$ is represented by an F_σ Baire set, and as such is determined by countably many coordinates. The set of *coordinates determining* $\mathcal{S}$ where $\mathcal{S} \subseteq \mathcal{R}_{(X)}$ is the union of the coordinates that determine each member of $\mathcal{S}$.

We say that a collection $\{\mathcal{S}_i : i \in I\}$ of subsets of $\mathcal{R}_{(X)}$ is *independently determined* if there is a family $\{A_i : i \in I\}$ of pairwise disjoint subsets of X such that x is A_i-determined for all $x \in \mathcal{S}_i$; i.e., $\mathcal{S}_i \subseteq \mathcal{R}_{(A_i)}$, for all $i \in I$. We also say that two elements $x, y \in \mathcal{R}_{(X)}$ are *independently determined* if the family $\big\{\{x\}, \{y\}\big\}$ is independently determined, and similarly we can say that x and $\mathcal{S} \subseteq \mathcal{R}_{(X)}$ are *independently determined* to indicate that $\big\{\{x\}, \mathcal{S}\big\}$ is independently determined.

Note that if $\mathcal{F}$ is an independently determined family of subsets, then $\mathcal{F}$ is stochastically independent; however, conversely, two stochastically independent members of the measure algebra may fail to be disjointly determined.

For a finite partial function $s : X \dashrightarrow \{0,1\}$, we let $[s] = [s]_{(X)} \in \mathcal{R}_{(X)}$ be the equivalence class of the basic clopen set

$$\big\{z \in \{0,1\}^X : z \supseteq s\big\}. \tag{19}$$

In some contexts, $[s]$ will denote the clopen set (19) instead. Let $\mathrm{Fin}(X, \{0,1\})$ denote the collection of all finite partial functions from X into $\{0,1\}$.

Lemma 2.2. *Let $(\mathcal{R}, \mu) = (\mathcal{R}_{(\theta)}, \mu_{(\theta)})$ for some infinite cardinal θ, and let $t \in \mathrm{Fin}(\theta, \{0,1\})$. Then every $x \in \mathcal{R}_{[t]}$ has a unique $\theta \setminus \mathrm{dom}(t)$-determined $y \in \mathcal{R}$ such that $x = y \wedge [t]$.*

Corollary 2.3. *There exists a unique measure algebra isomorphism $\pi : \mathcal{R}_{[t]} \to \mathcal{R}_{(\theta \setminus (\mathrm{dom}(t))}$ such that $\pi([s]) = [s \setminus t]$ for all $s \in \mathrm{Fin}(\theta, \{0,1\})$ compatible with t.*

Proof. Let $\pi(x)$ be the identification of the element of $\mathcal{R}$ given by Lemma 2.2 with an element of $\mathcal{R}_{(\theta\setminus(\mathrm{dom}(t))}$. □

Maharam's Theorem gives an isomorphism between $\mathcal{R}_{[t]}$ and $\mathcal{R}_{(\theta\setminus\mathrm{dom}(t))}$, but the point of the preceding corollary was to identify the natural one.

2.1. **A chain condition for conditional probabilities.** There is a classical chain condition for probability algebras due to Gillis [9], which entails that every uncountable subset $\mathcal{A}$ of a probability algebra, in which every element has measure greater than some δ, contains an uncountable subset $\mathcal{B} \subseteq \mathcal{A}$ where $\mu(x \wedge y) > \delta^2$ for every two elements x and y in $\mathcal{B}$. What we would like here is an analogue for conditional probabilities:

$$\mu(x \mid y) = \frac{\mu(x \wedge y)}{\mu(y)}; \tag{20}$$

i.e., for uncountable sequences $x_\alpha, y_\alpha \in \mathcal{R}^+$ $(\alpha < \omega_1)$ such that $\mu(x_\alpha \mid y_\alpha) > \delta$ for all α, there exists an uncountable $X \subseteq \omega_1$ such that $\mu(x_\alpha \wedge x_\beta \mid y_\alpha \wedge y_\beta) > \delta^2$ for all $\alpha \in X$. Although this is false as stated (cf. [21]), if the y_β's are of a simple enough form, then it becomes valid (Theorem 2.4).

Theorem 2.4. *Let $(\mathcal{R}, \mu) = (\mathcal{R}_{(\theta)}, \mu_{(\theta)})$ for some cardinal θ. If $\mathcal{A}$ is an uncountable family of pairs (x, s) where $x \in \mathcal{R}$, $s \in \mathrm{Fin}(\theta, \{0,1\})$, and $\mu(x \mid [s]) > \delta$ for all $(x, s) \in \mathcal{A}$, then there is an uncountable $\mathcal{B} \subseteq \mathcal{A}$ such that $\mu(x \wedge y \mid [s] \wedge [t]) > \delta^2$ for all $(x, s), (y, t) \in \mathcal{B}$.*

Proof. Let (x_α, s_α) $(\alpha < \omega_1)$ enumerate a subset of $\mathcal{A}$. Find an uncountable $X \subseteq \omega_1$ such that $\{\mathrm{dom}(s_\alpha) : \alpha \in X\}$ forms a Δ-system with root $\Gamma \subseteq \theta$, and

$$\begin{matrix} |s_\alpha \setminus \Gamma| = m \\ s_\alpha \restriction \Gamma = t \end{matrix} \quad \text{for all } \alpha \in X. \tag{21}$$

Denote $\mu_{(\theta\setminus\Gamma)}$ by ρ, and write $\pi = \pi_{[t]}$ for the isomorphism from Corollary 2.3. Since the scalar correction of μ to a probability measure on $\mathcal{R}_{[t]}$ is given by $\mu(\cdot)$ / $\mu([t])$, we have $\rho\big(\pi(x)\big) = \frac{\mu(x)}{\mu([t])}$ for all $x \in \mathcal{R}_{[t]}$. However, by cancellation, the scalar multiple of a measure has the same conditional probabilities as the original measure. Hence, it suffices to find an uncountable $X' \subseteq X$ such that

$$\rho\big(\pi(x_\alpha) \wedge \pi(x_\beta) \,\big|\, \pi([s_\alpha]) \wedge \pi([s_\beta])\big) > \delta^2 \quad \text{for all } \alpha, \beta \in X'. \tag{22}$$

Furthermore, the hypothesis translates to $\rho(\pi(x_\alpha \wedge [s_\alpha])) > \delta\rho(\pi([s_\alpha]))$ for all α. All of the $\rho(\pi([s_\alpha]))$'s for $\alpha \in X$ are equal to some fixed $\sigma > 0$; namely, $\sigma = 2^{-m}$ by (21). Thus by Gillis' Theorem, there is an uncountable $X' \subseteq X$ such that

$$\rho\big(\pi(x_\alpha \wedge [s_\alpha]) \wedge \pi(x_\beta \wedge [s_\beta])\big) > \delta^2\sigma^2 \quad \text{for all } \alpha, \beta \in X'. \tag{23}$$

But by the Δ-system construct, $\pi([s_\alpha]) = [s_\alpha \setminus t]$ is determined independently of $\pi([s_\beta]) = [s_\beta \setminus t]$ for all $\alpha \neq \beta$ in X, and thus by stochastic independence, $\rho(\pi([s_\alpha]) \wedge \pi([s_\beta])) = \sigma^2$. This establishes (22). □

2.2. **Continuous representatives.** In [15] we observed that every member of $L^0(\mu_{(\theta)})\,/\,\mathrm{Fin}$ has a continuous representative (actually the entire paper deals only with $\theta = \omega$, but the proof of this fact applies to arbitrary θ). Here we make additional specifications on the representatives. The existence of continuous representatives is essentially the fact that the collection of equivalence classes of clopen sets is dense in the metric topology on $\mathcal{R}_{(\theta)}$ (in other words, for every measurable set $A \subseteq \{0,1\}^\theta$ and every $\varepsilon > 0$ there exists a clopen $C \subseteq \{0,1\}^\theta$ such that $\mu(A \triangle C) < \varepsilon$).

Lemma 2.5. *For every pair of random variables $\dot a$,$\dot b$ on $\{0,1\}^\theta$ with codomain $\mathcal{P}(\mathbb{N})$, and every $h : \mathbb{N} \to [0,1]$ with*

$$\mu\big(\|n \in \dot a \cap \dot b\|\big) + h(n) > 0 \quad \text{for all } n, \tag{24}$$

there exist continuous functions $\dot c, \dot d : \{0,1\}^\theta \to \mathcal{P}(\mathbb{N})$ such that

(a) $\|\dot c \cap \dot d = \emptyset\| = 1$,

and for all n,

(b) $\|n \in \dot c\|$ *and* $\|n \in \dot d\|$ *are both determined by the set of coordinates which determines* $\big\{\|n \in \dot a\|, \|n \in \dot b\|\big\}$,
(c) $\mu\big(\|n \in \dot a \triangle \dot c\|\big) < \mu\big(\|n \in \dot a \cap \dot b\|\big) + h(n)$,
(d) $\mu\big(\|n \in \dot b \triangle \dot d\|\big) < \mu\big(\|n \in \dot a \cap \dot b\|\big) + h(n)$.

Proof. For each n: Let $A_n \subseteq \theta$ be the set of coordinates determining $\big\{\|n \in \dot a\|, \|n \in \dot b\|\big\}$. In case A_n is finite, $\|n \in \dot a\|$ and $\|n \in \dot b\|$ both have clopen representatives $C_n, D_n \subseteq \{0,1\}^\theta$, respectively. Otherwise, when A_n is infinite, working in the space $\{0,1\}^{A_n}$, we can find a measurable $E_n \subseteq \|n \in \dot a \cap \dot b\|$ which is A_n-determined with measure

$$\mu(E_n) = \frac{\mu\big(\|n \in \dot a \cap \dot b\|\big)}{2}. \tag{25}$$

Requirement (24) allows us to find A_n-determined clopen sets C_n and D'_n such that

$$\mu\Big(C_n \triangle \Big(\|n \in \dot a\| \setminus \big(\|n \in \dot b\| \setminus E_n\big)\Big)\Big) < \frac{\mu\big(\|n \in \dot a \cap \dot b\|\big) + h(n)}{6}, \tag{26}$$

$$\mu\Big(D'_n \triangle \big(\|n \in \dot b\| \setminus E_n\big)\Big) < \frac{\mu\big(\|n \in \dot a \cap \dot b\|\big) + h(n)}{6}; \tag{27}$$

i.e. (26) and (27) hold for any representatives of $\|n \in \dot a\|$ and $\|n \in \dot b\|$. Then since $\|n \in \dot a\| \cap \big(\|n \in \dot b\| \setminus E_n\big) = \|n \in \dot a \cap \dot b\| \setminus E_n$, which has measure $\mu\big(\|n \in \dot a \cap \dot b\|\big)\,/\,2$ by (25), recalling the metric d_μ (cf. page 22), the triangle inequality gives

$$\mu\big(C_n \triangle \|n \in \dot a\|\big) < \frac{2\mu\big(\|n \in \dot a \cap \dot b\|\big) + h(n)}{3}. \tag{28}$$

From (26) and (27) we have $\mu(C_n \cap D'_n) < \bigl[\mu\bigl(\|n \in \dot a \cap \dot b\|\bigr) + h(n)\bigr] \,/\, 3$, and thus letting $D_n = D'_n \setminus C_n$ yields

$$
\begin{aligned}
\mu\bigl(D_n \mathbin{\triangle} \|n \in \dot b\|\bigr) &\le \mu\bigl(D_n \mathbin{\triangle} \bigl(\|n \in \dot b\| \setminus E_n\bigr)\bigr) \\
&\qquad + \mu\bigl(\bigl(\|n \in \dot b\| \setminus E_n\bigr) \mathbin{\triangle} \|n \in \dot b\|\bigr) \\
&\le \mu(D_n \mathbin{\triangle} D'_n) + \mu\bigl(D'_n \mathbin{\triangle} \bigl(\|n \in \dot b\| \setminus E_n\bigr)\bigr) + \mu(E_n) \\
&< \left(\frac{1}{3} + \frac{1}{6} + \frac{1}{2}\right)\bigl[\mu\bigl(\|n \in \dot a \cap \dot b\|\bigr) + h(n)\bigr] \\
&= \mu\bigl(\|n \in \dot a \cap \dot b\|\bigr) + h(n),
\end{aligned} \tag{29}
$$

using the triangle inequality with d_μ.

Now the functions $\dot c, \dot d : \{0,1\}^\theta \to \mathcal{P}(\mathbb{N})$ given by $\dot c(x) = \{n : x \in C_n\}$ and $\dot d(x) = \{n : x \in D_n\}$ are continuous and as needed. $\square$

Remark 2.6. Note that if in Lemma 2.5, $\sum_{n=0}^\infty \mu\bigl(\|n \in \dot a \cap \dot b\|\bigr) < \infty$, then $\bigl\|[\dot a] = [\dot c]\bigr\| = 1$, i.e. $[\dot a] = [\dot c]$ in $L^0(\mu_{(\theta)}) \,/\, \mathrm{Fin}$, because

$$
\bigl\|[\dot a] \ne [\dot c]\bigr\| = \bigwedge_{k=0}^{\infty} \bigvee_{n=k}^{\infty} \|n \in \dot a \mathbin{\triangle} \dot c\|, \tag{30}
$$

and obviously $\bigl\|[\dot b] = [\dot d]\bigr\| = 1$ too.

2.3. **Characterization.** The following Theorem 3 identifies some Souslin random gaps.

Theorem 3. *Let $\mu = \mu_{(\theta)}$ for some cardinal θ. Suppose $\{\dot a_\alpha : \alpha < \omega_1\}$ and $\{\dot b_\alpha : \alpha < \omega_1\}$ are two families of representatives of members of $L^0(\mu) \,/\, \mathrm{Fin}$. If*

$$
\left\{\bigcup_{\alpha<\omega_1} \bigl\{\|n \in \dot a_\alpha\|,\ \|n \in \dot b_\alpha\|\bigr\} : n \in \mathbb{N}\right\} \text{ is independently determined,} \tag{31}
$$

and for some $h \in \ell^1$,

$$
\mu\bigl(\|n \in \dot a_\alpha \cap \dot b_\beta\|\bigr) \le h(n) \quad \text{for all } \alpha, \beta < \omega_1 \text{ and all } n \in \mathbb{N}, \tag{32}
$$

then $(\{[\dot a_\alpha] : \alpha < \omega_1\}, \{[\dot b_\alpha] : \alpha < \omega_1\})$ forms a Souslin pregap.

Note that condition (32) implies that $\{[\dot a_\alpha] : \alpha < \omega_1\}$ is orthogonal to $\{[\dot b_\alpha] : \alpha < \omega_1\}$ because $[\dot a_\alpha] \wedge [\dot b_\beta] = 0$ in $L^0(\mu) \,/\, \mathrm{Fin}$ is equivalent to $\bigl\|\dot a_\alpha \cap \dot b_\beta \text{ is infinite}\bigr\| = 0$, and

$$
\bigl\|\dot a_\alpha \cap \dot b_\beta \text{ is infinite}\bigr\| = \bigwedge_{k=0}^{\infty} \bigvee_{n=k}^{\infty} \|n \in \dot a_\alpha \cap \dot b_\beta\|. \tag{33}
$$

Condition (31) is a very demanding requirement. If we consider an (ω_1, ω_1) gap in $L^0(\mu) \,/\, \mathrm{Fin}$, i.e. a linear gap, then we can relax this to $\bigl(\bigl\{\|n \in \dot a_\alpha\|, \|n \in \dot b_\alpha\|\bigr\} : n \in \mathbb{N}\bigr)$ being an independently determined sequence for every α separately. We feel that such requirements are not very natural and that with the right probability theory (concerning countable collections of random variables) a natural characterization, say that $\bigl\{\bigl\{\|n \in \dot a_\alpha\|, \|n \in \dot b_\alpha\|\bigr\} : n \in \mathbb{N}\bigr\}$ is a stochastically independent family, or even some further weakening, is obtainable.

2.3.1. *Proof of Theorem* 3. The proof begins by choosing a sequence $(A^n : n \in \mathbb{N})$ of pairwise disjoint subsets of θ such that

$$\big\|n \in \dot{a}_\alpha\big\| \text{ and } \big\|n \in \dot{b}_\alpha\big\| \text{ are both } A^n\text{-determined} \quad \text{for all } n, \text{ for all } \alpha. \tag{34}$$

Without loss of generality assume that $h > 0$. Then for each α, we apply Lemma 2.5 to the pair $(\dot{a}_\alpha, \dot{b}_\alpha)$, with $h_\alpha(n) = h(n) - \mu(\|n \in \dot{a}_\alpha \cap \dot{b}_\alpha\|)$, to obtain a pair $(\dot{c}_\alpha, \dot{d}_\alpha)$ of continuous functions as in the conclusion of the lemma. Note that we have

$$\begin{aligned}\mu\big(\big\|n \in \dot{c}_\alpha \cap \dot{d}_\beta\big\|\big) \le \mu\big(\big\|n \in \dot{a}_\alpha \cap \dot{b}_\beta\big\|\big) + \mu\big(\big\|n \in \dot{c}_\alpha \setminus \dot{a}_\alpha\big\|\big)&\\ + \mu\big(\big\|n \in \dot{d}_\beta \setminus \dot{b}_\beta\big\|\big)&\\ < h(n) + \big(\mu\big(\big\|n \in \dot{a}_\alpha \cap \dot{b}_\alpha\big\|\big) + h_\alpha(n)\big)&\\ + \big(\mu\big(\big\|n \in \dot{a}_\beta \cap \dot{b}_\beta\big\|\big) + h_\beta(n)\big)&\\ = 3h(n)&\end{aligned} \tag{35}$$

for all n, for all $\alpha, \beta < \omega_1$. For each α, for each $n \in \mathbb{N}$, letting

$$y^n_\alpha = \big\|n \in \dot{c}_\alpha\big\| \quad \text{and} \quad z^n_\alpha = \big\|n \in \dot{d}_\alpha\big\|, \tag{36}$$

by continuity we can write $y^n_\alpha = \bigvee_{s \in C^n_\alpha}[s]$ and $z^n_\alpha = \bigvee_{s \in D^n_\alpha}[s]$ where $C^n_\alpha, D^n_\alpha \subseteq \operatorname{Fin}(A^n, \{0,1\})$ are both finite, using clause (b) of Lemma 2.5. Thus

$$B^n_\alpha = \bigcup_{s \in C^n_\alpha \cup D^n_\alpha} \operatorname{dom}(s) \subseteq A^n \tag{37}$$

is finite.

By Remark 2.6, it suffices to prove that $(\dot{c}_\alpha, \dot{d}_\alpha : \alpha < \omega_1)$ is Souslin. Suppose that $k \in \mathbb{N}$, $(x_\xi : \xi < \omega_1)$ is a sequence in $\mathcal{R}^+$ and $(\Gamma_\xi : \xi < \omega_1)$ is a sequence of finite subsets of ω_1 such that

$$x_\xi \wedge \bigvee_{\alpha, \beta \in \Gamma_\xi} \big\|\dot{c}_\alpha \cap \dot{d}_\beta \setminus k \neq \emptyset\big\| = 0 \quad \text{for all } \xi. \tag{38}$$

By going to an uncountable subsequence, we can assume that $|\Gamma_\xi| = m$ for all ξ.

Let $\varepsilon > 0$ be given. For each $\xi < \omega_1$, choose $s_\xi \in \operatorname{Fin}(\theta, \{0,1\})$ such that

$$\mu\big([s_\xi] \wedge x_\xi\big) > \sqrt{1 - \frac{\varepsilon}{2}} \cdot \mu([s_\xi]). \tag{39}$$

Since the A_n's are pairwise disjoint, if we choose a large enough $p_\xi \in \mathbb{N}$, then since $h \in \ell^1$ and by equation (37),

$$\sum_{n=p_\xi}^{\infty} h(n) < \frac{\varepsilon}{12m^2}, \tag{40}$$

$$\operatorname{dom}(s_\xi) \cap \bigcup_{\alpha \in \Gamma_\xi} \bigcup_{n=p_\xi}^{\infty} B^n_\alpha = \emptyset. \tag{41}$$

It clearly follows from (39) that there is a $t_\xi \supseteq s_\xi$ where

$$\operatorname{dom}(t_\xi) = \bigcup_{\alpha \in \Gamma_\xi} \bigcup_{n < p_\xi} B^n_\alpha, \tag{42}$$

$$\mu(x_\xi \mid [t_\xi]) > \sqrt{1 - \frac{\varepsilon}{2}}. \tag{43}$$

By going to an uncountable subset $X \subseteq \omega_1$, we arrange that $p_\xi = p$ for all $\xi \in X$. For each $\xi \in X$, put

$$\mathit{\Omega}_\xi = \bigcup_{\alpha \in \mathit{\Gamma}_\xi} \bigcup_{n<p} B^n_\alpha. \tag{44}$$

By going to an uncountable subsequence, we obtain $X' \subseteq X$ such that $\{\mathit{\Omega}_\xi : \xi \in X'\}$ forms a Δ-system, say with root $\mathit{\Omega}$. Let $(\gamma^i : i < o)$ be the strictly increasing enumeration of $\mathit{\Omega}$. By further refinement, we can furthermore assume that

$$\begin{aligned}(45)\quad &\big(\{(C^n_\alpha, D^n_\alpha : n < p) : \alpha \in \mathit{\Gamma}_\xi\}, t_\xi, (\gamma^i : i < o)\big)\\ &\qquad\qquad \cong \big(\{(C^n_\beta, D^n_\beta : n < p) : \beta \in \mathit{\Gamma}_\eta\}, t_\eta, (\gamma^i : i < o)\big)\end{aligned}$$

for all $\xi, \eta \in X'$, meaning that there is a finite partial injection g on ω_1 such that if every instance of each ordinal ζ appearing in the structure $\big(\{(C^n_\alpha, D^n_\alpha : n < p) : \alpha \in \mathit{\Gamma}_\xi\}, t_\xi, (\gamma^i : i < o)\big)$ is replaced with $g(\zeta)$, then $\big(\{(C^n_\beta, D^n_\beta : n < p) : \beta \in \mathit{\Gamma}_\eta\}, t_\eta, (\gamma^i : i < o)\big)$ is obtained; i.e. $\{(C_\beta, D_\beta : n < p) : \beta \in \mathit{\Gamma}_\eta\} = \big\{\big(\{s \circ g^{-1} : s \in C^n_\alpha\}, \{s \circ g^{-1} : s \in D^n_\alpha\} : n < p\big) : \alpha \in \mathit{\Gamma}_\xi\big\}$, $t_\eta = t_\xi \circ g^{-1}$ and $g(\gamma^i) = \gamma^i$ for all $i < o$.

Claim 2.7. $[t_\xi] \wedge [t_\eta] \wedge \big((y^n_\alpha \wedge z^n_\beta) \vee (y^n_\beta \wedge z^n_\alpha)\big) = 0$ for all $k \le n < p$, for all $\alpha \in \mathit{\Gamma}_\xi$ and all $\beta \in \mathit{\Gamma}_\eta$, for all $\xi, \eta \in X'$.

Proof. Fix $\xi, \eta \in X'$ and $k \le n < p$. For every $\alpha \in \mathit{\Gamma}_\xi$, since $\operatorname{dom}(t_\xi) \supseteq B^n_\alpha$, $[t_\xi] \wedge y^n_\alpha$ is either 0 or $[t_\xi]$, and $[t_\xi] \wedge z^n_\alpha$ is either 0 or $[t_\xi]$. Similarly, for every $\beta \in \mathit{\Gamma}_\eta$, $[t_\eta] \wedge y^n_\beta$ and $[t_\eta] \wedge z^n_\beta$ are both either 0 or $[t_\eta]$. Now fix $\alpha \in \mathit{\Gamma}_\xi$ and $\beta \in \mathit{\Gamma}_\eta$. The isomorphism witnessing (45) for ξ and η maps (C^n_δ, D^n_δ) to (C^n_β, D^n_β) for some $\delta \in \mathit{\Gamma}_\xi$, so that both

$$[t_\eta] \wedge y^n_\beta = 0 \quad \text{iff} \quad [t_\xi] \wedge y^n_\delta = 0, \tag{46}$$

$$[t_\eta] \wedge z^n_\beta = 0 \quad \text{iff} \quad [t_\xi] \wedge z^n_\delta = 0. \tag{47}$$

However, by (38), $x_\xi \wedge y^n_\alpha \wedge z^n_\delta = 0$ and $x_\xi \wedge y^n_\delta \wedge z^n_\alpha = 0$. Since $[t_\xi] \wedge x_\xi \ne 0$, this implies that at most one of $[t_\xi] \wedge y^n_\alpha$ and $[t_\xi] \wedge z^n_\delta$ is nonzero, and at most one of $[t_\xi] \wedge y^n_\delta$ and $[t_\xi] \wedge z^n_\alpha$ is nonzero. It follows that both $[t_\xi] \wedge y^n_\alpha \wedge [t_\eta] \wedge z^n_\beta = 0$ and $[t_\xi] \wedge z^n_\alpha \wedge [t_\eta] \wedge y^n_\beta = 0$, proving the claim. □

Note that t_ξ is compatible with t_η for all $\xi, \eta \in X'$ by (45), since $t_\zeta \subseteq \mathit{\Omega}_\zeta$ for all $\zeta \in X$ by (42) and (44).

Claim 2.8. $[t_\xi \cup t_\eta]$ and $\big\{\bigvee_{n=p}^\infty y^n_\alpha \wedge z^n_\beta, \bigvee_{n=p}^\infty y^n_\beta \wedge z^n_\alpha\big\}$ are independently determined for all $\alpha \in \mathit{\Gamma}_\xi$ and $\beta \in \mathit{\Gamma}_\eta$, for all ξ, η in X'.

Proof. By equation (42), $\operatorname{dom}(t_\xi)$ and $\operatorname{dom}(t_\eta)$ are both subsets of $A_0 \cup \cdots \cup A_{p-1}$. Hence $[t_\xi \cup t_\eta]$ is $A_0 \cup \cdots \cup A_{p-1}$-determined. On the other hand, y^n_α and z^n_α are both A_n-determined for all α, by equation (37). Thus both suprema are $A_p \cup A_{p+1} \cup \cdots$-determined. Since the A_n's are pairwise disjoint, the claim is proved. □

Now using (43) in Theorem 2.4, we obtain $\xi \ne \eta$ in X' such that

$$\mu\big([t_\xi \cup t_\eta] \wedge x_\xi \wedge x_\eta\big) > \Big(1 - \frac{\varepsilon}{2}\Big) \wedge \mu([t_\xi \cup t_\eta]). \tag{48}$$

For all $\alpha \in \Gamma_\xi$ and $\beta \in \Gamma_\eta$,

$$\begin{aligned}\mu\left([t_\xi \cup t_\eta] \wedge \bigvee_{n=k}^{\infty} y_\alpha^n \wedge z_\beta^n\right) &\le \mu\left([t_\xi \cup t_\eta] \wedge \bigvee_{n=k}^{p-1} y_\alpha^n \wedge z_\beta^n\right) \\ &\quad + \mu\left([t_\xi \cup t_\eta] \wedge \bigvee_{n=p}^{\infty} y_\alpha^n \wedge z_\beta^n\right) \\ &\le 0 + \mu([t_\xi \cup t_\eta]) \cdot \mu\left(\bigvee_{n=p}^{\infty} y_\alpha^n \wedge z_\beta^n\right) \\ &\le \mu\big([t_\xi \cup t_\eta]\big) \cdot \sum_{n=p}^{\infty} 3h(n) \\ &< \frac{\varepsilon \cdot \mu([t_\xi \cup t_\eta])}{4m^2},\end{aligned} \tag{49}$$

where Claims 2.7 and 2.8 are used for the second inequality, (35) for the third, and (40) is used for the fourth inequality; similarly,

$$\mu\left([t_\xi \cup t_\eta] \wedge \bigvee_{n=k}^{\infty} y_\beta^n \wedge z_\alpha^n\right) < \frac{\varepsilon \cdot \mu([t_\xi \cup t_\eta])}{4m^2}. \tag{50}$$

Thus the measure of $[t_\xi \cup t_\eta] \wedge \bigvee_{\alpha\in\Gamma_\xi} \bigvee_{\beta\in\Gamma_\eta} \bigvee_{n=k}^{\infty} (y_\alpha^n \wedge z_\beta^n) \vee (y_\beta^n \wedge z_\alpha^n)$ is less than $\frac{\varepsilon}{2} \cdot \mu([t_\xi \cup t_\eta])$, which with (48) tells us that

$$\mu\left([t_\xi \cup t_\eta] \wedge \left(x_\xi \wedge x_\eta - \bigvee_{\alpha\in\Gamma_\xi} \bigvee_{\beta\in\Gamma_\eta} \big\|(\dot c_\alpha \cap \dot d_\beta \setminus k) \cup (\dot c_\beta \cap \dot d_\alpha \setminus k) \ne \emptyset\big\|\right)\right) > (1-\varepsilon) \cdot \mu([t_\xi \cup t_\eta]), \tag{51}$$

and in particular the condition (15) is satisfied. This proves that $(\dot c_\alpha, \dot d_\alpha : \alpha < \omega_1)$ is Souslin, thereby completing the proof of Theorem 3.

3. (ω_1, ω_1) Souslin gap

We conclude the paper with a proof of Theorem 2, and thus Theorem 1, by constructing an (ω_1, ω_1) Souslin gap in $L^0(\mu)$ / Fin for μ a nonseparable measure. As we have seen, we can assume that $\mu = \mu_{(\theta)}$ for some uncountable cardinal θ. Write $\mathcal{R} = \mathcal{R}_{(\theta)}$.

Define a mapping $\phi : \theta \times \mathbb{N} \times \mathbb{Z} \to \mathrm{Fin}(\theta, \{0,1\})$, where $\mathrm{dom}(\phi(\alpha, i, j)) = [\alpha, \alpha + i)$ and the concatenation

$$\phi(\alpha,i,j)(\alpha+i-1)\phi(\alpha,i,j)(\alpha+i-2)\cdots\phi(\alpha,i,j)(\alpha) \tag{52}$$

is the base 2 representation of $j \mod 2^i$. This can be expressed equivalently as

$$\phi(\alpha,i,j)(\alpha+k) = \frac{j \mod 2^i}{2^k} \wedge 1 \quad \text{for all } k < i. \tag{53}$$

Define $s_\alpha, t_\alpha \in \mathrm{Fin}(\theta, \{0,1\})^{\mathbb{N}}$ by

$$s_\alpha(n) = \phi(\omega \cdot \alpha + n\lceil \log_2(n+1)\rceil, \lceil \log_2(n+2)\rceil, 0), \tag{54}$$

$$t_\alpha(n) = \phi(\omega \cdot \alpha + n\lceil \log_2(n+1)\rceil, \lceil \log_2(n+2)\rceil, 1). \tag{55}$$

Define random variables $\dot{c}_\alpha, \dot{d}_\alpha$ ($\alpha < \theta$) by

$$\big\|n \in \dot{c}_\alpha\big\| = [s_\alpha(n)], \tag{56}$$

$$\big\|n \in \dot{d}_\alpha\big\| = [t_\alpha(n)]. \tag{57}$$

First of all, note that since $\log_2(n+2) \geq 1$ for all n, the third parameter in ϕ ensures that

$$\big\|\dot{c}_\alpha \cap \dot{d}_\alpha = \emptyset\big\| = 1 \quad \text{for all } \alpha. \tag{58}$$

For all $\alpha \neq \beta$, by stochastic independence, $\mu\big(\big\|n \in \dot{c}_\alpha \cap \dot{d}_\beta\big\|\big) = 2^{-2\lceil \log_2(n+2)\rceil} \leq \frac{1}{(n+2)^2}$ for all n, and this sequence is summable, which implies that $[\dot{c}_\alpha] \wedge [\dot{d}_\beta] = 0$ in $L^0(\mu)\,/\,\mathrm{Fin}$ (i.e., $\dot{c}_\alpha \cap \dot{d}_\beta$ is finite with probability one).

On the other hand, the length $\lceil \log_2(n+2)\rceil$ of the basic elements is short enough so that $\mu\big(\big\|n \in \dot{c}_\alpha\big\|\big) = \mu\big(\big\|n \in \dot{d}_\alpha\big\|\big) = 2^{-\lceil \log_2(n+2)\rceil}$ and hence

$$\frac{1}{2n+4} < \mu\big(\big\|n \in \dot{c}_\alpha\big\|\big) = \mu\big(\big\|n \in \dot{d}_\alpha\big\|\big) \leq \frac{1}{n+2}, \tag{59}$$

which is used to prove the following.

Claim 3.1. With probability one, no subset of $\mathbb{N}$ interpolates the two families $\{\dot{c}_\alpha : \alpha < \theta\}$ and $\{\dot{d}_\alpha : \alpha < \theta\}$. In particular, they form a gap in $L^0(\mu)\,/\,\mathrm{Fin}$.

Proof. Suppose that $r \in \{0,1\}^\theta$ is an $\mathcal{R}$-generic object (over V). Suppose that $e \in V[r]$ is a subset of $\mathbb{N}$ such that $\dot{c}_\alpha[r] \subseteq^* e$ for all α. Choose a countable $J \subseteq \theta$ large enough so that $e \in V[r \restriction I]$, where $I = \bigcup_{\alpha \in J}[\omega \cdot \alpha, \omega \cdot \alpha + \omega)$. It follows from Kunen's Theorem [23] (stating that for any $K \subseteq \theta$ in V, $r \in \{0,1\}^\theta$ is an $\mathcal{R}_{(\theta)}$-generic object over V iff $r \restriction K$ is $\mathcal{R}_{(K)}$-generic over V and $r \restriction \theta \setminus K$ is $\mathcal{R}_{(\theta \setminus K)}$-generic over $V[r \restriction K]$) that $V[r]$ is a forcing extension of $V[r \restriction I]$ by the measure algebra $\mathcal{S} = \mathcal{R}_{(\theta\setminus I)}$ taken in $V[r \restriction I]$, and that for every $\alpha \notin J$, the $\mathcal{S}$-names for $\dot{c}_\alpha[r]$ and $\dot{d}_\alpha[r]$, have the same definitions as given in (56) and (57), respectively.

In $V[r \restriction I]$: Fix $\alpha \notin J$. Let $\mathcal{J}$ be the set of all $a \subseteq \mathbb{N}$ such that

$$\big\|a \cap \dot{c}_\alpha \text{ is finite}\big\| \neq 0. \tag{60}$$

Since

$$(n+1)\lceil \log_2(n+2)\rceil - n\lceil \log_2(n+1)\rceil \geq \lceil \log_2(n+2)\rceil \tag{61}$$

for all $n \in \mathbb{N}$, $\big\|n \in \dot{c}_\alpha\big\|$ ($n \in \mathbb{N}$) is a stochastically independent sequence, and thus by Cauchy's criterion for infinite products,

$$\begin{aligned} a \in \mathcal{J} \quad &\text{iff} \quad \bigvee_{k=0}^{\infty} \bigwedge_{n \in a\setminus k} -\big\|n \in \dot{c}_\alpha\big\| \neq 0 \\ &\text{iff} \quad \sup_{k \in \mathbb{N}} \prod_{n \in a \setminus k} 1 - \mu\big(\big\|n \in \dot{c}_\alpha\big\|\big) = 1 \\ &\text{iff} \quad \sum_{n \in a} \mu\big(\big\|n \in \dot{c}_\alpha\big\|\big) < \infty \\ &\text{iff} \quad \sum_{n \in a} (n+1)^{-1} < \infty \\ &\text{iff} \quad a \in \mathcal{I}_{\frac{1}{n}}, \end{aligned} \tag{62}$$

where $\mathcal{I}_{\frac{1}{n}}$ is the well-known analytic ideal of all subsets of $\mathbb{N}$ on which the function $1\,/\,(n+1)$ is summable. In particular, $\mathcal{J}$ is a proper ideal, which means that $\big\|\dot{c}_\alpha$ is infinite$\big\| = 1$.

Since $\big\|e \supseteq^* \dot{c}_\alpha\big\| \neq 0$, $e \notin \mathcal{J}$. However, $\mathcal{J}$ is also the ideal of all subsets $a \subseteq \mathbb{N}$ such that $\big\|a \cap \dot{d}_\alpha$ is finite$\big\| \neq 0$. Moreover, the calculation in (62) shows that in fact $\alpha \in \mathcal{J}$ iff $\big\|a \cap \dot{d}_\alpha$ is finite $\big\| = 1$. Hence the proof is complete, because we have shown that $e \cap \dot{d}_\alpha[r]$ is infinite. □

Notice that by (61), for every $n \in \mathbb{N}$, $\bigcup_{\alpha<\omega_1}\big\{\big\|n \in \dot{c}_\alpha\big\|, \big\|n \in \dot{d}_\alpha\big\|\big\}$ is determined by the coordinates $\bigcup_{\alpha<\omega_1}[\omega\cdot\alpha + n\lceil\log_2(n+1)\rceil, \omega\cdot\alpha + (n+1)\lceil\log_2(n+2)\rceil)$, and thus the families are independently determined for $m \neq n$, as condition (31) of Theorem 3 requires. Hence Theorem 3 entails that $(\dot{c}_\alpha, \dot{d}_\alpha : \alpha < \omega_1)$ is Souslin by letting $h \in \ell^1$ be given by $h(n) = \frac{1}{(n+2)^2}$, because for all $\alpha \neq \beta$, by equation (59) and stochastic independence,

$$\mu\big(\big\|n \in \dot{c}_\alpha \cap \dot{d}_\beta\big\|\big) = \mu\big(\big\|n \in \dot{c}_\alpha\big\|\big)\mu\big(\big\|n \in \dot{d}_\beta\big\|\big) \leq h(n), \tag{63}$$

which with (58) assures condition (32).

Now we build an (ω_1, ω_1) Souslin gap $(\dot{a}_\alpha, \dot{b}_\alpha : \alpha < \omega_1)$, where $[\dot{c}_\alpha] \subseteq^*_{\text{æ}} [\dot{a}_\alpha]$ and $[\dot{d}_\alpha] \subseteq^*_{\text{æ}} [\dot{b}_\alpha]$ in $L^0(\mu)\,/\,\mathrm{Fin}$ for all $\alpha < \omega_1$. Therefore, since $(\dot{c}_\alpha, \dot{d}_\alpha : \alpha < \omega_1)$ forms a gap by Claim 3.1, $(\dot{a}_\alpha, \dot{b}_\alpha : \alpha < \omega_1)$ will automatically be a gap. The idea is to limit the augmentation of the measures of $\big\|n \in \dot{c}_\alpha\big\|$ and $\big\|n \in \dot{d}_\alpha\big\|$ so that Theorem 3 still applies.

The (ω_1, ω_1) gap we have been striving towards is obtained from an "ascending tower" of sorts (especially conditions (i) and (ii) below). Namely, a sequence $T_\alpha \in \mathrm{Fin}(\omega_1)^{\mathbb{N}}$ $(\alpha < \omega_1)$, where $\mathrm{Fin}(\omega_1)$ denotes the collection of all finite subsets of ω_1, satisfying:

(i) $\alpha \in T_\alpha(n)$ for all n,
(ii) $T_\xi(n) \subseteq T_\alpha(n)$ for all but finitely many n, for all $\xi < \alpha$,
(iii) $|T_\alpha(n)| \leq \sqrt[3]{n+1}$ for all n.

Such a tower can easily be constructed by recursion on α, by adding the requirement

(iv) $\lim\limits_{n\to\infty} \dfrac{|T_\alpha(n)|}{\sqrt[3]{n+1}} = 0$

to carry the recursion through.

Now $(\dot{a}_\alpha, \dot{b}_\alpha : \alpha < \omega_1)$ is defined by

$$\big\|n \in \dot{a}_\alpha\big\| = \bigvee_{\xi\in T_\alpha(n)}[s_\xi(n)] \quad \text{and} \quad \big\|n \in \dot{b}_\alpha\big\| = \bigvee_{\xi\in T_\alpha(n)}[t_\xi(n)] \tag{64}$$

for all $n \in \mathbb{N}$, for all $\alpha < \omega_1$. Then in fact, $\big\|\dot{a}_\alpha \supseteq \dot{c}_\alpha\big\| = 1$ and $\big\|\dot{b}_\alpha \supseteq \dot{d}_\alpha\big\| = 1$ by (i), and with (ii) it follows that both $\{\dot{a}_\alpha : \alpha < \omega_1\}$ and $\{\dot{b}_\alpha : \alpha < \omega_1\}$ have order type ω_1. Applying Lemma 2.1 to the independently determined family $\big\{\{[s_\xi(n)], [t_\xi(n)]\} : \xi < \omega_1\big\}$, with both of the subsets A and B equal to $T_\alpha(n)$, and using (59) with condition (iii) yields

$$\begin{aligned} \mu(\|n \in \dot{a}_\alpha \cap \dot{b}_\beta\|) &\le \mu(\|n \in \dot{a}_\alpha\|)\mu(\|n \in \dot{b}_\beta\|) \\ &\le \left(\frac{\sqrt[3]{n+1}}{n+2}\right)^2 \\ &< (n+1)^{-\frac{4}{3}} \end{aligned} \tag{65}$$

for all $n \in \mathbb{N}$, for all $\alpha, \beta < \omega_1$. Since this extension of the original nonlinear gap still satisfies (31), Theorem 3 applies with the function in ℓ^1 given by[4]

$$n \mapsto (n+1)^{-\frac{4}{3}}. \tag{66}$$

Therefore $(\dot{a}_\alpha, \dot{b}_\alpha : \alpha < \omega_1)$ is an (ω_1, ω_1) Souslin gap in the Boolean algebra $(L^0(\mu) \,/\, \mathrm{Fin}, \subseteq^*_{æ})$, concluding the paper.

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