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SPIKE SOLUTIONS IN COUPLED NONLINEAR SCHRÖDINGER EQUATIONS WITH ATTRACTIVE INTERACTION

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Abstract. We consider the following elliptic system:

$$\begin{cases} \varepsilon^2\Delta u - \lambda_1 u + \mu_1 u^3 + \beta u v^2 = 0 \quad \text{in } \Omega, \\ \varepsilon^2\Delta v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0 \quad \text{in } \Omega, \\ u, v > 0 \ \text{ in } \Omega, \quad u = v = 0 \ \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \leq 3)$ is a smooth and bounded domain, $\varepsilon > 0$ is a small parameter, $\lambda_1, \lambda_2, \mu_1, \mu_2 > 0$ are positive constants and $\beta \neq 0$ is a coupling constant. We show that there exists an interval $I = [a_0, b_0]$ and a sequence of numbers $0 < \beta_1 < \beta_2 < ... < \beta_n < ...$ such that for any $\beta \in (0, +\infty) \setminus (I \cup \{\beta_1, ..., \beta_n, ...\})$, the above problem has a solution such that both u and v develop a spike layer at the innermost part of the domain. Central to our analysis is the nondegeneracy of radial solutions in $\mathbb{R}^N$.

1. Introduction

In this paper, we consider the coupled Gross-Pitaevskii equations, i.e., the coupled nonlinear Schrödinger equations,

$$\begin{cases} -\sqrt{-1}\frac{\partial}{\partial t}\Phi_1 = \varepsilon^2\Delta\Phi_1 + \mu_1|\Phi_1|^2\Phi_1 + \beta|\Phi_2|^2\Phi_1 \text{ for } \ y \in \Omega, t > 0, \\ -\sqrt{-1}\frac{\partial}{\partial t}\Phi_2 = \varepsilon^2\Delta\Phi_2 + \mu_2|\Phi_2|^2\Phi_2 + \beta|\Phi_1|^2\Phi_2 \text{ for } \ y \in \Omega, t > 0, \\ \Phi_j = \Phi_j(y,t) \in \mathbb{C}, \quad j = 1, 2, \\ \Phi_j(y,t) = 0 \ \text{ for } \ y \in \partial\Omega, t > 0, j = 1, 2, \end{cases} \tag{1.1}$$

where ϵ, μ_1, μ_2 are positive constants, Ω is a domain in $\mathbb{R}^N$, $N \leq 3$, and β is a coupling constant.

System (1.1) arises in many physical problems. When Ω is a bounded domain, problem (1.1) arises in the Hartree-Fock theory for a double condensate, i.e., a binary mixture of Bose-Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$ ([19]). Physically, $|\Phi_1|$ and $|\Phi_2|$ are the corresponding condensate amplitudes, and μ_j and β are the intraspecies and interspecies scattering lengths respectively. The sign of the scattering length β determines whether the interactions of states $|1\rangle$ and $|2\rangle$ are *repulsive* or *attractive*. When $\beta < 0$, the interactions of states $|1\rangle$ and $|2\rangle$ are *repulsive* ([34]). In contrast, when $\beta > 0$, the interactions of states $|1\rangle$ and $|2\rangle$ are *attractive*. For atoms of the single state $|j\rangle$, when $\mu_j > 0$, the interactions of the single state $|j\rangle$ are *attractive*.

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When $\Omega = \mathbb{R}^N$, system (1.1) also arises in the study of incoherent solitons in nonlinear optics. We refer to [29, 30] for experimental results and [2, 7, 21, 22, 23] for a comprehensive list of references. Physically, the solution Φ_j denotes the j-th component of the beam in Kerr-like photorefractive media. The positive constant μ_j is for self-focusing in the j-th component of the beam. The coupling constant β is the *interaction* between the first and the second component of the beam. As $\beta > 0$, the interaction is *attractive*, while the interaction is *repulsive* if $\beta < 0$.

In order to obtain solitary wave solutions of the system (1.1), we set $\Phi_1(x,t) = e^{\sqrt{-1}\lambda_j t}\, u(x)$, $\Phi_2(x,t) = e^{\sqrt{-1}\lambda_j t}\, v(x)$, and the system (1.1) is transformed to an elliptic system given by

$$\begin{cases} \varepsilon^2 \Delta u - \lambda_1 u + \mu_1 u^3 + \beta u v^2 = 0 \quad \text{in } \Omega, \\ \varepsilon^2 \Delta v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0 \quad \text{in } \Omega, \\ u, v > 0 \ \text{ in } \ \Omega, \\ u = v = 0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N (N \le 3)$ is a smooth and bounded domain, $\varepsilon > 0$ is a small parameter, $\lambda_1, \lambda_2, \mu_1, \mu_2 > 0$ are positive constants and $\beta \neq 0$ is a coupling constant.

When $\Omega = \mathbb{R}^N$, the existence of the least energy solution to (1.2) is studied in [1, 25, 33]. In [1, 3, 28, 33], the existence of bound states (i.e., solutions to (1.2)) when $\beta > 0$ is proved.

In [26], the following result was proved:

Theorem 1.1. *There exists a constant $\beta_0 = \beta_0(N, \lambda_1, \lambda_2, \mu_1, \mu u_2) \in (0, \sqrt{\mu_1 \mu_2})$ such that the following holds:*

(1) *For any $\beta \in (-\infty, \beta_0)$ and ε sufficiently small,* (1.2) *has a least energy solution $(u_\varepsilon, v_\varepsilon)$. Let P_ε be a local maximum point of u_ε and Q_ε be a local maximum point of v_ε.*

(2) *If $0 < \beta < \beta_0$, then $|P_\varepsilon - Q_\varepsilon|/\varepsilon \to 0$ and*

$$d(P_\varepsilon, \partial\Omega) \to \max_{P \in \Omega} d(P, \partial\Omega), \quad d(Q_\varepsilon, \partial\Omega) \to \max_{P \in \Omega} d(P, \partial\Omega). \tag{1.3}$$

Furthermore, $u_\varepsilon(x), v_\varepsilon(x) \to 0$ in $C^1_{loc}(\bar{\Omega} \backslash \{P_\varepsilon, Q_\varepsilon\})$ and let

$$U_\varepsilon(y) := \ u_\varepsilon(P_\varepsilon + \varepsilon y), \quad V_\varepsilon(y) := \ v_\varepsilon(P_\varepsilon + \varepsilon y).$$

Then as $\varepsilon \to 0$, $(U_\varepsilon, V_\varepsilon) \to (U_0, V_0)$, which is a least-energy solution of the following problem in $\mathbb{R}^N$:

$$\begin{cases} \Delta U_0 - \lambda_1 U_0 + \mu_1 U_0^3 + \beta U_0 V_0^2 = \ 0 \quad \text{in } \ \mathbb{R}^N, \\ \Delta V_0 - \lambda_2 V_0 + \mu_2 V_0^3 + \beta U_0^2 V_0 = \ 0 \quad \text{in } \ \mathbb{R}^N, \\ U_0(0) = \ \max_{y \in \mathbb{R}^N} U_0(y), V_0(0) = \ \max_{y \in \mathbb{R}^N} V_0(y), \\ U_0, V_0 > 0 \text{ in } \mathbb{R}^N, \ U_0, V_0 \to 0 \quad \text{as} \quad |y| \to +\infty. \end{cases} \tag{1.4}$$

(3) *If $\beta < 0$, then we have*

$$\varphi(P_\varepsilon, Q_\varepsilon) \to \max_{(P,Q) \in \Omega^2} \varphi(P, Q), \tag{1.5}$$

where

$$\varphi(P, Q) = \ \min\{\sqrt{\lambda_1}|P - Q|, \sqrt{\lambda_2}|P - Q|, \sqrt{\lambda_1} d(P, \partial\Omega), \sqrt{\lambda_2} d(Q, \partial\Omega)\}. \tag{1.6}$$

Furthermore, $u_\varepsilon(x), v_\varepsilon(x) \to 0$ in $C^1_{loc}(\bar{\Omega} \backslash \{P_\varepsilon, Q_\varepsilon\})$, and if we let

$$U_\varepsilon(y) := \ u_\varepsilon(P_\varepsilon + \varepsilon y), \quad V_\varepsilon(y) := \ v_\varepsilon(Q_\varepsilon + \varepsilon y),$$

then

$$U_\varepsilon(y) \to w_1(y), \quad V_\varepsilon(y) \to w_2(y),$$

where $w_i(y)$, $i = 1, 2$, *is the unique solution of*

(1.7)
$$\begin{cases} \Delta w_i - \lambda_i w_i + \mu_i w_i^3 = 0 \ \ in \ \ \mathbb{R}^N, \\ w_i(0) = \max_{y\in\mathbb{R}^N} w_i(y), i = 1, 2, \quad w_i > 0 \ in \ \mathbb{R}^N, \ w_i(y) \to 0 \ \ as \ \ |y| \to +\infty. \end{cases}$$

The case when $\beta > \beta_0$ was left open. In this paper, we solve this problem by proving the following result:

Theorem 1.2. *There exists an interval* $I = [a_0, b_0] \subset (0, +\infty)$ *and a sequence of numbers* $\beta_1 < \beta_2 < ... < \beta_n < ...$ *such that for any* $\beta \in (0, +\infty)\backslash(I \cup \{\beta_1, ..., \beta_n, ...\})$ *and* ε *sufficiently small,* (1.2) *has a solution* $(u_\varepsilon, v_\varepsilon)$. *Let* P_ε *be a local maximum point of* u_ε *and* Q_ε *be a local maximum point of* v_ε. *Then* $|P_\varepsilon - Q_\varepsilon|/\varepsilon \to 0$ *and*

$$d(P_\varepsilon, \partial\Omega) \to \max_{P\in\Omega} d(P, \partial\Omega), \quad d(Q_\varepsilon, \partial\Omega) \to \max_{P\in\Omega} d(P, \partial\Omega). \tag{1.8}$$

Furthermore, $u_\varepsilon(x), v_\varepsilon(x) \to 0$ *in* $C^1_{loc}(\bar{\Omega}\backslash\{P_\varepsilon, Q_\varepsilon\})$ *and let*

$$U_\varepsilon(y) := u_\varepsilon(P_\varepsilon + \varepsilon y), \quad V_\varepsilon(y) := v_\varepsilon(P_\varepsilon + \varepsilon y);$$

then as $\varepsilon \to 0$, $(U_\varepsilon, V_\varepsilon) \to (U_0, V_0)$, *which is a solution of the problem* (1.4).

Remark. 1. We can also construct solutions at local maximum points of the distance function.

2. The interval I is almost necessary for existence. In fact, let us suppose $\lambda_1 \le \lambda_2, \mu_2 < \mu_1$ and $\mu_2 < \beta < \mu_1$. Multiplying the equation (1.2) for u by v and the equation (1.2) for v by u, and then integrating by parts and subtracting together, we obtain

$$(\lambda_2 - \lambda_1)\int_\Omega uv + \int_\Omega [(\mu_1 - \beta)u^3 v + (\beta - \mu_2)uv^3] = 0 \tag{1.9}$$

which implies that $u, v \equiv 0$. This implies that there are no solutions to (1.2) if $\lambda_1 \le \lambda_2, \mu_2 < \beta < \mu_1$.

One attempt at proving Theorem 1.2 is to use the mountain-pass lemma and analyze the mountain-pass solution. The problem is that such a solution may become trivial (i.e., $u = 0, v = 0$) or semitrivial (i.e, $u = 0$ or $v = 0$). Furthermore, there is no simple characterization of mountain-pass solutions when β becomes large and also the solutions to (1.4) may not be unique.

Our proof of Theorem 1.2 is by the so-called "*Localized Energy Method*". That is, we first use a Liapunov-Schmidt method to reduce the problem to a finite-dimensional problem, and then use variational methods to find critical points of the reduced finite-dimensional problem. Such a method has been used successfully in many papers for the scalar equations, see e.g. [4], [5], [15], [13], [14], [16], [17] and [24]. (In particular, we follow [24].) However, as far as the authors know, this method has never been used for *strongly coupled* elliptic equations. One of the main difficulties in using this method is the *nondegeneracy* assumption which is difficult to prove for systems. For single scalar equations, the nondegeneracy can be proved by using the uniqueness of radial solutions (see Appendix C of [31]). However for systems, the uniqueness of the radial solutions seems out of reach at this moment. Here, we use an idea of the first author in [11] by showing that nondegeneracy holds for (1.4) except for isolated points of β. More precisely, let (U_1, U_2) be a solution

of (1.4). We say that (U_1, U_2) is *nondegenerate* if the solution set of the linearized equation

$$\begin{cases} \Delta\phi_1 - \lambda_1\phi_1 + 3\mu_1 U_1^2\phi_1 + \beta U_2^2\phi_1 + 2\beta U_1U_2\phi_2 = 0, \\ \Delta\phi_2 - \lambda_2\phi_2 + 3\mu_2 U_2^2\phi_2 + \beta U_1^2\phi_2 + 2\beta U_1U_2\phi_1 = 0, \\ |\phi_1| + |\phi_2| \le 1 \end{cases} \tag{1.10}$$

is exactly N-dimensional, namely,

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \sum_{j=1}^{N} a_j \begin{pmatrix} \frac{\partial U_1}{\partial z_j} \\ \frac{\partial U_2}{\partial z_j} \end{pmatrix} \tag{1.11}$$

for some constants a_j.

The following result is proved in this paper, and contains elements of independent interest.

Theorem 1.3. *There exists an interval $I = [a_0, b_0] \subset (0, +\infty)$ and a sequence of numbers $\beta_1 < \beta_2 < ... < \beta_n < ...$ such that for any $\beta \in (0, +\infty)\backslash(I \cup \{\beta_1, ..., \beta_n, ...\})$, problem* (1.4) *has a nondegenerate solution* (U_1, U_2).

Remark. 1. The set I is given explicitly in Section 2. See (2.7). As remarked before, there is not always existence if $\beta \in I$.

2. We believe that the set $\{\beta_1, \beta_2, ..., \beta_n, ...\}$ is finite. In particular, for β large, all solutions to (1.4) are nondegenerate.

3. Our proof of Theorem 1.3 also gives a new proof of existence of solutions to (1.4). Here we use bifurcation analysis, which is different from those of [1], [3] and [33] where variational or topological method is used.

The organization of this paper is as follows. In Section 2 and Section 3, we prove Theorem 1.3. Section 2 contains nondegeneracy in the space of radial functions while Section 3 contains nondegeneracy in the space of nonradial functions. From Section 4 to Section 6, we apply the localized energy method to prove Theorem 1.2. Section 4 studies a linear problem, Section 5 studies a nonlinear problem and Section 6 completes the proof of Theorem 1.2.

2. Nondegeneracy in the space of radial functions

Let (u, v) be a solution of (1.4). By the moving plane method, as in [35], u and v are both radially symmetric and strictly decreasing, i.e., $u = u(r), v = v(r), u^{'}(r) < 0, v^{'}(r) < 0$ for $r \neq 0$. We say that $(u(r), v(r))$ is locally unique if the linearized problem

$$\begin{cases} \Delta\phi_1 - \lambda_1\phi_1 + 3\mu_1 u^2\phi_1 + \beta v^2\phi_1 + 2\beta uv\phi_2 = 0, \\ \Delta\phi_2 - \lambda_2\phi_2 + 3\mu_2 v^2\phi_2 + \beta u^2\phi_2 + 2\beta uv\phi_1 = 0, \\ \phi_1 = \phi_1(r), \phi_2 = \phi_2(r) \end{cases} \tag{2.1}$$

admits only *trivial* decaying solutions. In this section, we prove the following theorem.

Theorem 2.1. *There exists $I = [a_0, b_0]$ and $\beta_1 < \beta_2 < ...$ such that problem* (1.4) *has a locally unique nondegenerate solution for $\beta \notin I_0 \cup \{\beta_1, ..., \beta_n, ...\}$.*

Before we prove Theorem 2.1, we need some definitions and lemmas. We consider

$$\begin{cases} \Delta u - \lambda_1 u + \mu_1 u^3 + \beta u v^2 = 0 \ \text{ in } \ \mathbb{R}^N, \\ \Delta v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0 \ \text{ in } \ \mathbb{R}^N, \\ u(0) = \max_{y\in\mathbb{R}^N} u(y), v(0) = \max_{y\in\mathbb{R}^N} v(y), \\ u = u(r), v = v(r) > 0 \text{ in } \mathbb{R}^N, u, v \to 0 \ \text{ as } \ |y| \to +\infty. \end{cases} \tag{2.2}$$

Note that equation (2.2) admits three **trivial** solutions

$$(0,0),\ (\bar{u},0),\ (0,\bar{v}) \tag{2.3}$$

where $\bar{u} = w_1, \bar{v} = w_2$, and w_j is the unique solution of (1.7). By a simple scaling

$$w_j(r) = \sqrt{\frac{\lambda_j}{\mu_j}} w(\sqrt{\lambda_j} r), \tag{2.4}$$

where w is the unique solution to (1.7) with $\lambda_j = \mu_j = 1$.

Let us define

$$a_0 = \inf_{\phi\in H^1} \frac{\int_{\mathbb{R}^N} (|\nabla\phi|^2 + \lambda_2\phi^2)}{\int_{\mathbb{R}^N} w_1^2\phi^2}, \quad b_0 = \inf_{\phi\in H^1} \frac{\int_{\mathbb{R}^N} (|\nabla\phi|^2 + \lambda_1\phi^2)}{\int_{\mathbb{R}^N} w_2^2\phi^2}. \tag{2.5}$$

Without loss of generality, we may assume that

$$a_0 \le b_0. \tag{2.6}$$

We then set

$$I := [a_0, b_0]. \tag{2.7}$$

When $\lambda_1 = \lambda_2$, problem (2.2) admits a bound state of the form

$$(u_0, v_0) = (\sqrt{\mu_1} c_1 w_1, \sqrt{\mu_2} c_2 w_2), \tag{2.8}$$

where

$$c_1 = \sqrt{\frac{\mu_2 - \beta}{\mu_1\mu_2 - \beta^2}},\ c_2 = \sqrt{\frac{\mu_1 - \beta}{\mu_1\mu_2 - \beta^2}}$$

as long as $(\mu_1 - \beta)(\mu_2 - \beta) > 0$, i.e., $\beta \notin [\min(\mu_1, \mu_2), \max(\mu_1, \mu_2)]$. Note also that by a simple scaling

$$a_0 = \mu_1 \inf_{\phi\in H^1} \frac{\int_{\mathbb{R}^N} (|\nabla\phi|^2 + \phi^2)}{\int_{\mathbb{R}^N} w^2\phi^2} = \mu_1.$$

(See the proof of Lemma 2.2 below.) Similarly $b_0 = \mu_2$. Thus in this case $I = [\min(\mu_1, \mu_2), \max(\mu_1, \mu_2)]$.

Our next lemma says that this solution is nondegenerate in the space of radial functions. (Note that in this case, by the Remark after Theorem 1.2, there is no existence to (1.4) for $\beta \in [\min(\mu_1, \mu_2), \max(\mu_1, \mu_2)]$.)

Lemma 2.2. *Let $\lambda_1 = \lambda_2$ and $\beta \notin [\min(\mu_1, \mu_2), \max(\mu_1, \mu_2)]$. Then the solution (u_0, v_0) defined by (2.8) is nondegenerate in the space of radial functions.*

Proof. We just need to study the eigenvalue problem (2.1) associated with (u_0, v_0). Let $\lambda_1 = \lambda_2 = \lambda_0$. Without loss of generality, we may assume that $\lambda_0 = 1$. Then $(u_0, v_0) = (c_1 w, c_2 w)$ and (2.1) becomes

$$\begin{cases} \Delta\phi_1 - \phi_1 + (3\mu_1 c_1^2 + \beta c_2^2) w^2 \phi_1 + 2\beta c_1 c_2 w^2 \phi_2 = 0, \\ \Delta\phi_2 - \phi_2 + (3\mu_2 c_2^2 + \beta c_1^2) \phi_2 + 2\beta c_1 c_2 w^2 \phi_1 = 0, \\ \phi_1 = \phi_1(r), \phi_2 = \phi_2(r). \end{cases} \tag{2.9}$$

By an orthonormal transformation, (2.9) can be transformed to two single equations

$$\Delta\Phi_1 - \Phi_1 + 3w^2\Phi_1 = 0, \Phi_1 = \Phi_1(r), \tag{2.10}$$

$$\Delta\Phi_2 - \Phi_2 + (3 - 2\beta(c_1^2 + c_2^2))w^2\Phi_2 = 0,\ \Phi_2 = \Phi_2(r). \tag{2.11}$$

Note that the eigenvalues of

$$\Delta\Phi - \Phi + \nu w^2\Phi = 0, \Phi \in H^1(\mathbb{R}^N) \tag{2.12}$$

are

$$\nu_1 = 1, \nu_2 = ... = \nu_{N+1} = 3, \nu_{N+2} > 3, \tag{2.13}$$

where the eigenfunction corresponding to ν_1 is cw, and the eigenfunctions corresponding to ν_2 are spanned by $\frac{\partial w}{\partial z_j}, j = 1, ..., N$. (See Lemma 4.1 of [36].)

Hence $\Phi_1 = 0$. $\Phi_2 = 0$ unless $3 - 2\beta(c_1^2 + c_2^2) = 1$. If $3 - 2\beta(c_1^2 + c_2^2) = 1$, then we have $\beta = \mu_1 = \mu_2$, since $\mu_1 c_1^2 + \beta c_2^2 = \mu_2 c_2^2 + \beta c_1^2 = 1$. This is impossible since $\beta \notin [\min(\mu_1, \mu_2), \max(\mu_1, \mu_2)]$.

In conclusion, we derive that $\Phi_1 = \Phi_2 = 0$ and hence $\phi_1 = \phi_2 = 0$. □

Next we show that a_0 or b_0 is a bifurcation point.

Lemma 2.3. *When $\beta = a_0, (u, v) = (\bar{u}, 0)$ or $\beta = b_0, (u, v) = (0, \bar{v})$, the linearized problem* (2.1) *has exactly a one-dimensional set of solutions.*

Proof. It is easy to see that a_0 can be attained by a radially symmetric and positive function denoted by $\bar{\psi}_0$ (normalized so that $\bar{\psi}_0(0) = 1$). Let $\beta = a_0, (u, v) = (\bar{u}, 0)$. Then (2.1) becomes

$$\begin{cases} \Delta\phi_1 - \lambda_1\phi_1 + 3\mu_1\bar{u}^2\phi_1 = 0, \\ \Delta\phi_2 - \lambda_2\phi_2 + a_0\bar{u}^2\phi_2 = 0, \\ \phi_1 = \phi_1(r), \phi_2 = \phi_2(r). \end{cases} \tag{2.14}$$

Note that the two equations in (2.14) are decoupled. By the same argument as those of Lemma 2.2, we have $\phi_1 = 0$. On the other hand, by the definition of a_0, we see that ϕ_2 is the principal eigenfunction, and hence $\phi_2 = c\bar{\psi}_0$ for some $c > 0$. □

Henceforth, we may assume that

$$\lambda_1 \neq \lambda_2. \tag{2.15}$$

We will work on the space $E = C_{r,0}(\mathbb{R}^N) \times C_{r,0}(\mathbb{R}^N)$, where $C_{r,0}(\mathbb{R}^N)$ denotes the space of continuous radial functions vanishing at ∞. We can write the system (2.2) on E as

$$u = (-\Delta + \lambda_1)^{-1}(\mu_1 u^3 + \beta uv^2), \quad v = (-\Delta + \lambda_2)^{-1}(\mu_2 v^3 + \beta u^2 v). \tag{2.16}$$

We also need the following lemma.

Lemma 2.4. *(1) For each fixed $\beta > 0$, there is a $C_\beta > 0$ such that*

$$\|u\|_{L^\infty(\mathbb{R}^N)} + \|v\|_{L^\infty(\mathbb{R}^N)} \leq C_\beta \tag{2.17}$$

for any nonnegative solution (u, v) of (2.16).

(2) Let β be bounded. Then the set of nonnegative solutions of (2.16) *is compact in E.*

Proof. (1). We proceed by contradiction, assuming that there is a sequence of solutions (u_n, v_n) to (1.4) with

$$\max_{x\in\mathbb{R}^N} u_n(x) + \max_{x\in\mathbb{R}^N} v_n(x) \to +\infty \qquad \text{as } n \to \infty. \tag{2.18}$$

We follow a blow up procedure introduced by Gidas and Spruck [20] for scalar equations. Since the method is standard, we only sketch the argument. Without loss of generality, we may assume that

$$M_n := u_n(0) = \max_{x\in\mathbb{R}^N} u_n(x) \geq v_n(0) = \max_{x\in\mathbb{R}^N} v_n(x). \tag{2.19}$$

Now we perform a rescaling, setting $x = \frac{y}{M_n}$ and defining functions $U_n, V_n : \mathbb{R}^N \to \mathbb{R}$ by

$$U_n(y) = \frac{u_n(\frac{y}{M_n})}{M_n}, \qquad V_n(y) = \frac{v_n(\frac{y}{M_n})}{M_n} \qquad \text{for } y \in \mathbb{R}^N. \tag{2.20}$$

Then

$$1 := \max_{y\in\mathbb{R}^N} U_n(y) \geq \max_{y\in\mathbb{R}^N} V_n(y), \tag{2.21}$$

and (U_n, V_n) solves the rescaled problem

$$\begin{cases} -\Delta U_n = \mu_1 U_n^3 + \beta U_n V_n^2 - \dfrac{\lambda_1}{M_n^2} U_n & \text{in } \mathbb{R}^N, \\ -\Delta V_n = \mu_1 V_n^3 + \beta V_n U_n^2 - \dfrac{\lambda_2}{M_n^2} V_n & \text{in } \mathbb{R}^N. \end{cases} \tag{2.22}$$

Passing to a subsequence if necessary, we see that $(U_n, V_n) \to (U_0, V_0)$, which is a nontrivial and nonnegative bounded radial solution of

$$-\Delta u = \mu_1 u^3 + \beta u v^2, \ -\Delta v = \mu_2 v^3 + \beta v u^2. \tag{2.23}$$

If u is nonnegative, then we have $-\Delta u \geq \mu_1 u^3$ on $\mathbb{R}^N$ which implies $u \equiv 0$ by standard results (if $N \leq 3$). See [20]. Similarly if v is nontrivial, we also obtain a contradiction. Hence we have the a priori bound (2.17).

(2) We now show that the set of nonnegative solutions to (2.16) cannot become noncompact in E for bounded β (while it remains nonnegative and bounded in E). To do this, we note that a bounded set T in $C_{r,0}(\mathbb{R}^N)$ is precompact (that is, it has compact closure) if and only if it is *equicontinuous* on bounded sets, and given $\epsilon > 0$, there is a $\mu > 0$ such that $|u(r)| \leq \epsilon$ if $r \geq \mu$ and $u \in T$. (This can most easily be seen if we identify $C_{r,0}(\mathbb{R}^N)$ with $\{w \in C[0,1]; w(1) = 0\}$ by mapping $[0,\infty)$ onto $[0,1]$ with ∞ corresponding to 1 and using the Arzela-Ascoli compactness theorem.) Now our branch is bounded in $C^2(\mathbb{R}^N)$ for bounded β. Thus we can only have trouble with the second condition, i.e., if there exists $\alpha > 0$ and $r_i \to +\infty$ and solutions (u_i, v_i) to (2.16) with $u_i(r_i) + v_i(r_i) = \alpha$. By changing the origin to r_i and passing to the limit, we will obtain a nontrivial solution (u, v) on $\mathbb{R}$ of the following problem:

$$-u'' = -\lambda_1 u + \mu_1 u^3 + \beta u v^2, \quad -v'' = -\lambda_2 v + \mu_2 v^3 + \beta v u^2 \tag{2.24}$$

with $u(0)+v(0) = \alpha, u, v \geq 0$ and bounded. Moreover u and v must be decreasing on $\mathbb{R}$ since nonnegative solutions of (2.2) are decreasing on $[0,\infty)$. Hence u and v have limits u_+, v_+ at ∞ (and limit (u_-, v_-) at $-\infty$). Thus (u_-, v_-) and (u_+, v_+) both solve $\lambda_1 u = \mu_1 u^3 + \beta u v^2, \lambda_2 v = \mu_2 v^3 + \beta u^2 v$. If we choose α small, $u_+ + v_+ \leq \alpha$, and hence $u_+ = v_+ = 0$. If u does not vanish identically, $u_- > 0$ and $\lambda_1 = u_-^2 + \beta v_-^2$.

Hence $u(-\lambda_1 + u^2 + \beta v^2) < 0$ on $\mathbb{R}$ (by the decreasing properties of u and v) and hence $u'' > 0$ on $\mathbb{R}$. Thus u is strictly convex and bounded on $\mathbb{R}$ which is impossible. Hence compactness of nonnegative solutions to (2.16) holds. □

We proceed to prove Theorem 2.1.

When $\beta = 0$, it is known that there exists a unique positive solution $(\bar{u}, \bar{v})$ which is the only nonnegative solution for $\beta = 0$ (except for $(\bar{u}, 0)$ and $(0, \bar{v})$). Note that $(\bar{u}, \bar{v})$ is nondegenerate for $\beta = 0$ in E since the system (2.16) is diagonal, and we can use the results for scalar equations. The operator obtained by linearizing the right hand side of (2.16) at $(\bar{u}, \bar{v})$ is compact on E, since (u, v) decays. (Note that the nonlinear mapping itself is not completely continuous on E.) Hence we can use the degree theory for C^2 Fredholm maps ([10]). (Note that the map is smooth in E and a number of authors have considered similar degrees.) For the moment, we need to consider changing-sign solutions of (2.16). By a slight variant of the theory in Section 1 of [10], there is a connected set $\tilde{C}$ of solutions of (2.16) in $E \times \mathbb{R}$ with $\beta \geq 0$ starting at $(\bar{u}, \bar{v})$ for $\beta = 0$, while it is noncompact in $E \times \mathbb{R}$ or contains another solution for $\beta = 0$. (This is a standard Rabinowitz type continuation result where we start from a nondegenerate solution rather than from the zero solution.) Moreover, $\tilde{C}$ can only leave the cone of nonnegative functions at $(\bar{u}, 0)$ for $\beta = a_0$ or at $(0, \bar{v})$ for $\beta = b_0$, where $a_0 \leq b_0$ are defined by (2.5). To see this, suppose that (u_n, v_n) are solutions of (2.16) such that (u_n, v_n, β_n) converges to (u_0, v_0, β_0) in $E \times \mathbb{R}$, where u_n changes sign and u_0 is positive on $(0, \infty)$. Then $u_n(r_n)$ must be small and r_n must be large whenever $u_n(r_n) \leq 0$, for large n. Thus any component Z_n of $\{r : u_n(r) < 0\}$ must be where r is large and $|u_n| + v_n|$ is small. (Recall that $|u_0| + |v_0| \to 0$ as $r \to +\infty$.) Hence $-\lambda_1 + \mu_1 u_n^2 + \beta v_n^2$ is negative on Z_n and thus, by the equation for u_n, $\Delta u_n < 0$ on Z_n. This contradicts the fact that u_n must have a minimum on Z_n. If we only assumed that u_0 was nonnegative on $[0, \infty)$, the maximum principle applied to the equation for u_0 implies that either $u_0 \equiv 0$ or $u_0(r) > 0$ on $[0, \infty)$. Hence we see that $\tilde{C}$ can only leave the cone of nonnegative functions if there exist solutions (u_n, v_n) for $\beta = \beta_n$ such that $(u_n, v_n) \to (0, v_0)$ in E and $\beta \to \beta_0$ as $n \to \infty$, where u_n, v_n and v_0 are positive solutions on $[0, \infty)$. (Note that $\|u_n\|_{L^\infty}$ and $\|v_n\|_{L^\infty}$ cannot both be small from the equation (2.16).) By the first equation, we easily see that $\frac{u_n}{\|u_n\|_{L^\infty}}$ must converge to a positive decreasing solution of $-\Delta h = (-\lambda_1 + \beta_0 v_0^2)h$ on $[0, \infty)$, at least through a subsequence. h must be nontrivial because u_n has its maximum at zero. Hence $\beta_0 = b_0$. Thus our claim is proved and the branch $\tilde{C}$ continues up to $\min(a_0, b_0) = a_0$ (where we can assume $a_0 \leq b_0$ without loss of generality).

We now note that our nonlinearity is easily seen to be real analytic on E (in fact it is polynomial). Since $\tilde{C}$ starts from $\beta = 0$ where there is a unique and nondegenerate positive solution, Theorem 1 of [11] implies that for all β in $[0, a_0)$ except at isolated points, there are an odd number of nondegenerate positive solutions in the component $\tilde{C}$ (nondegenerate in E). There is one minor point here. The results in [12] imply that C is locally connected and hence we can choose an open neighborhood $\mathcal{U}$ of $\tilde{C}$ in $E \times \mathbb{R}$ containing no other solutions of our equation except those in $\tilde{C}$. In fact, the exceptional points cannot accumulate at a_0 as we see below. Now by Lemma 2.3, it is easy to see that at $\beta = a_0$, there is bifurcation from a simple eigenvalue from the solution $(\bar{u}, 0)$ (which is a solution for all $\beta > 0$) in the sense of [8]-[9]. More precisely, the other solutions near $(\bar{u}, 0, a_0)$ form a real

analytic arc $u = u(\alpha), v = \alpha(\tilde{h} + \gamma(\alpha)), \beta = \phi(\alpha)$ for α small, where α is a scalar variable, $u(0) = \bar{u}, \gamma(0) = 0, \phi(0) = a_0$ and $v(\alpha)$ is in a complement to the span $\tilde{h}$ in $C_{r,0}(\mathbb{R}^N)$. Here $\tilde{h} > 0$ is the principal eigenfunction of $-\Delta\tilde{h} = -\lambda_2\tilde{h} + \beta\bar{u}^2\tilde{h}$. There is an analogous bifurcation at $(0, \bar{v})$ at $\beta = b_0$, where in this case $\beta = \psi(\tilde{\alpha})$ with ψ real analytic, $\psi(0) = b_0$ and $\tilde{\alpha}$ is a scalar parameter. We will prove below that either $\phi^{'}(\alpha) \neq 0$ for small nonzero α and $\psi^{'}(\tilde{\alpha}) \neq 0$ for small nonzero $\tilde{\alpha}$, or ϕ and ψ are constant functions $a_0 = b_0$ and there is an arc A of positive solutions of (2.16) for $\beta = a_0$ joining $(\bar{u}, 0)$ to $(0, \bar{v})$. We refer to this as **Claim A** and defer its proof to the end of the proof of Theorem 2.1. We refer to the second possibility as the exceptional case. If $\phi^{'}(\alpha) \neq 0$ for small nonzero α, Theorem 1.17 in [8] (or by [10]) implies $(u(\alpha), v(\alpha))$ is a nondegenerate solution for $\beta = \phi(\alpha)$ for small non-zero α, and an analogous result holds for $\beta = \psi(\tilde{\alpha})$. If $\phi(\alpha) < a_0$ for small positive α, we have a nondegenerate solution for $\beta \in (a_0, a_0 - \delta)$. If $\phi(\alpha) > a_0$, for small positive α, the argument in Theorem 1 of [11] applies all the way to a_0, and we have our claim.

If $(\bar{u}, 0)$ and $(0, \bar{v})$ do not belong to the closure of $\tilde{C}$, Theorem 1 in [11] implies that we have a nondegenerate positive solution for all $\beta > 0$ except at isolated points. If both $(\bar{u}, 0)$ and $(0, \bar{v})$ belong to the closure of $\tilde{C}$ and we are not in the exceptional case, Theorem 1 in [11] together with our remarks above implies that (except for isolated values of β) the number of nondegenerate positive solutions must change by 1 as β crosses a_0 and by 1 as β crosses b_0 (and by an even number as β crosses a_0 if $a_0 = b_0$). Thus for $\beta > b_0$, there are an odd number of nondegenerate positive solutions (except for isolated β). If $(\bar{u}, 0)$ is not in the closure of $\tilde{C}$ but $(0, \bar{v})$ is in the closure of $\tilde{C}$, the branch $\tilde{C}_1$ of positive solutions coming out of $(\bar{u}, 0)$ at $\beta = a_0$ must have an even number of nondegenerate positive solutions for $\beta < a_0$ (as usual except for isolated β) since $\tilde{C}_1$ does not continue to $\beta = 0$. (Note that the only positive solution for $\beta = 0$ lies in $\tilde{C}$.) Thus $\tilde{C}_1$ must have an odd number of nondegenerate positive solutions for all $\beta > a_0$ (as usual except for isolated β), and we are finished. A similar argument is valid if $(0, \bar{v})$ is in the closure of $\tilde{C}$ but $(\bar{u}, 0)$ is not. Finally in the exceptional case, the only solutions leaving the set of positive solutions lie in the arc A which consists of degenerate solutions (since each is nonisolated). These do not affect the argument in the proof of Theorem 1 in [11], and so, once again there is a positive nondegenerate solution for all $\beta > 0$ except for isolated β.

Lastly, we prove **Claim A**. First, note that, by real analyticity, either ϕ is constant or $\phi^{'}(\alpha) \neq 0$ for small nonzero α. Thus if the first possibility of Claim A fails, then ϕ is constant. By the proof of Theorem 1 in [11], $\tilde{C} = \bigcup_{i=1}^{\infty} D_i$, where D_i for $i \geq 2$ is an $i-$dimensional manifold, D_1 is closed and the D_i's are disjoint. Since the solution coming out of $(\bar{u}, 0)$ at $\beta = a_0$ is locally an arc, this must lie in D_1. By the theory in [11] and [12], at any point (z, μ) of D_1, D_1 is locally a finite union of closed arcs $W_i, i = 1, ..., k$, which intersect only at (z, μ), (z, μ) is an interior point of each arc W_i and each W_i can either be parametrized by β (for β in $(\mu - \delta, \mu + \delta)$ for $\delta > 0$) or W_i lies in $\beta = \mu$. (Different W_i can satisfy a different one of these alternatives.) By this and compactness, we see that $\mathcal{T}$–the set of nonnegative solutions in $D_1 \cap \{\beta = a_0\}$ –consists of a finite number of points and a finite number of disjoint arcs joining these points, i.e., a finite graph with an even number of edges at each vertex except at $(\bar{u}, 0)$ (and $(0, \bar{v})$ if $b_0 = a_0$). By elementary graph theory, this is impossible if $a_0 \neq b_0$ and, if $a_0 = b_0$, there is an

arc in $\mathcal{T}$ joining $(\bar{u}, 0)$ to $(0, \bar{v})$. Since this implies ψ is constant, we have proved Claim A and hence proved Theorem 2.1. $\square$

3. Nondegeneracy in the general case

Let (u, v) be the solution of (1.4). In this section, we show that for the linearization of equation (1.4) around the solution (u, v), the nonradial part of the kernel has exactly dimension N (thus comprising exactly the translational modes). This is summarized as follows:

Theorem 3.1. *Suppose that $(\phi, \psi) \in H^2(\mathbb{R}^N) \times H^2(\mathbb{R}^N)$ satisfies the following eigenvalue problem:*

$$\begin{cases} \Delta\phi - \lambda_1\phi + 3\mu_1 u^2\phi + \beta v^2\phi + 2\beta uv\psi = 0, \\ \Delta\psi - \lambda_2\psi + 3\mu_2 v^2\psi + \beta u^2\psi + 2\beta uv\phi = 0, \end{cases} \tag{3.1}$$

where $\beta \neq 0$. Then

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} \in span\left\{ \begin{pmatrix} \frac{\partial u}{\partial z_j} \\ \frac{\partial v}{\partial z_j} \end{pmatrix}, j = 1, ..., N \right\} \cup \left\{ \begin{pmatrix} \phi \\ \psi \end{pmatrix} \middle| \begin{array}{l} \phi = \phi(r), \psi = \psi(r), \\ (\phi, \psi) \text{ satisfies (2.1)} \end{array} \right\}. \tag{3.2}$$

Proof. We first recall that the eigenvalues of Δ_{S^N} are given by

$$\mu_1 = 0,\ \mu_2 = ... = \mu_{N+1} = N - 1,\ \mu_{N+1} < \mu_{N+2}, \dots . \tag{3.3}$$

Let $e_i(\theta)$ be the corresponding eigenfunctions, i.e.,

$$\Delta_{S^N} e_i + \mu_i e_i = 0, \quad i = 1, 2, \dots .$$

For any solution (ϕ, ψ) of (3.1) set

$$\phi_i(r) = \int_{S^N} \phi(r, \theta) e_i(\theta)\, d\theta, \quad \psi_i(r) = \int_{S^N} \psi(r, \theta) e_i(\theta)\, d\theta.$$

Recall that

$$\Delta\phi = \Delta_r\phi + \frac{\Delta_{S^N}\phi}{r^2}.$$

We obtain the following system:

$$\begin{cases} \Delta\phi_i - \lambda_1\phi_i - \frac{\mu_i}{r^2}\phi_i + 3\mu_1 u^2\phi_i + \beta v^2\phi_i + 2\beta uv\psi_i = 0, \\ \Delta\psi_i - \lambda_2\psi_i - \frac{\mu_i}{r^2}\psi_i + 3\mu_2 v^2\psi_i + \beta u^2\psi_i + 2\beta uv\phi_i = 0. \end{cases} \tag{3.4}$$

The proof will be finished by showing the following claims.

Claim 1. If $i \geq N + 2$, then $\phi_i \equiv \psi_i \equiv 0$.

Suppose this is not the case. We first multiply (3.4) by u' and v', respectively, where $' = \frac{\partial}{\partial r}$, and integrate over the ball B_r centered at the origin with radius r. Note that

$$\begin{cases} \Delta u' - \lambda_1 u' + 3\mu_1 u^2 u' + \beta v^2 u' + 2\beta uvu' = \frac{N-1}{r^2} u', \\ \Delta v' - \lambda_2 v' + 3\mu_2 v^2 v' + \beta u^2 v' + 2\beta uvv' = \frac{N-1}{r^2} v'. \end{cases} \tag{3.5}$$

Integration by parts gives

$$
\begin{aligned}
0 &= \int_{B_r} (\Delta\phi_i - \lambda_1\phi_i - \frac{\mu_i}{r^2}\phi_i + 3\mu_1 u^2\phi_i + \beta v^2\phi_i)u' + \int_{B_r} 2\beta uv\psi_i u' \\
0 &= \int_{\partial B_r} (u'\phi_i' - \phi_i u'') + \int_{B_r} \frac{N-1-\mu_i}{r^2} u'\phi_i \\
&\quad - \int_{B_r} 2\beta uv v'\phi_i + \int_{B_r} 2\beta uv u'\psi_i.
\end{aligned} \tag{3.6}
$$

Similarly, we get

$$
\begin{aligned}
0 &= \int_{\partial B_r} (v'\psi_i' - \psi_i v'') + \int_{B_r} \frac{N-1-\mu_i}{r^2} v'\psi_i \\
&\quad - \int_{B_r} 2\beta uv u'\psi_i + \int_{B_r} 2uv v'\phi_i.
\end{aligned} \tag{3.7}
$$

Adding (3.6) and (3.7) we get

$$
\begin{aligned}
0 &= \int_{\partial B_r} (u'\phi_i' - \phi_i u'') + \int_{\partial B_r} (v'\psi_i' - \psi_i v'') \\
&+ \int_{B_r} \frac{N-1-\mu_i}{r^2}(u'\phi_i + v'\psi_i) = I_1(r) + I_2(r) + I_3(r),
\end{aligned} \tag{3.8}
$$

where $I_j(r),\ j = 1, 2, 3$, are defined by the last equality. We now choose appropriate r and estimate each of the terms $I_j(r)$. By definition we have

$$\phi_i'(0) = \psi_i'(0) = 0.$$

Without loss of generality we assume that there is some $r_1 > 0$ such that $\phi_i(r) < 0$ for $0 < r < r_1$ and $\phi_i(r_1) = 0$. (We choose $r_1 = \infty$ if $\phi_i < 0$ in $(0, \infty)$.) Note that by standard ODE theory $\phi_i'(r_1) > 0$.

We claim that $\psi_i(r) < 0$ for r small. Suppose this is not the case. Let $\psi_i(r) > 0$ for $r \in (0, r_2)$ for some $r_2 > 0$. If $r_2 < r_1$, then we have

$$
\begin{aligned}
&\int_{\partial B_{r_2}} (v'\psi_i' - \psi_i v'') \geq 0, \\
&\int_{B_{r_2}} \frac{N-1-\mu_i}{r^2} v'\psi_i > 0, \\
&\int_{B_{r_2}} 2\beta uv u'\psi_i < 0, \\
&\int_{B_{r_2}} 2uv v'\phi_i > 0.
\end{aligned}
$$

Adding all the above together, we obtain a contradiction to (3.7).

If $r_2 \geq r_1$, we use (3.6) to get a contradiction.

Therefore $\psi_i(r) < 0$ for r small. This implies that there is some $r_2 > 0$ such that $\psi_i(r) < 0$ for $0 < r < r_2$ and $\phi_i(r_2) = 0$. (We choose $r_2 = \infty$ if $\psi_i < 0$ in $(0, \infty)$.) Note that necessarily $\psi_i'(r_2) > 0$.

From now on we distinguish three different cases.

Case 1.1. $r_1 = r_2$.

Set $r = r_1 = r_2$. We easily calculate

$$I_1(r) < 0, \quad I_2(r) < 0, \quad I_3(r) < 0.$$

By (3.8) this gives a contradiction.

Case 1.2. $r_2 < r_1$.

We easily calculate $I_3(r_2) < 0$ and $I_2(r_2) < 0$. It is more difficult to evaluate $I_1(r_2)$. We define

$$\Phi(r) = r^{N-1}\phi_i'(r)u'(r) - r^{n-1}u''(r)\phi_i(r).$$

Then for $r_2 < r < r_1$,

$$\Phi'(r) = (r^{N-1}\phi_i')'u' - (r^{n-1}u'')'\phi_i.$$

Now we use

$$\frac{1}{r^{N-1}}(r^{N-1}\phi_i')' - \lambda_1\phi_i + 3\mu_1 u^2\phi_i + \beta v^2\phi_i + 2\beta uv\psi_i = \frac{\mu_i}{r^2}\phi_i,$$
$$\frac{1}{r^{N-1}}(r^{N-1}u^{''})' - \lambda_1 u^{'} + 3\mu_1 u^2 u^{'} + \beta v^2 u^{'} + 2\beta uv v^{'} = \frac{N-1}{r^2}u^{'},$$

and get

$$\Phi'(r) = r^{N-3}(\mu_i - (N-1))\phi_i u' - 2r^{n-1}\beta uv u^{'}\psi_i + 2\beta r^{N-1}uvv'\phi_i > 0.$$

Here we have used the fact that for $r_2 < r < r_1$, $\psi_i(r) > 0$. In fact, if there is $r_2 < r_3 < r_1$ such that $\psi(r) > 0, \psi(r_3) = 0$. Then similar to before, we use (3.7) to deduce a contradiction.

Putting these two facts together we conclude

$$\begin{aligned} 0 &> r_1^{N-1}(u'(r_1)\phi_i'(r_1) - \phi_i(r_1)u''(r_1)) \\ &= \Phi(r_1) > \Phi(r_2) = \frac{1}{|S^{N-1}|}I_1(r_2). \end{aligned}$$

By (3.8) this gives a contradiction.

Case 1.3. $r_1 < r_2$.

The proof in this case is similar to Case 1.2. We omit the details.

In conclusion, we have proved that for $i \geq N+2$, $\phi_i(r) = \psi_i(r) \equiv 0$. This proves Claim 1.

Claim 2. For $i = 2, ..., N+1$, $(\phi_j, \psi_j) = c_j(u^{'}, v^{'})$ for some constant c_j.

We have to show that the solution set of

$$\begin{cases} \Delta\phi_i - \lambda_1\phi_i + 3\mu_1 u^2\phi_i + \beta v^2\phi_i + 2\beta uv\psi_i = \frac{N-1}{r^2}\phi_i, \\ \Delta\psi_i - \lambda_2\psi_i + 3\mu_2 v^2\psi_i + \beta u^2\psi_i + 2\beta uv\phi_i = \frac{N-1}{r^2}\psi_i, \\ \phi_i(r), \psi_i(r) \to 0 \text{ as } r \to +\infty \end{cases} \tag{3.9}$$

is one-dimensional.

Suppose that (ϕ_i, ψ_i) solve (3.9). We must have

$$\phi_i(0) = \psi_i(0) = 0.$$

Similar to the proof of Claim 1, we see that either $\phi_i = \psi_i \equiv 0$ or $\phi_i < 0, \psi_i < 0$ for all r or $\phi_i > 0, \psi_i > 0$ for all r. Since $(u^{'}, v^{'})$ satisfies (3.9), by linearity of (3.9),

$(\phi_i, \psi_i) - c(u', v')$ also satisfies (3.9). Let $c_0 = \frac{\phi_i(1)}{u'(1)}$. Then $\phi_i(1) - cu'(1) = 0$. Thus $(\phi_i, \psi_i) = c_0(u', v')$. This proves Claim 2.

Theorem 2.1 follows from Claim 1 and Claim 2. □

Remark. There is an alternative proof of this result for $\beta > 0$ which may be useful for generalizations. One can easily show by the variational characterization of eigenvalues that the result will follow if we prove that the smallest eigenvalue of the system (3.9) is zero (by using the variational structure). Since we can easily show that the least eigenvalue of (3.9) has a nonnegative eigenfunction (by showing that replacing a test function (h, k) by $(|h|, |k|)$ decreases the energy) and since we can use orthogonality to prove that there cannot be two distinct eigenvalues of the eigenvalue problem for (3.9) having nonnegative eigenfunctions, zero must be the least eigenvalue of the eigenvalue problem of (3.9) as claimed. (Note that $(-u'(r), -v'(r))$ is a nonnegative eigenfunction of (3.9) corresponding to the eigenvalue zero.)

Completion of the Proof of Theorem 1.3. Theorem 1.3 follows directly from Theorem 2.1 and Theorem 3.1. □

4. Approximate solutions and energy computations

In this section we introduce some notation and present some preliminary analysis on approximate solutions.

From now on, we assume that $\beta \notin I \cup \{\beta_1, ..., \beta_n, ...\}$ as in Theorem 1.3.

Without loss of generality, we may assume that $0 \in \Omega$. By the following rescaling:

$$x = \epsilon z, \ \ z \in \Omega_\epsilon := \{\epsilon z \in \Omega\}, \tag{4.1}$$

equation (1.2) becomes

$$\begin{cases} \Delta u - \lambda_1 u + \mu_1 u^3 + \beta u v^2 = 0 \quad \text{in} \ \ \Omega_\epsilon, \\ \Delta v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0 \quad \text{in} \ \ \Omega_\epsilon, \\ u, v > 0 \ \ \text{in} \ \ \Omega_\epsilon, \ \ \ u = v = 0 \ \ \text{on} \ \ \partial\Omega_\epsilon. \end{cases} \tag{4.2}$$

For $u, v \in H^2(\Omega_\epsilon) \cap H_0^1(\Omega_\epsilon)$, we put

$$S_\epsilon \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} S_1(u, v) \\ S_2(u, v) \end{pmatrix}$$

where $S_1(u, v) = \Delta u - \lambda_1 u + \mu_1 u^3 + \beta u v^2$ and $S_2(u, v) = \Delta v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v$.

Then solving system (4.2) is equivalent to

$$S_\epsilon \begin{pmatrix} u \\ v \end{pmatrix} = 0, \quad u \in H^2(\Omega_\epsilon) \cap H_0^1(\Omega_\epsilon), \quad v \in H^2(\Omega_\epsilon) \cap H_0^1(\Omega_\epsilon). \tag{4.3}$$

Associated with problem (4.2) is the following energy functional:

$$J_\epsilon[u] = \frac{1}{2} \int_{\Omega_\epsilon} (|\nabla u|^2 + \lambda_1 u^2 + |\nabla v|^2 + \lambda_2 v^2) - \int_{\Omega_\epsilon} [\frac{1}{4} u^4 + \frac{\beta}{2} u^2 v^2 + \frac{1}{4} v^4], \ \ u, v \in H_0^1(\Omega_\epsilon). \tag{4.4}$$

We define an inner product

$$\langle (u_1, u_2), (v_1, v_2) \rangle_\epsilon = \int_{\Omega_\epsilon} (u_1 v_1 + u_2 v_2), \ \text{for} \ u_i, v_i \in L^2(\Omega_\epsilon), i = 1, 2, \tag{4.5}$$

and a configuration space

$$\Lambda := \left\{ P \in \Omega \middle| d(P, \partial\Omega) > \delta \right\} \tag{4.6}$$

where δ is small.

Let (U_1, U_2) be the solution of (1.4) which satisfies Theorem 1.3. For each $i = 1, 2$, by [35], U_i is radially symmetric; $U_i(y) = U_i(|y|)$ and is strictly decreasing; $U_i^{'}(r) < 0$ for $r > 0, r = |y|$. Moreover, we have the following asymptotic behavior of U_i:

$$U_i(r) = A_i r^{-\frac{N-1}{2}} e^{-\sqrt{\lambda_i} r}(1+O(\frac{1}{r})), \quad U_i^{'}(r) = -A_i \sqrt{\lambda_i} r^{-\frac{N-1}{2}} e^{-r}(1+O(\frac{1}{r})), \tag{4.7}$$

for r large, where $A_i > 0$ is a constant.

For $Q \in \Omega$, we define $U_{1,\epsilon,Q}$ to be the unique solution of

$$\Delta u - \lambda_1 u + [\mu_1 U_1^3 + \beta U_1 U_2^2](\cdot - \frac{Q}{\epsilon}) = 0 \text{ in } \Omega_\epsilon, \quad u = 0 \text{ on } \partial\Omega_\epsilon. \tag{4.8}$$

Similarly, we set $U_{2,\epsilon,Q}$ to be the unique solution of

$$\Delta v - \lambda_2 v + [\mu_2 U_2^3 + \beta U_2 U_1^2](\cdot - \frac{Q}{\epsilon}) = 0 \text{ in } \Omega_\epsilon, \quad v = 0 \text{ on } \partial\Omega_\epsilon. \tag{4.9}$$

Without loss of generality, we may assume that

$$\lambda_1 \leq \lambda_2. \tag{4.10}$$

We first analyze $U_{1,\epsilon,Q}$. To this end, set

$$\varphi_{1,\epsilon,Q}(x) = U_1(\frac{|x-Q|}{\epsilon}) - U_{1,\epsilon,Q}(\frac{x}{\epsilon}).$$

Then $\varphi_{1,\epsilon,Q}$ satisfies

$$\Delta v - \lambda_2 v = 0 \text{ in } \Omega_\epsilon, \quad v = U_1(\frac{|x-Q|}{\epsilon}) \text{ on } \partial\Omega_\epsilon. \tag{4.11}$$

Using (4.7) and modifying the proof in Section 4 of [32], we obtain the following lemma

Lemma 4.1. (1)

$$-\varepsilon \log \varphi_{i,\varepsilon,Q}(Q) \to 2\sqrt{\lambda_i} d(Q, \partial\Omega) \quad \textit{as} \quad \varepsilon \to 0. \tag{4.12}$$

(2) *Let* $V_{\varepsilon,i}(y) = \frac{\varphi_{i,\varepsilon,Q}(Q+\varepsilon y)}{\varphi_{i,\varepsilon,Q}(Q)}$. *Then as* $\varepsilon \to 0$ *(up to a subsequence),* $V_{\varepsilon,i}(y) \to V_i(y)$, *where* $V_i(y)$ *is a solution of*

$$\begin{cases} \Delta V_i - \lambda_i V_i = 0 \quad \textit{in} \quad \mathbb{R}^N, \\ V_i(0) = 1, V_i > 0 \quad \textit{in} \quad \mathbb{R}^N. \end{cases} \tag{4.13}$$

(3)

$$\sup_{y \in \Omega_{\varepsilon,P}} \left| e^{-\sqrt{\lambda_i}(1+\sigma)|y|} V_{\varepsilon,i}(y) \right| \leq C \quad \textit{for any} \quad 0 < \sigma < 1. \tag{4.14}$$

Using (4.1), we obtain the following error and energy estimates.

Lemma 4.2. *Assume that $Q \in \Lambda$. Then we have*

(1)

$$|S_1(U_{1,\epsilon,Q}, U_{2,\epsilon,Q})| + |S_2(U_{1,\epsilon,Q}, U_{2,\epsilon,Q})| \le C(\varphi_{1,\epsilon,Q} + \varphi_{2,\epsilon,Q}), \tag{4.15}$$

(2)

$$J_\epsilon[U_{1,\epsilon,Q}, U_{2,\epsilon,Q}] = I_0 + (a_1 + o(1))\varphi_{1,\epsilon,Q}(Q) + (a_2 + o(1))\varphi_{2,\epsilon,Q}, \tag{4.16}$$

where $I_0 = \frac{1}{4}[\mu_1 U_1^4 + 2\beta U_1^2 U_2^2 + \mu_2 U_2^4], a_1 = \frac{1}{2}\int_{\mathbb{R}^N}(\mu_1 U_1^3 + \beta U_1 U_2^2)e^{\sqrt{\lambda_1}y_1}, a_2 = \frac{1}{2}\int_{\mathbb{R}^N}(\mu_2 U_2^3 + \beta U_2 U_1^2)e^{\sqrt{\lambda_2}y_1}$.

Proof. (1) Since $\lambda_1 \le \lambda_2$, we estimate $S_2(U_{1,\epsilon,Q}, U_{2,\epsilon,Q})$ first:

$$\begin{aligned} S_2(U_{1,\epsilon,Q}, U_{2,\epsilon,Q}) &= \Delta U_{2,\epsilon,Q} - \lambda_2 U_{2,\epsilon,Q} + \mu_2 U_{2,\epsilon,Q}^3 + \beta U_{1,\epsilon,Q}^2 U_{2,\epsilon,Q} \\ &= \mu_2(U_{2,\epsilon,Q}^3 - U_2^3) + \beta(U_{1,\epsilon,Q}^2 U_{2,\epsilon,Q}^2 - U_1 U_2^2). \end{aligned} \tag{4.17}$$

The first term in (4.17) can be estimated easily:

$$|U_{2,\epsilon Q}^3 - U_2^3| \le C U_2^2 \varphi_{2,\epsilon,Q} \le C\varphi_{2,\epsilon,Q}(Q)$$

by Lemma 4.1 (3). Similarly, we can estimate other terms in (4.17) except the term $U_1^2\varphi_{2,\epsilon,Q}$. If $\lambda_1 = \lambda_2$, we have $\varphi_{2,\epsilon,Q} \le \varphi_{1,\epsilon,Q}$. If $\lambda < \lambda_2$, we also have $\varphi_{2,\epsilon,Q} \le \varphi_{1,\epsilon,Q}$ by the comparison principle. In any case, we obtain

$$U_1^2\varphi_{2,\epsilon,Q} \le C U_1^2 \varphi_{1,\epsilon,Q} \le C\varphi_{1,\epsilon,Q}(Q).$$

Similarly, we can estimate $S_1(U_{1,\epsilon,Q}, U_{2,\epsilon,Q})$.

(2) Using the definitions of $U_{i,\epsilon,Q}$, we obtain

$$\begin{aligned} J_\epsilon[U_{1,\epsilon,Q}, U_{2,\epsilon,Q}] &= \frac{1}{2}\int_{\Omega_\epsilon}(\mu_1 U_1^3 + \beta U_1 U_2^2)(U_1 - \varphi_{1,\epsilon,Q}) \\ &\quad + \frac{1}{2}\int_{\Omega_\epsilon}(\mu_2 U_2^3 + \beta U_2 U_1^2)(U_2 - \varphi_{2,\epsilon,Q}) \\ &\quad - \frac{1}{4}\int_{\Omega_\epsilon}(\mu_1 U_{1,\epsilon,Q}^4 + 2\beta U_{1,\epsilon,Q}^2 U_{2,\epsilon,Q}^2 + \mu_2 U_{2,\epsilon,Q}^4) \\ &= \frac{1}{4}\int_{\mathbb{R}^N}[\mu_1 U_1^4 + 2\beta U_1^2 U_2^2 + \mu_2 U_2^4] \\ &\quad - \frac{1}{2}\int_{\Omega_\epsilon}[\mu_1 U_1^3 + \beta U_1 U_2^2]\varphi_{1,\epsilon,Q} - \frac{1}{2}\int_{\Omega_\epsilon}[\mu_2 U_2^3 + \beta U_1^2 U_1]\varphi_{2,\epsilon,Q} \\ &\quad + O(e^{-(2\sqrt{\lambda}+\delta)\frac{d(Q,\partial\Omega)}{\epsilon}}). \end{aligned}$$

By Lemma 4.1 (2), we have

$$\frac{1}{2}\int_{\Omega_\epsilon}[\mu_1 U_1^3 + \beta U_1 U_2^2]\varphi_{1,\epsilon,Q} = \varphi_{1,\epsilon,Q}(Q)\frac{1}{2}\int_{\Omega_\epsilon}[\mu_1 U_1^3 + \beta U_1 U_2^2]V_{\epsilon,1},$$

where

$$\int_{\Omega_\epsilon}[\mu_1 U_1^3 + \beta U_1 U_2^2]V_{\epsilon,i} \to \int_{\mathbb{R}^N}[\mu_1 U_1^3 + \beta U_1 U_2^2]V_1 = \int_{\mathbb{R}^N}[\mu_1 U_1^3 + \beta U_1 U_2^2]e^{\sqrt{\lambda_1}y_1}$$

by Lemma 4.7 of [32]. Similarly, we have

$$\int_{\Omega_\epsilon}[\mu_2 U_2^3 + \beta U_1^2 U_1]\varphi_{2,\epsilon,Q} = \varphi_{2,\epsilon,Q}(\int_{\mathbb{R}^N}[\mu_2 U_2^3 + \beta U_2 U_1^2]e^{\sqrt{\lambda_2}y_1} + o(1)).$$

This proves (4.16). □

5. Localized energy method and proof of Theorem 1.2

In this section we develop the linear theory which allows us to perform the finite-dimensional reduction procedure. Here we need the nondegeneracy result Theorem 1.3.

Fix $Q \in \Lambda$. We define the following functions:

$$Z_{i,j} = \frac{\partial U_i}{\partial z_j}\chi(\frac{4\epsilon}{\delta}|z - \frac{Q}{\epsilon}|), \qquad i = 1, 2,\ j = 1, ..., N, \tag{5.1}$$

$$Z_j = \begin{pmatrix} Z_{1,j} \\ Z_{2,j} \end{pmatrix}, \quad j = 1, 2, ..., N, \tag{5.2}$$

where $\chi(t)$ is a smooth cut-off function such that $\chi(t) = 1$ for $|t| < 1$ and $\chi(t) = 0$ for $|t| > 2$.

We first consider the following linear problem: Let

$$L_\epsilon \begin{pmatrix} \phi \\ \psi \end{pmatrix} := \begin{pmatrix} \Delta\phi - \lambda_1\phi + (3\mu_1 U_{1,\epsilon,Q}^2 + U_{2,\epsilon,Q}^2)\phi + 2\beta U_{1,\epsilon,Q}U_{2,\epsilon,Q}\psi \\ \Delta\psi - \lambda_1\psi + (3\mu_2 U_{2,\epsilon,Q}^3 + U_{1,\epsilon,Q}^2)\psi + 2\beta U_{1,\epsilon,Q}U_{2,\epsilon,Q}\phi \end{pmatrix}.$$

Given $h_1, h_2 \in L^2(\Omega_\epsilon)$, find a function $\begin{pmatrix} \phi \\ \psi \end{pmatrix}$ satisfying

$$\begin{cases} L_\epsilon \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \sum_{j=1}^N c_j Z_j, \\ \langle \begin{pmatrix} \phi \\ \psi \end{pmatrix}, Z_j\rangle_\epsilon = 0, j = 1, ..., N, \qquad \text{and} \quad \phi = \psi = 0 \quad \text{on } \partial\Omega_\epsilon, \end{cases} \tag{5.3}$$

for some constants $c_l, l = 1, ..., N$. To this purpose, we define two norms

$$\|\phi\|_* = \|\phi\|_{W^{2,q}(\Omega_\epsilon)}, \quad \|f\|_{**} = \|f\|_{L^q(\Omega_\epsilon)}, \tag{5.4}$$

where $q > \frac{N}{2}$ is a fixed number.

We have the following result:

Proposition 5.1. *Let ϕ satisfy* (5.3)*. Then for ϵ sufficiently small and $\mathbf{Q} \in \Lambda$, we have*

$$\|\phi\|_* + \|\psi\|_* \le C(\|h_1\|_{**} + \|h_2\|_{**}), \tag{5.5}$$

where C is a positive constant independent of ϵ and $Q \in \Lambda$.

Proof. We follow the proof of Proposition 3.1 of [24, p. 264]. Arguing by contradiction, assume that there a sequence $(\phi_n, \psi_n, h_{1n}, h_{2n})$ satisfying (5.3) such that

$$\|\phi_n\|_* + \|\psi_n\|_* = 1; \qquad \|h_{1n}\|_{**} + \|h_{2n}\| = o(1). \tag{5.6}$$

To avoid clumsy notations, we omit the dependence on n.

Multiplying (5.3) by Z_j, we obtain

$$\sum_{i=1}^N c_i\langle Z_i, Z_j\rangle_\epsilon = -\langle \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, Z_j\rangle_\epsilon + \langle L_\epsilon \begin{pmatrix} \phi \\ \psi \end{pmatrix}, Z_j\rangle,\ j = 1, ..., N. \tag{5.7}$$

Since $U_{i,\epsilon,Q} = U_i + O(e^{-\frac{\delta}{\epsilon}})$ and $\langle Z_i, Z_j\rangle_\epsilon = \langle Z_1, Z_1\rangle_\epsilon\delta_{ij}$, by integration by parts, we obtain

$$c_j = o(1), j = 1, ..., N. \tag{5.8}$$

Thus as $n \to +\infty$, we obtain $(\phi_n, \psi_n) \to (\phi_0, \psi_0)$ in C^0_{loc}, where (ϕ_0, ψ_0) is a solution of

$$\begin{cases} \Delta\phi_0 - \lambda_1\phi_0 + (3\mu_1 U_1^2 + U_2^2)\phi_0 + 2\beta U_1 U_2 \psi_0 = 0, \\ \Delta\psi_0 - \lambda_2\psi_0 + (3\mu_2 U_2^3 + U_1^2)\psi_0 + 2\beta U_1 U_2 \phi_0 = 0, \\ |\phi_0| + |\psi_0| \le 1. \end{cases} \tag{5.9}$$

By Theorem 1.3, $\phi_0 \equiv 0, \psi_0 \equiv 0$. This implies by the Lebesgue Dominated Convergence Theorem that

$$\|(3\mu_1 U_{1,\epsilon,Q}^2 + U_{2,\epsilon,Q}^2)\phi_\epsilon + 2\beta U_{1,\epsilon,Q} U_{2,\epsilon,Q}\psi_\epsilon\|_{L^q(\Omega_\epsilon)} = o(1).$$

By elliptic regularity, $\|\phi_\epsilon\|_{W^{2,q}(\Omega_\epsilon)} = o(1)$. Similarly, we obtain $\|\psi_\epsilon\|_{L^q(\Omega_\epsilon)} = o(1)$. A contradiction. □

By the Fredholm Alternative (see the proof of Proposition 3.2 of [24, p. 267]), we also obtain

Proposition 5.2. *There exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ the following property holds true. Given $h_1, h_2 \in W^{2,q}(\Omega_\epsilon)$, there exists a unique pair $(\phi, \psi, \mathbf{c}) = (\phi, \{c_j\}_{j=1,\dots,N})$ such that*

$$L_\epsilon \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \sum_j c_j Z_j, \tag{5.10}$$

$$\langle \begin{pmatrix} \phi \\ \psi \end{pmatrix}, Z_j \rangle_\epsilon = 0, j = 1, \dots, N, \ \phi = \psi = 0 \quad \text{on } \partial\Omega_\epsilon. \tag{5.11}$$

Moreover, we have

$$\|\phi\|_* + \|\psi\|_* \le C\|h_1\|_{**} + \|h_2\|_{**} \tag{5.12}$$

for some positive constant C.

Finally, we solve a nonlinear problem: For ϵ small and for $Q \in \Lambda$, we are going to find a function $\begin{pmatrix} \phi_{1,\epsilon,Q} \\ \phi_{2,\epsilon,Q} \end{pmatrix}$ such that for some constants $c_j, j = 1, \dots, N$, the following equation holds true:

$$\begin{cases} S_\epsilon(U_{1,\epsilon,Q} + \phi_{1,\epsilon,Q}, U_{2,\epsilon,Q} + \phi_{2,\epsilon,Q}) = \sum_{l=1}^N c_l Z_l \text{ in } \Omega_\epsilon, \\ \langle \begin{pmatrix} \phi_{1,\epsilon,Q} \\ \phi_{2,\epsilon,Q} \end{pmatrix}, Z_j \rangle_\epsilon = 0, j = 1, \dots, N, \quad \phi = \psi = 0 \text{ on } \partial\Omega_\epsilon. \end{cases} \tag{5.13}$$

Using Lemma 4.2 and the Contraction Mapping Principle, similar to the proof of Proposition 4.2 of [24, p. 268], we obtain

Proposition 5.3. *For $Q \in \Lambda$ and ϵ sufficiently small, there exists a unique $\begin{pmatrix} \phi_{1,\epsilon,Q} \\ \phi_{2,\epsilon,Q} \end{pmatrix}$ such that (5.13) holds. Moreover, $Q \mapsto \begin{pmatrix} \phi_{1,\epsilon,Q} \\ \phi_{2,\epsilon,Q} \end{pmatrix}$ is of class C^1 as a map into $W^{2,q}(\Omega_\epsilon)$, and we have*

$$\sum_{i=1}^2 \|\phi_{i,\epsilon,\mathbf{Q}}\|_* \le C(\varphi_{1,\epsilon,Q}(Q) + \varphi_{2,\epsilon,Q}(Q)) \tag{5.14}$$

for some constant $C > 0$.

6. Proof of Theorem 1.2

In this section, we prove Theorem 1.2.

We first present a reduction lemma. Fix $\mathbf{Q} \in \Lambda$. Let $(\phi_{1,\epsilon,Q}, \phi_{2,\epsilon,Q})$ be the solution given by Proposition 5.3. We define a new functional

$$\mathcal{M}_\epsilon(Q) = J_\epsilon[U_{1,\epsilon,Q} + \phi_{1,\epsilon,Q}, U_{2,\epsilon,Q} + \phi_{2,\epsilon,Q}] : \Lambda \to R. \tag{6.1}$$

Then we have (similar to the proof of Theorem 1.1 of [24, p. 271])

Lemma 6.1. *If Q_ϵ is a critical point of $\mathcal{M}_\epsilon(Q)$ in Λ, then $(U_{1,\epsilon,Q_\epsilon}+\phi_{1,\epsilon,Q_\epsilon}, U_{2,\epsilon,Q_\epsilon}+\phi_{2,\epsilon,Q_\epsilon})$ is a critical point of J_ϵ and hence a solution to* (4.2).

Therefore, the proof of Theorem 1.2 is finished after we have the following proposition:

Proposition 6.2. *For ϵ small, the minimization problem*

$$\min\{\mathcal{M}_\epsilon(Q) : Q \in \Lambda\} \tag{6.2}$$

has a solution $Q^\epsilon \in \Lambda^\circ$ the interior of Λ. Furthermore,

$$d(Q_\epsilon, \partial\Omega) \to \max_{P\in\Omega} d(P, \partial\Omega).$$

Proof. Since $J_\epsilon[U_{1,\epsilon,Q} + \phi_{1,\epsilon,Q}, U_{2,\epsilon,Q} + \phi_{2,\epsilon,Q}]$ is continuous in Q, the minimization problem has a solution. Let $\mathcal{M}_\epsilon(Q_\epsilon)$ be the minimum where $Q_\epsilon \in \Lambda$.

We claim that Q_ϵ must stay in the interior of Λ.

We first obtain an asymptotic formula for $\mathcal{M}_\epsilon(Q)$. In fact for any $Q \in \Lambda$, we have

$$\begin{aligned}\mathcal{M}_\epsilon(Q) &= J_\epsilon[U_{1,\epsilon,Q}, U_{2,\epsilon,Q}] \\ &+ \int_{\Omega_\epsilon} (-S_1[U_{1,\epsilon,Q}, U_{2,\epsilon,Q}]\phi_{1,\epsilon,Q} - S_2[U_{1,\epsilon,Q}, U_{2,\epsilon,Q}]\phi_{2,\epsilon,Q}) + O(\sum_{i=1}^{2} \|\phi_{i,\epsilon,\mathbf{Q}}\|_*^2) \\ &= I_0 + \sum_{i=1}^{2}(a_i + o(1))\varphi_{i,\epsilon,Q}(Q) \end{aligned} \tag{6.3}$$

by Lemma 4.2 and Proposition 5.3.

First, by choosing Q_0 such that $d(Q_0, \partial\Omega) = \max_{P\in\Omega} d(P, \partial\Omega)$, we obtain a lower bound for $\mathcal{M}_\epsilon$:

$$\mathcal{M}_\epsilon(Q_\epsilon) \le I_0 + \sum_{i=1}^{2}(a_i + o(1))\varphi_{i,\epsilon,Q_0}(Q_0), \tag{6.4}$$

which, by (6.3) and Lemma 4.1, gives

$$\lim_{\epsilon\to 0} d(Q_\epsilon, \partial\Omega) \ge d(Q_0, \partial\Omega). \tag{6.5}$$

Proposition 6.2 is thus proved. □

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