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HOMOTOPY ON SPATIAL GRAPHS AND THE SATO-LEVINE INVARIANT

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ABSTRACT. Edge-homotopy and vertex-homotopy are equivalence relations on spatial graphs which are generalizations of Milnor's link-homotopy. We introduce some edge (resp. vertex)-homotopy invariants of spatial graphs by applying the Sato-Levine invariant for the 2-component constituent algebraically split links and show examples of non-splittable spatial graphs up to edge (resp. vertex)-homotopy, all of whose constituent links are link-homotopically trivial.

1. INTRODUCTION

Throughout this paper we work in the piecewise linear category. Let G be a finite graph which does not have isolated vertices and free vertices. An embedding f of G into the 3-sphere S^3 is called a *spatial embedding of* G or simply a *spatial graph*. For a spatial embedding f and a subgraph H of G which is homeomorphic to the 1-sphere S^1 or a disjoint union of 1-spheres, we call $f(H)$ a *constituent knot* or a *constituent link* of f, respectively. A graph G is said to be *planar* if there exists an embedding of G into the 2-sphere S^2, and a spatial embedding of a planar graph is said to be *trivial* if it is ambient isotopic to an embedding of the graph into a 2-sphere in S^3. A spatial embedding f of a graph G is said to be *split* if there exists a 2-sphere S in S^3 such that $S \cap f(G) = \emptyset$ and each component of $S^3 - S$ has intersection with $f(G)$, and otherwise f is said to be *non-splittable*.

Two spatial embeddings of a graph G are said to be *edge-homotopic* if they are transformed into each other by *self crossing changes* and ambient isotopies, where a self crossing change is a crossing change on the same spatial edge, and *vertex-homotopic* if they are transformed into each other by crossing changes on two adjacent spatial edges and ambient isotopies.[1] These equivalence relations were introduced by Taniyama [19] as generalizations of Milnor's *link-homotopy* on links [8]; namely if G is homeomorphic to a disjoint union of 1-spheres, then these are none other than link-homotopy. There are many studies about link-homotopy. In particular, the link-homotopy classification was given for 2- and 3-component links

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[1]In [19], edge-homotopy and vertex-homotopy were called *homotopy* and *weak homotopy*, respectively.

by Milnor [8], for 4-component links by Levine [7] and for all links by Habegger and Lin [2]. On the other hand, there are very few studies about edge (resp. vertex)-homotopy on spatial graphs [18], [9], [13], [11].

In [18], Taniyama defined an edge (resp. vertex)-homotopy invariant of spatial graphs called the *α-invariant* by applying the *Casson invariant* (or equivalently the second coefficient of the *Conway polynomial*) of the constituent knots and showed that there exists a non-trivial spatial embedding f of a planar graph up to edge (resp. vertex)-homotopy, even in the case where f does not contain any constituent link. But the α-invariant cannot detect a non-splittable spatial embedding of a disconnected graph up to edge (resp. vertex)-homotopy. As far as the authors know, an example of a non-splittable spatial embedding of a disconnected graph up to edge (resp. vertex)-homotopy, all of whose constituent links are link-homotopically trivial, has not yet been demonstrated.

Our purpose in this paper is to study spatial embeddings of disconnected graphs up to edge (resp. vertex)-homotopy by applying the *Sato-Levine invariant* [14] (or equivalently the third coefficient of the Conway polynomial) for the constituent 2-component algebraically split links and show that there exist infinitely many non-splittable spatial embeddings of a certain disconnected graph up to edge (resp. vertex)-homotopy, all of whose constituent links are link-homotopically trivial. These examples show that edge (resp. vertex)-homotopy on spatial graphs behaves quite differently than link-homotopy on links. In the next section we give the definitions of our invariants and state their invariance up to edge (resp. vertex)-homotopy.

2. Definitions of invariants

We call a subgraph of a graph G a *cycle* if it is homeomorphic to the 1-sphere, and a cycle is called a *k-cycle* if it contains exactly k edges. For a subgraph H of G, we denote the set of all cycles of H by $\Gamma(H)$. We set $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$ for a positive integer m and $\mathbb{Z}_0 = \mathbb{Z}$. We regard $\mathbb{Z}_m$ as an abelian group in the obvious way. We call a map $\omega : \Gamma(H) \to \mathbb{Z}_m$ a *weight on $\Gamma(H)$ over $\mathbb{Z}_m$*. For an edge e of H, we denote the set of all cycles of H which contain the edge e by $\Gamma_e(H)$. For a pair of two adjacent edges e_1 and e_2 of H, we denote the set of all cycles of H which contain the edges e_1 and e_2 by $\Gamma_{e_1,e_2}(H)$. Then we say that a weight ω on $\Gamma(H)$ over $\mathbb{Z}_m$ is *weakly balanced* [2] *on an edge e* if

$$\sum_{\gamma \in \Gamma_e(H)} \omega(\gamma) = 0$$

in $\mathbb{Z}_m$ [10], and *weakly balanced on a pair of adjacent edges e_1 and e_2* if

$$\sum_{\gamma \in \Gamma_{e_1,e_2}(H)} \omega(\gamma) = 0$$

in $\mathbb{Z}_m$. Let $G = G_1 \cup G_2$ be a disjoint union of two connected graphs and $\omega_i : \Gamma(G_i) \to \mathbb{Z}_m$ a weight on $\Gamma(G_i)$ over $\mathbb{Z}_m$ $(i = 1, 2)$. Let f be a spatial embedding of G such that

$$\omega_1(\gamma)\omega_2(\gamma')\mathrm{lk}(f(\gamma), f(\gamma')) = 0$$

[2] A weight ω on $\Gamma(H)$ over $\mathbb{Z}_m$ is said to be *balanced on an edge e of H* if $\sum_{\gamma \in \Gamma_e(H)} \omega(\gamma)[\gamma] = 0$ in $H_1(H; \mathbb{Z}_m)$, where the orientation of γ is induced by the one of e [18].

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(G_2)$, where $\mathrm{lk}(L) = \mathrm{lk}(K_1, K_2)$ denotes the *linking number* of a 2-component oriented link $L = K_1 \cup K_2$. Then we define $\beta_{\omega_1,\omega_2}(f) \in \mathbb{Z}_m$ by

$$\beta_{\omega_1,\omega_2}(f) \equiv \sum_{\substack{\gamma\in\Gamma(G_1)\\ \gamma'\in\Gamma(G_2)}} \omega_1(\gamma)\omega_2(\gamma')a_3(f(\gamma), f(\gamma')) \pmod m,$$

where $a_3(L) = a_3(K_1, K_2)$ denotes the third coefficient of the Conway polynomial of a 2-component oriented link $L = K_1 \cup K_2$. We remark here that $a_3(L)$ coincides with the *Sato-Levine invariant* $\beta(L)$ of L if L is *algebraically split*, namely $\mathrm{lk}(K_1, K_2) = 0$ [1], [17]. Thus our $\beta_{\omega_1,\omega_2}(f)$ is also the modulo m reduction of the summation of Sato-Levine invariants for the constituent 2-component algebraically split links of f.

Remark 2.1. For a 2-component algebraically split link $L = K_1 \cup K_2$,

(1) The value of $a_3(L)$ does not depend on the orientations of K_1 and K_2. Actually we can check it easily by the original definition of the Sato-Levine invariant.
(2) The value of $a_3(L)$ is not a link-homotopy invariant of L (see also Lemma 3.1). For example, the Whitehead link L is link-homotopically trivial but $a_3(L) = 1$.

Now we state the invariance of $\beta_{\omega_1,\omega_2}$ up to edge (resp. vertex)-homotopy under some conditions on the graphs.

Theorem 2.2. *Let $G = G_1 \cup G_2$ be a disjoint union of two connected graphs and ω_i a weight on $\Gamma(G_i)$ over $\mathbb{Z}_m$ $(i = 1, 2)$. Let f be a spatial embedding of G such that*

$$\omega_1(\gamma)\omega_2(\gamma')\mathrm{lk}(f(\gamma), f(\gamma')) = 0$$

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(G_2)$. Then we have the following:

(1) *If ω_i is weakly balanced on any edge of G_i $(i = 1, 2)$, then $\beta_{\omega_1,\omega_2}(f)$ is an edge-homotopy invariant of f.*
(2) *If ω_i is weakly balanced on any pair of adjacent edges of G_i $(i = 1, 2)$, then $\beta_{\omega_1,\omega_2}(f)$ is a vertex-homotopy invariant of f.*

We prove Theorem 2.2 in the next section. In addition, by using an integer-valued invariant (Theorem 4.2), we show that there exist infinitely many non-splittable spatial embeddings of a certain disconnected graph up to edge-homotopy all of whose constituent links are link-homotopically trivial (Example 4.3). We also exhibit an infinite family of non-splittable spatial embeddings of a certain disconnected graph up to vertex-homotopy which can be distinguished by our integer-valued invariant (Example 4.4).

We note that if a graph G contains a connected component which is homeomorphic to the 1-sphere, then our invariants in Theorem 2.2 are useless. For such cases, we can define edge (vertex)-homotopy invariants that take values in $\mathbb{Z}_2$ on a weaker condition for weights than the one stated in Theorem 2.2. For a subgraph H of a graph G, we say that a weight ω on $\Gamma(H)$ over $\mathbb{Z}_2$ is *totally balanced* if

$$\sum_{\gamma\in\Gamma(H)} \omega(\gamma)[\gamma] = 0$$

in $H_1(H;\mathbb{Z}_2)$. We note that if a weight ω on $\Gamma(H)$ over $\mathbb{Z}_2$ is totally balanced, then it is weakly balanced on any edge e of H (Lemma 3.2), but not always weakly balanced on any pair of adjacent edges of H (Remark 3.3). Then we have the following:

Theorem 2.3. *Let $G = G_1 \cup G_2$ be a disjoint union of two connected graphs and ω_i a weight on $\Gamma(G_i)$ over $\mathbb{Z}_2$ $(i = 1, 2)$. Let f be a spatial embedding of G such that*

$$\omega_1(\gamma)\omega_2(\gamma')\mathrm{lk}(f(\gamma), f(\gamma')) = 0$$

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(G_2)$. Then we have the following:

(1) *If either ω_1 is totally balanced on $\Gamma(G_1)$ or ω_2 is totally balanced on $\Gamma(G_2)$, then $\beta_{\omega_1,\omega_2}(f)$ is an edge-homotopy invariant of f.*
(2) *If either ω_1 is totally balanced on $\Gamma(G_1)$ and weakly balanced on any pair of adjacent edges of G_1, or ω_2 is totally balanced on $\Gamma(G_2)$ and weakly balanced on any pair of adjacent edges of G_2, then $\beta_{\omega_1,\omega_2}(f)$ is a vertex-homotopy invariant of f.*

We also prove Theorem 2.3 in the next section and give some examples in Section 5. In particular, we show that there exist infinitely many non-splittable spatial embeddings of a certain disconnected graph up to vertex-homotopy, all of whose constituent links are link-homotopically trivial (Example 5.4). We remark here that the $\mathbb{Z}_2$-valued invariant in Theorem 2.3 cannot always be extended to an integer-valued one (Remark 5.5).

Theorems 2.2 and 2.3 do not work for spatial graphs as illustrated in Figure 2.1, for instance. In Section 6, we state a method to detect such non-splittable spatial graphs up to edge-homotopy by using a planar surface having a graph as a spine (Theorem 6.1). Actually we show that each of the spatial graphs as illustrated in Figure 2.1 is non-splittable up to edge-homotopy (Example 6.2).

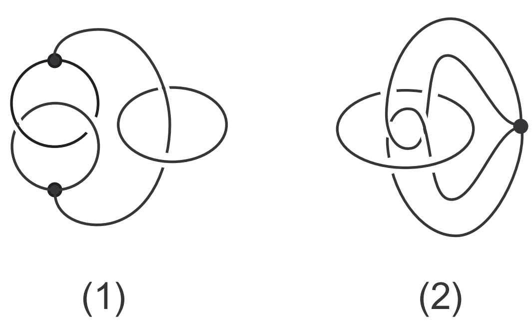

FIGURE 2.1

3. Proofs of Theorems 2.2 and 2.3

We first calculate the change in the third coefficient of the Conway polynomial of 2-component algebraically split links which differ by a single self crossing change.

Lemma 3.1. *Let L_+ and L_- be two 2-component oriented links and $L_0 = J_1 \cup J_2 \cup K$ a 3-component oriented link which are identical except inside the depicted regions as illustrated in Figure* 3.1. *Suppose that* $\mathrm{lk}(L_+) = \mathrm{lk}(L_-) = 0$. *Then it holds that*

$$a_3(L_+) - a_3(L_-) = -\mathrm{lk}(J_1, K)^2 = -\mathrm{lk}(J_2, K)^2.$$

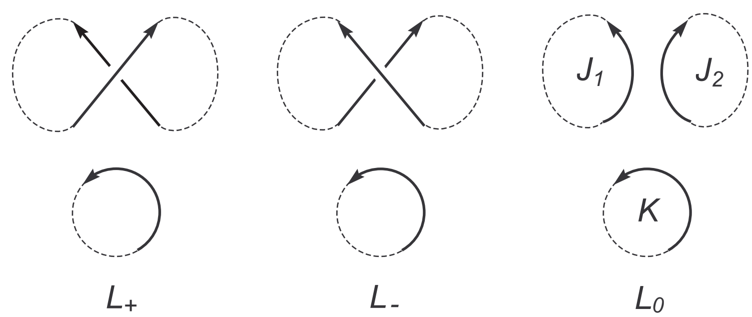

FIGURE 3.1

Proof. By the skein relation of the Conway polynomial and a well-known formula for the second coefficient of the Conway polynomial of a 3-component oriented link (cf. [4], [3], [5]), we have that

$$a_3(L_+) - a_3(L_-) = \mathrm{lk}(J_1, J_2)\mathrm{lk}(J_2, K) + \mathrm{lk}(J_2, K)\mathrm{lk}(J_1, K) + \mathrm{lk}(J_1, K)\mathrm{lk}(J_1, J_2). \tag{3.1}$$

We note that

$$\mathrm{lk}(J_1, K) + \mathrm{lk}(J_2, K) = 0 \tag{3.2}$$

by the condition $\mathrm{lk}(L_+) = \mathrm{lk}(L_-) = 0$. Thus by (3.1) and (3.2), we have that

$$\begin{aligned} a_3(L_+) - a_3(L_-) &= \mathrm{lk}(J_1, J_2)\{-\mathrm{lk}(J_1, K)\} + \mathrm{lk}(J_2, K)\mathrm{lk}(J_1, K) \\ &\quad + \mathrm{lk}(J_1, K)\mathrm{lk}(J_1, J_2) \\ &= \mathrm{lk}(J_2, K)\mathrm{lk}(J_1, K). \end{aligned}$$

Therefore by (3.2) we have the result. □

Proof of Theorem 2.2. (1) Let f and g be two spatial embeddings of G such that

$$\omega_1(\gamma)\omega_2(\gamma')\mathrm{lk}(f(\gamma), f(\gamma')) = 0 \tag{3.3}$$

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(G_2)$ and g is edge-homotopic to f. Then it also holds that

$$\omega_1(\gamma)\omega_2(\gamma')\mathrm{lk}(g(\gamma), g(\gamma')) = 0 \tag{3.4}$$

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(G_2)$ because the linking number of a 2-component constituent link of a spatial graph is an edge-homotopy invariant. First we show that if f is transformed into g by self crossing changes on $f(G_1)$ and ambient isotopies, then $\beta_{\omega_1,\omega_2}(f) = \beta_{\omega_1,\omega_2}(g)$. It is clear that any link invariant of a constituent link of a spatial graph is also an ambient isotopy invariant of the spatial graph. Thus we may assume that g is obtained from f by a single crossing change on $f(e)$ for an edge e of G_1 as illustrated in Figure 3.2. Moreover, by smoothing this crossing point we can obtain the spatial embedding h of G and the knot J_h as illustrated in Figure 3.2. Then by (3.3), (3.4), Lemma 3.1 and the assumption for ω_1 we have that

$$\begin{aligned}
\beta_{\omega_1,\omega_2}(f) - \beta_{\omega_1,\omega_2}(g) &\equiv \sum_{\substack{\gamma\in\Gamma(G_1)\\ \gamma'\in\Gamma(G_2)}} \omega_1(\gamma)\omega_2(\gamma')\left\{a_3(f(\gamma),f(\gamma')) - a_3(g(\gamma),g(\gamma'))\right\} \\
&= \sum_{\substack{\gamma\in\Gamma_e(G_1)\\ \gamma'\in\Gamma(G_2)}} \omega_1(\gamma)\omega_2(\gamma')\left\{a_3(f(\gamma),f(\gamma')) - a_3(g(\gamma),g(\gamma'))\right\} \\
&= -\sum_{\substack{\gamma\in\Gamma_e(G_1)\\ \gamma'\in\Gamma(G_2)}} \omega_1(\gamma)\omega_2(\gamma')\mathrm{lk}(h(\gamma'),J_h)^2 \\
&= -\left(\sum_{\gamma\in\Gamma_e(G_1)} \omega_1(\gamma)\right) \sum_{\gamma'\in\Gamma(G_2)} \omega_2(\gamma')\mathrm{lk}(h(\gamma'),J_h)^2 \\
&\equiv 0.
\end{aligned}$$

Therefore we have that $\beta_{\omega_1,\omega_2}(f) = \beta_{\omega_1,\omega_2}(g)$. In the same way we can show that if f is transformed into g by self crossing changes on $f(G_2)$ and ambient isotopies, then $\beta_{\omega_1,\omega_2}(f) = \beta_{\omega_1,\omega_2}(g)$. Thus we have that $\beta_{\omega_1,\omega_2}$ is an edge-homotopy invariant.

(2) By considering the triple of spatial embeddings as illustrated in Figure 3.3, we can prove (2) in a similar way as the proof of (1). We omit the details. □

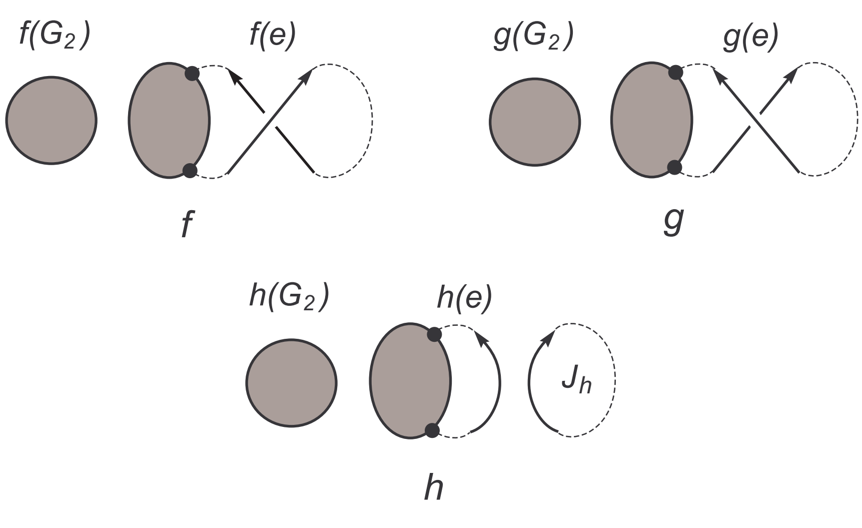

FIGURE 3.2

Next we prove Theorem 2.3. For a subgraph H of a graph G, we have the following.

Lemma 3.2. *A totally balanced weight ω on $\Gamma(H)$ over $\mathbb{Z}_2$ is weakly balanced on any edge e of H.*

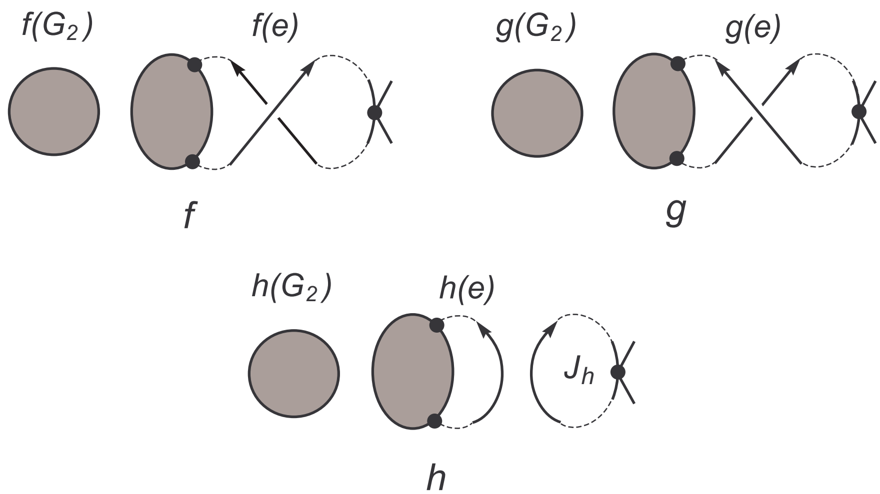

FIGURE 3.3

Proof. For an edge e of H, we can represent any $\gamma \in \Gamma_e(H)$ as $e + c_\gamma \in Z_1(H;\mathbb{Z}_2)$, where c_γ is a 1-chain in $C_1(H \setminus e;\mathbb{Z}_2)$. Then we have that

$$\begin{aligned} 0 &= \sum_{\gamma\in\Gamma(H)} \omega(\gamma)[\gamma] \\ &= \sum_{\gamma\in\Gamma_e(H)} \omega(\gamma)[e+c_\gamma] + \sum_{\gamma'\in\Gamma(H)\setminus\Gamma_e(H)} \omega(\gamma')[\gamma'] \end{aligned}$$

in $H_1(H;\mathbb{Z}_2)$. This implies that if ω is not weakly balanced on e, then ω is not totally balanced on $\Gamma(H)$ over $\mathbb{Z}_2$. □

Remark 3.3. A totally balanced weight ω on $\Gamma(H)$ over $\mathbb{Z}_2$ is not always weakly balanced on any pair of adjacent edges of H. For example, let ω be a weight on Θ_3 (see Example 4.3) over $\mathbb{Z}_2$ defined by $\omega(\gamma) = 1$ for any cycle $\gamma \in \Gamma(\Theta_3)$. It is easy to see that ω is totally balanced, but not weakly balanced, on each pair of adjacent edges of Θ_3.

Proof of Theorem 2.3. (1) Let f and g be two spatial embeddings of G which are edge-homotopic such that

$$\omega_1(\gamma)\omega_2(\gamma')\mathrm{lk}(f(\gamma), f(\gamma')) = \omega_1(\gamma)\omega_2(\gamma')\mathrm{lk}(g(\gamma), g(\gamma')) = 0$$

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(G_2)$. First we show that if f is transformed into g by self crossing changes on $f(G_1)$ and ambient isotopies, then $\beta_{\omega_1,\omega_2}(f) = \beta_{\omega_1,\omega_2}(g)$. In the same way as the proof of Theorem 2.2, we may consider three spatial embeddings f, g and h of G and the knot J_h as illustrated in Figure 3.2.

Then, by the same calculation in the proof of Theorem 2.2, we have that

$$\begin{aligned}\beta_{\omega_1,\omega_2}(f)-\beta_{\omega_1,\omega_2}(g) &\equiv -\left(\sum_{\gamma\in\Gamma_e(G_1)}\omega_1(\gamma)\right)\sum_{\gamma'\in\Gamma(G_2)}\omega_2(\gamma')\mathrm{lk}(h(\gamma'),J_h)^2\\ &\equiv \left(\sum_{\gamma\in\Gamma_e(G_1)}\omega_1(\gamma)\right)\sum_{\gamma'\in\Gamma(G_2)}\omega_2(\gamma')\mathrm{lk}(h(\gamma'),J_h)\\ &\equiv \left(\sum_{\gamma\in\Gamma_e(G_1)}\omega_1(\gamma)\right)\mathrm{lk}\left(\sum_{\gamma'\in\Gamma(G_2)}\omega_2(\gamma')h(\gamma'),J_h\right).\end{aligned}$$

If ω_1 is totally balanced on $\Gamma(G_1)$, then by Lemma 3.2 it is weakly balanced on any edge e of G_1. This implies that $\beta_{\omega_1,\omega_2}(f)=\beta_{\omega_1,\omega_2}(g)$. If ω_2 is totally balanced on $\Gamma(G_1)$, then we have that

$$\mathrm{lk}\left(\sum_{\gamma'\in\Gamma(G_2)}\omega_2(\gamma')h(\gamma'),J_h\right)\equiv \mathrm{lk}\left(0,J_h\right)=0.$$

Therefore this also implies that $\beta_{\omega_1,\omega_2}(f)=\beta_{\omega_1,\omega_2}(g)$. In the same way we can show that if f is transformed into g by self crossing changes on $f(G_2)$ and ambient isotopies, then $\beta_{\omega_1,\omega_2}(f)=\beta_{\omega_1,\omega_2}(g)$. Thus we have that $\beta_{\omega_1,\omega_2}$ is an edge-homotopy invariant.

(2) By considering the triple of spatial embeddings as illustrated in Figure 3.3, we can prove (2) in a similar way as the proof of (1). We also omit the details. □

Since the Conway polynomial of a split link is zero, our invariants take the value zero for any split (2-component) spatial graph. Therefore if the value of our invariant of a spatial graph is not zero, then it is non-splittable up to edge (resp. vertex)-homotopy.

4. Integer-valued invariants

Let G be a planar graph. An embedding $p:G\to S^2$ is said to be *cellular* if the closure of each of the connected components of $S^2-p(G)$ is homeomorphic to the disk. Then we regard the set of the boundaries of all of the connected components of $S^2-p(G)$ as a subset of $\Gamma(G)$ and denote it by $\Gamma_p(G)$. We say that G *admits a checkerboard coloring on* S^2 if there exists a cellular embedding $p:G\to S^2$ such that we can color all of the connected components of $S^2-p(G)$ by two colors (black and white) so that any of the two components which are adjacent by an edge have distinct colors; see Figure 4.1. We denote the subset of $\Gamma_p(G)$ which corresponds to the black (resp. white) colored components by $\Gamma_p^b(G)$ (resp. $\Gamma_p^w(G)$).

Proposition 4.1. *Let G be a planar graph which is not homeomorphic to S^1 and admits a checkerboard coloring on S^2 with respect to a cellular embedding $p:G\to S^2$. Let ω_p be a weight on $\Gamma(G)$ over $\mathbb{Z}$ defined by*

$$\omega_p(\gamma)=\begin{cases}1 & (\gamma\in\Gamma_p^b(G)),\\ -1 & (\gamma\in\Gamma_p^w(G)),\\ 0 & (\gamma\in\Gamma(G)\setminus\Gamma_p(G)).\end{cases}$$

Then ω_p is weakly balanced on any edge of G.

FIGURE 4.1

Proof. For any edge e of G, there exist exactly two cycles $\gamma \in \Gamma_p^b(G)$ and $\gamma' \in \Gamma_p^w(G)$ such that $e \subset \gamma$ and $e \subset \gamma'$. Thus we have the result. $\square$

We call the weight ω_p in Proposition 4.1 a *checkerboard weight*. Thus by Proposition 4.1 and Theorem 2.2 (1), we can obtain an integer-valued edge-homotopy invariant as follows.

Theorem 4.2. *Let $G = G_1 \cup G_2$ be a disjoint union of two connected planar graphs such that G_i is not homeomorphic to S^1 and admits a checkerboard coloring on S^2 with respect to a cellular embedding $p_i : G_i \to S^2$ $(i = 1, 2)$. Let ω_{p_i} be a checkerboard weight on $\Gamma(G_i)$ over $\mathbb{Z}$ $(i = 1, 2)$ and f a spatial embedding of G such that*

$$\omega_{p_1}(\gamma)\omega_{p_2}(\gamma')\mathrm{lk}(f(\gamma), f(\gamma')) = 0$$

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(G_2)$. Then $\beta_{\omega_{p_1},\omega_{p_2}}(f)$ is an integer-valued edge-homotopy invariant of f. $\square$

Example 4.3. Let Θ_n be a graph with two vertices u and v and n edges $e_1, e_2, \ldots, e_n$, each of which joins u and v. A spatial embedding of Θ_n is called a *(spatial) theta n-curve* or simply a *theta curve* if $n = 3$. For $n \geq 2$, we denote that a cycle of Θ_n consists of two edges e_i and e_j by γ_{ij} $(i < j)$. Then it is clear that Θ_n admits a cellular embedding $p : \Theta_n \to S^2$ so that

$$\Gamma_p(\Theta_n) = \{\gamma_{12}, \gamma_{23}, \ldots, \gamma_{n-1,n}, \gamma_{1n}\}.$$

Moreover, for $m \geq 1$, Θ_{2m} admits a checkerboard coloring on S^2 so that

$$\begin{aligned} \Gamma_p^b(\Theta_{2m}) &= \{\gamma_{12}, \gamma_{34}, \ldots, \gamma_{2m-1,2m}\}, \\ \Gamma_p^w(\Theta_{2m}) &= \{\gamma_{23}, \gamma_{45}, \ldots, \gamma_{2m-2,2m-1}, \gamma_{1,2m}\}. \end{aligned}$$

Now let G be a disjoint union of two copies of Θ_4, each of which admits a checkerboard coloring on S^2 with respect to the cellular embedding p as above. Let ω_p be a checkerboard weight on $\Gamma(\Theta_4)$ over $\mathbb{Z}$ and g_1 a spatial embedding of G as illustrated in Figure 4.2. We can see that any of the 2-component constituent links of g_1 has a zero linking number. More precisely, g_1 contains exactly one non-trivial 2-component link $L = g_1(\gamma_{14}) \cup g_1(\gamma'_{14})$ whose linking number is zero. Thus by Theorem 4.2 we have that $\beta_{\omega_p,\omega_p}(g_1)$ is an integer-valued edge-homotopy invariant of g_1. Then, by a direct calculation we have that $a_3(L) = 2$, namely $\beta_{\omega_p,\omega_p}(g_1) = 2$. Note that a 2-component link is link-homotopically trivial if and only if its linking number is zero [8]. This implies that g_1 is non-splittable up to edge-homotopy despite the fact that any of the constituent links of g_1 is link-homotopically trivial.

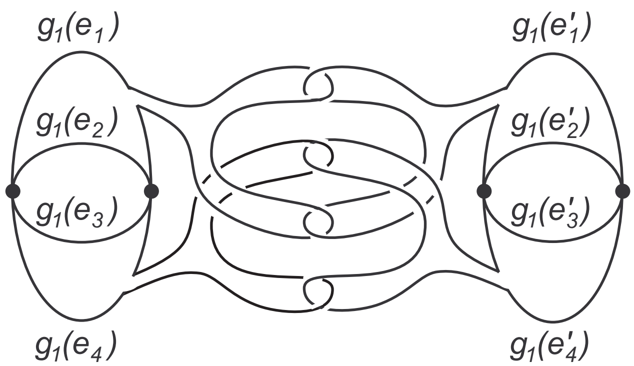

FIGURE 4.2

Moreover, for an integer m, let g_m be a spatial embedding of G as illustrated in Figure 4.3. If $m \neq 0$, we can see that g_m contains exactly one non-trivial 2-component link $L = g_m(\gamma_{14}) \cup g_m(\gamma'_{14})$ whose linking number is zero. Thus we also have that $\beta_{\omega_p,\omega_p}(g_m)$ is an integer-valued edge-homotopy invariant of g_m. Then, by a calculation we have that $a_3(L) = 2m$, namely $\beta_{\omega_p,\omega_p}(g_m) = 2m$. This implies that there exist infinitely many non-splittable spatial embeddings of G up to edge-homotopy, all of whose constituent links are link-homotopically trivial.

Example 4.4. Let H be a graph as illustrated in Figure 4.4. We denote the cycle of H which contains e_i and e_j by γ_{ij} $(i < j)$. Let G be a disjoint union of two copies of H and g_1 a spatial embedding of G as illustrated in Figure 4.5. This spatial embedding g_1 contains exactly one 4-component constituent link $L = g_1(\gamma_{12} \cup \gamma_{34} \cup \gamma'_{12} \cup \gamma'_{34})$. Note that if g_1 is split up to vertex-homotopy, then L is split up to link-homotopy. Since $|\mu_{1234}(L)| = 1$, where μ_{1234} denotes Milnor's *μ-invariant* of length 4 of 4-component links [8], we have that L is non-splittable up to link-homotopy. Therefore we have that g_1 is non-splittable up to vertex-homotopy.

We can also prove this fact by our integer-valued vertex-homotopy invariant as follows. Let ω be a weight on $\Gamma(H)$ over $\mathbb{Z}$ defined by $\omega(\gamma_{14}) = \omega(\gamma_{23}) = 1$, $\omega(\gamma_{13}) = \omega(\gamma_{24}) = -1$ and $\omega(\gamma) = 0$ if γ is a 2-cycle. Then it is easy to see that ω is weakly balanced on any pair of adjacent edges of H. We can see that g_1 contains exactly one non-trivial 2-component constituent link $M = g_1(\gamma_{14} \cup \gamma'_{14})$ with $\mathrm{lk}(M) = 0$ and $a_3(M) = 2$. Thus by Theorem 2.2 (2) we have that $\beta_{\omega,\omega}(g_1)$ is an integer-valued vertex-homotopy invariant of g_1 and $\beta_{\omega,\omega}(g_1) = 2$. This implies that g_1 is non-splittable up to vertex-homotopy.

Moreover, let g_m be a spatial embedding of G as illustrated in Figure 4.5, which can be constructed in the same way as in Example 4.3. Then we can see that $\beta_{\omega,\omega}(g_m)$ is an integer-valued vertex-homotopy invariant of g_m and $\beta_{\omega,\omega}(g_m) = 2m$. This implies that g_m is non-splittable up to vertex-homotopy for any integer $m \neq 0$ and g_i and g_j are not vertex-homotopic for any $i \neq j$.

5. MODULO TWO INVARIANTS

Proposition 5.1. *Let G be a planar graph which is not homeomorphic to S^1 and $p : G \to S^2$ a cellular embedding. Let $\omega_p : \Gamma(G) \to \mathbb{Z}_2$ be a weight on $\Gamma(G)$ over $\mathbb{Z}_2$*

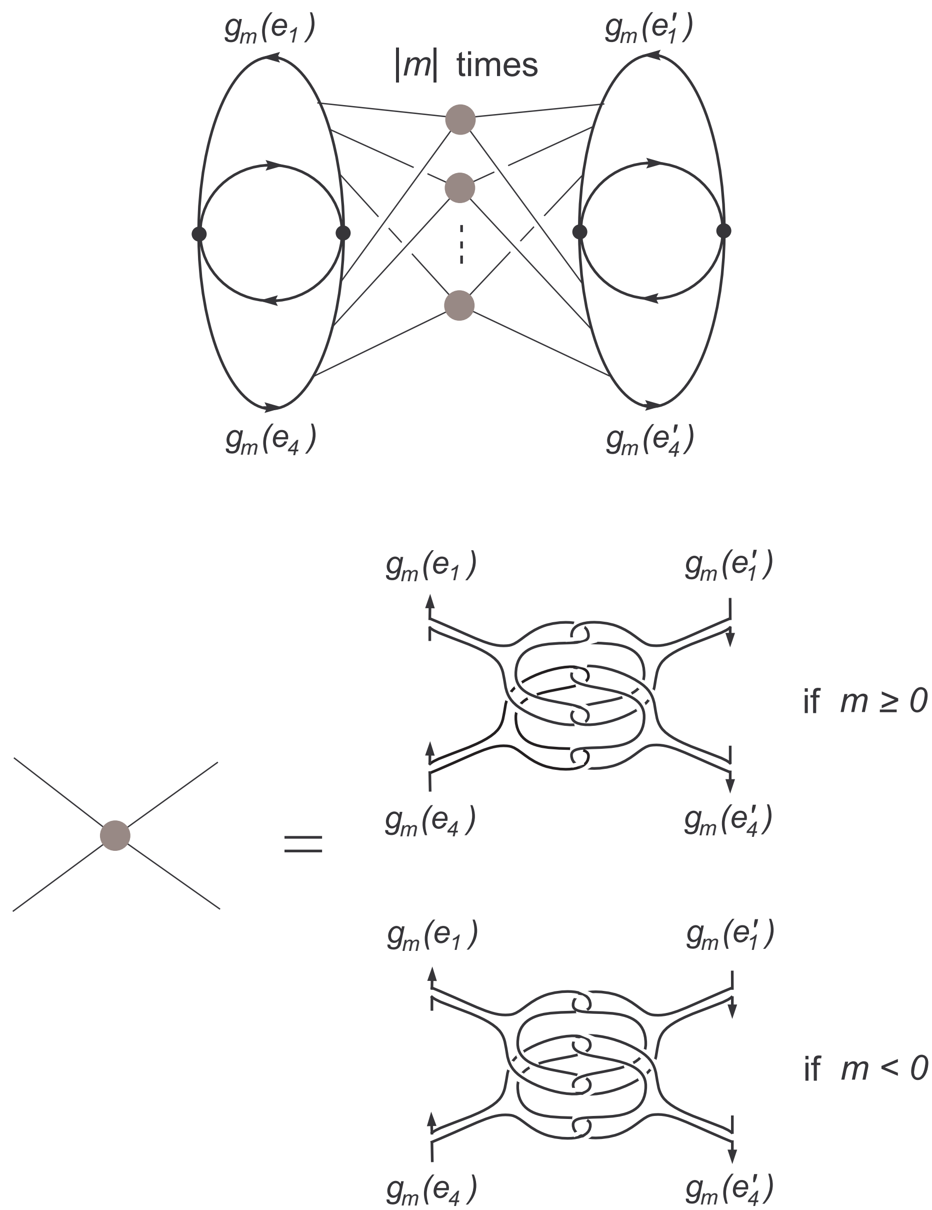

FIGURE 4.3

defined by

$$\omega_p(\gamma) = \begin{cases} 1 & (\gamma \in \Gamma_p(G)), \\ 0 & (\gamma \in \Gamma(G) \setminus \Gamma_p(G)). \end{cases}$$

Then ω_p *is totally balanced.*

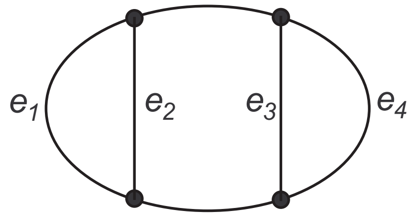

FIGURE 4.4

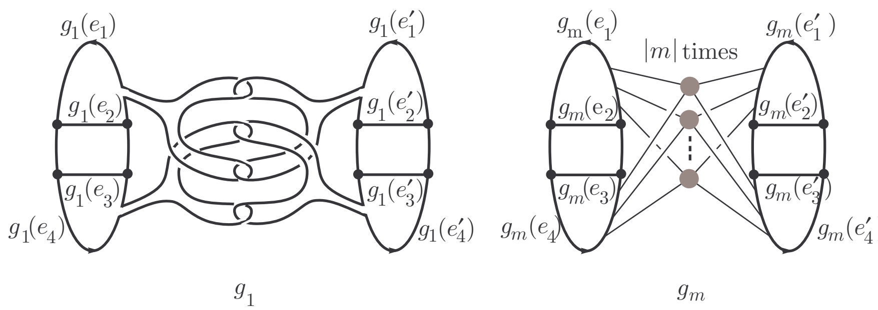

FIGURE 4.5

Proof. It holds that

$$\sum_{\gamma\in\Gamma(G)} \omega_p(\gamma)[\gamma] = \sum_{\gamma\in\Gamma_p(G)} [\gamma] = 2\left[\sum_{e\in E(G)} e\right] = 0$$

in $H_1(G;\mathbb{Z}_2)$, where $E(G)$ denotes the set of all edges of G. Thus we have the result. $\square$

Thus by Proposition 5.1 and Theorem 2.3 (1), we can obtain an edge-homotopy invariant as follows.

Theorem 5.2. *Let $G = G_1 \cup G_2$ be a disjoint union of two connected graphs such that G_1 is planar, not homeomorphic to S^1 and admits a cellular embedding $p_1 : G_1 \to S^2$. Let ω_{p_1} be a weight on $\Gamma(G_1)$ over $\mathbb{Z}_2$ as in Proposition* 5.1*, ω_2 a weight on $\Gamma(G_2)$ over $\mathbb{Z}_2$ and f a spatial embedding of G such that*

$$\omega_{p_1}(\gamma)\omega_2(\gamma')\mathrm{lk}(f(\gamma), f(\gamma')) = 0$$

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(G_2)$. Then $\beta_{\omega_{p_1},\omega_2}(f)$ is an edge-homotopy invariant of f. $\square$

Example 5.3. Let G be a disjoint union of Θ_3 and a circle γ. Let ω_p be a weight on $\Gamma(\Theta_3)$ over $\mathbb{Z}_2$ as in Proposition 5.1 with respect to a cellular embedding $p : \Theta_3 \to S^2$ as in Example 4.3, and ω a weight on $\Gamma(\gamma)$ over $\mathbb{Z}_2$ defined by $\omega(\gamma) = 1$. Let g be a spatial embedding of G as illustrated in Figure 5.1 (1). We can see that g contains exactly one non-trivial 2-component link $L = g(\gamma_{13}) \cup g(\gamma)$ which is the Whitehead link, so $\mathrm{lk}(L) = 0$ and $a_3(L) = 1$. Thus by Theorem 5.2 we have that

$\beta_{\omega_p,\omega}(g)$ is an edge-homotopy invariant of g and $\beta_{\omega_p,\omega}(g) = 1$. Namely g is non-splittable up to edge-homotopy despite the fact that any of the constituent links of g is link-homotopically trivial.

Example 5.4. Let G be a disjoint union of the complete bipartite graph on $3+3$ vertices $K_{3,3}$ and a circle γ. Let $\omega_{3,3}$ be a weight on $K_{3,3}$ over $\mathbb{Z}_2$ defined by $\omega_{3,3}(\gamma') = 1$ if γ' is a 4-cycle and 0 if γ' is a 6-cycle. Let ω be a weight on $\Gamma(\gamma)$ over $\mathbb{Z}_2$ defined by $\omega(\gamma) = 1$. Then it is not hard to see that $\omega_{3,3}$ is totally balanced and weakly balanced on any pair of adjacent edges of $K_{3,3}$. For a positive integer m, let g_m be a spatial embedding of G as illustrated in Figure 5.1 (2). Note that $g_i(K_{3,3})$ and $g_j(K_{3,3})$ are not vertex-homotopic for any $i \neq j$ [9]; namely g_i and g_j are not vertex-homotopic for any $i \neq j$. Since all of the 2-component constituent links of g_m are algebraically split, by Theorem 2.3 (2) we have that $\beta_{\omega_{3,3},\omega}(g)$ is a vertex-homotopy invariant of g_m. Moreover we can see that there exists exactly one 4-cycle γ' of $K_{3,3}$ so that $L = g_m(\gamma \cup \gamma')$ is non-trivial. Since L is the Whitehead link, we have that $\beta_{\omega_{3,3},\omega}(g_m) = 1$. Therefore g_m is non-splittable up to vertex-homotopy despite the fact that any of the constituent links of g is link-homotopically trivial.

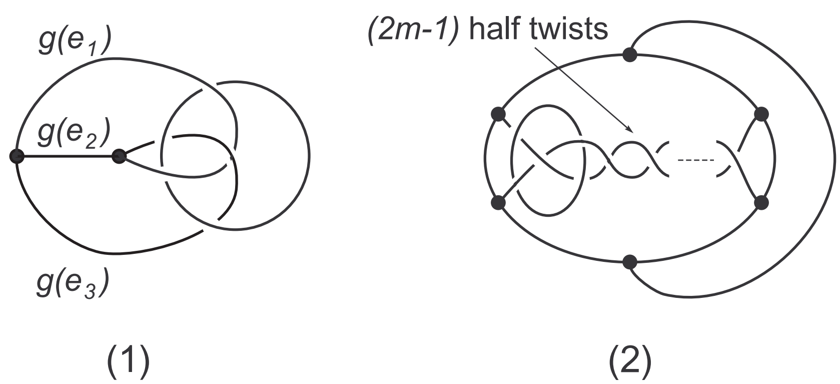

FIGURE 5.1

Remark 5.5. The $\mathbb{Z}_2$-valued invariant in Theorem 2.3 cannot always be extended to an integer-valued one. For example,

(1) Let us consider the graph G and the invariant $\beta_{\omega_p,\omega}$ as in Example 5.3. Let f be a spatial embedding of G as illustrated in Figure 5.2. We can see that f is edge-homotopic to the trivial spatial embedding h of G. But by a calculation we have that $\sum_{1\le i<j\le 3} a_3(f(\gamma_{ij}), f(\gamma)) = -2$.

(2) Let G be a disjoint union of Θ_4 and a circle γ. Let ω_p be a checkerboard weight on $\Gamma(\Theta_4)$ over $\mathbb{Z}$ as in Example 4.3. Note that the modulo two reduction of a checkerboard weight is totally balanced. So by Theorem 2.3 (1), the modulo two reduction of $\sum_{\gamma_{ij}\in\Gamma(\Theta_4)} \omega_p(\gamma_{ij}) a_3(f(\gamma_{ij} \cup \gamma))$ is an edge-homotopy invariant of a spatial embedding f of G. Moreover, we can see that the integer-value $\sum_{\gamma_{ij}\in\Gamma(\Theta_4)} \omega_p(\gamma_{ij}) a_3(f(\gamma_{ij} \cup \gamma))$ is invariant under the self crossing change on $f(\Theta_4)$ in the same way as in the proof of Theorem 2.2 (1). But this value may change under a self crossing change

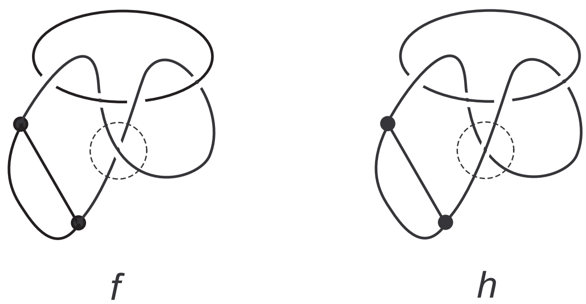

FIGURE 5.2

on $f(\gamma)$. For example, let f and g be two spatial embeddings of G as illustrated in Figure 5.3. We can see that f is edge-homotopic to g. But by a calculation we have that

$$\begin{aligned}\sum_{\gamma_{ij}\in\Gamma(\Theta_4)} \omega_p(\gamma_{ij})a_3(f(\gamma_{ij}), f(\gamma)) &= -1,\\ \sum_{\gamma_{ij}\in\Gamma(\Theta_4)} \omega_p(\gamma_{ij})a_3(g(\gamma_{ij}), g(\gamma)) &= 1.\end{aligned}$$

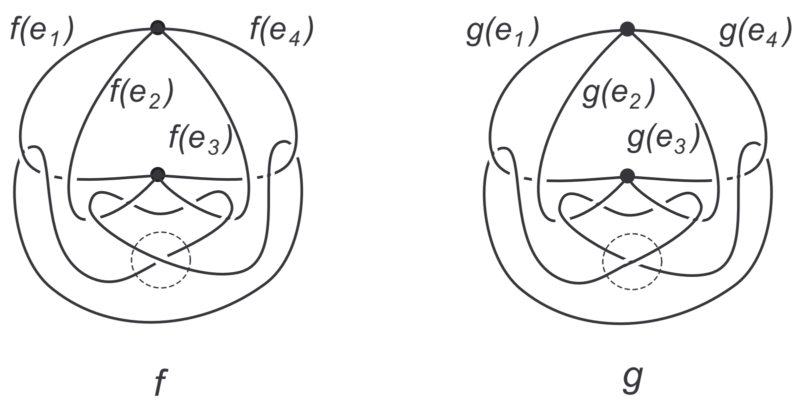

FIGURE 5.3

6. Applying the boundary of a planar surface

Let X be a disjoint union of a graph G and a planar surface F with boundary. Let ω be a weight on $\Gamma(G)$ over $\mathbb{Z}_2$ and φ an embedding of X into S^3 such that

$$\omega(\gamma)\mathrm{lk}(\varphi(\gamma), \varphi(\gamma')) = 0$$

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G)$ and $\gamma' \in \Gamma(\partial F)$. Then we define $\beta_\omega(\varphi) \in \mathbb{Z}_2$ by

$$\beta_\omega(\varphi) \equiv \sum_{\substack{\gamma\in\Gamma(G)\\ \gamma'\in\Gamma(\partial F)}} \omega(\gamma)a_3(\varphi(\gamma), \varphi(\gamma')) \pmod 2.$$

Let G be a disjoint union of a connected graph G_1 and a connected planar graph G_2. Let f be a spatial embedding of G and p an embedding of G_2 into S^2. We denote the regular neighborhood of $p(G_2)$ in S^2 by $F(G_2;p)$, which is a planar surface having $p(G_2)$ as a spine. Then the spatial embedding f induces an embedding $\tilde{f}_p$ of the disjoint union $G_1 \cup F(G_2;p)$ into S^3, so that $\tilde{f}_p(G_1) = f(G_1)$ and $\tilde{f}_p(F(G_2;p))$ has $f(G_2)$ as a spine in the natural way. Note that such an induced embedding $\tilde{f}_p$ is not unique up to ambient isotopy. Let ω be a weight on $\Gamma(G_1)$ over $\mathbb{Z}_2$ so that

$$\omega(\gamma)\mathrm{lk}(\tilde{f}_p(\gamma), \tilde{f}_p(\gamma')) = 0$$

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(\partial F(G_2;p))$. Then we have the following.

Theorem 6.1. *If f is split up to edge-homotopy, then $\beta_\omega(\tilde{f}_p) = 0$ for any induced embedding $\tilde{f}_p$ of $G_1 \cup F(G_2;p)$.*

Proof. By the assumption we have that f is transformed into a split spatial embedding u of G by self crossing changes and ambient isotopies. Then each of the self crossing changes induces a self crossing change on $\tilde{f}_p(G_1)$ or a *band-pass move* [6] (see Figure 6.1) on $\tilde{f}_p(F(G_2;p))$. Namely $\tilde{f}_p$ can be transformed into an induced embedding $\tilde{u}_p$ of $G_1 \cup F(G_2;p)$ by such moves and ambient isotopies. Let $\tilde{g}_p$ be an embedding of $G_1 \cup F(G_2;p)$ into S^3 obtained from $\tilde{f}_p$ by a single self crossing change on $\tilde{f}_p(G_1)$ or a single band-pass move on $\tilde{f}_p(F(G_2;p))$. Then it still holds that

$$\omega(\gamma)\mathrm{lk}(\tilde{g}_p(\gamma), \tilde{g}_p(\gamma')) = 0$$

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(\partial F(G_2;p))$.

Claim. $\beta_\omega(\tilde{f}_p) = \beta_\omega(\tilde{g}_p)$.

Assume that $\tilde{g}_p$ is obtained from $\tilde{f}_p$ by a single self crossing change on $\tilde{f}_p(G_1)$. Since it holds that

$$\sum_{\gamma' \in \Gamma(\partial F(G_2;p))} [\gamma'] = 0$$

in $H_1(F(G_2;p);\mathbb{Z}_2)$, we can see that $\beta_\omega(\tilde{f}_p) = \beta_\omega(\tilde{g}_p)$ in a similar way as the proof of Theorem 2.3 (1). Next we assume that $\tilde{g}_p$ is obtained from $\tilde{f}_p$ by a single band-pass move on $\tilde{f}_p(F(G_2;p))$. Then $\tilde{g}_p|_{G_1 \cup \partial F(G_2;p)}$ is obtained from $\tilde{f}_p|_{G_1 \cup \partial F(G_2;p)}$ by a single *pass move* [6] (see Figure 6.1) on $\tilde{f}_p(\partial F(G_2;p))$. We divide our situation into the following two cases.

Case 1. Four strings in the pass move belong to $\tilde{f}_p(\gamma'_1)$ and $\tilde{f}_p(\gamma'_2)$ for exactly two cycles γ'_1 and γ'_2 in $\Gamma(\partial F(G_2;p))$.

This pass move causes a single self crossing change on $\tilde{f}_p(\gamma'_1)$ and a single self crossing change on $\tilde{f}_p(\gamma'_2)$. Then the separated components that result from smoothing each of the self crossings are orientation-reversing parallel knots; see Figure 6.2. So the difference between $\beta_\omega(\tilde{f}_p)$ and $\beta_\omega(\tilde{g}_p)$ is cancelled out in a similar way as in the proof of Theorem 2.2 (1). Thus we have that $\beta_\omega(\tilde{f}_p) = \beta_\omega(\tilde{g}_p)$.

Case 2. Four strings in the pass move belong to $\tilde{f}_p(\gamma')$ for a cycle γ' in $\Gamma(\partial F(G_2;p))$.

It is known that a pass move on the same component of a proper link $L = J_1 \cup J_2 \cup \cdots \cup J_n$ preserves $\overline{\mathrm{Arf}}(L) \equiv \mathrm{Arf}(L) - \sum_{i=1}^n \mathrm{Arf}(J_i) \in \mathbb{Z}_2$ (cf. [16]).[3]

[3]The value of $\overline{\mathrm{Arf}}(L)$ is called the *reduced Arf invariant* of L [15].

Especially, if $n = 2$, then $a_3(L) \equiv \overline{\mathrm{Arf}}(L) \pmod 2$ [12, Lemma 3.5 (ii)]. Therefore in this case the pass move preserves $\omega(\gamma)a_3(\tilde{f}_p(\gamma), \tilde{f}_p(\gamma'))$ for any cycle $\gamma \in \Gamma(G_1)$. This implies that $\beta_\omega(\tilde{f}_p) = \beta_\omega(\tilde{g}_p)$.

Now by the argument above, we have that $\beta_\omega(\tilde{f}_p) = \beta_\omega(\tilde{u}_p)$. Then, each 2-component link $\tilde{u}_p(\gamma \cup \gamma')$ is split for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(\partial F(G_2; p))$ because u is split. Therefore we have that $\beta_\omega(\tilde{f}_p) = \beta_\omega(\tilde{u}_p) = 0$. This completes the proof. □

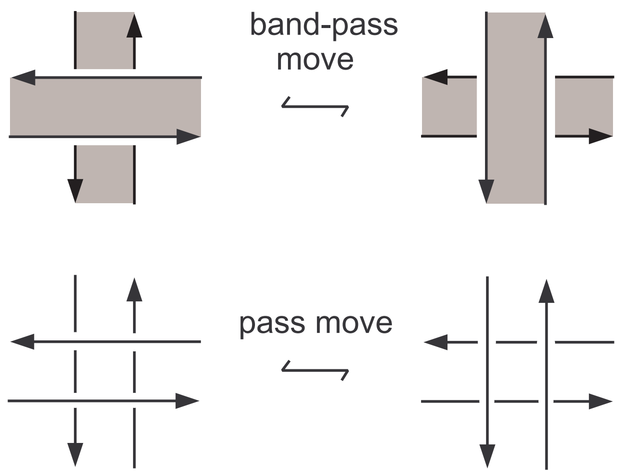

FIGURE 6.1

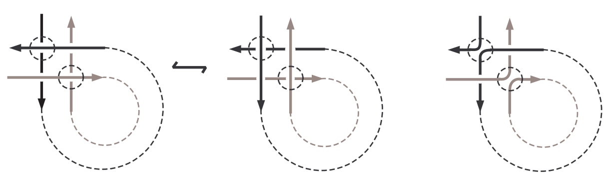

FIGURE 6.2

Example 6.2. Let G be a disjoint union of a circle γ and the *handcuff graph* (resp. 2-*bouquet*) G_2. Let ω be a weight on $\Gamma(\gamma)$ over $\mathbb{Z}_2$ defined by $\omega(\gamma) = 1$. We fix an embedding $p : G_2 \to S^2$ and take a regular neighborhood $F(G_2; p)$ as illustrated in Figure 6.3 (1) (resp. (2)).

Let f be a spatial embedding of G as illustrated in Figure 2.1 (1) (resp. (2)). Let us take an induced embedding $\tilde{f}_p : \gamma \cup F(G_2; p) \to S^3$ as illustrated in Figure 6.4 (1) (resp. (2)). Note that $\mathrm{lk}(\tilde{f}_p(\gamma), \tilde{f}_p(\gamma')) = 0$ for any $\gamma' \in \Gamma(\partial F(G_2; p))$. Then it can be calculated that $\beta_\omega(\tilde{f}_p) = 1$. Thus by Theorem 6.1 we have that f is non-splittable up to edge-homotopy.

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FIGURE 6.3

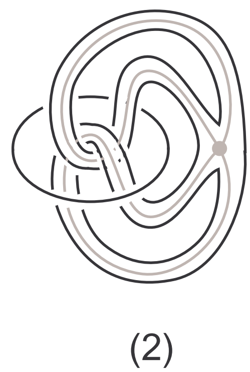

FIGURE 6.4

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