TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 361, Number 1, January 2009, Pages 223–236
S 0002-9947(08)04639-4
Article electronically published on August 19, 2008

THE EQUATION $x^p y^q = z^r$ AND GROUPS THAT ACT FREELY ON Λ-TREES

N. BRADY, L. CIOBANU, A. MARTINO, AND S. O ROURKE

Abstract. Let G be a group that acts freely on a Λ-tree, where Λ is an ordered abelian group, and let x, y, z be elements in G. We show that if $x^p y^q = z^r$ with integers p, q, $r \geq 4$, then x, y and z commute. As a result, the one-relator groups with $x^p y^q = z^r$ as relator, are examples of hyperbolic and CAT(-1) groups which do not act freely on any Λ-tree.

1. Introduction

There has recently been a great deal of interest in tree-free groups, that is, groups which act freely and without inversions by isometries on some Λ-tree. The principal source of this interest has been related to the solution of the Tarski problem, where limit groups, one of the main objects of study, have been shown to act freely on $\mathbb{Z}^n$-trees for some n. Groups that act freely on Λ-trees (so-called Λ-*free groups*) generalise free groups in the sense that $\mathbb{Z}$-free groups are precisely free groups. Moreover, for general Λ, these groups satisfy properties reminiscent of free groups. For example, they are torsion-free, closed under free products, and commutativity is a transitive relation on non-identity elements. In addition, all known examples of finitely generated Λ-free groups that contain no copy of $\mathbb{Z} \times \mathbb{Z}$ are hyperbolic.

The purpose of this article is to generalise a classical theorem in free groups to the broad class of tree-free groups. The result of Lyndon and Schützenberger ([14]) states that any elements x, y, and z of F, a free group, that satisfy the relation $x^p y^q = z^r$ for p, q, $r \geq 2$ commute. (See also [13], [4], [19], [18], [7].) Therefore all solutions to this equation are contained in a cyclic subgroup of F. Here we show,

Theorem 3.2. *Let G be a group that acts freely, and without inversions, by isometries on a Λ-tree, where Λ is an ordered abelian group, and let x, y, z be elements in G. If $x^p y^q = z^r$ with p, q, $r \geq 4$, then x, y and z commute.*

While the argument of Lyndon and Schützenberger relies on combinatorics of words in the free group, our argument relies on the information provided by the action via isometries of the group on the Λ-tree.

A Λ-metric space can be defined in the same way as a conventional metric space with $\mathbb{R}$ replaced by Λ. A Λ-tree can be characterised as a geodesically convex Λ-metric space (X, d) which is 0-hyperbolic and which satisfies $d(x, v) + d(y, v) - d(x, y) \in 2\Lambda$ for all $x, y, v \in X$ (see [5]). When the group Λ is Archimedean the free actions on Λ-trees are well understood. In particular, the finitely generated

Received by the editors December 6, 2006.
2000 *Mathematics Subject Classification.* Primary 20E08, 20F65.
Key words and phrases. Free actions, Λ-trees, hyperbolic groups, CAT(-1).

groups that act freely on $\mathbb{R}$-trees have been completely classified by Rips. They are the groups that can be written as a free product $G_1 \star G_2 \star \cdots \star G_n$ for some integer $n \geq 1$, where each G_i is either a finitely generated free abelian group or a non-exceptional surface group. In the non-Archimedean case, Martino and O Rourke (see [16]) have provided examples of $\mathbb{Z}^n$-free groups. Also, it is known that among the groups that act freely on $\mathbb{R}^n$-trees are the fully residually free groups, or *limit groups* ([12], [20], [10]). The fact that limit groups are exactly the groups with the same universal theory as free groups (see [17]) immediately implies that solutions of $x^p y^q = z^r$ commute in limit groups. We show that in addition to limit groups, the commutativity of solutions to $x^p y^q = z^r$ holds in all the groups that act freely on Λ-trees, with some restriction on the exponents.

We would like to point out one intriguing difference in the behaviour of the equation $x^2y^2 = z^2$ in free groups versus groups that act freely on Λ-trees. In free groups, if we have elements x, y and z such that $x^2y^2 = z^2$, then x, y and z commute, while for general Λ-free groups this is not true, since the exceptional surface group $\langle x, y, z, \mid x^2y^2z^2 = 1\rangle$ acts freely on a $\mathbb{Z}^2$-tree ([8]). By the Base-Change Functor Theorem (see Section 2) it follows that this group acts freely on any non-Archimedean tree.

One interesting question to ask is where the arguments for free groups and tree-free groups must diverge when considering equations. In fact, most of our techniques work in the various cases we consider for all equations of the type $x^p y^q = z^r$, with $p, q, r \geq 2$. The real difference seems to be that in free groups for the cases of small exponents one has to use inductive arguments on length which cannot work for a general Λ-tree, since there are generally infinitely many lengths less than any given one. For example, if $p = q = r = 3$, we can successfully employ the same techniques we used for larger values of p, q and r. However, we encountered difficulties in part (3) of our proof, when the intersection Δ of the axes A_x and A_y has exactly the same length as the shortest of the translations, y^q.

Nevertheless, one immediate consequence of our result is that one can construct many groups which cannot act freely on any Λ-tree. In particular, if we look at the one-relator groups, defined as follows,

$$G_{pqr} = \langle x, y, z \mid x^p y^q = z^r\rangle,$$

we get a family of groups which do not act freely on any Λ-tree, for $p, q, r \geq 4$. Moreover, these groups are all small-cancellation groups; they are $C(6) - T(4)$ for all $p, q, r \geq 2$, and so are word hyperbolic (see [9]). Therefore we obtain

Corollary 1.1. *The groups G_{pqr} form a family of word hyperbolic groups which cannot act freely, and without inversions, by isometries on any Λ-tree.*

We note that Chiswell ([6]) has produced a family of word hyperbolic groups with no *non-trivial* action on a Λ-tree; any such action by one of these groups has a global fixed point. In the same spirit, any group which satisfies Kazdan's property T has no non-trivial action on a Λ-tree, and so one would expect that generic hyperbolic groups will not be tree-free. However, the groups G_{pqr} do not have property T because their abelianisation is infinite, which implies that they act non-trivially on an $\mathbb{R}$-tree.

Note that by contrast the groups

$$\langle x_1, x_2, \ldots, x_n \mid x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} = 1\rangle$$

are expressible as amalgamated free products of free groups over maximal cyclic subgroups for $n \geq 4$ provided at least four α_i are non-zero. It follows that these groups are $\mathbb{Z}^2$-free (see [1], [16]). In fact, these groups are fully residually free groups ([2], [3]).

In the final section of the paper we show that these groups G_{pqr} have CAT(−1) structures. This question arises naturally since, intuitively, a result true for tree-like structures often has a weaker analogue for hyperbolic structures. Our aim, initially, was to try to provide examples of word hyperbolic groups which do not have CAT(−1) structures, using the sorts of length arguments we employ for the case of Λ-trees. While this naive approach doesn't seem to work, there is still some hope that the arguments may provide some restrictions to the possible CAT(−1) structures, in particular translation lengths. The aim would then be to construct multiple HNN-extensions from the groups G_{pqr} which are word hyperbolic via the Bestvina and Feighn Combination Theorem, on the one hand, and which violate the translation length restrictions for CAT(−1) structures, on the other hand.

2. Background

A complete account of Λ-trees is given in [5]. Here we recall the basic relevant definitions and results. An *ordered abelian group* is an abelian group Λ, together with a total ordering $\leq$ on Λ, compatible with addition. For $a \leq b$, we define $[a,b]_\Lambda = \{x \in \Lambda \mid a \leq x \leq b\}$. A Λ-*metric space* (X,d) can be defined in the same way as a conventional metric space. That is, $d : X \times X \to \Lambda$ is symmetric, satisfies the triangle inequality and satisfies $d(x,y) = 0$ if and only if $x = y$. A *segment* in X is the image of an isometry $\alpha : [a,b]_\Lambda \to X$ for some a, b in Λ, with $\alpha(a), \alpha(b)$ the endpoints of the segment. A Λ-metric space is *geodesic* if for all x and y in X there is a segment $[x,y]$ with endpoints x and y.

Definition 2.1. A Λ-tree is a geodesic Λ-metric space (X,d) such that:

(a) if two segments of (X,d) intersect in a single point, which is an endpoint of both, then their union is a segment;
(b) the intersection of two segments with a common endpoint is also a segment.

It follows that there is a unique segment having x and y as endpoints. We denote this segment by $[x,y]$. From now on, X shall denote a Λ-tree. We note that a *subtree* of X is a subset $A \subseteq X$ such that $x, y \in A$ implies $[x,y] \subseteq A$.

Isometries of Λ-trees are classified, just as those of ordinary trees, and come in three distinct types: *inversions, elliptic* and *hyperbolic* isometries. Elliptic isometries are those which fix some point in the tree, inversions are those which do not fix a point but whose square does and hyperbolic isometries are the remaining ones. It is usually convenient (and no loss of generality, see 2.4) to assume that all isometries are either elliptic or hyperbolic, as we shall now do.

For every isometry, g, of X, there is a well-defined translation length,

$$\|g\| = \min \{d(x, gx) \mid x \in X\}.$$

It can be shown that this minimum is always realised, so that it is equal to zero for elliptic elements and is strictly positive for hyperbolic ones. One then defines A_g, the *characteristic set* or *axis* of g, as

$$A_g = \{x \in X \mid d(x, gx) = \|g\|\}.$$

For an elliptic element this is simply the fixed subtree, and for a hyperbolic element it is the maximal invariant linear subtree of X on which the isometry acts by translation. For an isometry, g, of X the characteristic set is also equal to

$$A_g = \{p \in X \mid [g^{-1}p, p] \cap [p, gp] = \{p\}\}.$$

It is an easy exercise to verify the following.

Lemma 2.2. *If g is a hyperbolic isometry and n is a non-zero integer, then $\|g^n\| = |n|\|g\|$ and $A_g = A_{g^n}$.*

To better visualise axes of translation in Λ-trees, especially for hyperbolic elements, we provide Figure 1. We remark that if $[p, x] \cap A_g = \{x\}$, then p, x, gx and gp are collinear in this order.

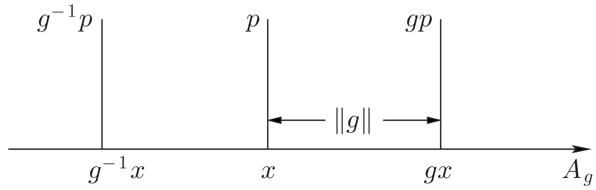

FIGURE 1. Axis of translation A_g

In a Λ-tree, X, every triple of points, p_1, p_2, p_3 has a Y-point, $Y(p_1, p_2, p_3)$, which uniquely lies on all segments $[p_i, p_j]$, for $i \neq j$. The characteristic set of an isometry g is also equal to $\{Y(g^{-1}p, p, gp) \mid p \in X\}$.

If g and h are hyperbolic isometries of X such that $A_g \cap A_h \neq \emptyset$, and g and h translate in the same direction along $A_g \cap A_h$, then we say that g and h meet *coherently*. If $A_g \cap A_h \neq \emptyset$ and g and h translate in different directions along $A_g \cap A_h$, then g and h meet *incoherently*.

Now let G be a group that acts on X via isometries. In this paper we consider only *free* actions, that is, actions without inversions in which no non-trivial element of G fixes a point in the tree. Thus all non-trivial isometries are hyperbolic.

One of the characteristics of free actions on Λ-trees is that for $gh \neq hg$ we have that $A_g \cap A_h$ is a segment of length not exceeding $\|g\| + \|h\|$, since otherwise the commutator of g and h would be an elliptic element, contradicting the freeness of the action. We state this formally, because of its importance, even though it amounts to a fairly trivial observation.

Lemma 2.3 ([5, Remark, page 111])**.** *Let G be a group acting freely without inversions on a Λ-tree, and let $g, h \in G$. Then if g and h do not commute, $A_g \cap A_h$ cannot contain a segment of length greater than or equal to $\|g\| + \|h\|$. Conversely, if g and h commute, they share an axis and hence $A_g \cap A_h$ will contain a segment of length greater than or equal to $\|g\| + \|h\|$.*

One property of Λ-free groups that we will use in this paper is that of *commutative-transitivity* of non-identity elements, which is equivalent to saying that centralisers of non-identity elements are abelian and follows from the fact that two non-identity elements commute if and only if they have the same axis. Therefore, if F is a non-abelian free group, $F \times \mathbb{Z}$ is not tree-free. However, letting T denote the Cayley graph of F, $F \times \mathbb{Z}$ acts on the contractible space $T \times \mathbb{R}$, which fails to be an $\mathbb{R}$-tree under the natural sum metric since property (a) of Definition 2.1 is not satisfied. However, we note that it is possible to make $T \times \mathbb{R}$ into an $\mathbb{R}$-tree (with a finer topology) analogous to the two-sided comb metric on $\mathbb{R}^2$; in this metric, $F \times \mathbb{Z}$ no longer acts isometrically.

Another useful fact about actions on Λ-trees is the device that relates actions on Λ_1-trees to actions on Λ_2-trees, as in the following theorem.

Theorem 2.4 ([15], [5], Corollary 2.4.9, page 76 (Base-Change Functor))**.** *Let $h : \Lambda_1 \to \Lambda_2$ be an order preserving homomorphism between ordered abelian groups and let G be a group acting by isometries on a Λ_1-tree, (X_1, d_1). Then there is a Λ_2-tree, (X_2, d_2) on which G acts by isometries and a mapping $\phi : X_1 \to X_2$ such that*

(i) *$d_2(\phi(x), \phi(y)) = h(d_1(x, y))$, for all $x, y \in X_1$,*
(ii) *$\phi(gx) = g\phi(x)$ for all $g \in G$ and $x \in X_1$,*
(iii) *$\|g\|_{X_2} = h(\|g\|_{X_1})$ for all $g \in G$.*

For the X_2 constructed in the proof of Theorem 2.4 we have that if the action of G on X_1 is free and h is injective, then the action of G on X_2 is also free. In particular we can construct the barycentric subdivision X' of X_1 by taking the endomorphism h to be $\lambda \to 2\lambda$; the resulting action is then without inversions.

Note also that since $\mathbb{Z}^n$ embeds in $\mathbb{R}^n$ every free action on a $\mathbb{Z}^n$-tree gives rise to a free action on an $\mathbb{R}^n$-tree.

3. The main theorem

The proof of our main theorem will be a length-based argument relying on the analysis of the various configurations of axes. Formally we shall argue by contradiction by assuming $x^p y^q = z^r$ in a tree-free group, for x, y, z which do not commute. By commutative-transitivity, this is equivalent to the assumption that no two of x, y, z commute, which is expressed in the proof via Lemma 2.3.

The proof itself is elementary from the point of view of Λ-tree theory, though the justification of the figures we provide would be rather technical from first principles. However, we should stress that the technical proofs we refer to are largely formal demonstrations that one's geometric intuition works perfectly well in the context of Λ-trees. The following lemma provides a useful tool for determining the position of axes, which is a crucial part of our proof.

Lemma 3.1. *Let u, v be distinct points in a Λ-tree, X, and g a hyperbolic isometry of X. Then if u, v, ug, vg are collinear in the order given, all the points u, v, ug, vg must lie on the axis of g.*

Informally, one chooses a point u which one wants to show is on the axis of g, and a point v, 'close' to u in the positive g direction, and checks collinearity of the images in the given order.

We shall now present three lemmas which describe the axis of a product of two elements, depending on how the original axes intersect. These are all standard results which we recall here so as to more easily refer to them in our main argument. The detailed proofs of each of these lemmas may be found in [6]; however we also provide more informal justifications based on Lemma 3.1.

Lemma 3.2 (Lemmas 3.2.2 and 3.3.1, [5])**.** *Let g and h be hyperbolic isometries of a Λ-tree (X, d).*

- *If $A_g \cap A_h = \emptyset$, then $||gh|| = ||g|| + ||h|| + 2d(A_g, A_h)$. (See Figure 2.)*

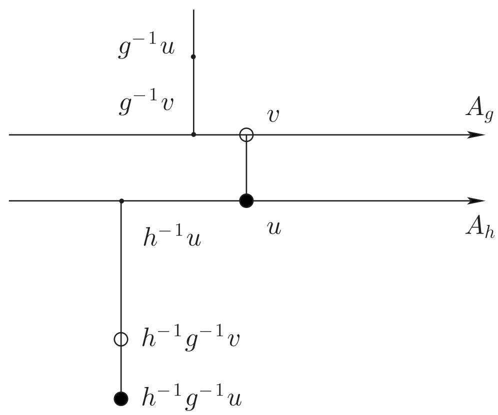

FIGURE 2. Disjoint axes

- *If A_g and A_h meet coherently, then $||gh|| = ||g|| + ||h||$. (See Figure 3 for a possible configuration of points.)*

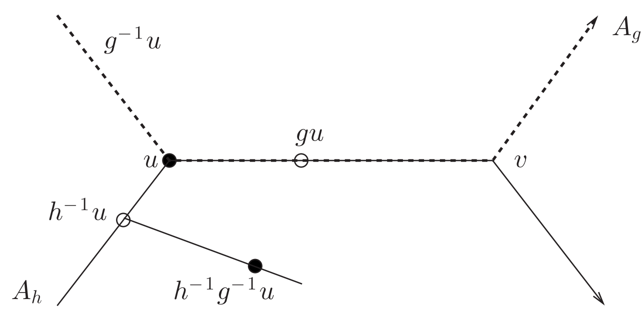

FIGURE 3. Coherent axes

Proof. To prove the first statement, we shall simply justify Figure 2 and appeal to Lemma 3.1, since it is clear that $d(h^{-1}g^{-1}u, u) = ||g|| + ||h|| + 2d(u, v)$.

The segment $[u, v]$ meets the A_g only at v; therefore $[g^{-1}u, g^{-1}v]$ is a segment that meets the A_g only at $g^{-1}v$. Hence $g^{-1}u, g^{-1}v, u$ are collinear, and $[g^{-1}u, u]$ meets the A_h only at u. So $h^{-1}g^{-1}u, h^{-1}g^{-1}v$ and $h^{-1}u$ are collinear and $[h^{-1}g^{-1}u, h^{-1}u]$ meets A_h only at $h^{-1}u$. One then applies Lemma 3.1 to the points $h^{-1}g^{-1}u, h^{-1}g^{-1}v, u, v$ to deduce that $h^{-1}g^{-1}u$ is on the axis of gh.

The second part of the lemma is justified similarly. If $\Delta \geq ||g|| + ||h||$, then $A_g = A_h$ and the stated equality is clear. Otherwise $A_g \cap A_h$ is a segment $[u, v]$ where $u \in [g^{-1}u, v]$ ([5]); that is, v lies in the positive A_g direction from u. Then $[g^{-1}u, u]$ meets A_h only at u, and $[h^{-1}g^{-1}u, h^{-1}u]$ meets A_h only at $h^{-1}u$. Since gu is in the positive A_g direction from u, we can apply Lemma 3.1 to the points $h^{-1}g^{-1}u, h^{-1}u, u, gu$ to show that $h^{-1}g^{-1}u$ is on the axis of gh. The only possible ambiguity is whether gu is on the left or right of v, but this makes no difference to the argument. □

Lemma 3.3 ([5, Lemma 3.3.3]). *Let g and h be hyperbolic isometries of a Λ-tree (X, d) which meet incoherently, and let $\Delta(g, h)$ be the intersection of A_g and A_h. If $|\Delta(g, h)| < ||h|| \leq ||g||$, then*

- *gh meets g, h coherently,*
- *$||gh|| = ||g|| + ||h|| - 2|\Delta(g, h)|$, and the configuration can be seen in Figure* 4.

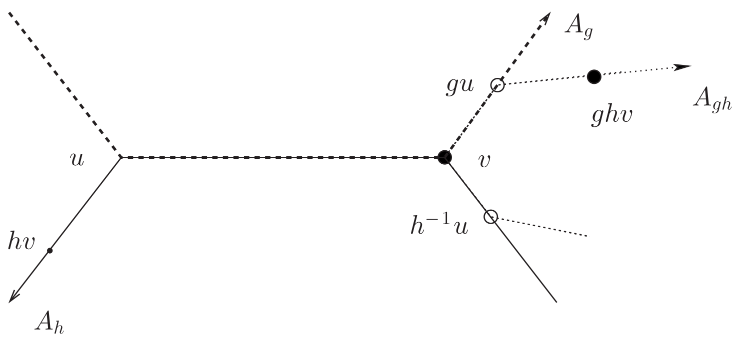

FIGURE 4. Incoherent axes - small intersection

Proof. We identify four ordered collinear points that will allow us to use Lemma 3.1. The positions of the points $h^{-1}u$ and gu can be immediately deduced. We only need to show that v, gu and ghv are collinear and occur in this order. It is clear that hv is in the positive direction of A_h from u. Since the segment $[u, hv]$ meets A_g only at u, $[gu, ghv]$ meets A_g only at gu, and we are done. □

Lemma 3.4 ([5, Lemma 3.3.4]). *Let g and h be hyperbolic isometries of a Λ-tree (X, d) which meet incoherently, and let $\Delta(g, h)$ be the intersection of A_g and A_h. If $||h|| < |\Delta(g, h)| \leq ||g||$, then*

- *gh meets g, h^{-1} coherently,*
- *$||gh|| = ||g|| - ||h||$, and the configuration can be seen in Figure* 5.

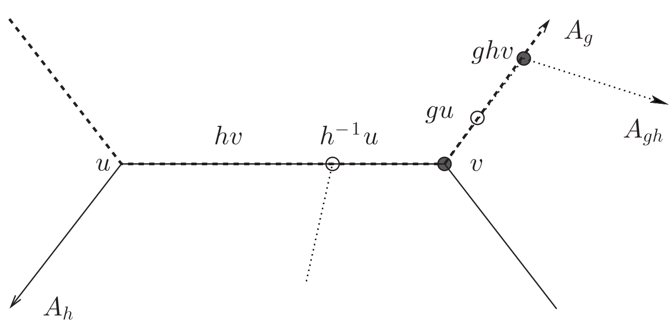

FIGURE 5. Incoherent axes - large intersection

Proof. One can easily establish that the points $[h^{-1}u, v, gu, ghv]$ are collinear and then apply Lemma 3.1. Arguments similar to those given in Lemma 3.3 can then be used to show that $A_{gh} \cap A_h = [h^{-1}u, v]$ and $A_{gh} \cap A_g = [v, ghv]$. □

Lemma 3.5 ([5, Lemma 3.3.5])**.** *Let g, h and gh be hyperbolic isometries of a Λ-tree (X, d) such that g and h meet incoherently, and let $\Delta(g, h)$ be the intersection of A_g and A_h. If $\Delta = |\Delta(g, h)| = ||h|| < ||g||$ and $w = Y(h^{-1}g^{-1}v, v, ghv)$, then*

- *gh meets g coherently. $A_{gh} \cap A_g = [w, ghw]$.*
- *$A_{gh} \cap A_h$ is either empty or a single point. The latter case occurs if and only if $w = v$, which implies $\{w\} = A_{gh} \cap A_h$.*
- *$||gh|| = ||g|| - ||h|| - 2d(v, w)$, and the configuration can be seen in Figure 6.*

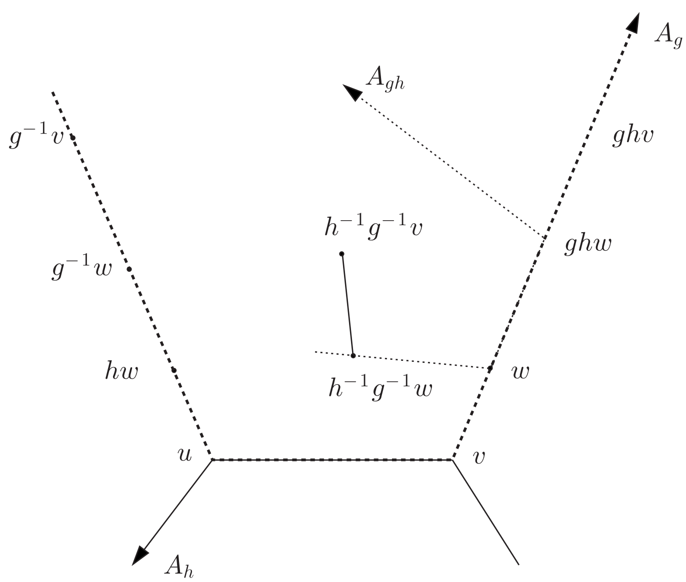

FIGURE 6. Incoherent axes - exact intersection

Proof. Since the axis of gh is equal to $\{Y(h^{-1}g^{-1}p, p, ghp) : p \in X\}$, it is immediate that $w \in A_{gh}$. It is clear that $ghv = gu$ is on A_g, in the positive A_g direction from v. Next note that $[g^{-1}v, u]$ is a segment of length $\|g\| - \Delta$ which meets A_h only in the point u. Therefore, $[h^{-1}g^{-1}v, v]$ is a segment of the same length which only meets A_h in the point v. Hence, if $h^{-1}g^{-1}v$ were to lie on A_g, it would have to be in the positive A_g direction from v and by comparing lengths it would have to be equal to $ghv = gu$, contradicting the fact that gh is hyperbolic. Thus $h^{-1}g^{-1}v$ does not lie on A_g.

Now observe that w is on the axis of g, since $Y(h^{-1}g^{-1}v, v, ghv) \in [v, ghv] = [v, gu] \subseteq A_g$; also note that $ghw = Y(v, ghv, (gh)^2v) \in [v, ghv] \subseteq A_g$. Thus $[v, ghv] = [v, w] \cup [w, ghv]$ and we shall prove that $ghw \in [w, ghv]$. We shall therefore argue by contradiction and assume that $ghw \in [v, w]$. Since $ghv, ghw \in A_g$ this would imply that $(gh)^2w \in A_g$. Moreover, as $w \in [v, ghv]$, ghv must be in the positive A_g direction from w. Clearly, $(gh)^2w \in [ghv, ghw]$ and so by comparing lengths, $(gh)^2w = w$, which contradicts the assumption that gh is hyperbolic.

To completely justify the picture the reader should also satisfy themselves that the segment $[w, v]$ can only meet A_{gh} at w. This is because we already know that A_{gh} contains $h^{-1}g^{-1}w, w, ghw$ and that it is a linear set. However, the only point on $[w, v]$ which can be collinear with all of $h^{-1}g^{-1}w, w$ and ghw (in some order) is w itself. Similarly, $[ghw, ghv] \cap A_{gh} = \{ghw\}$ and $[h^{-1}g^{-1}v, h^{-1}g^{-1}w] \cap A_{gh} = \{h^{-1}g^{-1}w\}$. Thus $A_g \cap A_{gh} = [w, ghw]$.

Now one can easily compute $||gh|| = d(hw, ghw) - 2d(v, w) - |\Delta(g, h)| = ||g|| - ||h|| - 2d(v, w)$, using the fact that w is on the axis of gh. □

We now proceed with the proof of the main theorem.

Theorem 3.6. *Let G be a group that acts freely, and without inversions, by isometries on a Λ-tree, where Λ is an ordered abelian group, and let x, y, z be elements in G. If $x^p y^q = z^r$ with p, q, $r \geq 4$, then x, y and z commute.*

Proof. Let us assume that x, y and z do not commute and let A_x, A_y and A_z be the axes of translation of x, y and z, respectively. By Lemma 2.2, we know that $A_x = A_{x^n}$ and $\|x^n\| = |n|\|x\|$ (the same is clearly true for y and z) for every non-zero integer n. We can assume without loss of generality that p, q, r are positive and

(1) $$r\|z\| \leq q\|y\| \leq p\|x\|.$$

If $A_x \cap A_y = \emptyset$, then by Lemma 3.2,

$$r\|z\| = \|z^r\| = \|x^p y^q\| > \|x^p\| + \|y^q\| = p\|x\| + q\|y\|,$$

which contradicts assumption (1).

Now let us assume that $A_x \cap A_y \neq \emptyset$. Let $\Delta(x, y)$ be the intersection of the two axes, and let $\Delta = |\Delta(x, y)| \in \Lambda$ be the length of this segment. Since we assume that x and y do not commute and the action is free, by Lemma 2.2,

(2) $$\Delta < \|x\| + \|y\|.$$

If A_x and A_y meet coherently, then by Lemma 3.2,

$$r\|z\| = \|z^r\| = \|x^p y^q\| = \|x^p\| + \|y^q\| = p\|x\| + q\|y\|,$$

which also contradicts assumption (1).

Now let us assume that A_x and A_y meet incoherently. Let $\Delta(x, z)$ and $\Delta(y, z)$ be the intersection of A_x and A_z, and A_y and A_z, respectively. Then we have three cases to consider, depending on the length of Δ relative to $\|y^q\|$.

(1) Let us first assume that the intersection of A_x and A_y is relatively small:

(3) $$\Delta < \|y^q\|.$$

Then by Lemma 3.3, setting $g = x^p$ and $h = y^q$, A_z meets both A_x and A_y coherently, and we have the configuration as in Figure 7.

Since $\|z^r\| \leq \|y^q\|$ and $\|z^r\| = \|x^p\| + \|y^q\| - 2\Delta$ we have

(4) $$\Delta \geq \frac{\|x^p\|}{2}.$$

Inequalities (2) and (4) give $(p-2)\|x\| < 2\|y\|$ and by using the assumption (1) we also get $(q-2)\|y\| < 2\|x\|$.

By putting these inequalities together we get that, if $p \geq 4$,

(5) $$(q-2)\|y\| < 2\|x\| \leq (p-2)\|x\| < 2\|y\|.$$

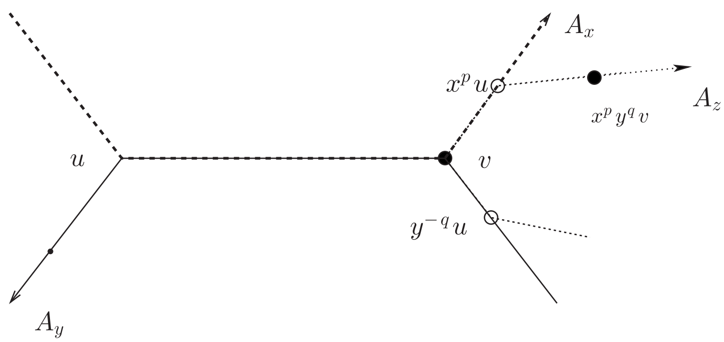

FIGURE 7. Incoherent axes - small intersection

This implies $q < 4$, which is not in our range, and so this configuration cannot happen.

(2) Let us now assume that

$$\Delta > \|y^q\|, \tag{6}$$

as in Figure 8.

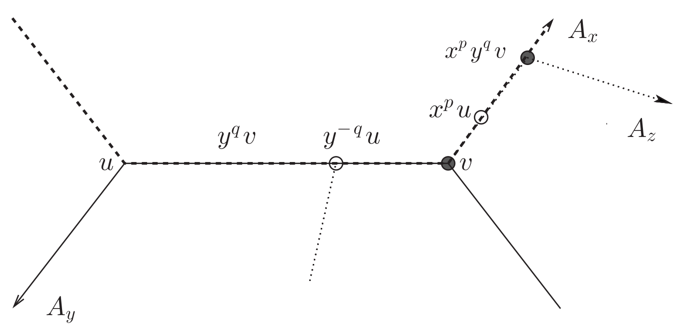

FIGURE 8. Incoherent axes - large intersection

Note that if $\Delta > \|x^p\|$, then $\Delta > \max\{\|x^p\|, \|y^q\|\} \geq \max\{2\|x\|, 2\|y\|\} \geq \|x\| + \|y\|$, contradicting our assumption (2). Therefore, $\|y^q\| < \Delta \leq \|x^p\|$.

This is exactly the situation of Lemma 3.4, so if we set $g = x^p$, $h = y^q$, then $A_z = A_{gh}$ meets A_x coherently and A_y incoherently, and

$$\|z^r\| = \|x^p\| - \|y^q\|. \tag{7}$$

Since x, y and z do not commute,

$$\|x\| + \|z\| > |\Delta(x, z)| = \|z^r\| \implies \|x\| > (r-1)\|z\| \tag{8}$$

and (2) together with (6) give

$$\|x\| > (q-1)\|y\|. \tag{9}$$

Since $p \geq 4$ we have

$$\|x^p\| = \|x^{p-4}\| + \|x^2\| + \|x^2\| > \|x^{p-4}\| + q\|y\| + r\|z\|$$

as $q \geq 3$ and $r \geq 3$ imply $2\|x\| > q\|y\|$ and $2\|x\| > r\|z\|$ by (8) and (9). This contradicts (7).

(3) The last case to consider is $\Delta = \|y^q\|$. Note that in this case, $\Delta < \|x^p\|$ by (1), since otherwise $y^{-q}x^p$ would be elliptic. Therefore, we are in the situation described in Lemma 3.5, setting $g = x^p$ and $h = y^q$. (See Figure 9.)

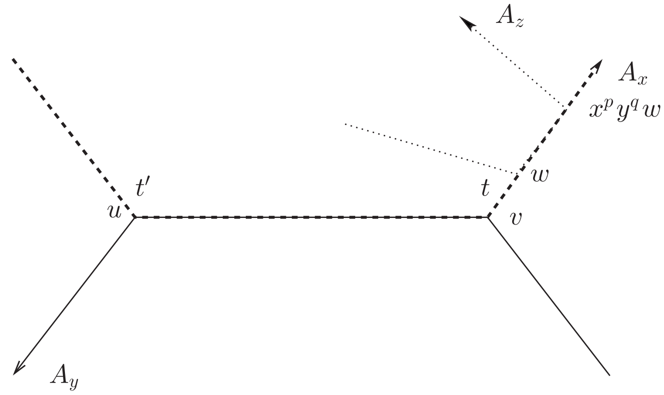

FIGURE 9. Incoherent axes - exact intersection

By (2) the above equality gives $(q-1)\|y\| < \|x\|$, which implies $\Delta = \|y^q\| < 2\|x\|$ since $q > 2$. Using the notation from Lemma 3.5, let w be $Y(y^{-q}x^{-p}v, v, x^p y^q v)$, and let $l = d(v, w)$. From Lemma 3.5, $[w, x^p y^q w]$ is the intersection of A_x and A_z and

$$|\Delta(x,z)| = \|z^r\| = \|x^p\| - \|y^q\| - 2l. \tag{10}$$

We will show that

$$\|x\| < \Delta < 2\|x\| - 2l. \tag{11}$$

Suppose that the second inequality does not hold. Then, passing to the barycentric subdivision if necessary, there exist points $t \in [v, w]$ and $t' \in [y^q w, u]$ such that $d(t, t') = 2\|x\|$ and $t' = y^q t$ (see Theorem 2.4). This implies $x^2 y^q t = t$ and so $x^2 y^q$ is an elliptic element, a situation possible only if $y^q = x^{-2}$. However, by commutative-transitivity this implies that x and y commute, which contradicts our initial assumption.

Now let us assume the first inequality of (11) does not hold, that is, $\Delta \leq \|x\|$. If $\Delta + 2l \geq \|x\|$, then we can repeat the previous argument to show that xy^q is an elliptic element, which implies that x and y commute, and we obtain a contradiction. So $\Delta + 2l < \|x\|$. Since x and z do not commute, we have $|\Delta(x,z)| < \|x\| + \|z\|$, which implies $\|z^r\| < \|x\| + \|z\|$. But $\Delta + 2l < \|x\|$ and (10) imply $\|z\| > (p-2)\|x\|$, which is false by (1). This concludes the proof of (11).

It follows from (10) and (11) that $\|z^r\| > (p-2)\|x\|$. Since we assume that x and z do not commute we get $|\Delta(x,z)| = \|z^r\| < \|x\| + \|z\|$, so in conclusion $(p-3)\|x\| < \|z\|$, which contradicts (1) if $p, q \geq 4$. □

4. CAT(−1) structures

In this section we show that the groups

$$G_{pqr} \;=\; \langle\, x, y, z \,|\, x^p y^q = z^r \,\rangle$$

are all CAT(−1). As mentioned in the introduction, the original motivation for studying these CAT(−1) structures, was to see if certain multiple HNN extensions (described in Remark 4.2 below) of G_{pqr} gave examples of hyperbolic groups which are not CAT(−1). We do not know whether or not these HNN groups are CAT(−1). However they should admit high-dimensional CAT(0) structures by the techniques of Hsu and Wise ([11]).

Proposition 4.1. *Let $p, q, r \geq 2$ be integers. The groups*

$$G_{pqr} \;=\; \langle\, x, y, z \,|\, x^p y^q = z^r \,\rangle$$

admit CAT(−1) structures corresponding to each isometry class of triangles in the hyperbolic plane.

Proof. In order to see that the groups G_{pqr} are all CAT(−1), fix an arbitrary triangle in the hyperbolic plane with positive angles α, β and γ. Subdivide the side opposite the angle α (respectively β, γ) into p (respectively q, r) subsegments of equal length, and label the subsegments by x (respectively y, z^{-1}) as shown on the left side of Figure 10.

The quotient space of this triangle obtained by isometrically identifying all the x-edges (respectively y-edges, z-edges) is a cell complex, with one vertex, three 1-cells (labelled x, y and z respectively) and a single 2-cell corresponding to the triangle. This cell complex is a presentation 2-complex for the group G_{pqr}. It is a piecewise hyperbolic 2-complex.

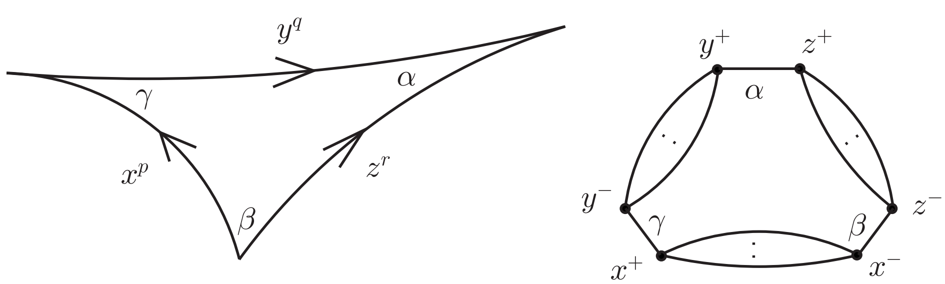

Figure 10. 2-cell of G_{pqr} presentation 2-complex and vertex link

The link of the single vertex is the metric graph shown on the right side of Figure 10. There are $p-1$ edges from x^+ to x^-, $q-1$ edges from y^+ to y^-, and $r-1$ edges from z^+ to z^-, all of length π. The remaining three edges have lengths α, β and γ as indicated in the figure. There are no non-trivial loops in the link of length less than 2π. Thus the link is a CAT(1) metric graph, and the 2-complex is a locally CAT(−1) presentation complex for G_{pqr}. □

Remark 4.1. There are 3-dimensional CAT(−1) structures for G_{pqr} too. One way to see this is to note that G_{pqr} is the fundamental group of the 2-complex obtained from a "thrice punctured sphere" (compact, orientable surface with three circle boundary components and with Euler characteristic -2) by wrapping one boundary circle p times around a target circle, wrapping another boundary circle q times around a second target circle, and wrapping the third boundary circle r times around a third target circle. This thickens up to give a compact, hyperbolic 3-manifold with boundary an orientable surface of genus 2.

Extracting the combinatorial information from the previous description, we can see that the G_{pqr} are the fundamental groups of graphs of groups with underlying graph a tripod, edge groups all infinite cyclic, valence 1 vertex groups all infinite cyclic, and valence 3 vertex group being free of rank 2. The inclusions from the edge groups to the valence 3 vertex group map to generators a, b and their product ab. The inclusion maps from the edge groups to the valence 1 vertex groups are just multiplication by p, q and r.

Remark 4.2. It is easy to produce hyperbolic groups from the G_{pqr} via multiple HNN extensions over infinite cyclic subgroups. For instance, one can add a stable letter which conjugates one generator to another, add another stable letter which conjugates a generator to a commutator of two generators, and so on. One uses the Bestvina-Feighn combination theorem after each HNN extension to ensure that the resulting groups are hyperbolic. The group G_{pqr} is a subgroup of these multiple HNN groups. Therefore, if these HNN groups are CAT(-1), one gets an action of G_{pqr} by semi-simple isometries on a CAT(-1) space with various restrictions on translation lengths.

The graph of free groups over infinite cyclic edge groups viewpoint of the G_{pqr} in Remark 4.1 leads to a large class of CAT(0) structures. The Sageev construction techniques being developed by Hsu and Wise will give lots of new CAT(0) cubical structures for the G_{pqr}. Their techniques should also apply to give CAT(0) cubical structures for the hyperbolic multiple HNN extensions of the G_{pqr}. But these appear to be very far from CAT(-1) structures.

Acknowledgments

The authors thank Warren Dicks for questions and remarks that led to the writing of this paper. They also thank the Centre de Recerca Matemàtica in Bellaterra for its hospitality.

The second-named author was supported by the Marie Curie Intra-European Fellowship number 515027. The third author would like to thank the Ministerio de Educación y Ciencia for support under grant number MTM2006-13544-C02.

References

[1] Hyman Bass. Group actions on non-Archimedean trees. In *Arboreal group theory (Berkeley, CA, 1988)*, volume 19 of *Math. Sci. Res. Inst. Publ.*, pages 69–131. Springer, New York, 1991. MR1105330 (93d:57003)

[2] Benjamin Baumslag. Residually free groups. *Proc. London Math. Soc. (3)*, 17:402–418, 1967. MR0215903 (35:6738)

[3] Gilbert Baumslag. On generalised free products. *Math. Z.*, 78:423–438, 1962. MR0140562 (25:3980)

[4] Gilbert Baumslag. On a problem of Lyndon. *J. London Math. Soc.*, 35:30–32, 1960. MR0111780 (22:2641)

[5] I. Chiswell. Introduction to Λ-trees. *World Scientific*, 2001. MR1851337 (2003e:20029)

[6] I. M. Chiswell. Some examples of groups with no nontrivial action on a Λ-tree. *Mathematika*, 42(1):214–219, 1995. MR1346688 (96g:20036)

[7] Warren Dicks and H. H. Glover. An algorithm for cellular maps of closed surfaces. *Enseign. Math. (2)*, 43(3-4):207–252, 1997. MR1489884 (99e:57001)
[8] A. M. Gaglione and D. Spellman. Generalizations of free groups: Some questions. *Comm. Algebra*, 22(8):3159–3169, 1994. MR1272379 (95c:20035)
[9] S. M. Gersten and H. B. Short. Small cancellation theory and automatic groups. *Invent. Math.*, 102(2):305–334, 1990. MR1074477 (92c:20058)
[10] V. Guirardel. Limit groups and groups acting freely on $\mathbb{R}^n$-trees. *Geometry & Topology*, 8:1427–1490, 2004. MR2119301 (2005m:20060)
[11] T. Hsu and D. Wise. Cubulating graphs of free groups with cyclic edge groups. In preparation, 2006.
[12] O. Kharlampovich and A. Myasnikov. Irreducible affine varieties over a free group I. Irreducibility of quadratic equations and Nullstellensatz. *Journal of Algebra* 200:517–570, 1998. MR1610664 (2000b:20032b)
[13] R. C. Lyndon. The equation $a^2b^2 = c^2$ in free groups. *Michigan Math. J*, 6:89–95, 1959. MR0103218 (21:1999)
[14] R. C. Lyndon and M. P. Schützenberger. The equation $a^M = b^N c^P$ in a free group. *Michigan Math. J.*, 9:289–298, 1962. MR0162838 (29:142)
[15] A. Martino and S. O Rourke. Free actions on $\mathbb{Z}^n$-trees: A survey. In *Geometric methods in group theory*, volume 372 of *Contemp. Math.*, pages 11–25. Amer. Math. Soc., Providence, RI, 2005. MR2139673 (2006g:20037)
[16] A. Martino and S. O Rourke. Some free actions on non-Archimedean trees. *Journal of Group Theory* 7:275–286, 2004. MR2049022 (2005b:20047)
[17] V. N. Remeslennikov. $\exists$-free groups. *Sibirsk. Mat. Zh.*, 30(6):193–197, 1989. MR1043446 (91f:03077)
[18] Eugene Schenkman. The equation $a^n b^n = c^n$ in a free group. *Ann. of Math. (2)*, 70:562–564, 1959. MR0104723 (21:3476)
[19] Marcel-Paul Schützenberger. Sur l'équation $a^{2+n} = b^{2+m} c^{2+p}$ dans un groupe libre. *C. R. Acad. Sci. Paris*, 248:2435–2436, 1959. MR0103219 (21:2000)
[20] Zlil Sela. Diophantine geometry over groups. I. Makanin-Razborov diagrams. *Publ. Math. Inst. Hautes Études Sci.*, (93):31–105, 2001. MR1863735 (2002h:20061)

Department of Mathematics, University of Oklahoma, Norman, Oklahoma 73019
E-mail address: nbrady@math.ou.edu

Department of Mathematics, University of Fribourg, CH-1700 Fribourg, Switzerland
E-mail address: laura.ciobanu@unifr.ch

Department of Mathematics, Universitat Politècnica de Catalunya, 08860 Castelldefels, Spain
E-mail address: Armando.Martino@upc.edu

Department of Mathematics, Cork Institute of Technology, Cork, Ireland
E-mail address: Shane.ORourke@cit.ie