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NONLINEAR STABILITY OF RAREFACTION WAVES FOR THE COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH LARGE INITIAL PERTURBATION

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Abstract. The expansion waves for the compressible Navier-Stokes equations have recently been shown to be nonlinear stable. The nonlinear stability results are called local stability or global stability depending on whether the H^1-norm of the initial perturbation is small or not. Up to now, local stability results have been well established. However, for global stability, only partial results have been obtained. The main purpose of this paper is to study the global stability of rarefaction waves for the compressible Navier-Stokes equations. For this purpose, we introduce a positive parameter t_0 in the construction of smooth approximations of the rarefaction wave solutions for the compressible Euler equations so that the quantity $\ell = \frac{t_0}{\delta}$ (δ denotes the strength of the rarefaction waves) is sufficiently large to control the growth induced by the nonlinearity of the system and the interaction of waves from different families. Then by using the energy method together with the continuation argument, we obtain some nonlinear stability results provided that the initial perturbation satisfies certain growth conditions as $\ell \to +\infty$. Notice that the assumption that the quantity ℓ can be chosen to be sufficiently large implies that either the strength of the rarefaction waves is small or the rarefaction waves of different families are separated far enough initially.

1. Introduction

In this paper, we are concerned with the one-dimensional compressible Navier-Stokes equations in Lagrangian coordinates:

$$\left\{\begin{aligned} v_t - u_x &= 0, \\ u_t + p_x &= \left(\mu \frac{u_x}{v}\right)_x, \\ \left(e + \frac{u^2}{2}\right)_t + (up)_x &= \left(\kappa \frac{\theta_x}{v} + \mu \frac{u u_x}{v}\right)_x, \end{aligned}\right. \tag{1.1}$$

where the unknowns $v > 0, u$, $\theta > 0$, $p > 0$, e, and s represent the specific volume, the velocity, the absolute temperature, the pressure, the internal energy, and the entropy of the gas, respectively. The coefficients of viscosity and heat-conductivity,

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μ and κ, are assumed to be positive constants. We assume, as usual in thermodynamics, that by using any given two of the five thermodynamical variables, v, p, e, θ, and s, the remaining three variables are functions of them.

The second law of thermodynamics asserts that

$$\theta ds = de + pdv,$$

from which, if we choose (v,θ), (v,s), or (v,e) as independent variables and write $(p,e,s) = (p(v,\theta), e(v,\theta), s(v,\theta))$, or $(p,e,\theta) = (\tilde{p}(v,s), \tilde{e}(v,s), \tilde{\theta}(v,s))$, or $(p,s,\theta) = (\hat{p}(v,e), \hat{s}(v,e), \hat{\theta}(v,e))$, respectively, then we can deduce that

$$\begin{cases} s_v(v,\theta) = p_\theta(v,\theta), \\ s_\theta(v,\theta) = \frac{e_\theta(v,\theta)}{\theta}, \\ e_v(v,\theta) = \theta p_\theta(v,\theta) - p(v,\theta), \end{cases} \tag{1.2}$$

$$\begin{cases} \tilde{e}_v(v,s) = -p(v,\theta), & \tilde{e}_s(v,s) = \theta, \\ \tilde{p}_v(v,s) = p_v(v,\theta) - \frac{\theta(p_\theta(v,\theta))^2}{e_\theta(v,\theta)}, & \tilde{p}_s(v,s) = \frac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)}, \\ \tilde{\theta}_v(v,s) = -\frac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)}, & \tilde{\theta}_s(v,s) = \frac{\theta}{e_\theta(v,\theta)}, \end{cases} \tag{1.3}$$

or

$$\begin{cases} \hat{s}_e(v,e) = \frac{1}{\theta}, & \hat{s}_v(v,e) = \frac{p(v,\theta)}{\theta}, \\ \hat{p}_e(v,e) = \frac{p_\theta(v,\theta)}{e_\theta(v,\theta)}, & \hat{p}_v(v,e) = \left(p_v(v,\theta) - \frac{\theta(p_\theta(v,\theta))^2}{e_\theta(v,\theta)}\right) + \frac{p(v,\theta)p_\theta(v,\theta)}{e_\theta(v,\theta)}, \\ \hat{\theta}_e(v,e) = \frac{1}{e_\theta(v,\theta)}, & \hat{\theta}_v(v,e) = \frac{p(v,\theta) - \theta p_\theta(v,\theta)}{e_\theta(v,\theta)}. \end{cases} \tag{1.4}$$

From (1.3) and (1.4), we get that

$$\tilde{p}_v(v,s) = \hat{p}_v(v,e) - p(v,\theta)\hat{p}_e(v,e). \tag{1.5}$$

Throughout this paper, the pressure function $p(v,\theta)$ and the internal energy $e(v,\theta)$ are assumed to satisfy

$$p_v(v,\theta) = \frac{\partial p(v,\theta)}{\partial v} < 0, \quad e_\theta(v,\theta) = \frac{\partial e(v,\theta)}{\partial \theta} > 0 \tag{H$_1$}$$

and

$$\tilde{p}_{vv}(v,s) = \frac{\partial^2 \tilde{p}(v,s)}{\partial v^2} > 0 \text{ and } \tilde{p}(v,s) \text{ is convex with respect to } (v,s). \tag{H$_2$}$$

From (1.3) and (H$_1$), we have

$$\tilde{p}_v(v,s) = p_v(v,\theta) - \frac{\theta(p_\theta(v,\theta))^2}{e_\theta(v,\theta)} < 0, \tag{1.6}$$

$$\begin{cases} \tilde{e}_{ss}(v,s) & = \frac{\theta}{e_\theta(v,\theta)} > 0, \\ \tilde{e}_{vs}(v,s) & = \frac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)}, \\ \tilde{e}_{vv}(v,s) & = -p_v(v,\theta) + \frac{\theta(p_\theta(v,\theta))^2}{e_\theta(v,\theta)} > 0, \end{cases} \tag{1.7}$$

and

$$\tilde{e}_{ss}(v,s)\tilde{e}_{vv}(v,s) - (\tilde{e}_{vs}(v,s))^2 = -\frac{\theta p_v(v,\theta)}{e_\theta(v,\theta)} > 0, \tag{1.8}$$

which means that $\tilde{e}(v,s)$ is convex with respect to (v,s).

We are interested in showing that the expansion waves for (1.1) are nonlinear stable. For this, it is convenient to work with the equations for the entropy s and the absolute temperature θ with the equations

$$s_t = \kappa\left(\frac{\theta_x}{v\theta}\right)_x + \kappa\frac{\theta_x^2}{v\theta^2} + \mu\frac{u_x^2}{v\theta} \tag{1.9}$$

and

$$\theta_t + \frac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)}u_x = \frac{\kappa}{e_\theta(v,\theta)}\left(\frac{\theta_x}{v}\right)_x + \frac{\mu}{e_\theta(v,\theta)}\frac{u_x^2}{v}. \tag{1.10}$$

In fact, for smooth solutions, equations $(1.1)_1$, $(1.1)_2$, $(1.1)_3$ are equivalent to equations $(1.1)_1$, $(1.1)_2$, (1.9) or $(1.1)_1$, $(1.1)_2$, (1.10). In what follows, we will consider $(1.1)_1$, $(1.1)_2$, (1.9) with the initial data

(1.11)

$$(v(t,x),u(t,x),s(t,x))|_{t=0} = (v_0(x),u_0(x),s_0(x)) \to (v_\pm,u_\pm,s_\pm) \quad \text{as} \quad x\to\pm\infty.$$

Here $v_\pm>0$, $u_\pm$, $s_\pm$ are constants. Since we will focus on the expansion waves to (1.1), we assume that $s_+=s_-=\overline{s}$ in the rest of this paper.

For expansion waves, the right hand side of (1.1) decays faster than each individual term on the left hand side. Therefore, the compressible Navier-Stokes equations (1.1) may be approximated, time-asymptotically, by the Riemann problem of the compressible Euler equations:

$$\begin{cases} v_t - u_x &= 0,\\ u_t + \tilde{p}(v,s)_x &= 0,\\ s_t &= 0, \end{cases} \tag{1.12}$$

with Riemann data

(1.13)

$$(v(t,x),u(t,x),s(t,x))|_{t=0} = \left(v_0^R(x),u_0^R(x),s_0^R(x)\right) = \begin{cases} (v_-,u_-,s_-), & x<0,\\ (v_+,u_+,s_+), & x>0. \end{cases}$$

We consider the case when the Riemann problem (1.12), (1.13) admits a unique global weak (rarefaction wave) solution $\left(V^R\left(\frac{x}{t}\right),U^R\left(\frac{x}{t}\right),S^R\left(\frac{x}{t}\right)\right)$ which consists of a rarefaction wave of the first family, denoted by $\left(V_1^R\left(\frac{x}{t}\right),U_1^R\left(\frac{x}{t}\right),\overline{s}\right)$, and another of the third family, denoted by $\left(V_3^R\left(\frac{x}{t}\right),U_3^R\left(\frac{x}{t}\right),\ \overline{s}\right)$. That is, there exists a unique constant state $(v_m,u_m)\in\mathbf{R}^2$ $(v_m>0)$ such that $(v_m,u_m)\in\mathbf{R}_1(v_-,u_-)$ and $(v_+,u_+)\in\mathbf{R}_3(v_m,u_m)$. Here

(1.14)

$$\begin{cases} \mathbf{R}_1(v_-,u_-,\bar{s}) = \left\{(v,u,s)\ \middle|\ u = u_- + \displaystyle\int_{v_-}^{v}\sqrt{-\tilde{p}_z(z,\bar{s})}dz,\ u\ge u_-, s=\bar{s}\right\},\\ \mathbf{R}_3(v_m,u_m,\bar{s}) = \left\{(v,u,s)\ \middle|\ u = u_m - \displaystyle\int_{v_m}^{v}\sqrt{-\tilde{p}_z(z,\bar{s})}dz,\ u\ge u_m, s=\bar{s}\right\}. \end{cases}$$

In other words, the unique weak solution $\left(V^R\left(\frac{x}{t}\right), U^R\left(\frac{x}{t}\right), S^R\left(\frac{x}{t}\right)\right)$ to the Riemann problem (1.12) and (1.13) is given by

$$
\begin{aligned}
&\left(V^R\left(\frac{x}{t}\right), U^R\left(\frac{x}{t}\right), S^R\left(\frac{x}{t}\right)\right)\\
=\ &\left(V_1^R\left(\frac{x}{t}\right)+V_3^R\left(\frac{x}{t}\right)-v_m, U_1^R\left(\frac{x}{t}\right)+U_3^R\left(\frac{x}{t}\right)-u_m, \bar{s}\right),
\end{aligned}
\tag{1.15}
$$

where $\left(V_i^R(t,x), U_i^R(t,x), S^R(t,x)\right)$ $(i=1,3)$ are determined by the following equations:

$$
\begin{cases}
S^R(t,x) = & \bar{s},\\
U_1^R(t,x) - \int^{V_1^R(t,x)} \sqrt{-\tilde{p}_z(z,\bar{s})}dz = & u_- - \int^{v_-} \sqrt{-\tilde{p}_z(z,\bar{s})}dz,\\
\lambda_{1x}\left(V_1^R(t,x),\bar{s}\right) > & 0,\\
\lambda_1(v,s) = & -\sqrt{-\tilde{p}_v(v,s)},\\
U_3^R(t,x) + \int^{V_1^R(t,x)} \sqrt{-\tilde{p}_z(z,\bar{s})}dz = & u_m + \int^{v_m} \sqrt{-\tilde{p}_z(z,\bar{s})}dz,\\
\lambda_{3x}\left(V_3^R(t,x),\bar{s}\right) > & 0,\\
\lambda_3(v,s) = & \sqrt{-\tilde{p}_v(v,s)}.
\end{cases}
\tag{1.16}
$$

To study the above problem, as in [26] and [27], we first construct a smooth approximation to the above Riemann solution (1.15). Let $w_i(t,x)$ $(i=1,3)$ be the unique global smooth solution to the following Cauchy problem:

$$
\begin{cases}
w_{it} + w_i w_{ix} = 0,\\
w_i(t,x)|_{t=0} = w_{i0}(x) = \frac{w_{i+}+w_{i-}}{2} + \frac{w_{i+}-w_{i-}}{2} K_q \int_0^{\epsilon x} (1+y^2)^{-q} dy,
\end{cases}
\tag{1.17}
$$

where $q > \frac{3}{2}$, $K_q = \left(\int_0^{+\infty} (1+y^2)^{-q} dy\right)^{-1}$, $\epsilon > 0$ is a positive constant which will be specified later, and

$$
\begin{cases}
w_{1-} = \lambda_1(v_-,\bar{s}) = -\sqrt{-\tilde{p}_v(v_-,\bar{s})},\\
w_{1+} = \lambda_1(v_m,\bar{s}) = -\sqrt{-\tilde{p}_v(v_m,\bar{s})},\\
w_{3-} = \lambda_3(v_m,\bar{s}) = \sqrt{-\tilde{p}_v(v_m,\bar{s})},\\
w_{3+} = \lambda_3(v_+,\bar{s}) = \sqrt{-\tilde{p}_v(v_+,\bar{s})}.
\end{cases}
$$

Then, by setting $\epsilon = \delta = |v_- - v_+| + |u_- - u_+|$, the smooth approximation of the rarefaction wave profile $(V(t,x), U(t,x), S(t,x))$ is constructed as follows:

(1.18)
$$\begin{aligned}&\Big(V(t,x), U(t,x), S(t,x)\Big)\\&\quad = \Big(V_1(t+t_0,x) + V_3(t+t_0,x) - v_m,\ U_1(t+t_0,x) + U_3(t+t_0,x) - u_m, \bar{s}\Big),\end{aligned}$$

where $t_0 > 0$ is a sufficiently large but fixed positive constant which will be determined later and $(V_i(t,x), U_i(t,x))\,(i = 1,3)$ are given by the following equations:

$$(1.19)\qquad \begin{cases} \lambda_i\left(V_i(t,x), \bar{s}\right) = & w_i(t,x), \quad i = 1,3, \\ \lambda_1(v,s) = & -\sqrt{-\tilde{p}_v(v,s)}, \\ \lambda_3(v,s) = & \sqrt{-\tilde{p}_v(v,s)}, \\ U_1(t,x) = & u_- + \displaystyle\int_{v_-}^{V_1(t,x)} \sqrt{-\tilde{p}_z(z,\bar{s})}dz, \\ U_3(t,x) = & u_m - \displaystyle\int_{v_m}^{V_3(t,x)} \sqrt{-\tilde{p}_z(z,\bar{s})}dz, \end{cases}$$

and Θ is defined by

$$(1.20)\qquad \Theta(t,x) = \widetilde{\theta}\left(V(t,x), \bar{s}\right).$$

The nonlinear stability problem is to compare the large-time behavior of the global solution $(v(t,x), u(t,x), s(t,x))$ of the Cauchy problem $(1.1)_1$, $(1.1)_2$, (1.9), (1.11) with the corresponding rarefaction wave solution $\left(V^R\left(\frac{x}{t}\right), U^R\left(\frac{x}{t}\right), S^R\left(\frac{x}{t}\right)\right)$ of the Riemann problem (1.12), (1.13). More precisely, if the initial data $(v_0(x), u_0(x), s_0(x))$ is an H^1-perturbation of $(V(0,x), U(0,x), \bar{s})$, then the problem is to show that the Cauchy problem $(1.1)_1$, $(1.1)_2$, (1.9), (1.11) admits a unique global solution $(v(t,x), u(t,x), s(t,x))$, which tends to $\left(V^R\left(\frac{x}{t}\right), U^R\left(\frac{x}{t}\right), S^R\left(\frac{x}{t}\right)\right)$ in $L^\infty(\mathbf{R})$ as $t \to \infty$. Recently, it has been shown that expansion waves for the compressible Navier-Stokes equations are nonlinear stable (cf. [11, 18, 21, 26, 27, 28, 29, 35]). These nonlinear stability results are called *local stability* or *global stability* depending on whether the H^1-norm of the initial perturbation is small or not. Up to now, local stability results have been well established in [18], [21], and [26] but not for global stability. The main purpose of our present paper is devoted to studying the global stability of rarefaction waves for the compressible Navier-Stokes equations.

We now state our main results in this paper. Denote

$$(1.21)\qquad \ell = \frac{t_0}{\delta},$$

where $\delta = |u_- - u_+| + |v_- - v_+|$ is the strength of the rarefaction waves.

The first result is concerned with the nonlinear stability results for a general gas satisfying the assumptions (H_1) and (H_2). Compared to the previous results which show that rarefaction waves are locally stable (cf. [18, 29]), the following result shows that the stability holds provided that the L^2-norm of the initial perturbation is small while the L^2-norm of the first derivative of the initial perturbation can be large. More precisely, we have

Theorem 1.1 (Nonlinear Stability for General Gas)**.** *Assume that* $(V^R(t,x), U^R(t,x), S^R(t,x))$ *is the solution to the Riemann problem of the compressible Euler equations* (1.12)*,* (1.13) *given by* (1.15) *and that the initial data* $(v_0(x), u_0(x), s_0(x))$ *of the compressible Navier-Stokes equations* $(1.1)_1$*,* $(1.1)_2$*,* (1.10) *satisfies* (1.11)*, and*

$$\begin{cases} 0 < 2\underline{V} \leq v_0(x), V(t,x) \leq \frac{1}{2}\overline{V}, \\ 0 < 2\underline{\Theta} \leq \theta_0(x), \Theta(t,x) \leq \frac{1}{2}\overline{\Theta} \end{cases} \tag{1.22}$$

for all $(t,x) \in \mathbf{R}_+ \times \mathbf{R}$ *and some positive constants* $\underline{V}, \overline{V}, \underline{\Theta}$ *and* $\overline{\Theta}$*, and*

$$\begin{cases} \Big\| (v_0(x) - V(0,x), u_0(x) - U(0,x), \theta_0(x) - \Theta(0,x)) \Big\|_{L^2}^2 \leq D_1 \ell^{-\alpha}, \\ \Big\| (v_{0x}(x) - V_x(0,x), u_{0x}(x) - U_x(0,x), \theta_{0x}(x) - \Theta_x(0,x)) \Big\|_{L^2}^2 \leq D_2 \left(1 + \ell^{\beta}\right), \end{cases} \tag{1.23}$$

where D_1*,* D_2 *are some positive constants independent of* t_0*,* ℓ *and* δ^{-1} *and* α*,* β *are positive constants satisfying*

$$\beta < \min\left\{ \frac{3}{4}\alpha, \frac{\alpha}{6} + \frac{1}{24}, \frac{3}{56}, \frac{2}{5}\alpha + \frac{1}{40} \right\}. \tag{1.24}$$

Then the Cauchy problem (1.1)*,* (1.11) *admits a unique global solution* $(v(t,x),\ u(t,x),\ s(t,x))$ *satisfying*

$$\lim_{t\to+\infty} \sup_{x\in\mathbf{R}} \left\{ \left| \left(v(t,x) - V^R(t,x), u(t,x) - U^R(t,x), s(t,x) - \overline{s}\right) \right| \right\} = 0 \tag{1.25}$$

provided that ℓ *is chosen suitably large and the assumptions* (H_1) *and* (H_2) *are satisfied.*

Remark 1.1. It is worth pointing out that the assumption $\alpha > \beta$ is to deduce a uniform lower bound for $v(t,x)$.

Remark 1.2. Since in the proof of Theorem 1.1, the constant D_2 can be any positive constant independent of ℓ, t_0 and δ^{-1}, the H^1-norm of the initial perturbation $(v_0(x) - V(0,x), u_0(x) - U(0,x), \theta_0(x) - \Theta(0,x))$ can indeed be large. However, due to the fact that the parameter $\ell > 0$ is assumed to be a sufficiently large constant, the assumption $(1.23)_1$ does imply that the L^2-norm of the initial perturbation is small. In some sense it is a generalization of the result obtained in [4, 10] for viscous hyperbolic conservation laws with the same end states, i.e., $(v_-, u_-, \theta_-) = (v_+, u_+, \theta_+)$ to the compressible Navier-Stokes equations with different end states, i.e., $(v_-, u_-, \theta_-) \neq (v_+, u_+, \theta_+)$.

In Theorem 1.1, even though the H^1-norm of the initial disturbance can be large, by using $\beta < \alpha$, one can conclude by employing Sobolev's inequality that the $L^\infty-$norm of the initial disturbance is small. This implies that the nonlinear stability result obtained in Theorem 1.1 is essentially a local stability one. Thus a natural question is how to get the global stability for large perturbations in both the H^1-norm and the $L^\infty-$norm. The second theorem shows that, for the ideal polytropic gas, such a stability result indeed holds for the case when the adiabatic exponent γ is close to 1. To state the result precisely, we recall that for the ideal polytropic gas, $p(v,\theta)$ and $e(v,\theta)$ have the following special constitutive relations:

$$p(v,\theta) = \frac{R\theta}{v} = Av^{-\gamma}\exp\left(\frac{\gamma-1}{R}s\right), \qquad e(v,\theta) = \frac{R\theta}{\gamma-1}. \tag{1.26}$$

Here $R > 0$ is the gas constant, $\gamma > 1$ the adiabatic constant and A a positive constant. For such a problem, it is proved in [29] that for any H^1 initial perturbation whose H^1-norm is bounded by any constant independent of ℓ, t_0 and/or δ^{-1}, the corresponding Cauchy problem admits a unique global solution $(v(t,x), u(t,x), s(t,x))$ satisfying (1.25). We note however that since two parameters t_0 and δ are introduced in the construction of $(V(t,x), U(t,x), \Theta(t,x), S(t,x))$ and the smooth approximation of the rarefaction wave profile $(V^R(\frac{x}{t}), U^R(\frac{x}{t}), \Theta^R(\frac{x}{t}), \bar{s})$, when we consider the global stability, the fact that the H^1-norm of the initial perturbation depends on t_0 and δ^{-1} should be taken into consideration. The following result shows that stability holds provided that the H^1-norm of the initial perturbation satisfies certain growth conditions as the parameter $\ell \to \infty$.

Theorem 1.2 (Nonlinear Stability for the Ideal Polytropic Gas)**.** *Assume that* $\left(V^R(t,x),\ U^R(t,x), \bar{s}\right)$ *is the solution of the Riemann problem of the compressible Euler equations* (1.12), (1.13) *defined by* (1.15) *and that* $(p(v,\theta), e(v,\theta))$ *satisfy the constitutive relations* (1.26). *Then for any* $(v_0(x) - V(0,x), u_0(x) - U(0,x), s_0(x) - \bar{s}) \in H^1(\mathbf{R})$ *satisfying* (1.22) *and*

$$\Big\| (v_0(x) - V(0,x), u_0(x) - U(0,x), \xi_0(x) - \bar{s}) \Big\|_{H^1} \leq D_3\Big(1 + h(\ell)\Big), \tag{1.27}$$

where D_3 *is any positive constant independent of* ℓ, t_0 *and* δ^{-1}, $h(\ell) = \frac{1}{2D_3}\left(\kappa_2^{-1} \circ \kappa_1^{-1}\right)(\ell^{\gamma_1})$ *with* $\gamma_1 < \frac{1}{8}$, *and* κ_1, κ_2 *are some continuous increasing functions which will be defined by* (4.23) *and* (4.24), *respectively. Then the corresponding Cauchy problem* (1.1), (1.11) *admits a unique global solution* $(v(t,x), u(t,x), s(t,x))$ *satisfying* (1.25) *provided that* $\gamma - 1 > 0$ *is sufficiently small and* ℓ *is suitably large.*

Remark 1.3. In [18], the global stability of weak rarefaction waves was established for the ideal polytropic gas when the adiabatic exponent γ is close to one and the main idea there is to use the smallness of the rarefaction waves to control the possible growth induced by the nonlinearities of the equations and the interaction of waves from different families. From the above discussions, the result given in [18] was obtained under the additional assumption that the H^1-norm of the initial perturbation is bounded by a constant independent of δ^{-1}.

In the proof of Theorem 1.2, the assumption that γ is close to 1 is used to close the a priori assumption $\underline{\Theta} \leq \theta(t,x) \leq \overline{\Theta}$ for $(t,x) \in [0,T] \times \mathbf{R}$. Hence, one can imagine that for the isentropic gas, such a smallness assumption can be removed. The third theorem is to give the nonlinear stability of rarefaction waves for $p-$systems with viscosity for a general pressure $p = p(v)$ with large initial perturbation. Recall that the isentropic compressible Navier-Stokes equations in Lagrangian coordinates can be written as

$$\begin{cases} v_t - u_x &= 0, \\ u_t + p(v)_x &= \mu\left(\frac{u_x}{v}\right)_x \end{cases} \tag{1.28}$$

with the initial data

$$(v(t,x),u(t,x))|_{t=0}=(v_0(x),u_0(x))\to(v_\pm,u_\pm)\quad\text{as}\quad x\to\pm\infty. \tag{1.29}$$

Here $v_\pm>0$ and $u_\pm$ are given constants so that the Riemann problem of the isentropic compressible Euler equations

$$\begin{cases} v_t-u_x &=0,\\ u_t+p(v)_x &=0\end{cases} \tag{1.30}$$

with the Riemann data

$$(v(t,x),u(t,x))|_{t=0}=\left(\overline{v}_0^R(x),\overline{u}_0^R(x)\right)=\begin{cases}(v_-,u_-), & x<0,\\ (v_+,u_+), & x>0\end{cases} \tag{1.31}$$

admits a unique global weak (rarefaction wave) solution $\left(\overline{V}^R\left(\frac{x}{t}\right),\overline{U}^R\left(\frac{x}{t}\right)\right)$ containing a rarefaction wave of the first family, denoted by $\left(\overline{V}_1^R\left(\frac{x}{t}\right),\overline{U}_1^R\left(\frac{x}{t}\right)\right)$, and another of the second family denoted by $\left(\overline{V}_2^R\left(\frac{x}{t}\right),\overline{U}_2^R\left(\frac{x}{t}\right)\right)$. That is, there exists a unique constant state $(v_m,u_m)\in\mathbf{R}^2$ $(v_m>0)$ such that $(v_m,u_m)\in\mathbf{R}_1(v_-,u_-)$ and $(v_+,u_+)\in\mathbf{R}_2(v_m,u_m)$. Here

$$\begin{cases}\mathbf{R}_1(v_-,u_-)=\left\{(v,u)\;\middle|\; u=u_-+\displaystyle\int_{v_-}^{v}\sqrt{-p'(z)}dz,\ u\geq u_-\right\},\\[2ex] \mathbf{R}_2(v_m,u_m)=\left\{(v,u)\;\middle|\; u=u_m-\displaystyle\int_{v_m}^{v}\sqrt{-p'(z)}dz,\ u\geq u_m\right\}.\end{cases} \tag{1.32}$$

In other words, the weak solution $\left(\overline{V}^R\left(\frac{x}{t}\right),\overline{U}^R\left(\frac{x}{t}\right)\right)$ to the Riemann problem (1.30), (1.31) is given by

$$\left(\overline{V}^R\left(\frac{x}{t}\right),\overline{U}^R\left(\frac{x}{t}\right)\right)=\left(\overline{V}_1^R\left(\frac{x}{t}\right)+\overline{V}_2^R\left(\frac{x}{t}\right)-v_m,\overline{U}_1^R\left(\frac{x}{t}\right)+\overline{U}_2^R\left(\frac{x}{t}\right)-u_m\right). \tag{1.33}$$

To study the stability problem, we first construct a smooth approximation to the above Riemann solution (1.33). As in [27, 28], let $w_i(t,x)$ $(i=1,2)$ be the unique global smooth solution to the following Cauchy problem:

$$\begin{cases} w_{it}+w_iw_{ix}=0,\\ w_i(t,x)|_{t=0}=w_{i0}(x)=\frac{w_{i+}+w_{i-}}{2}+\frac{w_{i+}-w_{i-}}{2}K_q\displaystyle\int_0^{\epsilon x}(1+y^2)^{-q}dy,\end{cases} \tag{1.34}$$

where $q>\frac{3}{2}$, $K_q=\left(\displaystyle\int_0^{+\infty}(1+y^2)^{-q}dy\right)^{-1}$ and

$$\begin{cases} w_{1-}=\lambda_1(v_-)=-\sqrt{-p'(v_-)},\\ w_{1+}=\lambda_1(v_m)=-\sqrt{-p'(v_m)},\\ w_{2-}=\lambda_2(v_m)=\sqrt{-p'(v_m)},\\ w_{2+}=\lambda_2(v_+)=\sqrt{-p'(v_+)}.\end{cases}$$

Then, if we set $\epsilon = \delta$, $\left(\overline{V}(t,x), \overline{U}(t,x)\right)$, the smooth approximation of the rarefaction waves profile, is defined by

$$\begin{aligned}&\left(\overline{V}(t,x), \overline{U}(t,x)\right)\\&\quad = \Big(\overline{V}_1(t+t_0,x) + \overline{V}_2(t+t_0,x) - v_m,\ \overline{U}_1(t+t_0,x) + \overline{U}_2(t+t_0,x) - u_m\Big),\end{aligned} \tag{1.35}$$

where $t_0 > 0$ is a sufficiently large but fixed positive constant which will be determined later and $\left(\overline{V}_i(t,x), \overline{U}_i(t,x)\right)$ $(i = 1,2)$ are given by the following equations:

$$\begin{cases} \lambda_i\left(\overline{V}_i(t,x)\right) = w_i(t,x),\ \lambda_i(v) = (-1)^i\sqrt{-p'(v)}, \quad i = 1,2, \\ \overline{U}_1(t,x) = u_- + \displaystyle\int_{v_-}^{\overline{V}_1(t,x)} \sqrt{-p'(s)}ds, \\ \overline{U}_2(t,x) = u_m - \displaystyle\int_{v_m}^{\overline{V}_2(t,x)} \sqrt{-p'(s)}ds. \end{cases} \tag{1.36}$$

We only assume that $p(v)$ is a positive smooth function for $v > 0$ and satisfies

$$p'(v) < 0, \quad p''(v) > 0 \ \text{ for } \ v > 0, \tag{1.37}$$

which means that the system (1.30) is strictly hyperbolic and both characteristic fields are genuinely nonlinear.

Under the above assumptions, we have the following theorem.

Theorem 1.3 (Nonlinear Stability for General Isentropic Gas)**.** *Assume that* $\left(\overline{V}^R(t,x),\ \overline{U}^R(t,x)\right)$ *is the solution of the Riemann problem* (1.30), (1.31) *defined by* (1.33). *Then for any* $p(v)$ *satisfying* (1.37) *and* $\left(v_0(x) - \overline{V}(0,x), u_0(x) - \overline{U}(0,x)\right) \in H^1(\mathbf{R})$ *satisfying* $0 < 2\underline{V} \le v_0(x), \overline{V}(t,x) \le \frac{1}{2}\overline{V}$ *for all* $(t,x) \in \mathbf{R}_+ \times \mathbf{R}$ *and some positive constants* $\underline{V}$, $\overline{V}$ *and*

$$\left\| \left(v_0(x) - \overline{V}(0,x), u_0(x) - \overline{U}(0,x)\right) \right\|_{H^1} \le D_4\Big(1 + \overline{h}(\ell)\Big), \tag{1.38}$$

the Cauchy problem (1.28), (1.29) *admits a unique global solution* $(v(t,x), u(t,x))$ *satisfying*

$$\lim_{t\to+\infty} \sup_{x\in\mathbf{R}} \left\{ \left| \Big(v(t,x) - \overline{V}^R(t,x), u(t,x) - \overline{U}^R(t,x)\Big) \right| \right\} = 0 \tag{1.39}$$

provided that ℓ *is chosen sufficiently large. Here* $D_4 > 0$ *is any given positive constant independent of* ℓ, t_0 *and* δ^{-1}, $\overline{h}(\ell) = \frac{1}{2D_5}\left(\kappa_5 \circ \kappa_4\right)^{-1}\left(\ell^{\overline{\gamma}}\right)$, $\overline{\gamma} < \frac{1}{3}$, *and* κ_4, κ_5 *are two continuous increasing functions defined by* (5.20) *and* (5.28), *respectively.*

Remark 1.4. Notice that in the above theorems, the assumption that the quantity ℓ is sufficiently large implies that either the strength of the rarefaction waves is small or the rarefaction waves of different families are separated far enough initially. D_i $(i = 1,2,3,4)$ being independent of ℓ, t_0 and/or δ^{-1} means that if we consider the case when the strength of the rarefaction wave is small (or the rarefaction waves of different families are separated far enough initially), then such a constant does not depend on δ^{-1} (or t_0).

Remark 1.5. Theorem 1.2 and Theorem 1.3 imply that even if the H^1-norm of the initial perturbation depends on t_0, δ^{-1}, a nonlinear stability result can still be established provided that the initial perturbation satisfies certain growth conditions as $\ell \to \infty$. Therefore, Theorem 1.2 and Theorem 1.3 generalize the corresponding results obtained in [29]. We believe that the assumptions (1.27) and (1.38) are due to the limitation of our method, and it is interesting to see if these assumptions can be removed or relaxed.

Without loss of generality, when we consider the case when the strength δ of the rarefaction waves is sufficiently small or the rarefaction waves are separated far enough initially, we can assume that $\delta < 1$ or $t_0 > 1$. Thus, for the simplicity of presentation, we will not write explicitly the dependence of the constants on δ and t_0^{-1}.

Remark 1.6. The assumptions that the positive constants α, β in (1.23), γ_1 in (1.27) and $\overline{\gamma}$ in (1.38) satisfy

$$\beta < \min\left\{\frac{3}{4}\alpha, \frac{\alpha}{6} + \frac{1}{24}, \frac{3}{56}, \frac{2}{5}\alpha + \frac{1}{40}\right\}, \quad \gamma_1 < \frac{1}{8}, \quad \overline{\gamma} < \frac{1}{3}$$

show that the method is far from being optimal.

Remark 1.7. As to the regularity of the global solutions obtained in Theorems 1.1–1.3, if we assume further that the initial perturbation belongs to $H^s(\mathbf{R})$ $(s \geq 1)$, then the global solution we obtained in Theorems 1.1–1.3 belongs also to $C\left([0,\infty), H^s(\mathbf{R})\right) \cap C^j\left([0,\infty), H^{s-2j}(\mathbf{R})\right)$. Thus for s suitably large, the global solution obtained above is indeed a global classical solution.

Now let's outline the main ideas used in the proofs. The analysis is based on the continuation argument together with the energy method. Recall that for the local stability result obtained in [18, 21, 26], the smallness of the rarefaction waves and the smallness of the H^1-norm of the initial perturbation are used to control the growth induced by the nonlinearity of the system and/or by the interaction of waves from different families. Since the main purpose here is to study the global stability, the above analysis cannot be used any more. To overcome this difficulty, as in [29], we introduce the quantity ℓ in the construction of a smooth approximation to the Riemann solution $\left(V^R\left(\frac{x}{t}\right), U^R\left(\frac{x}{t}\right), S^R\left(\frac{x}{t}\right)\right)$ to control the above-mentioned possible growth. The main problem, as shown in the proofs of Theorems 1.1, 1.2 and 1.3, is how to choose such an ℓ which is a suitably large but fixed constant.

For the nonlinear stability result for a general gas, we first deduce certain energy estimates based on the $L^\infty-$norm a priori assumption (3.3) and then use the continuation argument to show that such an a priori assumption can indeed be closed. For this purpose, we need to assume that the initial perturbation satisfies (1.23). It is worth pointing out that since uniform lower and upper bounds for $v(t,x)$ and $\theta(t,x)$ are derived through energy estimates and Sobolev's inequality, we need to require the constants α and β in (1.23) to satisfy (1.24). This implies that although the H^1-norm of the initial perturbation can be large, its $L^\infty-$norm must be small. As to the nonlinear stability results obtained in Theorems 1.2 and 1.3, if the H^1-norm of the initial perturbation is independent of ℓ, then a straightforward modification of the energy method can result in a nonlinear stability result as those obtained in [29]. However if the H^1-norm of the initial perturbation depends on ℓ, this argument cannot be used any longer. By employing the continuation argument and the energy method, we can indeed show that if the initial perturbation satisfies certain growth condition as $\ell \to \infty$, then a nonlinear stability result can also be obtained. This argument has been used in [2] and [30] to deduce certain nonlinear stability of rarefaction waves for the $p-$system with artificial viscosity and the Jin-Xin relaxation approximation of the $p-$system with large initial perturbation respectively. Note that the uniform lower bound, which is independent of ℓ, for $v(t,x)$ is obtained by employing the theory of positively invariant regions (cf. [1], [32], [33]) and plays an essential role in the analysis in [2] and [30]. Since the compressible Navier-Stokes equations are a hyperbolic-parabolic coupled system, we cannot apply the theory of positively invariant regions to deduce such a uniform lower bound for $v(t,x)$. On the other hand, by the special structure of the Navier-Stokes equations, we can use J. Kanel's method (cf. [15]) to deduce a time-independent lower bound on $v(t,x)$. Although this lower bound depends also on ℓ, a careful analysis together with the continuation argument can lead to the stability results in Theorems 1.2 and 1.3.

Before concluding this section, we recall that the study on the large-time behavior of solutions to the compressible Navier-Stokes equations has a long history; cf. [3], [5], [7], [8], [11], [12], [13], [17], [18], [14], [15], [16], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [31], [33], [35] and the references therein. When the initial data $(v_0(x), u_0(x), s_0(x))$ is a small perturbation of a nonvacuum constant state, i.e., $v_- = v_+ > 0$, $u_- = u_+$, $s_- = s_+$, satisfactory results have been obtained; cf. [16] and [23]. In the case when the far fields of the initial data are different, i.e., $(v_-, u_-, s_-) \neq (v_+, u_+, s_+)$, many interesting results have been obtained. For example, when the solutions to the corresponding Riemann problem consist of only shock waves, the nonlinear stability of traveling wave solutions has been established in [17], [20], [25], etc., and the nonlinear stability of rarefaction waves is studied in [11], [18], [21], [26], [27], [28], [29], [35]. Finally the nonlinear stability of contact discontinuity was recently studied in [7] and [8].

This paper is arranged as follows: we will give some basic estimates in Section 2. The proofs of Theorems 1.1, 1.2 and 1.3 are given in Sections 3, 4, 5, respectively.

Notation. Throughout the rest of this paper, C, D or $O(1)$ will be used to denote a generic positive constant independent of t, t_0 and x and $C_i(\cdot,\cdot)$ or $D_i(\cdot,\cdot)$ $(i \in \mathbf{Z}_+)$ stands for some generic constants depending only on the quantities listed in the parentheses. Note that all these constants may vary from line to line.

For two functions $f(x)$ and $g(x)$, $f(x) \sim g(x)$ as $x \to a$ means that there exists a positive constant $C > 0$ such that $C^{-1}f(x) \leq g(x) \leq Cf(x)$ in a neighborhood of a. $H^l(\mathbf{R})$ $(l \geq 0)$ denotes the usual Sobolev space with norm $\|\cdot\|_l$, and $\|\cdot\|_0 = \|\cdot\|$ will be used to denote the usual L^2-norm. For $1 \leq p \leq +\infty$, $f(x) \in L^p(\mathbf{R}, \mathbf{R}^n)$, $\|f\|_{L^p} = \left(\int_{\mathbf{R}} |f(x)|^p dx\right)^{\frac{1}{p}}$. It is easy to see that $\|f\|_{L^2} = \|\cdot\|$. Finally, $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_{L^\infty_{t,x}}$ are used to denote $\|\cdot\|_{L^\infty(\mathbf{R})}$ and $\|\cdot\|_{L^\infty([0,t]\times\mathbf{R})}$ respectively.

2. Preliminaries

In this section, we give some basic estimates which will be used in proving our main results. First, we list some basic properties of the global smooth functions $(V(t,x), U(t,x), S(t,x), \Theta(t,x))$ constructed in (1.17) and (1.18) whose proof can be found in [27].

According to (1.1), (1.16), (1.18), we know that $(V(t,x),U(t,x),S(t,x))$ solves the following problem:

$$\left\{\begin{aligned} V_t-U_x=&\ \ 0,\\ U_t+p(V,\Theta)_x=&\ \ g(V,\Theta)_x,\\ \left(e(V,\Theta)+\tfrac{U^2}{2}\right)_t+(Up(V,\Theta))_x=&\ \ q(V,\Theta),\\ \Theta_t+\tfrac{\Theta p_\Theta(V,\Theta)}{e_\Theta(V,\Theta)}U_x=&\ \ r(V,\Theta),\\ S_t(V,\Theta)=&\ \ 0,\end{aligned}\right. \tag{2.1}$$

where

$$\left\{\begin{aligned} g(V,\Theta)=&\ \ p(V,\Theta)-p(V_1,\Theta_1)-p(V_2,\Theta_2)-p(v_m,\theta_m),\\ q(V,\Theta)=&\ \ (e(V,\Theta)-e(V_1,\Theta_1)-e(V_2,\Theta_2))_t+\left(\tfrac{U^2}{2}-\tfrac{U_1^2}{2}-\tfrac{U_2^2}{2}\right)_t\\ &\ \ +\left(Up(V,\Theta)-U_1p(V_1,\Theta_1)-U_2p(V_2,\Theta_2)\right)_x,\\ r(V,\Theta)=&\ \ \tfrac{\Theta p_\Theta(V,\Theta)}{e_\Theta(V,\Theta)}U_x-\tfrac{\Theta_1 p_\Theta(V_1,\Theta_1)}{e_\Theta(V_1,\Theta_1)}U_{1x}-\tfrac{\Theta_2 p_\Theta(V_2,\Theta_2)}{e_\Theta(V_2,\Theta_2)}U_{2x},\end{aligned}\right.$$

and $\theta_m=\tilde\theta(v_m,\overline{s})$.

Since $w_{i0}(x)$ is a strictly increasing function, we have the following lemma.

Lemma 2.1. *For each $i\in\{1,3\}$, the Cauchy problem* (1.17) *admits a unique global smooth solution $w_i(t,x)$ which satisfies the following properties:*

(i): $w_-<w_i(t,x)<w_+,\quad w_{ix}(x,t)>0$ *for each* $(t,x)\in\mathbf{R}_+\times\mathbf{R}$.

(ii): *For any p with $1\le p\le\infty$, there exists a constant $C_{p,q}$, depending only on p,q, such that*

$$\left\{\begin{aligned} &\|w_{ix}(t)\|^p_{L^p}\le C_{p,q}\min\left\{\epsilon^{p-1}\tilde w_i^p,\tilde w t^{-p+1}\right\},\\ &\|w_{ixx}(t)\|^p_{L^p}\le C_{p,q}\ \min\left\{\epsilon^{2p-1}\tilde w_i^p,\ \epsilon^{(p-1)\left(1-\frac{1}{2q}\right)}\tilde w_i^{-\frac{p-1}{2q}}t^{-p-\frac{p-1}{2q}}\right\}.\end{aligned}\right.$$

(iii): *If $0<w_{i-}(<w_{i+})$ and q is suitably large, then*

$$\left\{\begin{aligned} &|w_i(t,x)-w_{i-}|\le C\tilde w_i(1+(\epsilon x)^2)^{-\frac{q}{3}}(1+(\epsilon w_{i-}t)^2)^{-\frac{q}{3}},\ x\le 0,\\ &|w_{ix}(t,x)|\le c\epsilon\tilde w_i(1+(\epsilon x)^2)^{-\frac12}(1+(\epsilon w_{i+}t)^2)^{-\frac{q}{2}},\ x\le 0.\end{aligned}\right.$$

(iv): *If $(w_{i-})<w_{i+}\le 0$ and q is suitably large, then*

$$\left\{\begin{aligned} &|w_i(t,x)-w_{i+}|\le C\tilde w_i(1+(\epsilon x)^2)^{-\frac{q}{3}}(1+(\epsilon w_{i-}t)^2)^{-\frac{q}{3}},\ x\ge 0,\\ &|w_{ix}(t,x)|\le c\epsilon\tilde w_i(1+(\epsilon x)^2)^{-\frac12}(1+(\epsilon w_{i+}t)^2)^{-\frac{q}{2}},\ x\ge 0.\end{aligned}\right.$$

(v): $\lim\limits_{t\to+\infty}\sup\limits_{x\in\mathbf{R}}\left|w_i(t,x)-w_i^R\left(\frac{x}{t}\right)\right|=0$. *Here $\tilde w_i=w_{i+}-w_{i-}>0$ and $w_i^R(\frac{x}{t})$ is the rarefaction wave solution of the corresponding Riemann problem of $(1.17)_1$, i.e.,*

$$w_i^R(z)=\left\{\begin{array}{ll} w_{i-}, & z\le w_{i-},\\ z, & w_{i-}\le z\le w_{i+},\\ w_{i+}, & z\ge w_{i+}.\end{array}\right.$$

Based on the results obtained in Lemma 2.1 and from (1.18) and (1.19), we can deduce that

Lemma 2.2. *Letting $\epsilon = \delta$, $q = 2$, the smooth approximations $(V(t,x), U(t,x), \Theta(t,x))$ constructed in* (1.18) *and* (1.19) *have the following properties:*

(i): $V_t(t,x) = U_x(t,x) > 0$ *for each* $(t,x) \in \mathbf{R}_+ \times \mathbf{R}$.

(ii): *For any p with $1 \le p \le \infty$ there exists a constant C_p, depending only on p, such that*

$$\begin{cases} \left\| \left(V_x, U_x, \Theta_x\right)(t) \right\|_{L^p}^p \le C_p \min\left\{\delta^{2p-1},\ \delta (t+t_0)^{-p+1}\right\}, \\ \left\| \left(V_{xx}, U_{xx}, \Theta_{xx}\right)(t) \right\|_{L^p}^p \le C_p \min\left\{\delta^{3p-1},\ \delta^{\frac{p-1}{2}} (t+t_0)^{-\frac{5p-1}{4}}\right\}. \end{cases}$$

It is obvious that $\|V_x(t)\|_{L^2}^2$ is not integrable with respect to t; however we can get for any $r > 0$ and $p > 1$ that

$$\begin{cases} \displaystyle\int_0^\infty \left\| \left(V_x, U_x, \Theta_x\right)(t) \right\|_{L^{2+r}}^{2+r} dt \le C(r)\delta t_0^{-r}, \\ \displaystyle\int_0^\infty \left\| \left(V_{xx}, U_{xx}, \Theta_{xx}\right)(t) \right\|_{L^p} dt \le O(1) \left(\frac{t_0}{\delta}\right)^{-\frac{1}{4}\left(1-\frac{1}{p}\right)}. \end{cases}$$

(iii): *For each $p \ge 1$,*

$$\left\| \left(g(V,\Theta)_x, r(V,\Theta), q(V,\Theta)\right)(t) \right\|_{L^p} \le C(p)\delta^{\frac{2}{3}}(t+t_0)^{-\frac{4}{3}}.$$

Especially,

$$\int_0^\infty \left\| \left(g(V,\Theta)_x, r(V,\Theta), q(V,\Theta)\right)(t) \right\|_{L^p} dt \le C(p) \left(\frac{t_0}{\delta}\right)^{-\frac{1}{3}}.$$

(iv): $\displaystyle\lim_{t\to+\infty} \sup_{x\in\mathbf{R}} \left| (V(t,x), U(t,x), \Theta(t,x)) - \left(V^R\left(\tfrac{x}{t}\right), U^R\left(\tfrac{x}{t}\right), \Theta^R\left(\tfrac{x}{t}\right)\right) \right| = 0.$

(v): $\left| \Big(V_t(t,x), U_t(t,x), \Theta_t(t,x)\Big) \right| \le O(1) \left| (V_x(t,x), U_x(t,x), \Theta_x(t,x)) \right|.$

Remark 2.1. Notice that the quantities $g(V,\Theta), r(V,\Theta)$, and $q(V,\Theta)$ represent the interaction of waves from different families. From (iii) of Lemma 2.2 we can deduce that by introducing the parameter $\ell = \frac{t_0}{\delta}$ in the construction of a smooth approximation of the rarefaction wave profile, $g(V,\Theta), r(V,\Theta)$, and $q(V,\Theta)$ can be controlled suitably if ℓ is chosen sufficiently large. This is one of the reasons why we introduce such a parameter ℓ.

Now we turn to construct a convex entropy to the compressible Navier-Stokes equations (1.1) around the smooth rarefaction wave profile $(V(t,x), U(t,x), \Theta(t,x), S(t,x))$. For this purpose, we notice that (1.7) together with (1.8) imply that $\tilde{e}(v,s)$ is convex with respect to v and s. Consequently $\tilde{e}(v,s) + \frac{1}{2}u^2$ is a strictly convex function of (v,u,s). Now we can construct the following normalized entropy $\eta(v,u,s;V,U,S)$ around $(V(t,x), U(t,x), S(t,x))$:

$$\begin{aligned} \eta(v,u,s;V,U,S) = {} & \left(e(v,\theta) + \tfrac{u^2}{2}\right) - \left(e(V,\Theta) + \tfrac{U^2}{2}\right) \\ & - \Big\{ -p(V,\Theta)(v-V) + U(u-U) + \Theta(s-S) \Big\}. \end{aligned} \tag{2.2}$$

Here we have used the fact that $\tilde{e}_v(v,s) = -p(v,\theta)$, $\tilde{e}_s(v,s) = \theta$.

Setting

$$\begin{aligned} & (\varphi(t,x),\psi(t,x),\phi(t,x),\xi(t,x)) \\ = \; & (v(t,x)-V(t,x),u(t,x)-U(t,x),\theta(t,x)-\Theta(t,x),s(t,x)-\overline{s})\,, \end{aligned} \tag{2.3}$$

we can deduce that $(\varphi(t,x),\psi(t,x),\phi(t,x),\xi(t,x))$ solves

$$\begin{cases} \varphi_t-\psi_x= & 0, \\ \psi_t+[p(v,\theta)-p(V,\Theta)]_x= & \mu(\frac{u_x}{v})_x-g(V,\Theta)_x, \\ \phi_t+\frac{\theta p_\theta(V,\theta)}{e_\theta(v,\theta}\psi_x+\left(\frac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)}-\frac{\Theta p_\Theta(V,\Theta)}{e_\Theta(V,\Theta)}\right)U_x= & \frac{1}{e_\theta(v,\theta)}\left\{\kappa\left(\frac{\theta_x}{v}\right)_x+\mu\frac{u_x^2}{v}\right\}-r(V,\Theta), \\ \xi_t= & \kappa\left(\frac{\theta_x}{v\theta}\right)+\kappa\frac{\theta_x^2}{v\theta^2}+\mu\frac{u_x^2}{v\theta}, \end{cases} \tag{2.4}$$

with initial data

$$\begin{aligned} & (\varphi(t,x),\psi(t,x),\phi(t,x),\xi(t,x))\,|_{t=0} \\ = \; & (\varphi_0(x),\psi_0(x),\phi_0(x),\xi_0(x)) \\ = \; & \left(v_0(x)-V(0,x),u_0(x)-U(0,x),\theta_0(x)-\Theta(0,x),s_0(x)-\overline{S}\right). \end{aligned} \tag{2.5}$$

From (2.1)–(2.4), one easily verifies that the following entropy identity holds:

$$\begin{aligned} & \eta_t(v,u,\theta;V,U,\Theta)+\Big((p(v,\theta)-P(V,\Theta))\psi\Big)_x+\left\{\mu\Theta\frac{\psi_x^2}{v\theta}+\kappa\Theta\frac{\phi_x^2}{v\theta^2}\right\} \\ & +\Big(\tilde{p}(v,s)-\tilde{p}(V,\overline{s})-\tilde{p}_v(V,\overline{s})\psi-\tilde{p}_s(V,\overline{s})\xi\Big)U_x \\ = \; & \left(\mu\frac{\psi\psi_x}{v}+\kappa\frac{\phi\phi_x}{v\theta}\right)_x+\left(-\mu\frac{U_x\psi\psi_x}{v^2}+2\mu\frac{U_x\phi\psi_x}{v\theta}-\kappa\frac{\Theta_x\phi\varphi_x}{v^2\theta}+\kappa\frac{\Theta_x\phi\phi_x}{v\theta^2}\right) \\ & +\left(\mu\frac{U_{xx}\psi}{v}+\kappa\frac{\Theta_{xx}\phi}{v\theta}\right)+\left(-\mu\frac{V_xU_x\psi}{v^2}+\mu\frac{U_x^2\phi}{v\theta}-\kappa\frac{V_x\Theta_x\phi}{v^2\theta}\right) \\ & -q(V,\Theta)-g(V,\Theta)_xU-p_\theta(V,\Theta)r(V,\Theta)\psi+g(V,\Theta)_x\psi+r(V,\Theta)\xi. \end{aligned} \tag{2.6}$$

Before concluding this section, we cite the following result which will be used in proving Theorem 1.3 whose proof can be found in [29].

Lemma 2.3. *If $p(v)$ satisfies* (1.37), *then there exists a positive constant $D_5>0$ such that*

$$\Phi(V,\ v-V)\geq D_5\frac{(v-V)^2}{v+V},\ v\in(0,\infty). \tag{2.7}$$

Here

$$\Phi(V,Z)=p(V)Z-\int_V^{V+Z}p(s)ds. \tag{2.8}$$

3. Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. For convenience of presentation, in what follows we will choose (v,θ) as independent variables and for some fixed $T>0$, we define the solution space of (2.4) and (2.5) by

$$X(0,T):=\left\{(\varphi,\psi,\phi)(t,x)\;\middle|\;\begin{array}{l}(\varphi,\psi,\phi)(t,x)\in C^0\left(0,T;H^1(\mathbf{R})\right)\\ (\psi_x,\phi_x)(t,x)\in L^2\left(0,T;H^1(\mathbf{R})\right)\end{array}\right\}.$$

Under the assumptions stated in Theorem 1.1, we can get the following local existence result (cf. [14]).

Lemma 3.1. *Under the assumptions stated in Theorem 1.1, the Cauchy problem* (2.4)*,* (2.5) *admits a unique solution* $(\varphi(t,x),\psi(t,x),\phi(t,x)) \in X(0,t_1)$ *for some sufficiently small* $t_1>0$*, and* $(\varphi(t,x),\psi(t,x),\phi(t,x))$ *satisfies*

$$\left\{\begin{aligned} &0<\underline{V}\le \varphi(t,x)+V(t,x)\le \overline{V},\\ &0<\underline{\Theta}\le \phi(t,x)+\Theta(t,x)\le \overline{\Theta}\end{aligned}\right. \tag{3.1}$$

and

$$\left\{\begin{aligned} \|\,(\varphi(t,x),\psi(t,x),\xi(t,x))\,\|^2 \le{}& 2\|\,(\varphi_0(x),\psi_0(x),\xi_0(x))\,\|^2,\\ \|\,(\varphi_x(t,x),\psi_x(t,x),\xi_x(t,x))\,\|^2 \le{}& 2\|\,(\varphi_{0x}(x),\psi_{0x}(x),\xi_{0x}(x))\,\|^2.\end{aligned}\right. \tag{3.2}$$

Here t_1 *depends only on* $\|\,(\varphi_0(x),\psi_0(x),\phi_0(x))\,\|_{H^1}$*.*

To extend the local solution obtained in Lemma 3.1 globally, we need only to get H^1-norm a priori estimates on the solution. For this purpose, supposing that $(\varphi(t,x),\psi(t,x),\phi(t,x))$ obtained in Lemma 3.1 has been extended to the time $t=T\ge t_1$, i.e., $(\varphi(t,x),\psi(t,x),\phi(t,x))\in X(0,T)$, we now deduce certain energy estimates based on the following a priori assumption:

$$\underline{V}\le v(t,x)\le \overline{V},\ \underline{\Theta}\le\theta(t,x)\le\overline{\Theta},\quad \forall (t,x)\in[0,T]\times\mathbf{R}. \tag{3.3}$$

Lemma 3.2. *Under the assumptions stated in Theorem 1.1, let* $(\varphi(t,x),\psi(t,x),\phi(t,x))\in X(0,T)$ *be a solution to the Cauchy problem* (2.4)*,* (2.5) *satisfying the a priori assumption* 3.3*. Then we have for* $0\le t\le T$ *that*

$$\begin{aligned}\|(\varphi,\psi,\phi)(t)\|^2 &+ \int_0^t\int_{\mathbf{R}} U_x(\tau,x)|(\varphi,\xi,\phi)(\tau,x)|^2dxd\tau + \int_0^t \|(\psi_x,\phi_x)(\tau)\|^2d\tau\\ &\le D_6\left(\ell^{-\alpha}+\ell^{-\frac18}\left(1+\ell^\beta\right)\right),\end{aligned} \tag{3.4}$$

$$\|\varphi_x(\tau)\|^2+\int_0^t\int_{\mathbf{R}}\varphi_x^2(\tau,x)dxd\tau\le D_7\left(1+\ell^\beta\right), \tag{3.5}$$

$$\|\psi_x(t)\|^2+\int_0^t\int_{\mathbf{R}}\psi_{xx}^2(\tau,x)dxd\tau\le D_8\left(1+\ell^{3\beta-\frac18}+\ell^\beta+\ell^{2\beta-\alpha}\right), \tag{3.6}$$

and

$$\|\phi_x(t)\|^2+\int_0^t\int_{\mathbf{R}}\phi_{xx}^2(\tau,x)dxd\tau\le D_9\left(1+\ell^{6\beta-\frac14}+\ell^{2\beta}+\ell^{4\beta-2\alpha}\right) \tag{3.7}$$

provided that $\ell=\frac{t_0}{\delta}\ge\ell_1$*. Here* ℓ_1, D_i $(i=6,7,8,9)$ *are some positive constants depending only on* $\underline{V},\overline{V},\underline{\Theta}$ *and* $\overline{\Theta}$*.*

Proof. Under the a priori assumption (3.3), we can easily deduce that

$$\eta(v,u,s;V,U,S)=C\left(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta}\right)|(\varphi,\psi,\xi)|^2.$$

Based on this observation, the assumption that $\tilde{p}(v,s)$ is a convex function of v and s, and the fact that $|(\varphi,\psi,\xi)|^2$ is equivalent to $|(\varphi,\psi,\phi)|^2$, we have by integrating the entropy identity (2.6) with respect to t and x over $[0,t]\times\mathbf{R}$ that

$$\begin{aligned}
&\|(\varphi,\psi,\phi)(t)\|^2+\int_0^t\|(\psi_x,\phi_x)(\tau)\|^2d\tau\\
&+\int_0^t\int_{\mathbf{R}}\Big(\tilde p(v,s)-\tilde p(V,\overline{s})-\tilde p_v(V,\overline{s})\psi-\tilde p_s(V,\overline{s})\xi\Big)(\tau,x)U_x(\tau,x)dxd\tau\\
\le\ & C_1\left(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta}\right)\left\{\|(\varphi_0,\psi_0,\phi_0)\|^2+\sum_{j=1}^{14}I_j\right\},
\end{aligned}\tag{3.8}$$

where I_j $(j=1,2,\cdots,14)$ denote the corresponding terms related to those of (2.6), which, by employing Lemma 2.2, the a priori assumption (3.3) and Cauchy-Schwarz's inequality, can be estimated as follows:

$$\begin{aligned}
I_1 &= \int_0^t\int_{\mathbf{R}}\left|\left(\frac{U_x\psi\varphi_x}{v^2}\right)(\tau,x)\right|dxd\tau\\
&\le \ell^{-\frac18}\int_0^t\int_{\mathbf{R}}|\varphi_x(\tau,x)|^2dxd\tau+\ell^{\frac18}\int_0^t\int_{\mathbf{R}}\|U_x(\tau)\|^2_{L^\infty}\left(\frac{\psi_x^2}{v^4}\right)(\tau,x)dxd\tau\\
&\le \ell^{-\frac18}\int_0^t\int_{\mathbf{R}}|\varphi_x(\tau,x)|^2dxd\tau+\underline{V}^{-4}\ell^{\frac18}\|U_x\|^{\frac12}_{L^\infty_{t,x}}\int_0^t\|U_x(\tau)\|^{\frac32}_{L^\infty}\int_{\mathbf{R}}\psi^2(\tau,x)dxd\tau\\
&\le \ell^{-\frac18}\int_0^t\|\varphi_x(\tau)\|^2d\tau+O(1)\underline{V}^{-4}\ell^{-\frac18}\int_0^t(1+\tau)^{-\frac32}\|\psi(\tau)\|^2d\tau,\\
I_2 &= \int_0^t\int_{\mathbf{R}}\left|\left(\frac{U_x\phi\psi_x}{v\theta}\right)(\tau,x)\right|dxd\tau\\
&\le \ell^{-\frac18}\int_0^t\int_{\mathbf{R}}\psi_x^2(\tau,x)dxd\tau+\ell^{\frac18}\int_0^t\int_{\mathbf{R}}\left(\frac{|U_x|^2}{v^2\theta^2}\phi^2\right)(\tau,x)dxd\tau\\
&\le \ell^{-\frac18}\int_0^t\|\psi_x(\tau)\|^2d\tau+O(1)\underline{V}^{-2}\underline{\Theta}^{-2}\ell^{-\frac18}\int_0^t(1+\tau)^{-\frac32}\|\phi(\tau)\|^2d\tau,\\
I_3 &= \int_0^t\int_{\mathbf{R}}\left|\left(\frac{\Theta_x\phi\varphi_x}{v^2\theta}\right)(\tau,x)\right|dxd\tau\\
&\le \frac12\ell^{-\frac18}\int_0^t\int_{\mathbf{R}}\varphi_x^2(\tau,x)dxd\tau+\frac12\ell^{\frac18}\underline{V}^{-4}\underline{\Theta}^{-2}\int_0^t\|\Theta_x(\tau)\|^2_{L^\infty}\int_{\mathbf{R}}\phi^2(\tau,x)dxd\tau\\
&\le \frac12\ell^{-\frac18}\int_0^t\|\varphi_x(\tau)\|^2d\tau+\underline{V}^{-4}\underline{\Theta}^{-2}\ell^{-\frac18}\int_0^t(1+\tau)^{-\frac32}\|\phi(\tau)\|^2d\tau,\\
I_4 &= \int_0^t\int_{\mathbf{R}}\left|\left(\frac{\Theta_x\phi\phi_x}{v\theta^2}\right)(\tau,x)\right|dxd\tau\\
&\le \ell^{-\frac18}\int_0^t\|\phi_x(\tau)\|d\tau+\underline{V}^{-2}\underline{\Theta}^{-4}\ell^{-\frac18}\int_0^t(1+\tau)^{-\frac32}\|\phi(\tau)\|^2d\tau,\\
I_5 &= \int_0^t\int_{\mathbf{R}}\left|\left(\frac{U_{xx}\psi}{v}\right)(\tau,x)\right|dxd\tau\le\int_0^t\|U_{xx}(\tau)\|\left\|\left(\frac{\psi}{v}\right)(\tau)\right\|d\tau\\
&\le \int_0^t\|U_{xx}(\tau)\|d\tau+\underline{V}^{-1}\int_0^t\|U_{xx}(\tau)\|\|\psi(\tau)\|^2d\tau\\
&\le O(1)\ell^{-\frac18}+O(1)\underline{V}^{-1}\ell^{-\frac1{16}}\int_0^t(1+\tau)^{-\frac{17}{16}}\|\psi(\tau)\|^2d\tau,
\end{aligned}$$

$$
\begin{aligned}
I_6 &= \int_0^t \int_{\mathbf{R}} \left| \left(\frac{\Theta_{xx}\phi}{v\theta} \right)(\tau,x) \right| dx d\tau \le \int_0^t \|\Theta_{xx}(\tau)\| \left\| \left(\frac{\phi}{v\theta} \right)(\tau) \right\| d\tau \\
&\le \ell^{-\frac{1}{8}} + O(1)\underline{V}^{-1}\underline{\Theta}^{-1}\ell^{-\frac{1}{16}} \int_0^t (1+\tau)^{-\frac{17}{16}} \|\phi(\tau)\|^2 d\tau, \\
\sum_{j=7}^{9} I_j &= \int_0^t \int_{\mathbf{R}} \left| \left(\frac{V_x U_x \psi}{v^2} + \frac{U_x^2 \phi}{v\theta} + \frac{V_x U_x \phi}{v^2 \theta} \right)(\tau,x) \right| dx d\tau \\
&\le \int_0^t \int_{\mathbf{R}} U_x^{\frac{5}{2}}(\tau,x) dx d\tau + O(1)\left(\underline{V}^{-4} + \underline{\Theta}^{-2}\right)\ell^{-\frac{1}{8}} \int_0^t (1+\tau)^{-\frac{5}{4}} \|(\psi,\phi)(\tau)\|^2 d\tau \\
&\le O(1)\ell^{-\frac{1}{2}} + O(1)\left(\underline{V}^{-4} + \underline{\Theta}^{-2}\right)\ell^{-\frac{1}{8}} \int_0^t (1+\tau)^{-\frac{5}{4}} \|(\psi,\phi)(\tau)\|^2 d\tau, \\
\sum_{j=10}^{11} I_j &\le O(1) \int_0^t \int_{\mathbf{R}} |(q(V,\Theta), g(V,\Theta)_x)(\tau,x)| \, dx d\tau \le O(1)\ell^{-\frac{1}{3}}, \\
\sum_{j=12}^{14} I_j &\le O(1) \int_0^t \int_{\mathbf{R}} |(r(V,\Theta), g(V,\Theta)_x)(\tau,x)| \, |(\varphi,\psi,\xi)(\tau,x)| \, dx d\tau \\
&\le O(1) \int_0^t \|(r(V,\Theta), g(V,\Theta)_x)(\tau)\| \, \|(\varphi,\psi,\xi)(\tau)\| \, d\tau \\
&\le O(1) \left\{ \int_0^t \|(r(V,\Theta), g(V,\Theta)_x)(\tau)\| \, dx \right. \\
&\qquad \left. + \int_0^t \|(r(V,\Theta), g(V,\Theta)_x)(\tau)\| \, \|(\varphi,\psi,\xi)(\tau)\|^2 \, d\tau \right\} \\
&\le O(1) \left\{ \ell^{-\frac{1}{3}} + \ell^{-\frac{1}{3}} \int_0^t (1+\tau)^{-\frac{4}{3}} \|(\varphi,\psi,\phi)(\tau)\|^2 d\tau \right\}.
\end{aligned}
$$

If we choose $\ell > 0$ sufficiently large such that

(3.9)
$$
\begin{cases}
C_1\left(\underline{V}, \overline{V}, \underline{\Theta}, \overline{\Theta}\right)\ell^{-\frac{1}{8}} < \frac{1}{2}, \\
C_1\left(\underline{V}, \overline{V}, \underline{\Theta}, \overline{\Theta}\right)\left(\left(\underline{V}^{-4} + \underline{V}^{-2}\underline{\Theta}^{-2} + \underline{V}^{-4}\underline{\Theta}^{-2} + \underline{V}^{-2}\underline{\Theta}^{-4} + \underline{\Theta}^{-2}\right)\ell^{-\frac{1}{8}} + \underline{V}^{-1}\ell^{-\frac{1}{16}}\right) \le 1,
\end{cases}
$$

we can get by substituting the above estimates into (3.8) and by exploiting Gronwall's inequality that

(3.10)
$$
\begin{aligned}
&\|(\varphi,\psi,\phi)(t)\|^2 + \int_0^t \int_{\mathbf{R}} \left(\tilde{p}(v,s) - \tilde{p}(V,\overline{s}) - \tilde{p}_v(V,\overline{s})\varphi - \tilde{p}_s(V,\overline{s})\xi \right)(\tau,x) U_x(\tau,x) dx d\tau \\
&\qquad + \int_0^t \|(\psi_x,\phi_x)(\tau)\|^2 d\tau \\
&\quad \le C_2\left(\underline{V}, \overline{V}, \underline{\Theta}, \overline{\Theta}\right) \left\{ \|(\varphi_0,\psi_0,\phi_0)\|^2 + \ell^{-\frac{1}{8}} + \ell^{-\frac{1}{8}} \int_0^t \|\varphi_x(\tau)\|^2 d\tau \right\}.
\end{aligned}
$$

Now, we turn to evaluate the term $\int_0^t \|\varphi_x(\tau)\|^2 d\tau$. To do so, we have from (2.4) and (2.5) that

$$
\begin{aligned}
&\left\{\frac{1}{2}\left(\frac{\varphi_x}{v}\right)^2-\frac{\varphi_x}{v}\psi\right\}_t-p_v(v,\theta)\frac{\varphi_x^2}{v}-\left(\frac{\psi_x^2}{v}+\frac{p_\theta(v,\theta)\varphi_x\phi_x}{v}\right)+\left(\frac{\psi\psi_x}{v}\right)_x\\
=\ &\left\{V_x\Big(p_v(v,\theta)-p_v(V,\Theta)\Big)\frac{\varphi_x}{v}+\Theta_x\Big(p_\theta(v,\theta)-p_\theta(V,\Theta)\Big)\frac{\varphi_x}{v}\right.\\
&\left.+\frac{U_x\psi\varphi_x}{v^2}-\frac{V_x\psi\psi_x}{v^2}\right\}+\mu\frac{V_x\psi_x\varphi_x}{v^3}-\mu U_{xx}\varphi_x v^2+\mu\frac{V_xU_x\varphi_x}{v^3}+g(V,\Theta)_x\frac{\varphi_x}{v}.
\end{aligned}\tag{3.11}
$$

Integrating (3.11) with respect to t and x over $[0,t]\times\mathbf{R}$, we have from $p_v(v,\theta)<0$ that

$$
\begin{aligned}
&\left\|\left(\frac{\varphi_x}{v}\right)(t)\right\|^2+\int_0^t\int_{\mathbf{R}}\left(-p_v(v,\theta)\frac{\varphi_x^2}{v}\right)(\tau,x)dxd\tau\\
&\leq O(1)\left\{\left\|\left(\frac{\varphi_{0x}}{v_0},\psi_0\right)\right\|^2+\|\psi(t)\|^2+\int_0^t\int_{\mathbf{R}}\left\{\frac{\psi_x^2}{v}+\frac{p_\theta(v,\theta)\varphi_x\phi_x}{v}\right.\right.\\
&+V_x\left[p_v(v,\theta)-p_v(V,\Theta)\right]\frac{\varphi_x}{v}+\Theta_x\left[p_\theta(v,\theta)-p_\theta(V,\Theta)\right]\frac{\varphi_x}{v}+\frac{U_x\psi\varphi_x}{v^2}+\frac{V_x\psi\psi_x}{v^2}\\
&\qquad\left.\left.+\frac{V_x\psi_x\varphi_x}{v^3}-\frac{U_{xx}\varphi_x}{v^2}+\frac{V_xU_x\varphi_x}{v^3}+g(V,\Theta)_x\frac{\varphi_x}{v}\right\}(\tau,x)dxd\tau\right\}\\
&=O(1)\left\{\left\|\left(\frac{\varphi_{0x}}{v_0},\psi_0\right)\right\|^2+\|\psi(t)\|^2+\sum_{j=15}^{24}I_j\right\},
\end{aligned}\tag{3.12}
$$

where I_j $(j=15,\cdots,24)$ denote the corresponding terms in the above inequality.

Now we estimate I_j $(j=15,\cdots,24)$ term by term. For this purpose, we have from Lemma 2.2, the a priori assumption (3.3) and Cauchy-Schwarz's inequality that

$$
\begin{aligned}
|I_{15}|&\leq \underline{V}^{-1}\int_0^t\int_{\mathbf{R}}\psi_x^2(\tau,x)dxd\tau,\\
|I_{16}|&\leq -\frac{1}{8}\int_0^t\int_{\mathbf{R}}\left(\frac{p_v(v,\theta)}{v}\varphi_x^2\right)(\tau,x)dxd\tau+C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\int_0^t\|\phi_x(\tau)\|^2d\tau,\\
|I_{17}|&\leq -\frac{1}{8}\int_0^t\int_{\mathbf{R}}\left(\frac{p_v(v,\theta)}{v}\varphi_x^2\right)(\tau,x)dxd\tau+C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\ell^{-1}\int_0^t\int_{\mathbf{R}}\Big(\tilde{p}(v,s)\\
&\qquad-\tilde{p}(V,\overline{s})-\tilde{p}_v(V,\overline{s})\psi-\tilde{p}_s(V,\overline{s})\xi\Big)(\tau,x)U_x(\tau,x)dxd\tau,\\
|I_{18}|&\leq -\frac{1}{8}\int_0^t\int_{\mathbf{R}}\left(\frac{p_v(v,\theta)}{v}\varphi_x^2\right)(\tau,x)dxd\tau+C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\ell^{-1}\int_0^t\int_{\mathbf{R}}\Big(\tilde{p}(v,s)\\
&\qquad-\tilde{p}(V,\overline{s})-\tilde{p}_v(V,\overline{s})\psi-\tilde{p}_s(V,\overline{s})\xi\Big)(\tau,x)U_x(\tau,x)dxd\tau,\\
|I_{19}|&\leq -\frac{1}{8}\int_0^t\int_{\mathbf{R}}\left(\frac{p_v(v,\theta)}{v}\varphi_x^2\right)(\tau,x)dxd\tau\\
&\qquad\qquad+C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\ell^{-\frac{1}{4}}\int_0^t(1+\tau)^{-\frac{7}{4}}\|\psi(\tau)\|^2d\tau,
\end{aligned}
$$

$$
\begin{aligned}
|I_{20}| &\leq \int_0^t \int_{\mathbf{R}} \psi_x^2(\tau,x)dxd\tau + C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\ell^{-\frac{1}{4}} \int_0^t (1+\tau)^{-\frac{7}{4}} \|\psi(\tau)\|^2 d\tau,\\
|I_{21}| &\leq -\frac{1}{8}\int_0^t \int_{\mathbf{R}} \left(\frac{p_v(v,\theta)}{v}\varphi_x^2\right)(\tau,x)dxd\tau + C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\ell^{-2} \int_0^t \|\psi_x(\tau)\|^2 d\tau,\\
|I_{22}| &\leq -\frac{1}{8}\int_0^t \int_{\mathbf{R}} \left(\frac{p_v(v,\theta)}{v}\varphi_x^2\right)(\tau,x)dxd\tau + C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta}) \int_0^t \int_{\mathbf{R}} U_{xx}^2(\tau,x)dxd\tau\\
&\leq -\frac{1}{8}\int_0^t \int_{\mathbf{R}} \left(\frac{p_v(v,\theta)}{v}\varphi_x^2\right)(\tau,x)dxd\tau + C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\ell^{-\frac{5}{4}},\\
|I_{23}| &\leq -\frac{1}{8}\int_0^t \int_{\mathbf{R}} \left(\frac{p_v(v,\theta)}{v}\varphi_x^2\right)(\tau,x)dxd\tau + C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta}) \int_0^t \int_{\mathbf{R}} V_x^4(\tau,x)dxd\tau\\
&\leq -\frac{1}{8}\int_0^t \int_{\mathbf{R}} \left(\frac{p_v(v,\theta)}{v}\varphi_x^2\right)(\tau,x)dxd\tau + C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\ell^{-\frac{1}{4}},\\
|I_{24}| &\leq \int_0^t \|g(V,\Theta)_x(\tau)\|\, d\tau + \int_0^t \|g(V,\Theta)_x(\tau)\| \left\|\left(\frac{\varphi_x}{v}\right)(\tau)\right\|^2 d\tau\\
&\leq O(1)\ell^{-\frac{1}{3}} + C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\ell^{-\frac{4}{3}} \int_0^t \|\varphi_x(\tau)\|^2 d\tau.
\end{aligned}
$$

Inserting the above estimates into (3.12), we can get that

$$
\begin{aligned}
&\|\varphi_x(t)\|^2 + \int_0^t \|\varphi_x(\tau)\|^2 d\tau\\
\leq\ & C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\Bigg\{ \|(\varphi_{0x},\psi_0)\|^2 + \|\psi(t)\|^2 + \ell^{-\frac{1}{3}}\\
&+ \int_0^t \|(\psi_x(\tau),\phi_x(\tau))\|^2 d\tau + \ell^{-\frac{4}{3}} \int_0^t \|\varphi_x(\tau)\|^2 d\tau\\
&+ \ell^{-1} \int_0^t \int_{\mathbf{R}} \Big(\tilde{p}(v,s) - \tilde{p}(V,\overline{s}) - \tilde{p}_v(V,\overline{s})\psi - \tilde{p}_s(V,\overline{s})\xi\Big)(\tau,x)U_x(\tau,x)dxd\tau\\
&+ \ell^{-\frac{1}{4}} \int_0^t (1+\tau)^{-\frac{7}{4}} \|\psi(\tau)\|^2 d\tau \Bigg\}.
\end{aligned} \tag{3.13}
$$

(3.10) together with (3.13) imply

$$
\begin{aligned}
&\|\varphi_x(t)\|^2 + \int_0^t \|\varphi_x(\tau)\|^2 d\tau\\
&\qquad \leq C_3(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta}) \left(\|(\varphi_{0x},\varphi_0,\psi_0,\phi_0)\|^2 + \ell^{-\frac{1}{8}} \int_0^t \|\varphi_x(\tau)\|^2 d\tau \right).
\end{aligned} \tag{3.14}
$$

Having obtained (3.14), if we choose ℓ sufficiently large such that

$$
C_3(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\ell^{-\frac{1}{8}} < \frac{1}{2}, \tag{3.15}
$$

we can get from the assumption (1.23) that

$$
\|\varphi_x(t)\|^2 + \int_0^t \|\varphi_x(\tau)\|^2 d\tau \leq C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta}) \left(1+\ell^{\beta}\right). \tag{3.16}
$$

This proves (3.5).

With (3.16) in hand, we can get from (3.10), (3.16) and the assumption (1.23) that

(3.17)
$$\|(\varphi,\psi,\phi)(t)\|^2+\int_0^t\int_{\mathbf{R}}\Big(\tilde{p}(v,s)-\tilde{p}(V,\overline{s})-\tilde{p}_v(V,\overline{s})\varphi-\tilde{p}_s(V,\overline{s})\xi\Big)(\tau,x)U_x(\tau,x)dxd\tau$$
$$+\int_0^t\|(\psi_x,\phi_x)(\tau)\|^2d\tau\le C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\left(\ell^{-\alpha}+\ell^{-\frac{1}{8}}\left(1+\ell^{\beta}\right)\right),$$

which implies that (3.2) holds.

To complete the proof of Lemma 3.2, we need to deduce an estimate on $\|(\psi_x,\phi_x)(t)\|^2$. We first estimate $\|\psi_x(t)\|^2$. To this end, multiplying $(2.4)_2$ by $-\psi_{xx}(t,x)$, we have

$$\begin{aligned}
&\left(\frac{\psi_x^2}{2}\right)_t+\mu\frac{\psi_{xx}^2}{v}-(\psi_t\psi_x)_x\\
=\;&(p_v(v,\theta)\varphi_x+p_\theta(v,\theta)\phi_x)\psi_{xx}+\mu\frac{\varphi_x\psi_x\psi_{xx}}{v^2}+U_x\Big(p_v(v,\theta)-p_v(V,\Theta)\Big)\psi_{xx}\\
&+\Theta_x\Big(p_\theta(v,\theta)-p_\theta(V,\Theta)\Big)\psi_{xx}+\left(\mu\frac{V_x\psi_x\psi_{xx}}{v^2}+\mu\frac{U_x\varphi_x\psi_{xx}}{v^2}\right)-\mu\frac{U_{xx}\psi_{xx}}{v}\\
&+\mu\frac{U_xV_x\psi_{xx}}{v^2}+g(V,\Theta)_x\psi_{xx}.
\end{aligned}$$

Integrating the above identity with respect to t and x over $[0,t]\times\mathbf{R}$, we can get that

(3.18)
$$\begin{aligned}
&\|\psi_x(t)\|^2+\mu\int_0^t\int_{\mathbf{R}}\psi_{xx}^2(\tau,x)dxd\tau\\
&\le O(1)\Bigg\{\|\psi_{0x}\|^2+\int_0^t\int_{\mathbf{R}}\Big(p_v(v,\theta)\varphi_x\psi_{xx}+p_\theta(v,\theta)\phi_x\psi_{xx}+\mu\varphi_x\psi_x\psi_{xx}\\
&\qquad+V_x\Big(p_v(v,\theta)-p_v(V,\Theta)\Big)\psi_{xx}+\mu V_x\psi_x\psi_{xx}+\mu U_x\varphi_x\psi_{xx}-\mu U_{xx}\psi_{xx}\\
&\qquad\qquad+\mu V_xU_x\psi_{xx}+g(V,\Theta)_x\psi_{xx}\Big)(\tau,x)dxd\tau\Bigg\}\\
&\le O(1)\left\{\|\psi_{0x}\|^2+\sum_{j=25}^{34}I_j\right\},
\end{aligned}$$

where I_j $(j=25,\cdots,34)$ denote the corresponding terms in the above inequality.

By employing (3.2), (3.5) and Lemma 2.2, we have from Cauchy-Schwarz's inequality that

$$\begin{aligned}
I_{25}&\le\frac{\mu}{12}\int_0^t\int_{\mathbf{R}}\psi_{xx}^2(\tau,x)dxd\tau+C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\int_0^t\int_{\mathbf{R}}\varphi_x^2(\tau,x)dxd\tau\\
&\le\frac{\mu}{12}\int_0^t\int_{\mathbf{R}}\psi_{xx}^2(\tau,x)dxd\tau++C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\left(1+\ell^{\beta}\right),
\end{aligned}$$

$$
\begin{aligned}
I_{26} &\le \frac{\mu}{12}\int_0^t\int_{\mathbf{R}}\psi_{xx}^2(\tau,x)dxd\tau + C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\int_0^t\|\phi_x(\tau)\|^2 d\tau\\
&\le \frac{\mu}{12}\int_0^t\int_{\mathbf{R}}\psi_{xx}^2(\tau,x)dxd\tau + C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\left(\ell^{-\alpha}+\ell^{-\frac{1}{8}}\left(1+\ell^{\beta}\right)\right),\\
I_{27} &\le \mu\int_0^t\|\varphi_x(\tau)\|\,\|\psi_x(\tau)\|^{\frac{1}{2}}\,\|\psi_{xx}(\tau)\|^{\frac{3}{2}}\,d\tau\\
&\le \frac{\mu}{12}\int_0^t\int_{\mathbf{R}}\psi_{xx}^2(\tau,x)dxd\tau + O(1)\int_0^t\|\varphi_x(\tau)\|^4\,\|\psi_x(\tau)\|^2 d\tau\\
&\le \frac{\mu}{12}\int_0^t\int_{\mathbf{R}}\psi_{xx}^2(\tau,x)dxd\tau + C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\sup_{\tau\in[0,t]}\left\{\|\varphi_x(\tau)\|^4\right\}\int_0^t\|\psi_x(\tau)\|^2 d\tau\\
&\le \frac{\mu}{12}\int_0^t\int_{\mathbf{R}}\psi_{xx}^2(\tau,x)dxd\tau + C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\left(\ell^{2\beta-\alpha}+\ell^{3\beta-\frac{1}{8}}\right),\\
I_{28} &\le \frac{\mu}{12}\int_0^t\int_{\mathbf{R}}\psi_{xx}^2(\tau,x)dxd\tau + C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\ell^{-1}\int_0^t\int_{\mathbf{R}}U_x(\tau,x)|(\varphi,\xi)(\tau,x)|^2dxd\tau\\
&\le \frac{\mu}{12}\int_0^t\int_{\mathbf{R}}\psi_{xx}^2(\tau,x)dxd\tau + C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\ell^{-1}\left(\ell^{-\alpha}+\ell^{-\frac{1}{8}}\left(1+\ell^{\beta}\right)\right),\\
I_{29} &\le \frac{\mu}{12}\int_0^t\int_{\mathbf{R}}\psi_{xx}^2(\tau,x)dxd\tau + C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\ell^{-1}\int_0^t\int_{\mathbf{R}}U_x(\tau,x)|(\varphi,\xi)(\tau,x)|^2dxd\tau\\
&\le \frac{\mu}{12}\int_0^t\int_{\mathbf{R}}\psi_{xx}^2(\tau,x)dxd\tau + C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\ell^{-1}\left(\ell^{-\alpha}+\ell^{-\frac{1}{8}}\left(1+\ell^{\beta}\right)\right),\\
I_{30} &\le \frac{\mu}{12}\int_0^t\int_{\mathbf{R}}\psi_{xx}^2(\tau,x)dxd\tau + C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\ell^{-2}\int_0^t\|\psi_x(\tau)\|^2 d\tau\\
&\le \frac{\mu}{12}\int_0^t\int_{\mathbf{R}}\psi_{xx}^2(\tau,x)dxd\tau + C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\ell^{-2}\left(\ell^{-\alpha}+\ell^{-\frac{1}{8}}\left(1+\ell^{\beta}\right)\right),\\
I_{31} &\le \frac{\mu}{12}\int_0^t\int_{\mathbf{R}}\psi_{xx}^2(\tau,x)dxd\tau + C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\ell^{-2}\int_0^t\|\varphi_x(\tau)\|^2 d\tau\\
&\le \frac{\mu}{12}\int_0^t\int_{\mathbf{R}}\psi_{xx}^2(\tau,x)dxd\tau + C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\ell^{-2}\left(1+\ell^{\beta}\right),\\
I_{32} &\le \frac{\mu}{12}\int_0^t\int_{\mathbf{R}}\psi_{xx}^2(\tau,x)dxd\tau + O(1)\int_0^t\int_{\mathbf{R}}U_{xx}^2(\tau,x)dxd\tau\\
&\le \frac{\mu}{12}\int_0^t\int_{\mathbf{R}}\psi_{xx}^2(\tau,x)dxd\tau + O(1)\ell^{-\frac{5}{4}},\\
I_{33} &\le \frac{\mu}{12}\int_0^t\int_{\mathbf{R}}\psi_{xx}^2(\tau,x)dxd\tau + O(1)\int_0^t\int_{\mathbf{R}}\left(V_x^2U_x^2\right)(\tau,x)dxd\tau\\
&\le \frac{\mu}{12}\int_0^t\int_{\mathbf{R}}\psi_{xx}^2(\tau,x)dxd\tau + O(1)\ell^{-2},
\end{aligned}
$$

and

$$
\begin{aligned}
I_{34} &\le \frac{\mu}{12}\int_0^t\int_{\mathbf{R}}\psi_{xx}^2(\tau,x)dxd\tau + \int_0^t\int_{\mathbf{R}}|g(V,\Theta)_x(\tau,x)|^2dxd\tau\\
&\le \frac{\mu}{12}\int_0^t\int_{\mathbf{R}}\psi_{xx}^2(\tau,x)dxd\tau + O(1)\ell^{-\frac{1}{3}}.
\end{aligned}
$$

Putting the above estimates into (3.18), we have

(3.19)
$$\|\psi_x(t)\|^2 + \int_0^t \int_{\mathbf{R}} \psi_{xx}^2(\tau,x)dxd\tau \le C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})\left(1+\ell^{3\beta-\frac{1}{8}}+\ell^{\beta}+\ell^{2\beta-\alpha}\right).$$

This proves (3.6).

As to the estimate of $\|\phi_x(t)\|^2$, due to

$$\begin{aligned}
&\left(\frac{\phi_x^2}{2}\right)_t + \frac{\kappa}{ve_\theta(v,\theta)}\phi_{xx}^2 - (\phi_x\phi_t)_x = \\
&\frac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)}\psi_x\phi_{xx} + \left\{\frac{\kappa}{e_\theta(v,\theta)}\frac{\varphi_x\phi_x}{v^2} - \frac{\mu}{e_\theta(v,\theta)}\frac{\psi_x^2}{v} + U_x\left[\frac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)} - \frac{\Theta p_\theta(V,\Theta)}{e_\theta(V,\Theta)}\right]\right\}\phi_{xx} \\
&+\left\{\frac{\kappa}{e_\theta(v,\theta)}\left(\frac{V_x\phi_x\phi_{xx}}{v^2}+\frac{\Theta_x\varphi_x\phi_{xx}}{v^2}\right) - \frac{2\mu}{e_\theta(v,\theta)}\frac{V_x\psi_x\phi_{xx}}{v}\right\} - \frac{\kappa}{e_\theta(v,\theta)}\frac{\Theta_{xx}\phi_{xx}}{v} \\
&+\left\{\frac{\kappa}{e_\theta(v,\theta)}\frac{V_x\Theta_x\phi_{xx}}{v^2} + \frac{\mu}{e_\theta(v,\theta)}\frac{U_x^2\phi_{xx}}{v}\right\} + r(V,\Theta)\phi_{xx},
\end{aligned}$$

(3.7) can be proved by repeating the argument used above. We omit the details for brevity. This completes the proof of Lemma 3.2.

Now we turn to prove Theorem 1.1. The main idea is to combine the energy estimates obtained in Lemma 3.2 with the continuation argument to extend the local solution step by step to a global one. Our main trick is to use the parameter $\ell = \frac{t_0}{\delta}$ introduced in constructing the smooth approximation of the rarefaction wave solution to control the growth of the solution caused by the nonlinearity of the equation and/or by the interaction of waves from different families. It is worth pointing out that to do so we need to require the initial perturbation to satisfy the assumption (1.23).

Since the initial data $(\varphi_0(x),\psi_0(x),\phi_0(x))$ is assumed to satisfy (1.22) and (1.23), we have from the local existence result Lemma 3.1 that there exists a sufficiently small positive constant $t_1>0$ such that the Cauchy problem (2.4), (2.5) admits a unique solution $(\varphi(t,x),\psi(t,x),\phi(t,x))\in X(0,t_1)$ on the strip $[0,t_1]\times\mathbf{R}$ and for all $0\le t\le t_1, x\in\mathbf{R}$, $(\varphi(t,x),\psi(t,x),\phi(t,x))$ satisfies

$$\left\{\begin{array}{l} \left\|(\varphi,\psi,\phi)(t)\right\|^2 \le 2D_1\ell^{-\alpha}, \\ \left\|(\varphi_x,\psi_x,\phi_x)(t)\right\|^2 \le 2D_2\left(1+\ell^\beta\right). \end{array}\right. \tag{3.20}$$

(3.20) together with Sobolev's inequality imply

$$\begin{aligned}
\|(\varphi(t),\phi(t))\|_{L_x^\infty} &\le \|(\varphi(t),\phi(t))\|^{\frac12}\|(\varphi_x(t),\phi_x(t))\|^{\frac12} \\
&\le \sqrt[4]{4D_1D_2}\,\ell^{-\frac{\alpha}{4}}\left(1+\ell^{\frac{\beta}{4}}\right), \quad 0\le t\le t_1.
\end{aligned}\tag{3.21}$$

Consequently, since $\alpha>\beta$, we can choose $\ell>\ell_2$ with ℓ_2 suitably large such that

$$\sqrt[4]{4D_1D_2}\,\ell_2^{-\frac{\alpha}{4}}\left(1+\ell_2^{\frac{\beta}{4}}\right) < \min\left\{\underline{V},\frac{\overline{V}}{2},\underline{\Theta},\frac{\overline{\Theta}}{2}\right\}. \tag{3.22}$$

Then from the assumptions (1.22) and (3.21), we can deduce that

$$\underline{V}\le v(t,x)\le\overline{V},\ \underline{\Theta}\le\theta(t,x)\le\overline{\Theta},\quad \forall (t,x)\in[0,t_1]\times\mathbf{R}. \tag{3.23}$$

Having obtained (3.23), for $t\in[0,t_1], x\in\mathbf{R}$, we can deduce from Lemma 3.2 that there exists a sufficiently large constant $\ell_1>0$ such that for all $\ell\ge\ell_1$, the solution $(\varphi(t,x),\psi(t,x),\phi(t,x))$ to the Cauchy problem (2.4), (2.5) obtained above satisfies

$$\|(\varphi,\psi,\phi)(t)\|^2 \le D_6\left(\ell^{-\alpha}+\ell^{-\frac{1}{8}}\left(1+\ell^{\beta}\right)\right), \tag{3.24}$$

$$\|\varphi_x(t)\|^2 \le D_7\left(1+\ell^{\beta}\right), \tag{3.25}$$

$$\|\psi_x(t)\|^2 \le D_8\left(1+\ell^{3\beta-\frac{1}{8}}+\ell^{\beta}+\ell^{2\beta-\alpha}\right), \tag{3.26}$$

and

$$\|\phi_x(t)\|^2 \le D_9\left(1+\ell^{6\beta-\frac{1}{4}}+\ell^{2\beta}+\ell^{4\beta-2\alpha}\right). \tag{3.27}$$

Now taking $(\psi(t_1,x),\varphi(t_1,x),\phi(t_1,x))$ as initial data, we get from (3.24)–(3.27) that

$$\|(\varphi,\psi,\phi)(t_1)\|^2 \le D_6\left(\ell^{-\alpha}+\ell^{-\frac{1}{8}}\left(1+\ell^{\beta}\right)\right), \tag{3.28}$$

$$\|\varphi_x(t_1)\|^2 \le D_7\left(1+\ell^{\beta}\right), \tag{3.29}$$

$$\|\psi_x(t_1)\|^2 \le D_8\left(1+\ell^{3\beta-\frac{1}{8}}+\ell^{\beta}+\ell^{2\beta-\alpha}\right), \tag{3.30}$$

and

$$\|\phi_x(t_1)\|^2 \le D_9\left(1+\ell^{6\beta-\frac{1}{4}}+\ell^{2\beta}+\ell^{4\beta-2\alpha}\right). \tag{3.31}$$

By exploiting the local existence result Lemma 3.1 again, we can deduce that there exists a sufficiently small positive constant $t_2>0$ such that the solution $(\psi(t,x),\varphi(t,x),\phi(t,x))$ obtained above can be extended to the time $t=t_1+t_2$ and for $t_1\le t\le t_1+t_2$, $(\psi(t,x),\varphi(t,x),\phi(t,x))$ satisfies

$$\|(\varphi,\psi,\phi)(t)\|^2 \le 2D_6\left(\ell^{-\alpha}+\ell^{-\frac{1}{8}}\left(1+\ell^{\beta}\right)\right), \tag{3.32}$$

$$\|\varphi_x(t)\|^2 \le 2D_7\left(1+\ell^{\beta}\right), \tag{3.33}$$

$$\|\psi_x(t)\|^2 \le 2D_8\left(1+\ell^{3\beta-\frac{1}{8}}+\ell^{\beta}+\ell^{2\beta-\alpha}\right), \tag{3.34}$$

and

$$\|\phi_x(t)\|^2 \le 2D_9\left(1+\ell^{6\beta-\frac{1}{4}}+\ell^{2\beta}+\ell^{4\beta-2\alpha}\right). \tag{3.35}$$

For each $(t,x)\in[t_1,t_1+t_2]\times\mathbf{R}$, (3.32)–(3.35) together with Sobolev's inequality imply

$$\begin{aligned}\|(\varphi(t),\phi(t))\|_{L^\infty_x} &\le \|(\varphi(t),\phi(t))\|^{\frac{1}{2}}\|(\varphi_x(t),\phi_x(t))\|^{\frac{1}{2}}\\ &\le \sqrt[4]{4D_6D_7}\left(\ell^{-\frac{\alpha}{4}}+\ell^{\frac{\beta}{4}-\frac{1}{32}}\right)\left(1+\ell^{\frac{\beta}{4}}\right)\\ &\quad+\sqrt[4]{4D_6D_9}\left(\ell^{-\frac{\alpha}{4}}+\ell^{\frac{\beta}{4}-\frac{1}{32}}\right)\left(1+\ell^{\frac{3}{2}\beta-\frac{1}{16}}+\ell^{\beta-\frac{\alpha}{2}}\right).\end{aligned} \tag{3.36}$$

If α and β satisfy the following condition,

$$\beta<\min\left\{\frac{3}{4}\alpha,\frac{\alpha}{6}+\frac{1}{24},\frac{3}{56},\frac{2}{5}\alpha+\frac{1}{40}\right\}, \tag{3.37}$$

and $\ell \geq \ell_3$ with ℓ_3 suitably large such that

(3.38)
$$\begin{cases} \sqrt[4]{4D_6D_7}\left(\ell_3^{-\frac{\alpha}{4}}+\ell_3^{\frac{\beta}{4}-\frac{1}{32}}\right)\left(1+\ell_3^{\frac{\beta}{4}}\right)<\frac{1}{2}\min\left\{\underline{V},\frac{\overline{V}}{2},\underline{\Theta},\frac{\overline{\Theta}}{2}\right\}, \\ \sqrt[4]{4D_6D_9}\left(\ell_3^{-\frac{\alpha}{4}}+\ell_3^{\frac{\beta}{4}-\frac{1}{32}}\right)\left(1+\ell_3^{\frac{3}{2}\beta-\frac{1}{16}}+\ell_3^{\beta-\frac{\alpha}{2}}\right)<\frac{1}{2}\min\left\{\underline{V},\frac{\overline{V}}{2},\underline{\Theta},\frac{\overline{\Theta}}{2}\right\}, \end{cases}$$

then we can deduce from (3.38), (3.23) and the assumption (1.22) that

$$\underline{V} \leq v(t,x) \leq \overline{V},\ \underline{\Theta} \leq \theta(t,x) \leq \overline{\Theta}, \quad \forall (t,x) \in [0, t_1+t_2] \times \mathbf{R}. \tag{3.39}$$

Once we have obtained (3.39), we can get from Lemma 3.2 that there exists a sufficiently large $\ell_1 > 0$ such that if $\ell \geq \ell_1$, the following estimates hold for $0 \leq t \leq t_1 + t_2$:

$$\|(\varphi,\psi,\phi)(t)\|^2 \leq D_6\left(\ell^{-\alpha}+\ell^{-\frac{1}{8}}\left(1+\ell^{\beta}\right)\right), \tag{3.40}$$

$$\|\varphi_x(t)\|^2 \leq D_7\left(1+\ell^{\beta}\right), \tag{3.41}$$

$$\|\psi_x(t)\|^2 \leq D_8\left(1+\ell^{3\beta-\frac{1}{8}}+\ell^{\beta}+\ell^{2\beta-\alpha}\right), \tag{3.42}$$

and

$$\|\phi_x(t)\|^2 \leq D_9\left(1+\ell^{6\beta-\frac{1}{4}}+\ell^{2\beta}+\ell^{4\beta-2\alpha}\right). \tag{3.43}$$

Now taking $(\varphi(t_1+t_2,x), \phi(t_1+t_2,x), \psi(t_1+t_2,x))$ as initial data, since

$$\|(\varphi,\psi,\phi)(t_1+t_2)\|^2 \leq D_6\left(\ell^{-\alpha}+\ell^{-\frac{1}{8}}\left(1+\ell^{\beta}\right)\right), \tag{3.44}$$

$$\|\varphi_x(t_1+t_2)\|^2 \leq D_7\left(1+\ell^{\beta}\right), \tag{3.45}$$

$$\|\psi_x(t_1+t_2)\|^2 \leq D_8\left(1+\ell^{3\beta-\frac{1}{8}}+\ell^{\beta}+\ell^{2\beta-\alpha}\right), \tag{3.46}$$

and

$$\|\phi_x(t_1+t_2)\|^2 \leq D_9\left(1+\ell^{6\beta-\frac{1}{4}}+\ell^{2\beta}+\ell^{4\beta-2\alpha}\right), \tag{3.47}$$

we can deduce, by exploiting the local existence result Lemma 3.1, once more that the solution $(\varphi(t,x), \psi(t,x), \phi(t,x))$ obtained above can be extended to the time $t = t_1 + 2t_2$ and for $t_1 + t_2 \leq t \leq t_1 + 2t_2$, $(\varphi(t,x), \psi(t,x), \phi(t,x))$ satisfies (3.32)–(3.35).

Repeating the above procedure and noticing that the constants D_6, D_7, D_8 and D_9 are independent of each time step, we can thus extend the solution $(\varphi(t,x), \phi(t,x), \psi(t,x))$ step by step to the whole $\mathbf{R}_+$ provided that

$$\ell \geq \max\{\ell_1, \ell_2, \ell_3\}. \tag{3.48}$$

As a byproduct, one can deduce that if we let ℓ satisfy (3.48), the Cauchy problem (2.4), (2.5) admits a unique global solution $(\varphi(t,x),\psi(t,x),\phi(t,x))$ which satisfies (3.2)–(3.7) for all $t\in\mathbf{R}_+$. Having obtained these estimates, the estimate (1.25) follows immediately by employing the standard method. This completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

This section is devoted to proving Theorem 1.2. For this purpose, note that for the ideal polytropic gas, $p(v,\theta)$ and $e(v,\theta)$ satisfy the special constitutive relations (1.26). From (1.26) and the assumptions listed in Theorem 1.2, we can get

$$\|\phi_0\|_1\le C(\underline{V},\overline{V},\underline{\Theta},\overline{\Theta})(\gamma-1)\|(\varphi_0,\xi_0)\|_1. \tag{4.1}$$

Moreover, the entropy $\eta(v,u,\theta;V,U,\Theta)$ defined by (2.2) takes the form

$$\eta(v,u,\theta;V,U,\Theta)=R\Theta\Phi\left(\frac{v}{V}\right)+\frac{1}{2}(u-U)^2+\frac{R\Theta}{\gamma-1}\Phi\left(\frac{\theta}{\Theta}\right) \tag{4.2}$$

with

$$\Phi(s)=s-\ln s-1. \tag{4.3}$$

We look for a solution $(\varphi(t,x),\psi(t,x),\phi(t,x))$ of (2.4), (2.5) in the solution space

$$X(0,\infty):=\bigcup_{0<\underline{m}<\overline{m}<\infty}\ \bigcap_{0\le t_1<t_2<\infty}X_{\underline{m},\overline{m},M}\left(t_1,t_2;\underline{\Theta},\overline{\Theta}\right),$$

where for $t_1,\ t_2$ $(0\le t_1<t_2<\infty)$ and $\underline{m},\ \overline{m},\ M$ $(0<\underline{m}<\overline{m}<\infty,\ 0<M<\infty)$,

$$X_{\underline{m},\overline{m},M}(t_1,t_2;\underline{\Theta},\overline{\Theta})=\left\{(\varphi,\psi,\phi)(t,x)\ \middle|\ \begin{array}{c}(\varphi,\psi,\phi)(t,x)\in C^0([t_1,t_2];H^1(\mathbf{R}))\\ (\psi,\phi)(t,x)\in L^2(t_1,t_2;H^1(\mathbf{R}))\\ 0<\underline{\Theta}\le\phi(t,x)+\Theta(t,x)\le\overline{\Theta}\\ 0<\underline{m}\le\varphi(t,x)+V(t,x)\le\overline{m}\\ \sup\limits_{[t_1,t_2]}\|(\varphi,\psi,\phi)(t)\|_1\le M\end{array}\right\}.$$

Under the assumptions listed in Theorem 1.2, the local existence result (Lemma 3.1) tells us that the Cauchy problem (2.4), (2.5) admits a unique solution $(\varphi,\psi,\phi)(t,x)\in X_{\underline{V},\overline{V},M}(0,t_0;\underline{\Theta},\overline{\Theta})$ for some sufficiently small positive constant $t_0>0$ with $M=2\|(\varphi_0,\psi_0,\phi_0)\|_1$. Now suppose that such a solution has been extended to the time step $t=T$ with $(\varphi,\psi,\phi)(t,x)\in X_{M_1^{-1},M_1,M}(0,T;\underline{\Theta},\overline{\Theta})$ for some $T\ge t_0, M_1>0$ and $M>0$. That is, for each $(t,x)\in[0,T]\times\mathbf{R}$, $(v(t,x),\theta(t,x))$ satisfies the following a priori assumption:

$$\begin{cases}0<\frac{1}{M_1}\le v(t,x)=\varphi(t,x)+V(t,x)\le M_1,\\ 0<\underline{\Theta}\le\theta(t,x)=\phi(t,x)+\Theta(t,x)\le\overline{\Theta}.\end{cases} \tag{4.4}$$

We now deduce certain energy estimates on the solution $(\varphi(t,x),\psi(t,x),\phi(t,x))$.

For this purpose, we have from (2.2), (2.6), (4.2) and (4.3) that

$$
\begin{aligned}
&\int_{\mathbf{R}}\left\{R\Theta\Phi\left(\frac{v}{V}\right)+\frac{1}{2}\psi^2+\frac{R\Theta}{\gamma-1}\Phi\left(\frac{\theta}{\Theta}\right)\right\}(t,x)dx\\
&+\int_0^t\int_{\mathbf{R}}\left\{\mu\Theta\frac{\psi_x^2}{v\theta}+\kappa\Theta\frac{\phi_x^2}{v\theta^2}\right\}(\tau,x)dxd\tau\\
&+\int_0^t\int_{\mathbf{R}}\Big(\tilde{p}(v,s)-\tilde{p}(V,\overline{s})-\tilde{p}_v(V,\overline{s})\psi-\tilde{p}_s(V\tilde{s})\xi\Big)(\tau,x)U_x(\tau,x)dxd\tau\\
=&\int_{\mathbf{R}}\left\{R\Theta\Phi\left(\frac{v}{V}\right)+\frac{1}{2}\psi^2+\frac{R\Theta}{\gamma-1}\Phi\left(\frac{\theta}{\Theta}\right)\right\}(0,x)dx+\sum_{j=1}^4 R_j.
\end{aligned}\tag{4.5}
$$

Here

$$
\begin{cases}
R_1=\displaystyle\int_0^t\int_{\mathbf{R}}\left\{-\mu\frac{U_x\psi\varphi_x}{v^2}+2\mu\frac{U_x\psi_x\phi}{v\theta}-\kappa\frac{\Theta_x\phi\varphi_x}{v\theta^2}+\kappa\frac{\Theta_x\phi\phi_x}{v\theta^2}\right\}(\tau,x)dxd\tau,\\
R_2=\displaystyle\int_0^t\int_{\mathbf{R}}\left(\mu\frac{U_{xx}\varphi}{v}+\kappa\frac{\Theta_{xx}\phi}{v\theta}\right)(\tau,x)dxd\tau,\\
R_3=\displaystyle\int_0^t\int_{\mathbf{R}}\left(-\mu\frac{V_xU_x\psi}{v^2}+\mu\frac{U_x^2\phi}{v\theta}+\frac{V_x\Theta_x\phi}{v^2\theta}\right)(\tau,x)dxd\tau,\\
R_4=\displaystyle\int_0^t\int_{\mathbf{R}}\Big(-q(V,\Theta)-g(V,\Theta)_xU-p_\theta(V,\Theta)r(V,\Theta)\psi\\
\qquad\qquad+g(V,\Theta)_x\psi+r(V,\Theta)\xi\Big)(\tau,x)dxd\tau.
\end{cases}
$$

From the a priori assumptions (4.4), we have from (4.5) that

$$
\begin{aligned}
&\int_{\mathbf{R}}\left\{\Phi\left(\frac{v}{V}\right)+\psi^2+\frac{\phi^2}{\gamma-1}\right\}(t,x)dx+\int_0^t\int_{\mathbf{R}}\left\{\frac{\psi_x^2}{v}+\frac{\phi_x^2}{v}\right\}(\tau,x)dxd\tau\\
&\qquad\le C_4(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})\left\|\left(\varphi_0,\psi_0,\frac{\phi_0}{\sqrt{\gamma-1}}\right)\right\|^2+\sum_{j=1}^4 R_j'.
\end{aligned}\tag{4.6}
$$

Here $R_j'=C_4(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})R_j$ $(j=1,2,3,4)$ and note that the positive constant $C_4(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})$ in (4.6) is independent of M_1 .

Now we estimate R_j' $(j=1,2,3,4)$ term by term. In fact, from the a priori assumption (4.4), Cauchy-Schwarz's inequality and Lemma 2.2, we have

$$
\begin{aligned}
R_1'\le&\int_0^t\int_{\mathbf{R}}\left(\ell^{-\frac{1}{8}}\frac{\varphi_x^2}{v^3}+\frac{1}{2}\frac{\psi_x^2}{v}+\frac{1}{2}\frac{\phi_x^2}{v}\right)(\tau,x)dxd\tau\\
&+O(1)M_1\ell^{\frac{1}{8}}\int_0^t\|U_x(\tau)\|_{L^\infty}^2\left\|\left(\psi,\frac{\phi}{\sqrt{\gamma-1}}\right)(\tau)\right\|^2d\tau\\
\le&\ \ell^{-\frac{1}{8}}\int_0^t\int_{\mathbf{R}}\frac{\varphi_x^2}{v^3}(\tau,x)dxd\tau+\int_0^t\int_{\mathbf{R}}\left(\frac{1}{2}\frac{\psi_x^2}{v}+\frac{1}{2}\frac{\phi_x^2}{v}\right)(\tau,x)dxd\tau\\
&+O(1)M_1\ell^{-\frac{1}{8}}\max_{\tau\in[0,t]}\left\{\left\|\left(\psi,\frac{\phi}{\sqrt{\gamma-1}}\right)(\tau)\right\|^2\right\},
\end{aligned}\tag{4.7}
$$

$$
\begin{aligned}
(4.8)\quad R_2' &\le O(1)M_1\int_0^t\Big(\|U_{xx}(\tau)\|\|\psi(\tau)\|+\|\Theta_{xx}(\tau)\|\|\phi(\tau)\|\Big)d\tau\\
&\le \int_0^t\|(U_{xx},\Theta_{xx})(\tau)\|d\tau\\
&\qquad\qquad +O(1)M_1^2\int_0^t\Big(\|U_{xx}(\tau)\|\|\psi(\tau)\|^2+\|\Theta_{xx}(\tau)\|\|\phi(\tau)\|^2\Big)d\tau\\
&\le O(1)\ell^{-\frac12}+O(1)M_1^2\ell^{-\frac14}\max_{\tau\in[0,t]}\left\{\left\|\left(\psi,\frac{\phi}{\sqrt{\gamma-1}}\right)(\tau)\right\|^2\right\},
\end{aligned}
$$

$$
\begin{aligned}
(4.9)\quad R_3' &\le O(1)\int_0^t\left(\|V_x\|_{L^\infty}\|U_x\|^{\frac12}_{L^\infty}+\|U_x\|^{\frac32}_{L^\infty}+\|V_x\|_{L^\infty}\|\Theta\|^{\frac12}_{L^\infty}\right)(\tau)d\tau\\
&\quad +O(1)\int_0^t\int_{\mathbf{R}}\left(\|V_x\|^{\frac52}_{L^\infty}\frac{\psi^2}{v^4}+\|V_x\|^{\frac52}_{L^\infty}\frac{\phi^2}{v^2}+\|\Theta_x\|^{\frac52}_{L^\infty}\frac{\phi^2}{v^4}\right)(\tau,x)dxd\tau\\
&\le O(1)\ell^{-\frac14}+O(1)\left(M_1^2+M_1^4\right)\ell^{-\frac18}\max_{\tau\in[0,t]}\left\{\left\|\left(\psi,\frac{\phi}{\sqrt{\gamma-1}}\right)(\tau)\right\|^2\right\}.
\end{aligned}
$$

As to R_4', since $\xi=\frac{R}{\gamma-1}\ln\frac{\theta}{\Theta}+R\ln\frac{v}{V}$, we have from the a priori assumption (4.4) and Lemma 2.2 that

(4.10)

$$
\begin{aligned}
R_4' &\le O(1)(1+|\ln M_1|)\int_0^t\Big\|(q(V,\Theta),g(V,\Theta)_x,r(V,\Theta))(\tau)\Big\|_{L^1}d\tau\\
&\quad +O(1)\int_0^t\Big\|(q(V,\Theta),g(V,\Theta)_x,r(V,\Theta))(\tau)\Big\|\left(1+\left\|\left(\psi,\frac{\phi}{\sqrt{\gamma-1}}\right)(\tau)\right\|^2\right)d\tau\\
&\le O(1)\ell^{-\frac13}(1+|\ln M_1|)+O(1)\ell^{-\frac13}\max_{\tau\in[0,t]}\left\{\left\|\left(\psi,\frac{\phi}{\sqrt{\gamma-1}}\right)(\tau)\right\|^2\right\}.
\end{aligned}
$$

Inserting (4.7)–(4.10) into (4.6), we can get from the a priori assumption (4.4) that

$$
\begin{aligned}
&\left\|\left(\sqrt{\Phi\left(\frac{v}{V}\right)},\psi,\frac{\phi}{\sqrt{\gamma-1}}\right)(\tau)\right\|^2+\int_0^t\int_{\mathbf{R}}\left(\frac{\psi_x^2+\phi_x^2}{v}\right)(\tau,x)dxd\tau\\
&\quad\le C_5(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})\Bigg\{\left\|\left(\varphi_0,\psi_0,\frac{\phi_0}{\sqrt{\gamma-1}}\right)\right\|^2+1+\ell^{-\frac13}(1+|\ln M_1|)\\
(4.11)\quad&\qquad +\ell^{-\frac18}\int_0^t\int_{\mathbf{R}}\frac{\varphi_x^2}{v^3}(\tau,x)dxd\tau+\left(1+M_1^4\right)\ell^{-\frac18}\max_{\tau\in[0,t]}\left\{\left\|\left(\psi,\frac{\phi}{\sqrt{\gamma-1}}\right)(\tau)\right\|^2\right\}\Bigg\}.
\end{aligned}
$$

If we choose $\ell_4>1$ sufficiently large such that

$$
(4.12)\qquad \begin{cases} (1+|\ln M_1|)\ell^{-\frac13}\le 1,\\ C_5(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})\left(1+M_1^4\right)\ell^{-\frac18}\le\frac12,\end{cases}
$$

we have from (4.11) that for $\ell\ge\ell_4$,

$$
\begin{aligned}
&\left\|\left(\sqrt{\Phi\left(\frac{v}{V}\right)}, \psi, \frac{\phi}{\sqrt{\gamma-1}}\right)(\tau)\right\|^2 + \int_0^t\int_{\mathbf{R}}\left(\frac{\psi_x^2+\phi_x^2}{v}\right)(\tau,x)dxd\tau \\
&\quad \le C_6(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})\left\{\left\|\left(\varphi_0,\psi_0,\frac{\phi_0}{\sqrt{\gamma-1}}\right)\right\|^2 + 1 + \ell^{-\frac{1}{8}}\int_0^t\int_{\mathbf{R}}\left(\frac{\varphi_x^2}{v^3}\right)(\tau,x)dxd\tau\right\}.
\end{aligned}
\tag{4.13}
$$

Now we turn to control the term $\int_0^t\int_{\mathbf{R}}\frac{\varphi_x^2}{v^3}(\tau,x)dxd\tau$. To this end, since for the ideal polytropic gas, $p_v(v,\theta) = -\frac{R\theta}{v^2}$, $p_\theta(v,\theta)=\frac{R}{v}$, we have from (3.11) and the a priori assumption (4.4) that

$$
\begin{aligned}
&\int_{\mathbf{R}}\left(\frac{\varphi_x^2}{v^2}\right)(t,x)dx + \int_0^t\int_{\mathbf{R}}\left(\frac{\varphi_x^2}{v^3}\right)(\tau,x)dxd\tau \\
&\quad \le C(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})\left\{\|(\varphi_{0x},\psi_0)\|^2 + \|\psi(t)\|^2 + \int_0^t\int_{\mathbf{R}}\left(\frac{\psi_x^2}{v}+\frac{\phi_x^2}{v}\right)(\tau,x)dxd\tau\right\} \\
&\qquad + \sum_{j=5}^7 R_j.
\end{aligned}
\tag{4.14}
$$

Here

$$
\begin{cases}
R_5 = C(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})\displaystyle\int_0^t\int_{\mathbf{R}}\left\{V_x\left[-\frac{R\theta}{v^2}+\frac{R\Theta}{V^2}\right]\frac{\varphi_x}{v} + \Theta_x\left[\frac{R}{v}-\frac{R}{V}\right]\frac{\varphi_x}{v}\right. \\
\qquad\qquad \left. + \frac{U_x\psi\varphi_x}{v^2} - \frac{V_x\psi\psi_x}{v^2}\right\}(\tau,x)dxd\tau, \\
R_6 = C(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})\mu\displaystyle\int_0^t\int_{\mathbf{R}}\left(\frac{U_{xx}\varphi_x}{v^2} - \frac{V_x\varphi_x\psi_x}{v^3} - \frac{V_xU_x\varphi_x}{v^2}\right)(\tau,x)dxd\tau, \\
R_7 = C(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})\displaystyle\int_0^t\int_{\mathbf{R}}\left(g(V,\Theta)_x\frac{\varphi_x}{v}\right)(\tau,x)dxd\tau.
\end{cases}
$$

Now we estimate R_5, R_6 and R_7 term by term. For this purpose, we can deduce by the Cauchy-Schwarz inequality, Lemma 2.2, and the a priori assumption (4.4) that

$$
\begin{aligned}
R_5 &\le O(1)\int_0^t\int_{\mathbf{R}}\left\{\frac{|V_x\varphi_x|}{v^3}\left(|\varphi|^2+|\varphi|+|\phi|\right) + \frac{|\varphi_x(\Theta_x\varphi+U_x\psi)|}{v^2}\right. \\
&\qquad\qquad \left. + \frac{|V_x\psi\psi_x|}{v^2}\right\}(\tau,x)dxd\tau \\
&\le \frac{1}{3}\int_0^t\int_{\mathbf{R}}\left(\frac{\varphi_x^2}{v^3}+\frac{\psi_x^2}{v}\right)(\tau,x)dxd\tau \\
&\quad + O(1)\int_0^t\int_{\mathbf{R}}\left\{\frac{V_x^2}{v^3}\left(|\varphi|^4+|\varphi|^2+|\phi|^2\right) + \frac{\Theta_x^2\varphi^2+U_x^2\psi^2}{v} + \frac{V_x^2\psi^2}{v^3}\right\}(\tau,x)dxd\tau \\
&\le \frac{1}{3}\int_0^t\int_{\mathbf{R}}\left(\frac{\varphi_x^2}{v^3}+\frac{\psi_x^2}{v}\right)(\tau,x)dxd\tau \\
&\quad + O(1)\left(|\ln M_1| + M_1^3\right)\int_0^t\|U_x(\tau)\|_{L^\infty}^2\left\|\left(\sqrt{\Phi\left(\frac{v}{V}\right)},\psi,\frac{\phi}{\sqrt{\gamma-1}}\right)(\tau)\right\|^2 d\tau
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{3}\int_0^t\int_{\mathbf{R}}\left(\frac{\varphi_x^2}{v^3}+\frac{\psi_x^2}{v}\right)(\tau,x)dxd\tau\\
&\quad+O(1)\left(|\ln M_1|+M_1^3\right)\ell^{-\frac{1}{8}}\max_{\tau\in[0,t]}\left\{\left\|\left(\sqrt{\Phi\left(\frac{v}{V}\right)},\psi,\frac{\phi}{\sqrt{\gamma-1}}\right)(\tau)\right\|^2\right\},
\end{aligned}
\tag{4.15}
$$

$$
\begin{aligned}
R_6 &\leq \frac{1}{3}\int_{\mathbf{R}}\left(\frac{\varphi_x^2}{v^3}+\frac{\psi_x^2}{v}\right)(\tau,x)dxd\tau+O(1)\int_0^t\int_{\mathbf{R}}\left\{\frac{V_x^2\varphi_x^2}{v^5}+\frac{U_{xx}^2}{v}+\frac{U_x^4}{v^3}\right\}(\tau,x)dxd\tau\\
&\leq \frac{1}{3}\int_{\mathbf{R}}\left(\frac{\varphi_x^2}{v^3}+\frac{\psi_x^2}{v}\right)(\tau,x)dxd\tau\\
&\quad+O(1)\left(M_1+M_1^5\right)\ell^{-\frac{1}{8}}\left\{\max_{\tau\in[0,t]}\left\{\left\|\left(\frac{\varphi_x}{v}\right)(\tau)\right\|^2\right\}+1\right\},
\end{aligned}
\tag{4.16}
$$

and

$$
\begin{aligned}
R_7 &\leq \int_0^t\left\|g(V,\Theta)_x(\tau)\right\|\left\|\left(\frac{\varphi_x}{v}\right)(\tau)\right\|d\tau\\
&\leq \int_0^t\left\|g(V,\Theta)_x(\tau)\right\|d\tau+\int_0^t\left\|g(V,\Theta)_x(\tau)\right\|\left\|\left(\frac{\varphi_x}{v}\right)(\tau)\right\|^2d\tau\\
&\leq O(1)\ell^{-\frac{1}{3}}+O(1)\ell^{-\frac{1}{3}}\max_{\tau\in[0,t]}\left\{\left\|\left(\frac{\varphi_x}{v}\right)(\tau)\right\|^2\right\}.
\end{aligned}
\tag{4.17}
$$

Substituting (4.15)–(4.17) into (4.14), we arrive at

$$
\begin{aligned}
&\int_{\mathbf{R}}\left(\frac{\varphi_x^2}{v^2}\right)(t,x)dx+\int_0^t\int_{\mathbf{R}}\left(\frac{\varphi_x^2}{v^3}\right)(\tau,x)dxd\tau\\
\leq\ & O(1)\left\{\|(\varphi_{0x},\psi_0)\|^2+\|\psi(t)\|^2+\int_0^t\int_{\mathbf{R}}\left(\frac{\psi_x^2}{v}+\frac{\phi_x^2}{v}\right)(\tau,x)dxd\tau\right\}\\
&+O(1)\left(1+M_1^5\right)\ell^{-\frac{1}{8}}\left\{1+\max_{\tau\in[0,t]}\left\{\left\|\left(\sqrt{\Phi\left(\frac{v}{V}\right)},\psi,\frac{\phi}{\sqrt{\gamma-1}},\frac{\varphi_x}{v}\right)(\tau)\right\|^2\right\}\right\}.
\end{aligned}
\tag{4.18}
$$

Multiplying (4.13) by a suitably large positive constant which is independent of M_1 and adding the resulting inequality into (4.18) yield

$$
\begin{aligned}
&\left\|\left(\sqrt{\Phi\left(\frac{v}{V}\right)},\psi,\frac{\phi}{\sqrt{\gamma-1}},\frac{\varphi_x}{v}\right)(t)\right\|^2+\int_0^t\int_{\mathbf{R}}\left(\frac{\psi_x^2}{v}+\frac{\phi_x^2}{v}+\frac{\varphi_x^2}{v^3}\right)(\tau,x)dxd\tau\\
&\quad\leq C_7(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})\left\{\left\|\left(\varphi_{0x},\psi_0,\varphi_0,\frac{\phi_0}{\sqrt{\gamma-1}}\right)\right\|^2+1+\ell^{-\frac{1}{8}}\int_0^t\int_{\mathbf{R}}\left(\frac{\varphi_x^2}{v^3}\right)(\tau,x)dxd\tau\right.\\
&\qquad\left.+\left(1+M_1^5\right)\ell^{-\frac{1}{8}}\left\{1+\max_{\tau\in[0,t]}\left\{\left\|\left(\sqrt{\Phi\left(\frac{v}{V}\right)},\psi,\frac{\phi}{\sqrt{\gamma-1}},\frac{\varphi_x}{v}\right)(\tau)\right\|^2\right\}\right\}\right\}.
\end{aligned}
\tag{4.19}
$$

Thus if we choose $\ell_5\geq 1$ sufficiently large such that

$$
\begin{cases}
C_7(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})\ell_5^{-\frac{1}{8}}<\frac{1}{2},\\
C_7(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})\left(1+M_1^5\right)\ell_5^{-\frac{1}{8}}<\frac{1}{2},
\end{cases}
\tag{4.20}
$$

we can deduce that

$$\begin{aligned}
&\left\|\left(\sqrt{\Phi\left(\frac{v}{V}\right)},\psi,\frac{\phi}{\sqrt{\gamma-1}},\frac{\varphi_x}{v}\right)(t)\right\|^2+\int_0^t\int_{\mathbf{R}}\left(\frac{\psi_x^2}{v}+\frac{\phi_x^2}{v}+\frac{\varphi_x^2}{v^3}\right)(\tau,x)dxd\tau\\
&\leq C_8(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})\left\{\left\|\left(\varphi_{0x},\psi_0,\varphi_0,\frac{\phi_0}{\sqrt{\gamma-1}}\right)\right\|^2+1\right\}
\end{aligned}\tag{4.21}$$

holds for $0\leq t\leq T$ provided that $\ell\geq\ell_5$.

To exploit J. Kanel's method to deduce a lower bound for $v(t,x)$, we need to deduce the L^2-norm estimate on $\tilde{v}=\frac{v}{V}$. For this purpose, noticing that

$$\Phi\left(\frac{v}{V}\right)\geq\frac{\varphi^2}{V(v+V)},$$

and by choosing $\ell\geq\max\{\ell_4,\ell_5\}$, we have from (4.12) and (4.20) that

$$\begin{aligned}
\left|\tfrac{V_x\varphi}{vV}\right| &\leq \frac{\|V_x\|_{L^\infty_{t,x}}\sqrt{v+V}}{v\sqrt{V}}\sqrt{\Phi\left(\tfrac{v}{V}\right)}\\
&\leq O(1)\ell^{-1}M_1^{\frac{3}{2}}\sqrt{\Phi\left(\tfrac{v}{V}\right)}\\
&\leq O(1)\sqrt{\Phi\left(\tfrac{v}{V}\right)}.
\end{aligned}$$

Consequently, we get from

$$\frac{\tilde{v}_x}{\tilde{v}}=\frac{\varphi_x}{v}-\left(\frac{V_x}{v}-\frac{V_x}{V}\right)=\frac{\varphi_x}{v}-\frac{V_x\varphi}{vV},$$

and the estimate (4.21) that for $\ell\geq\max\{\ell_4,\ell_5\}$,

$$\begin{aligned}
&\left\|\left(\sqrt{\Phi\left(\tilde{v}\right)},\psi,\frac{\phi}{\sqrt{\gamma-1}},\frac{\tilde{v}_x}{\tilde{v}}\right)(t)\right\|^2+\int_0^t\int_{\mathbf{R}}\left(\frac{\psi_x^2}{v}+\frac{\phi_x^2}{v}+\frac{\varphi_x^2}{v^3}\right)(\tau,x)dxd\tau\\
&\leq C_9(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})\left\{\left\|\left(\varphi_{0x},\psi_0,\varphi_0,\frac{\phi_0}{\sqrt{\gamma-1}}\right)\right\|^2+1\right\},\quad 0\leq t\leq T.
\end{aligned}\tag{4.22}$$

It is worth pointing out that the most important thing is that the constant $C_9(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})$ on the right hand side of (4.22) is independent of M_1.

Thus we have obtained the following result.

Lemma 4.1. *Under the assumptions listed above, let $\kappa_1(x)=1+x^5$. If we assume that there exists a positive constant $\ell_6\geq 1$ such that*

$$\kappa_1(M_1)\ell_6^{-\frac{1}{8}}\leq C_{10}(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V}),\quad \kappa_1(M_1)=1+M_1^5\tag{4.23}$$

for some positive constant $C_{10}(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})$ satisfying

$$C_{10}(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})<\min\left\{\frac{1}{2C_7(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})},\frac{1}{2C_5(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})},\frac{1}{2}\right\},$$

then (4.22) holds for any $0\leq t\leq T$ provided that $\ell\geq\ell_6$. It is easy to see that such a constant $C_{10}(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})$ depends only on $\underline{\Theta},\overline{\Theta},\underline{V}$ and $\overline{V}$.

A direct corollary of Lemma 4.1 is

Corollary 4.1. *Under the assumptions stated in Lemma 4.1, there exists a continuous increasing function $\kappa_2(x) > 0$, which is defined on $x > 0$ and satisfies $\lim_{x\to\infty} \kappa_2(x) = \infty$, such that for $(t,x) \in [0,T] \times \mathbf{R}$,*

$$0 < 2\kappa_2 \left(\|(\varphi_0,\psi_0,\xi_0)\|_1\right)^{-1} \le v(t,x) \le \frac{1}{2}\kappa_2 \left(\|(\varphi_0,\psi_0,\xi_0)\|_1\right). \tag{4.24}$$

Moreover, we have that there exists a constant $\kappa_3 > 0$ which depends on $\|(\varphi_0,\psi_0,\xi_0)\|_1$, $\underline{\Theta}, \overline{\Theta}, \underline{V}$ and $\overline{V}$ such that the following estimates hold:

$$\|(\varphi,\psi,\xi)(t)\|_1 \le \quad \kappa_3 \left(\|(\varphi_0,\psi_0,\xi_0)\|_1\right). \tag{4.25}$$

Proof. We use the method of J. Kanel$'$ to deduce the desired bounds on $v(t,x)$ (cf. [15]). To this end, let

$$\Psi(\tilde{v}) = \int_1^{\tilde{v}} \frac{\sqrt{\Phi(\eta)}}{\eta} d\eta, \qquad \Phi(\eta) = \eta - \ln\eta - 1.$$

Since

$$\Psi(\tilde{v}) \to \begin{cases} -\infty & \text{as} \quad \tilde{v} \to 0_+, \\ +\infty & \text{as} \quad \tilde{v} \to +\infty, \end{cases}$$

and

$$\begin{aligned} |\Psi(\tilde{v}(t,x))| &= \left|\int_{-\infty}^x \frac{\partial}{\partial y}\Psi(\tilde{v}(t,x))dy\right| \le \frac{1}{2}\int_{\mathbf{R}} \left(\Phi\left(\frac{v}{V}\right) + \left(\frac{\tilde{v}_x}{\tilde{v}}\right)^2\right)(t,x)dx \\ &\le C_9(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})\left\{\left\|\left(\varphi_{0x},\psi_0,\varphi_0,\frac{\phi_0}{\sqrt{\gamma-1}}\right)\right\|^2 + 1\right\}, \end{aligned} \tag{4.26}$$

we can get the estimates (4.24) immediately.

Having obtained (4.24), the estimate (4.25) follows immediately from the a priori assumption $(4.4)_2$, (4.21), (4.23) and the standard energy estimates. This completes the proof of Corollary 4.1.

With the above preparation in hand, we now turn to prove Theorem 1.2. The standard continuation argument tells us that if we can get an H^1-norm estimate on the solution $(\varphi(t,x),\psi(t,x),\phi(t,x))$ on the domain $[0,T]\times\mathbf{R}$ in which it exists, then we can extend the local solution constructed in Lemma 3.1 globally. (The asymptotic behavior (1.25) is a direct consequence of the H^1-norm estimates.) As pointed out before, our main trick is to use the quantity ℓ introduced in the construction of the smooth approximation to the rarefaction wave profile to control the growth of the solution caused by the nonlinearity of the equations and/or the interaction of waves from different families. In fact, Lemma 4.1 and Corollary 4.1 imply that if we can find a fixed constant ℓ_6 such that (4.23) holds, then we can get the estimates (4.22), (4.24) and (4.25). But the problem is how to determine such an ℓ_6. When $\|(\varphi_0,\psi_0,\phi_0)\|_1$, the H^1-norm of the initial perturbation, is independent of ℓ, the estimate (4.24) implies that we can deduce a uniform lower and upper bound for $v(t,x)$ which is independent of ℓ and thus the existence of such an ℓ_6 is easily to be verified. Thus to prove Theorem 1.2, the only problem left is to close the a priori assumption $(4.4)_2$ that we imposed on the temperature $\theta(t,x)$.

Such a problem can be solved as follows by assuming that $\gamma-1>0$ is sufficiently small. In fact, from (4.22), (4.24) and (4.25), we can get from (4.1) that

$$\|\phi(t)\|_1 \le \sqrt{\gamma-1}C_{11}(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})\kappa_3\left(\|(\varphi_0,\psi_0,\xi_0)\|_1\right). \tag{4.27}$$

Consequently,

$$\begin{aligned}\sup_{[0,T]}\|\phi(t)\|_\infty &\le \sup_{t\in[0,T]}\left\{\|\phi(t)\|^{\frac{1}{2}}\|\phi_x(t)\|^{\frac{1}{2}}\right\}\\ &\le \sqrt{\gamma-1}C_{11}(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})\kappa_3\left(\|(\varphi_0,\psi_0,\xi_0)\|_1\right).\end{aligned} \tag{4.28}$$

By choosing $\gamma-1>0$ sufficiently small such that

$$\sqrt{\gamma-1}C_{11}(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})\kappa_3\left(\|(\varphi_0,\psi_0,\xi_0)\|_1\right) < \min\left\{\frac{\overline{\Theta}}{2},\underline{\Theta}\right\}, \tag{4.29}$$

we have for each $(t,x)\in[0,T]\times\mathbf{R}$ that

$$\theta(t,x)=\Theta(t,x)+\phi(t,x)\le\frac{\overline{\Theta}}{2}+\sqrt{\gamma-1}C_{11}(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})\kappa_3\left(\|(\varphi_0,\psi_0,\xi_0)\|_1\right)<\overline{\Theta} \tag{4.30}$$

and

$$\theta(t,x)=\Theta(t,x)+\phi(t,x)\ge 2\underline{\Theta}-\sqrt{\gamma-1}C_{11}(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})\kappa_3\left(\|(\varphi_0,\psi_0,\xi_0)\|_1\right)>\underline{\Theta}. \tag{4.31}$$

This proves $(4.4)_2$. This proves Theorem 1.2 for the case when $\|(\varphi_0,\psi_0,\phi_0)\|_1$, the H^1-norm of the initial perturbation, is independent of ℓ.

We note, however, that the above argument cannot be used directly to deal with the case when the H^1-norm of the initial perturbation depends on ℓ, the main reason being that in this case the lower bound and upper bound of $v(t,x)$ obtained in (4.24) depend also on ℓ and it is not so clear why one can choose ℓ sufficiently large such that (4.23) holds. In the following we will show that when the H^1-norm of the initial perturbation satisfies certain growth condition as $\ell\to\infty$, we can indeed find such an ℓ_6 that (4.23) holds.

Now let's turn to consider the case when the H^1-norm of the initial perturbation depends on ℓ. Recall that $h(\ell)=\frac{1}{2D_3}(\kappa_2\circ\kappa_1)^{-1}(\ell^{\gamma_1})$ with $\gamma_1<\frac{1}{8}$ and κ_1,κ_2 defined by (4.23) and (4.24) respectively. We know that $h(\ell)\to\infty$ as $\ell\to\infty$, and thus we can choose $\ell_7\ge 1$ sufficiently large such that

$$h(\ell)\ge 1,\quad \ell\ge l_7. \tag{4.32}$$

Consequently from the assumption (1.27), we can get that

$$\|(\varphi_0,\psi_0,\phi_0)\|_1\le 2D_3h(\ell),\quad \ell\ge\ell_7. \tag{4.33}$$

With (4.33) in hand, we can deduce from the local existence result (Lemma 3.1) that there exists a sufficiently small positive constant $t_1>0$ such that the Cauchy problem (2.4), (2.5) admits a unique solution $(\varphi(t,x),\psi(t,x),\phi(t,x))$ defined on the strip $\Pi_{t_1}:=\{(t,x):\ 0\le t\le t_1,\ x\in\mathbf{R}\}$ and $(\varphi(t,x),\psi(t,x),\phi(t,x))$ satisfies

$$\begin{cases}\|(\varphi,\psi,\phi)(t)\|_1\le 2\|(\varphi_0,\psi_0,\phi_0)\|_1, & 0\le t\le t_1,\\ \underline{V}\le v(t,x)\le\overline{V}, & (t,x)\in[0,t_1]\times\mathbf{R},\\ \underline{\Theta}\le\theta(t,x)\le\overline{\Theta}, & (t,x)\in[0,t_1]\times\mathbf{R}.\end{cases} \tag{4.34}$$

Having obtained (4.34), we can now employ Lemma 4.1 and Corollary 4.1 with $T = t_1$ and $M_1 = \max\left\{\underline{V}^{-1}, \overline{V}\right\}$ to get that if we choose $\ell_8 \geq 1$ sufficiently large such that

$$\kappa_1\left(\max\left\{\underline{V}^{-1}, \overline{V}\right\}\right)\ell_8^{-\frac{1}{8}} \leq C_{10}(\underline{\Theta}, \overline{\Theta}, \underline{V}, \overline{V}), \tag{4.35}$$

then for $\ell \geq \ell_8$, $(t,x) \in [0,t_1] \times \mathbf{R}$, we can deduce from (4.24) and (4.25) that

$$2\kappa_2\left(\|(\varphi_0, \psi_0, \xi_0)\|_1\right)^{-1} \leq v(t,x) \leq \frac{1}{2}\kappa_2\left(\|(\varphi_0, \psi_0, \xi_0)\|_1\right), \tag{4.36}$$

and

$$\|(\varphi, \psi, \xi)(t)\|_1 \leq \quad \kappa_3\left(\|(\varphi_0, \psi_0, \xi_0)\|_1\right). \tag{4.37}$$

Moreover by choosing $\overline{\gamma} > 0$ sufficiently small such that

$$\sqrt{\overline{\gamma}}C_{11}(\underline{\Theta}, \overline{\Theta}, \underline{V}, \overline{V})\kappa_3\Big(D_3\left(1 + h(\ell_8)\right)\Big) < \min\left\{\frac{\overline{\Theta}}{2}, \underline{\Theta}\right\}, \tag{4.38}$$

then from (1.27), (4.37) and by repeating the argument used to prove (4.30) and (4.31), we have that for $1 < \gamma \leq 1 + \overline{\gamma}$,

$$\underline{\Theta} < \theta(t,x) < \overline{\Theta}, \quad \forall (t,x) \in [0,t_1] \times \mathbf{R}. \tag{4.39}$$

Now taking $(\varphi(t_1,x), \psi(t_1,x), \phi(t_1,x))$ as initial data, we can deduce, by employing the local existence result (Lemma 3.1) again, that the solution $(\varphi(t,x), \psi(t,x),$ $\phi(t,x))$ obtained above can be extended to the time step $t = t_1 + t_2$, and for $t_1 \leq t \leq t_1 + t_2, x \in \mathbf{R}$, $(\varphi(t,x), \psi(t,x), \phi(t,x))$ satisfies

$$\|(\varphi, \psi, \xi)(t)\|_1 \leq \quad 2\kappa_3\left(\|(\varphi_0, \psi_0, \xi_0)\|_1\right) \tag{4.40}$$

and

$$\begin{cases} \kappa_2\left(\|(\varphi_0, \psi_0, \xi_0)\|_1\right)^{-1} \leq v(t,x) \leq \kappa_2\left(\|(\varphi_0, \psi_0, \xi_0)\|_1\right), \\ \underline{\Theta} \leq \theta(t,x) \leq \overline{\Theta}. \end{cases} \tag{4.41}$$

Since without loss of generality, we can assume that

$$\begin{cases} \|(\varphi_0, \psi_0, \xi_0)\|_1 \leq \kappa_3\left(\|(\varphi_0, \psi_0, \xi_0)\|_1\right), \\ \max\left\{\underline{V}^{-1}, \overline{V}\right\} \leq \kappa_2\left(\|(\varphi_0, \psi_0, \xi_0)\|_1\right), \end{cases} \tag{4.42}$$

then (4.40) and (4.41) hold for all $(t,x) \in [0, t_1 + t_2] \times \mathbf{R}$.

With the above estimates in hand, we now try to apply Lemma 4.1 and Corollary 4.1 with $M_1 = \kappa_2\left(\|(\varphi_0, \psi_0, \xi_0)\|_1\right)$ and $T = t_1 + t_2$. For this purpose, as pointed out before, the main problem is to see if we can find a positive $\ell_9 \geq \max\{\ell_7, \ell_8\}$ such that, for $(t,x) \in [0, t_1 + t_2] \times \mathbf{R}$ and $\ell \geq \ell_9$, (4.23) holds with $M_1 = \kappa_2\left(\|(\varphi_0, \psi_0, \xi_0)\|_1\right)$, i.e.,

$$\kappa_1\left(\kappa_2\Big(\|(\varphi_0, \psi_0, \xi_0)\|_1\Big)\right)\ell^{-\frac{1}{8}} \leq C_{10}(\underline{\Theta}, \overline{\Theta}, \underline{V}, \overline{V}). \tag{4.43}$$

In fact since $h(\ell) = \frac{1}{2D_3}\left(\kappa_2^{-1} \circ \kappa_1^{-1}\right)(\ell^{\gamma_1})$, we can get from (1.27) and (4.33) that

$$\kappa_1\left(\kappa_2\Big(D_3(1 + h(\ell))\Big)\right) \leq \kappa_1\left(\kappa_2\Big(2D_3 h(\ell)\Big)\right) = \kappa_1\left(\kappa_2\left(\kappa_2^{-1} \circ \kappa_1^{-1}\right)(\ell^{\gamma_1})\right) = \ell^{\gamma_1}.$$

Thus if $\gamma_1 < \frac{1}{8}$, we can indeed deduce that there exists a sufficiently large $\ell_9 \geq \max\{\ell_7, \ell_8\}$ such that for $\ell \geq \ell_9$, (4.43) holds and Lemma 4.1 and Corollary 4.1 tell us that (4.36) and (4.37) hold for $(t,x)\in[0,t_1+t_2]\times\mathbf{R}$. Furthermore, if we choose $\overline{\overline{\gamma}}>0$ sufficiently small such that

$$\sqrt{\overline{\overline{\gamma}}}C_{11}(\underline{\Theta},\overline{\Theta},\underline{V},\overline{V})\kappa_3\Big(D_3\left(1+h(\ell_9)\right)\Big)<\min\left\{\frac{\overline{\Theta}}{2},\underline{\Theta}\right\}, \tag{4.44}$$

then from (1.27), (4.37) and by repeating the argument used to prove (4.30) and (4.31), we have that for $1<\gamma\le 1+\overline{\overline{\gamma}}$,

$$\underline{\Theta}<\theta(t,x)<\overline{\Theta},\quad \forall(t,x)\in[0,t_1+t_2]\times\mathbf{R}. \tag{4.45}$$

Now taking $(\varphi(t_1+t_2,x),\psi(t_1+t_2,x),\xi(t_1+t_2,x))$ as initial data and by exploiting the local existence result (Lemma 3.1) again, $(\varphi(t,x),\psi(t,x),\phi(t,x))$ can be extended to the time step $t=t_1+2t_2$ and $(\varphi(t,x),\psi(t,x),\phi(t,x))$ satisfies (4.40) and (4.41) for all $(t,x)\in[0,t_1+t_2]\times\mathbf{R}$.

Repeating the above process and noticing that the constants C_9,κ_2 and κ_3 are independent of each time step, we can deduce from the continuation argument that if $\ell=\ell_9$ and $\gamma\in\left(1,1+\min\{\overline{\gamma},\overline{\overline{\gamma}}\}\right)$, then (4.24) and (4.25) hold for each $(t,x)\in[0,\infty)\times\mathbf{R}$. This completes the proof of Theorem 1.2. □

5. Proof of Theorem 1.3

To prove Theorem 1.3, we first notice that for an isentropic gas, (2.4) is reduced to

$$\begin{cases}\varphi_t-\psi_x=0,\\ \psi_t+\Big(p(v)-p(\overline{V})\Big)_x-\mu\left(\frac{u_x}{v}-\frac{\overline{U}_x}{\overline{V}}\right)_x=\mu\left(\frac{\overline{U}_x}{\overline{V}}\right)_x-g(\overline{V})_x,\end{cases} \tag{5.1}$$

with initial data

$$(\varphi(t,x),\psi(t,x))\mid_{t=0}=(\varphi_0(x),\psi_0(x))=\left(v_0(x)-\overline{V}(t_0,x),u_0(x)-\overline{U}(t_0,x)\right). \tag{5.2}$$

Here

$$\begin{cases}\varphi(t,x)=v(t,x)-\overline{V}(t,x),\\ \psi(t,x)=u(t,x)-\overline{U}(t,x),\\ g(\overline{V})_x=\left(p(\overline{V})-p(\overline{V}_1)-p(\overline{V}_2)+p(v_m)\right)_x,\end{cases}$$

and $g(\overline{V})_x$ inherits the properties of $g(V,\Theta)_x$ in Lemma 2.2. Similar to the proof of Theorem 1.2, all we need to show is that for some $T>0$, under the a priori assumption

$$0<\frac{1}{M_2}\le v(t,x)=\varphi(t,x)+\overline{V}(t,x)\le M_2,\quad\forall(t,x)\in[0,T]\times\mathbf{R}, \tag{5.3}$$

one can indeed deduce that there exists a positive constant $C_{12}(\underline{V},\overline{V})>0$ which depends only on the initial data $(\varphi_0(x),\psi_0(x))$ and the system but is independent of M_2 such that

$$\left(C_{12}\left(\|(\varphi_0,\psi_0)\|_1\right)\right)^{-1}\le v(t,x)=\varphi(t,x)+\overline{V}(t,x)\le C_{12}\left(\|(\varphi_0,\psi_0)\|_1\right). \tag{5.4}$$

To prove (5.4), we first perform some energy estimates. First, multiplying $(5.1)_1$ by $p(V)-p(V+\varphi)$, $(5.1)_2$ by ψ, adding the resultant two identities, and integrating it with respect to t and x over $[0,T]\times\mathbf{R}$, we have from the assumptions listed in Theorem 1.3 and some integrations by parts that

$$\begin{aligned}
&\left\|\left(\sqrt{\Phi(v,\overline{V})},\psi\right)(t)\right\|^2+\int_0^t\left\|\left(\frac{\psi_x}{\sqrt{v}}\right)(\tau)\right\|^2d\tau\\
&\le C(\underline{V},\overline{V})\left\{\|(\varphi_0,\psi_0)\|^2\right.\\
&\qquad\left.+\int_0^t\int_{\mathbf{R}}\left(\left|\frac{\overline{V}_x\psi_x\varphi}{v}\right|+|\psi|\left(|\overline{U}_{xx}|+|\overline{U}_x^2|+|g(\overline{V})_x|\right)\right)(\tau,x)dxd\tau\right\}.
\end{aligned}\tag{5.5}$$

Here

$$\Phi(v,\overline{V})=p(\overline{V})\varphi-\int_{\overline{V}}^v p(s)ds.$$

By employing the same argument as in proving Theorem 1.2, we can easily get

$$\begin{aligned}
&\left\|\left(\sqrt{\Phi(v,\overline{V})},\psi\right)(t)\right\|^2+\frac{1}{2}\int_0^t\left\|\left(\frac{\psi_x}{\sqrt{v}}\right)(\tau)\right\|^2d\tau\\
&\le C_{13}(\underline{V},\overline{V})\left\{\|(\varphi_0,\psi_0)\|^2\right.\\
&\qquad\left.+M_2\ell^{-\frac{1}{3}}\left\{1+\int_0^t(1+\tau)^{-\frac{7}{6}}\left\|\left(\sqrt{\Phi(v,\overline{V})},\psi\right)(\tau)\right\|^2d\tau\right\}\right\}.
\end{aligned}\tag{5.6}$$

Thus, if we can find ℓ sufficiently large such that

$$C_{13}(\underline{V},\overline{V})M_2\ell^{-\frac{1}{3}}\le 1,\tag{5.7}$$

then by Gronwall's inequality, we can get from (5.6) that

$$\left\|\left(\sqrt{\Phi(v,\overline{V})},\psi\right)(t)\right\|^2+\int_0^t\left\|\left(\frac{\psi_x}{\sqrt{v}}\right)(\tau)\right\|^2d\tau\le C_{14}(\underline{V},\overline{V})\left\{1+\|(\varphi_0,\psi_0)\|^2\right\}.\tag{5.8}$$

Second, similar to that of (3.12), we have from (5.1) that

$$\begin{aligned}
&\left[\frac{\mu}{2}\left(\frac{\varphi_x}{v}\right)^2-\frac{\varphi_x}{v}\psi\right]_t-\frac{\psi_x^2}{v}-p'(\varphi+\overline{V})\frac{\varphi_x^2}{v}\\
=\;&\left[p'(\varphi+\overline{V})-p'(\overline{V})\right]\frac{\overline{V}_x\varphi_x}{v}-\mu\frac{\overline{U}_{xx}\varphi_x}{v^2}+\mu\frac{\overline{V}_x\varphi_x\psi_x}{v^3}+\mu\frac{\overline{V}_x\overline{U}_x\varphi_x}{v^3}\\
&-\left(\frac{\psi\psi_x}{v}\right)_x-\frac{\psi\psi_x\overline{V}_x-\psi\varphi_x\overline{U}_x}{v^2}+g(\overline{V})_x\frac{\varphi_x}{v}.
\end{aligned}\tag{5.9}$$

Integrating (5.9) with respect to t and x over $[0,T]\times\mathbf{R}$, we have by some integrations by parts that

$$\begin{aligned}
&\int_{\mathbf{R}}\left\{\frac{\mu}{2}\left(\frac{\varphi_x}{v}\right)^2-\frac{\varphi_x}{v}\psi\right\}(t,x)dx-\int_0^t\int_{\mathbf{R}}\left(p'(\overline{V}+\varphi)\frac{\varphi_x^2}{v}\right)(\tau,x)dxd\tau\\
&=\int_{\mathbf{R}}\left\{\frac{\mu}{2}\left(\frac{\varphi_x}{v}\right)^2-\frac{\varphi_x}{v}\psi\right\}(0,x)dx+\int_0^t\int_{\mathbf{R}}\left(\frac{\psi_x^2}{v}\right)(\tau,x)dxd\tau\\
&\quad+\int_0^t\int_{\mathbf{R}}\left(\left[p'(\overline{V}+\varphi)-p'(\overline{V})\right]\frac{\overline{V}_x\varphi_x}{v}\right)(\tau,x)dxd\tau-\mu\int_{\mathbf{R}}\left(\frac{\overline{U}_{xx}\varphi_x}{v^2}\right)(\tau,x)dxd\tau\\
&\quad+\mu\int_0^t\int_{\mathbf{R}}\left(\frac{\overline{V}_x\varphi_x\psi_x}{v^3}\right)(\tau,x)dxd\tau+\mu\int_0^t\int_{\mathbf{R}}\left(\frac{\overline{V}_x\overline{U}_x\varphi_x}{v^3}\right)(\tau,x)dxd\tau
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^t\int_{\mathbf{R}}\left(\frac{\psi\psi_x\overline{V}_x-\psi\psi_x\overline{U}_x}{v^2}\right)(\tau,x)dxd\tau+\int_0^t\int_{\mathbf{R}}g(\overline{V})_x\frac{\varphi_x}{v}(\tau,x)dxd\tau \\
&:=\int_{\mathbf{R}}\left\{\frac{\mu}{2}\left(\frac{\varphi_x}{v}\right)^2-\frac{\varphi_x}{v}\psi\right\}(0,x)dx+\int_0^t\int_{\mathbf{R}}\left(\frac{\psi_x^2}{v}\right)(\tau,x)dxd\tau+\sum_{i=8}^{13}R_j. \qquad (5.10)
\end{aligned}
$$

Now we estimate R_j $(j=8,\cdots,13)$ term by term. Noticing the a priori assumption (5.3), we have from the Cauchy-Schwarz inequality and Lemma 2.2 that

$$
\begin{aligned}
R_8 &\le C(M_2)\ell^{-\frac{1}{3}}\int_0^t(1+\tau)^{-\frac{2}{3}}\|\varphi(\tau)\|\|\varphi_x(\tau)\|d\tau \\
&\le C(M_2)\ell^{-\frac{1}{3}}\int_0^t(1+\tau)^{-\frac{2}{3}}\left\|\sqrt{\Phi(v,\overline{V})(\tau)}\right\|\left\|\left(\sqrt{\frac{-p'(\overline{V}+\varphi)}{v}}\varphi_x\right)(\tau)\right\|d\tau \\
&\le \frac{1}{6}\int_0^t\left\|\left(\sqrt{\frac{-p'(\overline{V}+\varphi)}{v}}\varphi_x\right)(\tau)\right\|^2d\tau \\
&\quad + C(M_2)\ell^{-\frac{2}{3}}\int_0^t(1+\tau)^{-\frac{4}{3}}\left\|\sqrt{\Phi(v,\overline{V})(\tau)}\right\|^2d\tau, \qquad (5.11)
\end{aligned}
$$

$$
\begin{aligned}
|R_9| &\le C(M_2)\int_0^t\left\|\left(\sqrt{\frac{-p'(\overline{V}+\varphi)}{v}}\varphi_x\right)(\tau)\right\|\|\overline{U}_{xx}(\tau)\|d\tau \\
&\le \frac{1}{6}\int_0^t\left\|\left(\sqrt{\frac{-p'(\overline{V}+\varphi)}{v}}\varphi_x\right)(\tau)\right\|^2d\tau+C(M_2)\ell^{-\frac{1}{2}}, \qquad (5.12)
\end{aligned}
$$

$$
\begin{aligned}
R_{10} &\le C(M_2)\int_0^t\|\overline{V}_x(\tau)\|_{L^\infty}\left\|\left(\frac{\psi_x}{\sqrt{v}}\right)(\tau)\right\|\left\|\left(\frac{\varphi_x}{v}\right)(\tau)\right\|d\tau \\
&\le \int_0^t\left\|\left(\frac{\psi_x}{\sqrt{v}}\right)(\tau)\right\|^2d\tau+C(M_2)\ell^{-\frac{1}{3}}\int_0^t(1+\tau)^{-\frac{7}{6}}\left\|\left(\frac{\varphi_x}{v}\right)(\tau)\right\|^2d\tau, \qquad (5.13)
\end{aligned}
$$

$$
\begin{aligned}
R_{11} &\le C(M_2)\int_0^t\left\|\left(\sqrt{\frac{-p'(\overline{V}+\varphi)}{v}}\varphi_x\right)(\tau)\right\|\left\|\left(\overline{V}_x\overline{U}_x\right)(\tau)\right\|d\tau \\
&\le \frac{1}{6}\int_0^t\left\|\left(\sqrt{\frac{-p'(\overline{V}+\varphi)}{v}}\varphi_x\right)(\tau)\right\|^2d\tau+C(M_2)\ell^{-1}, \qquad (5.14)
\end{aligned}
$$

$$
\begin{aligned}
|R_{12}| &\le C(M_2)\ell^{-\frac{1}{3}}\int_0^t(1+\tau)^{-\frac{2}{3}}\left(\|\psi_x(\tau)\|\|\psi(\tau)\|+\|\varphi_x(\tau)\|\|\psi(\tau)\|\right)d\tau \\
&\le C(M_2)\ell^{-\frac{1}{3}}\int_0^t(1+\tau)^{-\frac{2}{3}}\left(\left\|\left(\frac{\psi_x}{\sqrt{v}}\right)(\tau)\right\|\|\psi(\tau)\|\right. \\
&\qquad\left.+\left\|\left(\sqrt{\frac{-p'(\overline{V}+\varphi)}{v}}\varphi_x\right)(\tau)\right\|\|\psi(\tau)\|\right)d\tau
\end{aligned}
$$

$$\le \frac{1}{6}\int_0^t \left\|\left(\sqrt{\frac{-p'(\overline{V}+\varphi)}{v}}\varphi_x\right)(\tau)\right\|^2 d\tau + \int_0^t\int_{\mathbf{R}}\left(\frac{\psi_x^2}{v}\right)(\tau)dxd\tau$$
$$+C(M_2)\ell^{-\frac{2}{3}}\int_0^t(1+\tau)^{-\frac{4}{3}}\|\psi(\tau)\|^2d\tau, \tag{5.15}$$

and

$$\begin{aligned} R_{13} &\le C(M_2)\int_0^t \left\|\left(\sqrt{\frac{-p'(\overline{V}+\varphi)}{v}}\varphi_x\right)(\tau)\right\| \left\|g(\overline{V})_x(\tau)\right\| d\tau \\ &\le \frac{1}{6}\int_0^t \left\|\left(\sqrt{\frac{-p'(\overline{V}+\varphi)}{v}}\varphi_x\right)(\tau)\right\|^2 d\tau + C(M_2)\ell^{-\frac{1}{3}}. \end{aligned} \tag{5.16}$$

Substituting (5.11)–(5.16) into (5.10), we arrive at

$$\begin{aligned} &\int_{\mathbf{R}}\left\{\frac{\mu}{2}\left(\frac{\varphi_x}{v}\right)^2-\frac{\varphi_x}{v}\psi\right\}(t,x)dx-\int_0^t\int_{\mathbf{R}}\left(p'(\overline{V}+\varphi)\frac{\varphi_x^2}{v}\right)(\tau,x)dxd\tau \\ &\quad\le \int_{\mathbf{R}}\left\{\frac{\mu}{2}\left(\frac{\varphi_x}{v}\right)^2-\frac{\varphi_x}{v}\psi\right\}(0,x)dx \\ &\qquad + C_{15}(M_2)\ell^{-\frac{2}{3}}\int_0^t(1+\tau)^{-\frac{4}{3}}\left\|\left(\sqrt{\Phi(v,\overline{V})(\tau)},\psi(\tau)\right)\right\|^2 d\tau \\ &\qquad + \int_0^t\int_{\mathbf{R}}\left(\frac{\psi_x^2}{v}\right)(\tau,x)dxd\tau + C_{15}(M_2)\ell^{-\frac{1}{3}}\left\{1+\int_0^t(1+\tau)^{-\frac{7}{6}}\left\|\frac{\varphi_x}{v}(\tau)\right\|^2d\tau\right\}. \end{aligned} \tag{5.17}$$

From (5.8), (5.17) and Gronwall's inequality, we see that if we can find ℓ sufficiently large such that

$$C_{15}(M_2)\ell^{-\frac{1}{3}} \le 1, \tag{5.18}$$

we can get that

$$\begin{aligned} \left\|\left(\sqrt{\Phi(v,\overline{V})},\psi,\frac{\varphi_x}{v}\right)(t)\right\|^2 + \int_0^t\left\|\left(\frac{\psi_x}{\sqrt{v}},\sqrt{\frac{-p'(\overline{V}+\varphi)}{v}}\varphi_x\right)(\tau)\right\|^2 d\tau \\ \le C_{16}(\underline{V},\overline{V})\left(1+\|(\varphi_0,\psi_0)\|_1^2\right). \end{aligned} \tag{5.19}$$

Setting

$$\kappa_4(x)=\max\{x,C_{15}(x)\},\quad \tilde{v}=\frac{v}{\overline{V}} \tag{5.20}$$

and noticing

$$\frac{\tilde{v}_x}{\tilde{v}}=\frac{\varphi}{v}-\left(\frac{\overline{V}_x}{v}-\frac{v_x}{V}\right)=\frac{\varphi_x}{v}-\frac{\varphi\overline{V}_x}{v\overline{V}}, \tag{5.21}$$

we can get from (5.7), (5.8), (5.18), (5.19), (5.21) and Lemma 2.2 that if we choose

ℓ sufficiently large such that

$$\kappa_4(M_2)\ell^{-\frac{1}{3}} \leq 1, \tag{5.22}$$

then we have

$$\begin{aligned} &\left\|\left(\sqrt{\Phi(v,\overline{V})}, \psi, \frac{\varphi_x}{v}, \frac{\tilde{v}_x}{\tilde{v}}\right)(t)\right\|^2 + \int_0^t \left\|\left(\frac{\psi_x}{\sqrt{v}}, \sqrt{\frac{-p'(\overline{V}+\varphi)}{v}}\varphi_x\right)(\tau)\right\|^2 d\tau \\ &\leq C_{17}(\underline{V},\overline{V})\left(1+\|(\varphi_0,\psi_0)\|_1^2\right). \end{aligned} \tag{5.23}$$

Here notice also that the constant $C_{17}(\underline{V},\overline{V})$ on the right hand side of (5.23) is independent of M_2.

Now we try to use (5.23) and a modified version of the J. Kanel′ method, cf. [29, 30], to deduce a lower bound and an upper bound for $v(t,x)$. For this purpose, we have from Lemma 2.3 that

$$\Phi(v,\overline{V}) \geq D_5\frac{(1-\tilde{v})^2}{1+\tilde{v}} := D_5\overline{\Phi}(\tilde{v}). \tag{5.24}$$

(5.23) together with (5.24) imply

$$\int_{\mathbf{R}} \overline{\Phi}(\tilde{v}(t,x))dx \leq C_{18}(\underline{V},\overline{V})\left\{1+\|(\varphi_0,\psi_0)\|_1^2\right\}. \tag{5.25}$$

Having obtained (5.25), similar to the proof of Theorem 1.2, if we set

$$\overline{\Psi}(\tilde{v}) = \int_1^{\tilde{v}} \frac{\sqrt{\overline{\Phi}(\eta)}}{\eta}d\eta, \tag{5.26}$$

we can get from the definition of $\overline{\Phi}(\eta)$ that

$$\overline{\Psi}(\tilde{v}) \to \begin{cases} -\infty, & \text{as } \tilde{v}\to 0_+, \\ +\infty, & \text{as } \tilde{v}\to +\infty. \end{cases} \tag{5.27}$$

The above observation together with the fact that

$$\begin{aligned} \left|\overline{\Psi}(\tilde{v}(t,x))\right| &= \left|\int_{-\infty}^x \frac{\partial\overline{\Psi}(\tilde{v}(t,y))}{\partial y}dy\right| \leq \left\|\sqrt{\overline{\Phi}(\tilde{v}(t))}\right\| \left\|\left(\frac{\tilde{v}_x}{\tilde{v}}\right)(t)\right\| \\ &\leq C_{19}(\underline{V},\overline{V})\left\{1+\|(\varphi_0,\psi_0)\|_1^2\right\} \end{aligned}$$

imply that there exists a positive constant κ_5 depending only on $\underline{V},\overline{V}$ and $\|(\varphi_0,\psi_0)\|_1$ such that

$$2\kappa_5\left(\|(\varphi_0,\psi_0)\|_1\right)^{-1} \leq v(t,x) \leq \frac{1}{2}\kappa_5\left(\|(\varphi_0,\psi_0)\|_1\right). \tag{5.28}$$

The above analysis together with a standard energy estimate on $\|\psi_x(t)\|$ yield the following result.

Lemma 5.1. *Let $(\varphi(t,x),\psi(t,x))$ be a solution to the Cauchy problem* (5.1)*,* (5.2) *defined on the strip Π_T for some $T>0$ and assume that the a priori assumption*

(5.3) *holds. If we can find* ℓ *sufficiently large such that* (5.22) *is satisfied, then for all* $(t,x)\in[0,T]\times\mathbf{R}$, $(\varphi(t,x),\psi(t,x))$ *satisfies* (5.23) *and* (5.28).

Moreover we can deduce that there exists a positive constant κ_6 *depending only on* $\underline{V},\overline{V}$ *and* $\|(\varphi_0,\psi_0)\|_1$ *such that*

$$\left\|(\varphi,\psi)(t)\right\|_1\le\kappa_6\left(\|(\varphi_0,\psi_0)\|_1\right). \tag{5.29}$$

Once we have obtained Lemma 5.1, the proof of Theorem 1.3 is completely the same as that of Theorem 1.2. We thus omit the details for brevity. This completes the proof of Theorem 1.3.

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