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SMALL PRINCIPAL SERIES AND EXCEPTIONAL DUALITY FOR TWO SIMPLY LACED EXCEPTIONAL GROUPS

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Abstract. We use the notion of rank defined in an earlier paper (2007) to introduce and study two correspondences between small irreducible unitary representations of the split real simple Lie groups of types $\mathbf{E}_n$, where $n \in \{6, 7\}$, and two reductive classical groups. We show that these correspondences classify all of the unitary representations of rank two (in the sense of our earlier paper) of these exceptional groups. We study our correspondences for a specific family of degenerate principal series representations in detail.

1. Introduction

Construction and classification of small irreducible unitary representations of noncompact semisimple Lie groups are challenging problems. Small unitary representations are important because they are natural candidates for being unipotent representations. In fact, many classes of small unitary representations are actually automorphic representations. The most outstanding small unitary representation of semisimple groups is probably the oscillator (or the Segal-Shale-Weil) representation. It is the smallest (nontrivial) unitary representation of the metaplectic group.

Questions about small unitary representations become much harder for exceptional groups. (See [GS] and [To] where minimal representations of simple Lie groups are extensively studied.)

This paper is a continuation of the author's work in [Sa1]. In [Sa1], the main goal of the author is to define a new notion of rank for unitary representations of a semisimple group over a local field of characteristic zero. In principle, this is a generalization of one of the main results of [Li2] in a fashion that it includes both the classical and the exceptional groups at the same time. Having an analogous theory of rank, one naturally expects that a general classification theorem similar to [Li2, Theorem 4.5] should exist. Our first main goal is to extend this classification theorem to exceptional Lie groups. This goal is achieved, at least for two real split simply laced groups, by Theorem 2.1.

Theorem 2.1 provides a correspondence between certain small irreducible unitary representations of an exceptional group G and irreducible unitary representations of a reductive subgroup S_1 of G. Here G is the real split group of type $\mathbf{E}_6$ or $\mathbf{E}_7$. Our second main goal is to understand this correspondence. One important

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$$\begin{array}{ccccccccc} 6 & - & 5 & - & 4 & - & 3 & - & 1 \\ & & & & | & & & & \\ & & & & 2 & & & & \end{array} \qquad \begin{array}{ccccccccccc} 7 & - & 6 & - & 5 & - & 4 & - & 3 & - & 1 \\ & & & & & & | & & & & \\ & & & & & & 2 & & & & \end{array}$$

Figure 1

common feature of the split groups of types $\mathbf{E}_6$ and $\mathbf{E}_7$ is the existence of certain degenerate principal series representations which are of rank two. This fact turns out to be a key ingredient in our analysis. See [Ws], where the author's motivation for choosing these groups is somewhat similar.

2. Statement of main results

Let $\mathbb{G}$ be a complex, simply connected, absolutely simple algebraic group of type $\mathbf{E}_n$, $n \in \{6,7\}$, which is defined and split over $\mathbb{R}$, and let G be the group of $\mathbb{R}$-rational points of $\mathbb{G}$. Let $\Pi_2(G)$ denote the set of irreducible unitary representations of G of rank two, where rank of an irreducible unitary representation is defined in [Sa1, Definition 5.3.3]. Section 3 contains a brief overview of the notion of rank in [Sa1].

Let S_1 be a reductive group given by[1]

$$S_1 = \begin{cases} \mathbb{R}^\times \ltimes Spin(3,4) & \text{if } G \text{ is of type } \mathbf{E}_6, \\ SL_2(\mathbb{R}) \times Spin(4,5) & \text{if } G \text{ is of type } \mathbf{E}_7, \end{cases}$$

and let $\Pi(S_1)$ denote the unitary dual of S_1. (In (5.12) we show how S_1 becomes a subgroup of G.)

Theorem 2.1. *There exists an injection $\Psi : \Pi_2(G) \mapsto \Pi(S_1)$ which is described in terms of Mackey theory.*

The map Ψ, which should potentially identify all of the unitary representations of G of rank two, is an analogue of the correspondences for classical groups which appear in [Ho], [Li2], and [Sca].

Let $\mathfrak{g}$ denote the Lie algebra of G and K be a maximal compact subgroup of G. Let $\mathbb{A}$ be a maximal split torus of $\mathbb{G}$ which is defined over $\mathbb{R}$, and let $A = \mathbb{A} \cap G$. Let Δ be the root system of $\mathbb{G}$ associated to $\mathbb{A}$. The root system Δ induces a root system of G. Choose a positive system Δ^+, and let B be the corresponding Borel subgroup of G. Let $N_B = [B,B]$. For every $\alpha \in \Delta$, let $\mathfrak{g}_\alpha$ be the (one-dimensional) root space of $\mathfrak{g}$ corresponding to α.

Let $\Delta_B = \{\alpha_1, ..., \alpha_n\}$ be a basis for Δ^+. The labelling of the Dynkin diagram of $\mathbf{E}_n$ by the α_i's is compatible with those given in [Bo, Planches] and [Kn1, Appendix C]. This labelling can be described by the diagrams in Figure 1.

Let $\mathfrak{a}_\mathbb{C}$ be the Lie algebra of $\mathbb{A}$. For any $i \in \{1, ..., n\}$, let $\varpi_i \in \mathrm{Hom}_\mathbb{C}(\mathfrak{a}_\mathbb{C}, \mathbb{C})$ denote the fundamental weight corresponding to the node labelled by α_i, and let e^{ϖ_i} denote the corresponding character of $\mathbb{A}$. For any $a \in A$, we have $e^{\varpi_i}(a) \in \mathbb{R}^\times$. Any $a \in A$ is uniquely identified by the values of the $e^{\varpi_i}(a)$.

Let P be the standard maximal parabolic subgroup of G which corresponds to the node labelled by α_n. Let the standard Levi factorisation of P be $P = L \ltimes N$. The group N is commutative. Let $\mathfrak{p}, \mathfrak{l}$ and $\mathfrak{n}$ denote the Lie algebras of P, L and N, respectively.

[1] When G is of type $\mathbf{E}_6$, the semidirect product $\mathbb{R}^\times \ltimes Spin(3,4)$ is explicitly described in the proof of Proposition 5.13.

Let $\overline{P}$ denote the parabolic subgroup opposite to P. Note that there exists an automorphism Υ of G such that $\Upsilon(P) = \overline{P}$: when G is of type $\mathbf{E}_7$, Υ is the conjugation by the longest element of the Weyl group and when G is of type $\mathbf{E}_6$, Υ is a composition of this conjugation with a diagram automorphism. For any unitary character χ of $\overline{P}$, we consider the degenerate principal series representation $\pi_\chi = \mathrm{Ind}_{\overline{P}}^{G}\chi$ of G induced (unitarily) from $\overline{P}$. The unitary representation π_χ is irreducible unless G is of type $\mathbf{E}_7$ and χ is the nontrivial character of $\overline{P}$ which is trivial on the connected component of the identity of $\overline{P}$. (This fact follows from the proof of Theorem 2.2 as well.)

Theorem 2.2. *If χ is a unitary character of $\overline{P}$ such that π_χ is irreducible, then $\pi_\chi \in \Pi_2(G)$ and*

A. *When G is of type $\mathbf{E}_6$, $\Psi(\pi_\chi)$ is a unitary character of S_1.*
B. *When G is of type $\mathbf{E}_7$, $\Psi(\pi_\chi)$ is isomorphic to the tensor product of a principal series representation of $SL_2(\mathbb{R})$ and the trivial representation of $Spin(4,5)$.*

Theorem 2.2 can also be easily extended to split groups of types $\mathbf{E}_6$ and $\mathbf{E}_7$ over complex and p-adic fields.

Fix a nontrivial positive multiplicative character e^{Λ_0} of L. For convenience, when G is of type $\mathbf{E}_7$, we assume

$$\Lambda_0 = \frac{1}{18}\sum_{\mathfrak{g}_\alpha \subset \mathfrak{n}} \alpha.$$

In fact, in this case Λ_0 is the linear functional introduced in [BSZ, Definition 2.4]. One can extend the character e^{Λ_0} trivially on N to a character of P. Let δ_P denote the modular function of P. For any $s \in \mathbb{C}$ one can define a degenerate principal series representation $I_P(s)$ of G as follows:

$$I_P(s) = \{f \in C^\infty(G) \mid \forall p \in P, \forall g \in G : f(gp) = \delta_P(p)^{-\frac{1}{2}} e^{-s\Lambda_0}(p) f(g)\}. \tag{2.3}$$

The action of G on $I_P(s)$ is by left translation:

$$(g \cdot f)(g_1) = f(g^{-1}g_1).$$

Reducibility points and unitarizability of subquotients of the family $I_P(s)$ of degenerate principal series representations are studied extensively in [Sah], [BSZ] and [Zh]. For $0 \leq s < 1$, one obtains a family of irreducible unitary representations of G (complementary series) which we denote by π_s. The representation $I_P(1)$ has an irreducible unitarizable subquotient, and this unitary representation will be denoted by π°. It is worth mentioning that the minimal representation (respectively, the trivial representation) of G is a subquotient of $I_P(5)$ (respectively, of $I_P(9)$). For more details, see [BSZ, Prop. 7.8, Part (2)]. We use the results in [Sah] and [BSZ] to prove the next theorem.

Theorem 2.4. *If G is of type $\mathbf{E}_7$, then π° and the π_s ($0 \leq s < 1$) belong to $\Pi_2(G)$. Moreover, $\Psi(\pi^\circ)$ is the trivial representation of $SL_2(\mathbb{R}) \times Spin(4,5)$.*

In the rest of this section, we introduce our notation and recall several facts about representations. Let $Q = M \ltimes H$ be the standard Levi factorisation of the standard Heisenberg parabolic subgroup of G (where H is a Heisenberg group) and let $R = S \ltimes U$ be the standard parabolic subgroup of G such that the root system of $[S,S]$ is of type $\mathbf{D}_4$ when G is of type $\mathbf{E}_6$ and of type $\mathbf{A}_1 \times \mathbf{D}_5$ when G is of type $\mathbf{E}_7$, respectively. Note that the group U is two-step nilpotent.

Let $P_\Gamma = L_\Gamma \ltimes N_\Gamma$ be the standard parabolic subgroup introduced in [Sa1, §3.2]. When G is of type $\mathbf{E}_6$, $[L_\Gamma, L_\Gamma]$ is equal to the $SL_2(\mathbb{R})$ which corresponds to α_4, and when G is of type $\mathbf{E}_7$, $[L_\Gamma, L_\Gamma]$ is equal to the product of the $SL_2(\mathbb{R})$'s which correspond to $\alpha_2, \alpha_3, \alpha_5$ and α_7. The group N_Γ is a tower of semidirect products of Heisenberg groups, i.e.,

$$N_\Gamma = N_1 \ltimes N_2 \ltimes N_3 \tag{2.5}$$

where the N_i's are Heisenberg groups (see [Sa1, Prop. 3.2.6]). Note that $N_3 = H$. Let $\mathfrak{n}_i$ denote the Lie algebra of N_i.

Let β_1 be the highest root in Δ^+, and let β_2 be the highest root of the root system of $[M, M]$. (Obviously we are using the positive system for $[M, M]$ which is induced by Δ^+.)

We denote the Lie algebras of the groups $Q, M, H, R, S, U, P_\Gamma, L_\Gamma$ and N_Γ by $\mathfrak{q}, \mathfrak{m}, \mathfrak{h}, \mathfrak{r}, \mathfrak{s}, \mathfrak{u}, \mathfrak{p}_\Gamma, \mathfrak{l}_\Gamma$ and $\mathfrak{n}_\Gamma$, respectively.

The trivial representation of any group is denoted by "1". The center of a group or a Lie algebra is denoted by $\mathcal{Z}(\cdot)$. For any Hilbert space $\mathcal{H}$, the algebra of bounded operators from $\mathcal{H}$ to itself is denoted by $\mathrm{End}(\mathcal{H})$.

If G_1 and G_2 are Lie groups where G_1 is a Lie subgroup of G_2, and if for any $i \in \{1, 2\}$, π_i is a unitary representation of G_i, then $\mathrm{Res}_{G_1}^{G_2} \pi_2$ and $\mathrm{Ind}_{G_1}^{G_2} \pi_1$ denote restriction and (unitary) induction. When there is no ambiguity about G_2, we may use $\pi_{2|G_1}$ instead of $\mathrm{Res}_{G_1}^{G_2} \pi_2$. Throughout this paper, we will use two properties of induction and restriction which we would like to remind the reader of. The first property is Mackey's subgroup theorem, as stated in [Mac]. The second property is the "projection formula", which states that

$$\mathrm{Ind}_{G_1}^{G_2}(\pi_1 \otimes (\mathrm{Res}_{G_1}^{G_2} \pi_2)) \approx (\mathrm{Ind}_{G_1}^{G_2} \pi_1) \otimes \pi_2.$$

If G_1 is a group acting on a set X, then for any $x \in X$ the stabilizer of x inside G_1 is denoted by $\mathrm{Stab}_{G_1}(x)$.

Needless to say, let $\mathbb{R}^\times$ and $\mathbb{R}^+$ denote (any group naturally isomorphic to) the multiplicative groups of nonzero and positive real numbers, respectively.

This paper is organized as follows. Section 3 is devoted to recalling the notion of rank in [Sa1] and the main result of [Sa1] on rank. Section 4 contains a suitable adaptation of a result of Scaramuzzi's [Sca]. In section 5 we prove Theorem 2.1 and describe the correspondence Ψ. In section 6 we prove that the π_χ, the π_s, and π° are of rank two. Sections 7, 8, and 9 are devoted to finding an explicit description of $\Psi(\pi_\chi)$ and $\Psi(\pi^\circ)$. Section 10 contains tables concerning root systems of types $\mathbf{E}_6$ and $\mathbf{E}_7$.

Remark. The reader should note that:

1. From the proof of Theorem 2.1 it can be seen that one can extend the theorem to imply the existence of many maps analogous to Ψ for several other exceptional groups over complex and p-adic fields. The only difficulties in the proof are certain technicalities with exceptional groups similar to those dealt with in [Sa1, §§5.1, 5.2]. However, for the sake of brevity, and because the main point of this paper is Theorem 2.2, we have stated Theorem 2.1 for the special case of real split groups of types $\mathbf{E}_6$ and $\mathbf{E}_7$.

2. When G is of type $\mathbf{E}_7$, it is an interesting problem to understand the "inverse image" of the unitary dual of $SL_2(\mathbb{R})$ under the map Ψ. For the principal series this follows from Theorem 2.2 and for the trivial representation this follows from Theorem 2.4. For any s such that $0 \leq s < 1$, $\Psi(\pi_s)$ should be the tensor product of a complementary series representation of G and the trivial representation of $Spin(4,5)$. For the case of the discrete series of $SL_2(\mathbb{R})$, Nolan Wallach has suggested a method of construction of small representations of G based on the idea of transfer [Wa]. The cases of the complementary series and the discrete series will hopefully be addressed in a subsequent paper.

3. We could have used the fancier language of Jordan triple systems to work with groups more coherently and probably include some classical groups as well. (Similar successful attempts of using Jordan algebras along these lines were made, for example, in [BSZ], [Sah], and [DS].) However, for classical groups our results are not new, and for the exceptional groups that we are going to study it is easier to do things more explicitly.

3. Representations of small rank

Recall from (2.5) that $N_\Gamma = N_1 \ltimes N_2 \ltimes N_3$. For any $i \in \{1,2,3\}$, let ρ_i be an arbitrary infinite-dimensional irreducible unitary representation of N_{4-i}. By means of the oscillator representation, one can extend each ρ_i to a unitary representation $\tilde{\rho}_i$ of N_Γ. The recipe for extension is given in [Sa1, §4.1]. We call any unitary representation of N_Γ of the form $\tilde{\rho}_1$, $\tilde{\rho}_1 \otimes \tilde{\rho}_2$ or $\tilde{\rho}_1 \otimes \tilde{\rho}_2 \otimes \tilde{\rho}_3$ a *rankable* representation of N_Γ of rank one, two or three, respectively (see [Sa1, Def. 4.1.1]). The trivial representation of N_Γ is said to be rankable of rank zero. One can see that any rankable representation of N_Γ is irreducible. The following theorem is essential to this work.

Theorem 3.1 ([Sa1, Theorem 5.3.2])**.** *Let π be an irreducible unitary representation of G. Then the restriction of π to N_Γ is supported on rankable representations of N_Γ of rank r, for a fixed number $r \in \{0,1,2,3\}$, which only depends on π.*

Using Theorem 3.1 one can define a notion of rank for unitary representations of G. A (possibly reducible) unitary representation π of G is said to be of rank r if the restriction of π to N_Γ is supported on rankable representations of N_Γ of rank r. Theorem 3.1 implies that for irreducible unitary representations, rank is well defined.

4. Representations of $GL_m(\mathbb{R})$ of rank one

In this section we prove a result which provides a suitable adaptation of [Sca, Theorem II.1.1] and will be used in the proof of Proposition 5.4. The proof of Proposition 4.1 is lengthy, but easy, and could be omitted. However, to make this manuscript self-contained, we would like to give its proof in detail.

Fix an integer $m > 4$. Let $GL_m(\mathbb{R})^+$ denote the connected component of the identity in $GL_m(\mathbb{R})$. Let π be a unitary representation of $GL_m(\mathbb{R})^+$ whose restriction to $SL_m(\mathbb{R})$ is of rank one in the sense of [Sa1, Definition 5.3.3]. Let Q_m be the standard Heisenberg parabolic subgroup of $GL_m(\mathbb{R})$ and $Q_m^+ = Q_m \cap GL_m(\mathbb{R})^+$.

Proposition 4.1. *The von Neumann algebra generated by $\pi(Q_m^+)$ is equal to the von Neumann algebra generated by $\pi(GL_m(\mathbb{R})^+)$.*

Proof. This proposition follows immediately from Lemma 4.4 below. □

Obviously $GL_m(\mathbb{R}) = \{\pm 1\} \ltimes GL_m(\mathbb{R})^+$ for a suitable subgroup $\{\pm 1\}$ of the diagonal matrices. (When m is odd, the semidirect product is actually a direct product.) Observe that from [Sa1, §6] it follows that a unitary representation of $GL_m(\mathbb{R})$ is of rank one in the sense of [Sca] if and only if the restriction of π to $SL_m(\mathbb{R})$ is of rank one in the sense of [Sa1]. From now on, a unitary representation of $GL_m(\mathbb{R})^+$ is said to be of rank one whenever its restriction to $SL_m(\mathbb{R})$ is of rank one.

Lemma 4.2. *Every irreducible unitary representation σ of rank one of $GL_m(\mathbb{R})^+$ is equal to the restriction of a unitary representation of $GL_m(\mathbb{R})$.*

Proof. If σ does not extend to a unitary representation of $GL_m(\mathbb{R})$, then $\mathrm{Ind}_{GL_m(\mathbb{R})^+}^{GL_m(\mathbb{R})}\sigma$ will be an irreducible unitary representation of $GL_m(\mathbb{R})$ of rank one. On the other hand, the representation

$$\mathrm{Res}_{GL_m(\mathbb{R})^+}^{GL_m(\mathbb{R})}\mathrm{Ind}_{GL_m(\mathbb{R})^+}^{GL_m(\mathbb{R})}\sigma$$

is reducible. (In fact, it is a direct sum of σ and $\overline{\sigma}$, where $\overline{\sigma}$ is a unitary representation of $GL_m(\mathbb{R})^+$ defined by $\overline{\sigma}(g) = \sigma(-1 \cdot g \cdot -1)$.) But from the description of irreducible representations of rank one of $GL_m(\mathbb{R})$ in [Sca, Theorem II.3.1] as induced representations, it follows that the restriction of any irreducible unitary representation of rank one of $GL_m(\mathbb{R})$ to $GL_m(\mathbb{R})^+$ remains irreducible. Therefore every irreducible unitary representation of $GL_m(\mathbb{R})^+$ of rank one should extend to a unitary representation of $GL_m(\mathbb{R})$, which will clearly be of rank one as well. □

Let H_m be the unipotent radical of Q_m. Pick a nontrivial element $g \in \mathcal{Z}(H_m)$ and let J be the centralizer of g in Q_m. Note that all of these centralizers are equal. The group J can be explicitly described in matrix form as

$$J = \{[x_{i,j}] \in GL_m(\mathbb{R}) \mid x_{1,1} = x_{m,m} \text{ and for any } 1 < i \leq m : x_{i,1} = x_{m,i-1} = 0\}.$$

In fact, J is the subgroup of Q_m given in [Sca, §I, Equation (18)]. (We advise the reader that in the notation of [Sca] our Q_m is in fact denoted by Q_1. We do not feel obliged to obey the notation of [Sca] since the author does not use it coherently throughout the paper.)

Let ρ_{tr} be an irreducible, infinite-dimensional, unitary representation of H_m. (The notation ρ_{tr} is chosen to keep our presentation as close to [Sca, §II.2] as possible. In our situation the indeterminate k of [Sca, §II, Equation (39)] is equal to one. In such a simple case, the detailed definition of ρ_{tr} in [Sca] is not needed.) By means of the oscillator representation, ρ_{tr} can be extended to a unitary representation ω_{tr} of J. An explicit formula for ω_{tr} is given in [Sca, §II.2, Equation (39)]. Let $J^+ = J \cap GL_m(\mathbb{R})^+$. Obviously ω_{tr} can be considered as a representation of J^+ too.

Let J_1 be the subgroup of J consisting of invertible $m \times m$ diagonal matrices of the form

$$\begin{bmatrix} t & 0 & 0 \\ 0 & \mathrm{I}_{m-2} & 0 \\ 0 & 0 & t \end{bmatrix},$$

where $t \in \mathbb{R}^\times$ and the middle block I_{m-2} is the identity matrix of size $m-2$, and let J_2 be the subgroup of J consisting of all block-diagonal matrices of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $X \in GL_{m-2}(\mathbb{R})$.

There are obvious isomorphisms $J_1 \approx \mathbb{R}^\times$ and $J_2 \approx GL_{m-2}(\mathbb{R})$. If χ_1 and χ_2 are unitary multiplicative characters of $\mathbb{R}^\times$, then by means of these isomorphisms one can think of χ_1 as a unitary character of J_1 and $\chi_2 \circ \det$ as a unitary character of J_2. One can extend the unitary character $\chi_1 \otimes \chi_2 \circ \det$ of the subgroup $J_1 \times J_2$ of J trivially on H_m to a unitary character of J. Therefore we obtain a unitary representation of Q_m given by induction from J as

$$\mathrm{Ind}_J^{Q_m}((\chi_1 \otimes \chi_2 \circ \det) \otimes \omega_{tr}).$$

Let P_m be the standard parabolic subgroup of $GL_m(\mathbb{R})$ whose standard Levi subgroup is

$$GL_1(\mathbb{R}) \times GL_{m-1}(\mathbb{R}).$$

If χ_1 and χ_2 are unitary multiplicative characters of $\mathbb{R}^\times$, then we can consider χ_1 as a unitary character of $GL_1(\mathbb{R})$ and $\tilde{\chi}_2 = \chi_2 \circ \det$ as a unitary character of $GL_{m-1}(\mathbb{R})$. Therefore $\chi_1 \otimes \tilde{\chi}_2$ is a unitary character of the Levi subgroup of P_m. We can extend $\chi_1 \otimes \tilde{\chi}_2$ trivially on the unipotent radical of P_m to a unitary character of P_m. As in [Sca, §II.3], define the unitary representation $\chi_1 \times \chi_2$ of $GL_m(\mathbb{R})$ as

$$\chi_1 \times \chi_2 = \mathrm{Ind}_{P_m}^{GL_m(\mathbb{R})}(\chi_1 \otimes \tilde{\chi}_2).$$

The next proposition follows from [Sca, Theorem II.2.1] and [Sca, Theorem II.3.1]. It will be used in the proof of Lemma 4.4 below.

Proposition 4.3. *Let $m > 4$ and π be an irreducible unitary representation of $GL_m(\mathbb{R})$ of rank one. Then there exist unitary multiplicative characters χ_1, χ_2 of $\mathbb{R}^\times$, such that*

$$\pi_{|Q_m} = \mathrm{Ind}_J^{Q_m}((\chi_1 \otimes \chi_2 \circ \det) \otimes \omega_{tr}).$$

The pair (χ_1, χ_2) is uniquely determined by π, and the correspondence $\pi \mapsto (\chi_1, \chi_2)$ is bijective, with inverse $(\chi_1, \chi_2) \mapsto \chi_2^{-1}\chi_1 \times \chi_2 \circ \det$.

Remark. Note that in Proposition 4.3, χ_1 and χ_2 play the roles of σ and χ of [Sca, Theorem II.2.1].

Lemma 4.4. *Let $m > 4$.*

a. *If π is an irreducible unitary representation of $GL_m(\mathbb{R})^+$ of rank one, then the restriction of π to Q_m^+ remains irreducible.*
b. *If π, π' are two distinct irreducible unitary representations of $GL_m(\mathbb{R})^+$ of rank one, then their restrictions to Q_m^+ are nonisomorphic representations.*

Proof. If π is as in Lemma 4.4a above, then π extends to a unitary representation $\hat{\pi}$ of $GL_m(\mathbb{R})$. By Proposition 4.3, there exist unitary multiplicative characters χ_1, χ_2 of $\mathbb{R}^\times$ such that

$$\mathrm{Res}_{Q_m}^{GL_m(\mathbb{R})}\hat{\pi} = \mathrm{Ind}_J^{Q_m}((\chi_1 \otimes \chi_2 \circ \det) \otimes \omega_{tr}).$$

By Mackey's subgroup theorem one can see that

$$\mathrm{Res}_{Q_m^+}^{GL_m(\mathbb{R})^+}\pi = \mathrm{Ind}_{J^+}^{Q_m^+}((\chi_1 \otimes \chi_2 \circ \det) \otimes \omega_{tr}).$$

Recall that H_m is the unipotent radical of Q_m. The restriction of the unitary representation ω_{tr} to H_m is irreducible. Therefore ω_{tr} is an irreducible unitary representation of J^+ too. Standard Mackey theory tells us that for any irreducible unitary representation ν of J^+/H_m, $\mathrm{Ind}_{J^+}^{Q_m^+}(\nu \otimes \omega_{tr})$ is an irreducible unitary representation of Q_m^+. Therefore $\pi_{|Q_m^+}$ is irreducible. This proves Lemma 4.4a.

Next we prove Lemma 4.4b. By Lemma 4.2 it suffices to show that if $\hat{\pi}$ and $\hat{\pi}'$ are two rank one unitary representations of $GL_m(\mathbb{R})$ whose restrictions to Q_m^+ are isomorphic, then the restrictions of $\hat{\pi}$ and $\hat{\pi}'$ to $GL_m(\mathbb{R})^+$ are isomorphic as well. By Proposition 4.3, there are unitary multiplicative characters $\chi_1, \chi_2, \chi_1', \chi_2'$ of $\mathbb{R}^\times$ such that

$$\mathrm{Res}_{Q_m}^{GL_m(\mathbb{R})}\hat{\pi} = \mathrm{Ind}_{J}^{Q_m}((\chi_1 \otimes \chi_2 \circ \det) \otimes \omega_{tr})$$

and

$$\mathrm{Res}_{Q_m}^{GL_m(\mathbb{R})}\hat{\pi}' = \mathrm{Ind}_{J}^{Q_m}((\chi_1' \otimes \chi_2' \circ \det) \otimes \omega_{tr}).$$

Obviously we have

$$\mathrm{Res}_{Q_m^+}^{GL_m(\mathbb{R})}\hat{\pi} = \mathrm{Ind}_{J^+}^{Q_m^+}((\chi_1 \otimes \chi_2 \circ \det) \otimes \omega_{tr})$$

and

$$\mathrm{Res}_{Q_m^+}^{GL_m(\mathbb{R})}\hat{\pi}' = \mathrm{Ind}_{J^+}^{Q_m^+}((\chi_1' \otimes \chi_2' \circ \det) \otimes \omega_{tr}).$$

By standard Mackey theory, if the restrictions of $\hat{\pi}$ and $\hat{\pi}'$ to Q_m^+ are isomorphic, then we have

$$\chi_1(a) = \chi_1'(a) \text{ for every } a \in \mathbb{R}^\times \text{ and } \chi_2(a) = \chi_2'(a) \text{ for every } a \in \mathbb{R}^+. \tag{4.5}$$

Let $P_m^+ = P_m \cap GL_m(\mathbb{R})^+$. From Proposition 4.3 it follows that

$$\hat{\pi} = \mathrm{Ind}_{P_m}^{GL_m(\mathbb{R})}(\chi_2^{-1}\chi_1 \times \chi_2 \circ \det) \qquad \text{and} \qquad \hat{\pi}' = \mathrm{Ind}_{P_m}^{GL_m(\mathbb{R})}({\chi_2'}^{-1}\chi_1' \times \chi_2' \circ \det).$$

Consequently, from Mackey's subgroup theorem it follows that

$$\hat{\pi}_{|GL_m(\mathbb{R})^+} = \mathrm{Ind}_{P_m^+}^{GL_m(\mathbb{R})^+}(\chi_2^{-1}\chi_1 \times \chi_2 \circ \det)$$

and

$$\hat{\pi}'_{|GL_m(\mathbb{R})^+} = \mathrm{Ind}_{P_m^+}^{GL_m(\mathbb{R})^+}({\chi_2'}^{-1}\chi_1' \times \chi_2' \circ \det).$$

Equalities in (4.5) imply that $\chi_2^{-1}\chi_1 \times \chi_2 \circ \det$ and ${\chi_2'}^{-1}\chi_1' \times \chi_2' \circ \det$ are identical characters of P_m^+; i.e., restrictions of $\hat{\pi}$ and $\hat{\pi}'$ to $GL_m(\mathbb{R})^+$ are isomorphic. This proves Lemma 4.4b. $\square$

5. Proof of Theorem 2.1

It is easy to see that the only irreducible unitary representation of G of rank zero is the trivial representation. It was shown in [Sa2, Prop. 4] that the only irreducible unitary representation of G of rank one is the *minimal* representation of G. Our concentration throughout the rest of this manuscript will be on irreducible unitary representations of G of rank two.

It was shown in [KS] that the minimal representation of G is irreducible when restricted to the Heisenberg parabolic subgroup. Our next task is to prove a similar, but much stronger version of this fact, for irreducible unitary representations of rank two. Namely, we will show that for irreducible unitary representations of rank two, the restriction of π to the parabolic subgroup R determines π uniquely. Our method of proof is an adaptation of an idea originally due to Howe [Ho].

The parabolic subgroup Q can be expressed as

$$Q = (\mathbb{R}^+ \times (\{\pm 1\} \ltimes [M, M])) \ltimes H, \tag{5.1}$$

where $\mathbb{R}^+$ and $\{\pm 1\}$ are appropriate subgroups of A. (Note the positions of direct and semidirect products.) The element $-1 \in \{\pm 1\}$ corresponds to the element $a \in A$ such that for G of type $\mathbf{E}_6$ we have $e^{\varpi_2}(a) = -1$ and $e^{\varpi_j}(a) = 1$ for any $j \neq 2$, and for G of type $\mathbf{E}_7$ we have $e^{\varpi_1}(a) = -1$ and $e^{\varpi_j}(a) = 1$ for any $j \neq 1$.

Throughout this section, we assume that π is a (possibly reducible) unitary representation of G of rank two. Let ρ be any infinite-dimensional irreducible unitary representation of H. Note that ρ extends to a unitary representation $\hat{\rho}$ of $[Q, Q]$, and in fact this extension is unique since $[M, M]$ is a perfect group. Since the action of Q on the center of H has only one open orbit and the stabilizer of every point of this orbit is $[Q, Q]$, it follows from [Sa1, §5.2] that one can express the restriction of π to Q as

$$\pi_{|Q} = \mathrm{Ind}_{[Q,Q]}^{Q}(\nu \otimes \hat{\rho}),$$

where ν is a unitary representation of $[M, M] = [Q, Q]/H$. Let Q^+ be the connected component of the identity in Q. Then we have

$$Q^+ = (\mathbb{R}^+ \times [M, M]) \ltimes H.$$

One can extend ν to the unitary representation $1 \otimes \nu$ of $\mathbb{R}^+ \times [M, M]$ and therefore obtain an extension $\hat{\nu}$ of ν to Q^+. (Note that it may not necessarily be possible to extend ν to all of Q.) Using the projection formula we have

$$\mathrm{Ind}_{[Q,Q]}^{Q}(\nu \otimes \hat{\rho}) = \mathrm{Ind}_{Q^+}^{Q}\mathrm{Ind}_{[Q,Q]}^{Q^+}(\nu \otimes \hat{\rho}) = \mathrm{Ind}_{Q^+}^{Q}(\hat{\nu} \otimes \mathrm{Ind}_{[Q,Q]}^{Q^+}\hat{\rho}).$$

Let $\tau = \hat{\nu} \otimes \mathrm{Ind}_{[Q,Q]}^{Q^+}\hat{\rho}$ and $\eta = \mathrm{Ind}_{[Q,Q]}^{Q^+}\hat{\rho}$. If $\mathcal{H}_1$ and $\mathcal{H}_2$ denote the Hilbert spaces of the unitary representations $\hat{\nu}$ and η, then τ is a unitary representation with Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Lemma 5.2. *The von Neumann algebra generated by $\tau(Q^+ \cap R)$ is equal to the von Neumann algebra generated by $\tau(Q^+)$.*

Proof. The key point is that the restriction of η to the subgroup $\mathbb{R}^+ \ltimes H$ of Q^+ is irreducible (see [KS]). The idea is that the H-spectrum of η is multiplicity-free and $\mathbb{R}^+$ acts on it transitively. Since the restriction of $\hat{\nu}$ to $\mathbb{R}^+ \ltimes H$ is trivial, it follows that the von Neumann algebra generated by $\tau(\mathbb{R}^+ \ltimes H)$ contains every operator of the form $I \otimes T$ inside $\mathrm{End}(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

Let $\mathcal{A}$ be the von Neumann algebra generated by $\tau(Q^+ \cap R)$. Since $Q^+ \cap R \supset \mathbb{R}^+ \ltimes H$, every operator of the form $I \otimes T$ inside $\mathrm{End}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ lies inside $\mathcal{A}$ as well. Moreover, for any $q \in Q^+ \cap R$, we have $\tau(q) = \hat{\nu}(q) \otimes \eta(q)$. Consequently, $\mathcal{A}$ contains all of the operators of the form $\hat{\nu}(q) \otimes T$ for $q \in Q^+ \cap R$, where T is any arbitrary element of $\mathrm{End}(\mathcal{H}_2)$.

Next we observe that $\hat{\nu}$ is actually a unitary representation of rank one of the reductive group $M^+ = Q^+/H$. But for classical groups it can be shown that the two notions of rank in [Li2] and in [Sa1] are essentially equivalent (see [Sa1, §6]). Therefore from [Li2, Theorem 4.5] and Lemma 4.1 it follows that the von Neumann algebra generated by $\hat{\nu}(Q^+ \cap R)$ is equal to the von Neumann algebra generated by $\hat{\nu}(Q^+)$. (When G is of type $\mathbf{E}_6$, the latter statement follows immediately from Proposition 4.1. When G is of type $\mathbf{E}_7$, it follows from the fact that the von Neumann algebra generated by $\hat{\nu}(Q^+ \cap R)$ contains the von Neumann algebra generated by

$$\hat{\nu}([Q,Q] \cap R),$$

which by [Li2, Theorem 4.5] is equal to the von Neumann algebra generated by $\hat{\nu}([Q,Q])$.) This implies that $\mathcal{A}$ contains all the operators of the form $\hat{\nu}(q) \otimes T$ for every $q \in Q^+$, which proves Lemma 5.2. □

Lemma 5.3. *The von Neumann algebra generated by $\pi(Q)$ is equal to the von Neumann algebra generated by $\pi(Q \cap R)$.*

Proof. Let $\sigma = \mathrm{Ind}_{Q^+}^{Q} \tau$. We have $\pi_{|Q} = \sigma$. By the double commutant theorem, it suffices to show that every $Q \cap R$-intertwining operator for σ is actually a Q-intertwining operator. Let $\mathcal{H}_\tau$ be the Hilbert space of τ. Recall that $-1 \in \{\pm 1\} \subset Q$ (see equation (5.1)). One can realize σ on $\mathcal{H}_\tau \oplus \mathcal{H}_\tau$ as

$$\begin{aligned} \sigma(q)(v \oplus w) &= \tau(q)v \oplus \overline{\tau}(q)w \qquad \text{for any} \quad q \in Q^+, \\ \sigma(-1)(v \oplus w) &= w \oplus v, \end{aligned}$$

where $\overline{\tau}$ is the unitary representation of Q^+ on $\mathcal{H}_\tau$ obtained by twisting by -1, i.e.,

$$\overline{\tau}(q) = \tau(-1 \cdot q \cdot -1) \quad \text{for any} \quad q \in Q^+.$$

Any element of $\mathrm{End}(\mathcal{H}_\tau \oplus \mathcal{H}_\tau)$ can be represented by a matrix

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix},$$

where T_1, T_2, T_3, T_4 are elements of $\mathrm{End}(\mathcal{H}_\tau)$. Let T be a $Q \cap R$-intertwining operator. The fact that T commutes with $\sigma(-1)$ implies that $T_1 = T_4$ and $T_2 = T_3$. The fact that T commutes with the action of $\sigma(q)$ for $q \in Q^+ \cap R$ implies that T_2 is a $Q^+ \cap R$-interwining operator between τ and $\overline{\tau}$ and T_1 is a $Q^+ \cap R$-intertwining operator of τ. However, the restrictions of τ and $\overline{\tau}$ to H are disjoint (one of $\tau, \overline{\tau}$ is supported on unitary representations of H with "positive" central character whereas the other one is supported on unitary representations with "negative" central character). Therefore T_2 should be zero. On the other hand, since by Lemma 5.2 the von Neumann algebras generated by $\tau(Q^+ \cap R)$ and $\tau(Q^+)$ are the same, it follows that T_1 is a Q^+-intertwining operator of τ as well. Consequently, T is of the form

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_1 \end{bmatrix},$$

which implies that T is a Q-intertwining operator for σ. □

Proposition 5.4. *Let π be a (possibly reducible) unitary representation of G of rank two on a Hilbert space $\mathcal{H}$. Then the von Neumann algebra (inside* $\mathrm{End}(\mathcal{H})$*) generated by $\pi(R)$ is identical to the von Neumann algebra generated by $\pi(G)$.*

Proof. The von Neumann algebra generated by $\pi(R)$ contains the von Neumann algebra generated by $\pi(Q \cap R)$, which by Lemma 5.3 is equal to the von Neumann algebra generated by $\pi(Q)$. Since Q and R are both maximal parabolic subgroups, the group generated by them is equal to G. Therefore the von Neumann algebra generated by $\pi(R)$ contains the von Neumann algebra generated by $\pi(G)$. □

Corollary 5.5. *If π is an irreducible unitary representation of G of rank two, then the restriction of π to R is irreducible and uniquely determines π.*

We are now able to apply the standard machinery of Mackey [Mac] to our situation. Let π be an irreducible unitary representation of G of rank two. Recall that the Levi factorisation of R is $R = S \ltimes U$. The group $[S,S]$ is the set of $\mathbb{R}$-points of a complex simple simply connected algebraic group which is defined and split over $\mathbb{R}$ and whose root system is of type $\mathbf{D}_4$ when G is of type $\mathbf{E}_6$ and of type $\mathbf{A}_1 \times \mathbf{D}_5$ when G is of type $\mathbf{E}_7$. Therefore we have

$$[S,S] = \begin{cases} Spin(4,4) & \text{if } G \text{ is of type } \mathbf{E}_6, \\ SL_2(\mathbb{R}) \times Spin(5,5) & \text{if } G \text{ is of type } \mathbf{E}_7. \end{cases}$$

Our next task is to understand the restriction of π to U. Recall that U is two-step nilpotent. Let $\mathcal{Z}(U)$ denote the center of U. Let σ be an irreducible unitary representation of U. For every element $z \in \mathcal{Z}(U)$, $\sigma(z)$ is a scalar. If $\mathcal{Z}(U) \subseteq \ker(\sigma)$, then σ should be one-dimensional. Now suppose $\mathcal{Z}(U) \nsubseteq \ker(\sigma)$. The group $\mathcal{Z}(U)$ is invariant under the action of S, and one can see that the action of the spin factor of $[S,S]$ on the Lie algebra $\mathcal{Z}(\mathfrak{u})$ is in fact identical to the standard representation $\mathbb{R}^{n-2,n-2}$ of $Spin(n-2,n-2)$. When G is of type $\mathbf{E}_7$, the factor $SL_2(\mathbb{R})$ of $[S,S]$ acts on $\mathcal{Z}(\mathfrak{u})$ trivially.

From the existence of a nondegenerate $Spin(n-2,n-2)$-invariant bilinear form $\langle \cdot,\cdot \rangle$ on $\mathbb{R}^{n-2,n-2}$, it follows that for any unitary character χ of $\mathcal{Z}(U)$ there exists an element $v \in \mathcal{Z}(\mathfrak{u})$ such that

$$\chi(x) = e^{\langle v, \log x \rangle \sqrt{-1}} \qquad \text{for any } x \in \mathcal{Z}(U). \tag{5.6}$$

Here "$\log x$" means the inverse of the exponential map $\exp : \mathcal{Z}(\mathfrak{u}) \mapsto \mathcal{Z}(U)$. From $\mathcal{Z}(U) \nsubseteq \ker(\sigma)$ it follows that $v \neq 0$. The action of S on $\mathcal{Z}(\mathfrak{u})$ has three orbits: an open orbit, the set of nonzero isotropic vectors, and the origin (see [HT] and [Ka]).

Definition 5.7. Consider an irreducible unitary representation of U whose restriction to $\mathcal{Z}(U)$ acts by a character of the form given in (5.6) for some $v \neq 0$. We call this representation small if v is a nonzero isotropic vector, and we call it big if v belongs to the open orbit.

Suppose G is of type $\mathbf{E}_n$. If σ is a big representation of U, then $U/\ker(\sigma)$ is a Heisenberg group of dimension $2^{n-2}+1$, whereas if σ is small, then $U/\ker(\sigma)$ is a direct product of the additive group of $\mathbb{R}^{2^{n-3}}$ and a Heisenberg group of dimension $2^{n-3}+1$.

Recall that $\mathfrak{u}$ and $\mathfrak{n}_3$ represent the Lie algebras of U and N_3, respectively. As a vector space, we can write $\mathfrak{u}$ as a direct sum

$$\mathfrak{u} = \mathfrak{X} \oplus \mathfrak{Y} \oplus \mathcal{Z}(\mathfrak{u}) \tag{5.8}$$

where:

a. Each of $\mathfrak{X}$ and $\mathfrak{Y}$ is a direct sum of certain root spaces $\mathfrak{g}_\alpha$.
b. $\mathfrak{X} = \mathfrak{u} \cap \mathfrak{l}$.
c. $\mathcal{Z}(\mathfrak{u})$ is the center of $\mathfrak{u}$.

Note that these conditions identify $\mathfrak{Y}$ uniquely. For an explicit description of this decomposition, see section 10. Similarly, we can write $\mathfrak{n}_3$ as a direct sum

$$\mathfrak{n}_3 = \mathfrak{W} \oplus \mathfrak{W}^* \oplus \mathcal{Z}(\mathfrak{n}_3)$$

where:

a. Each of $\mathfrak{W}$ and $\mathfrak{W}^*$ is a direct sum of certain root spaces $\mathfrak{g}_\alpha$.
b. $\mathfrak{W} = \mathfrak{n}_3 \cap \mathfrak{l}$.
c. $\mathcal{Z}(\mathfrak{n}_3)$ is the center of $\mathfrak{n}_3$.

Again $\mathfrak{W}^*$ is uniquely identified (see section 10). In fact $\mathfrak{W}$ and $\mathfrak{W}^*$ correspond to a polarization of the symplectric vector space $\mathfrak{n}_3/\mathcal{Z}(\mathfrak{n}_3)$.

The proof of the next lemma is given in section 10.

Lemma 5.9. *The Lie algebra $\mathcal{Z}(\mathfrak{u})$ contains $\mathfrak{g}_{\beta_1}$ and $\mathfrak{g}_{\beta_2}$. For any root $\alpha \notin \{\beta_1, \beta_2\}$, if $\mathfrak{g}_\alpha \subseteq \mathcal{Z}(\mathfrak{u})$, then $\mathfrak{g}_\alpha \subseteq \mathfrak{W}^*$. Moreover, there exist bases*

$$\{e_1, ..., e_{n-2}, e_{-1}, ..., e_{-(n-2)}\} \text{ and } \{f_2, ..., f_{n-2}, f_{-2}, ..., f_{-(n-2)}\}$$

of $\mathcal{Z}(\mathfrak{u})$ and $\mathfrak{W} \cap \mathfrak{s}$, respectively, such that

a. *For any $i \in \{-(n-2), ..., n-2\}$, there exist $\alpha_1, \alpha_2 \in \Delta^+$ such that $e_i \in \mathfrak{g}_{\alpha_1}$ and $f_i \in \mathfrak{g}_{\alpha_2}$.*
b. *$e_1 \in \mathfrak{g}_{\beta_1}$ and $e_{-1} \in \mathfrak{g}_{\beta_2}$.*
c. *For any $i, j \in \{2, ..., n-2\}$, $[e_{\pm i}, f_{\pm j}] = \delta_{\pm i, \pm j} e_1$.*
d. *For any $i, j \in \{1, ..., n-2\}$, we have*

$$\langle e_i, e_j \rangle = \langle e_{-i}, e_{-j} \rangle = 0 \qquad \text{and} \qquad \langle e_i, e_{-j} \rangle = \delta_{ij}.$$

e. *For any root α, if $\mathfrak{g}_\alpha \subseteq \mathfrak{n}_3$ and $\mathfrak{g}_\alpha \nsubseteq \mathrm{Span}_{\mathbb{R}}\{e_i, f_i \mid -(n-2) \le i \le n-2\}$, then $[e_{-1}, \mathfrak{g}_\alpha] = 0$.*

Proposition 5.10. *Let π be an irreducible unitary representation of G of rank two. Then the restriction of π to U is supported on big representations of U.*

Proof. Since $\mathcal{Z}(U) \subset N_\Gamma$, it suffices to prove that the restriction of any rankable representation of N_Γ of rank two to $\mathcal{Z}(U)$ is a direct integral of characters of the form given in equation (5.6) for which v belongs to the open S-orbit of $\mathcal{Z}(\mathfrak{u})$. Let $\tilde{\rho}_1$ be a rankable representation of N_Γ of rank one, obtained by extending an irreducible unitary representation of $N_3 = H$. By Lemma 5.9 for any $i, j \in \{\pm 2, \dots, \pm(n-2)\}$ we have

$$\langle e_j, [e_{-1}, f_i] \rangle = -\langle e_j, [f_i, e_{-1}] \rangle = \langle [f_i, e_j], e_{-1} \rangle = \langle -\delta_{i,j} e_1, e_{-1} \rangle = -\delta_{i,j},$$

which implies that $[e_{-1}, f_i] = -e_{-i}$. Using Lemma 5.9e and the formulas which describe the oscillator representation (see [Ho, §1]), one can see that the restriction of $\tilde{\rho}_1$ to $\mathcal{Z}(U)$ is a direct integral of characters of the form given in (5.6) for

$$v = te_1 + \sum_{i=2}^{n-2} (ta_i e_i + ta_{-i} e_{-i}) - (\sum_{i=2}^{n-2} a_i a_{-i} t) e_{-1},$$

where t and the a_i's are real numbers and $t \neq 0$. Obviously $\langle v, v \rangle = 0$; i.e., v is an isotropic vector. From Lemma 5.9e and the oscillator representation it also follows that the restriction of a rankable representation of rank two of N_Γ to $\mathcal{Z}(U)$ is a direct integral of characters for which v is given by

$$v = te_1 + \sum_{i=2}^{n-2} (ta_i e_i + ta_{-i} e_{-i}) - (s + \sum_{i=2}^{n-2} a_i a_{-i} t) e_{-1},$$

where t and the a_i's are as above and s is a nonzero real number. Obviously $\langle v, v \rangle \neq 0$. $\square$

Let π be an irreducible unitary representation of G of rank two. Let σ be the irreducible unitary representation of U with central character given by (5.6) where $v = e_1 + e_{-1}$. By Mackey theory, one can write the restriction of π to R as

$$\pi_{|R} = \mathrm{Ind}_{R_1}^{R} \eta,$$

where $R_1 = \mathrm{Stab}_S(e_1 + e_{-1}) \ltimes U$ and η is an irreducible unitary representation of R_1 with the property that for some $m \in \{0, 1, 2, \ldots, \infty\}$, $\mathrm{Res}_U^{R_1} \eta = m\sigma$. If σ can be extended to a unitary representation $\hat{\sigma}$ of R_1, then η can be written as a tensor product $\eta = \tau \otimes \hat{\sigma}$, where $\hat{\sigma}$ is the extension of σ to R_1 and τ is an irreducible unitary representation of $\mathrm{Stab}_S(e_1 + e_{-1})$ which is extended (trivially on U) to R_1. The unitary representation τ is uniquely determined by $\pi_{|R}$, and by Corollary 5.5 we obtain an injection

$$\Psi : \pi \mapsto \Psi(\pi), \tag{5.11}$$

where $\Psi(\pi) = \tau$, from $\Pi_2(G)$ into the unitary dual of $\mathrm{Stab}_S(e_1 + e_{-1})$.

From now on, we set

$$S_1 = \mathrm{Stab}_S(e_1 + e_{-1}). \tag{5.12}$$

The next proposition justifies the existence of the extension $\hat{\sigma}$.

Proposition 5.13. *Let σ be the big representation of U associated to the character given by* (5.6)*, where $v = e_1 + e_{-1}$. Then σ can be extended to a unitary representation $\hat{\sigma}$ of R_1. The extension is unique when G is of type $\mathbf{E}_7$.*

Proof. It is well known that for the action of $Spin(2k, \mathbb{C})$ on $\mathbb{C}^{2k}$, if $w \in \mathbb{C}^{2k}$ lies outside the variety of isotropic vectors, then

$$\mathrm{Stab}_{Spin(2k,\mathbb{C})}(w) = Spin(2k-1, \mathbb{C}). \tag{5.14}$$

From (5.14) and some elementary calculations it follows that when G is of type $\mathbf{E}_6$,

$$S_1 = \mathbb{R}^{\times} \ltimes Spin(3, 4),$$

where the element $-1 \in \mathbb{R}^{\times}$ is an element of A given by $e^{\varpi_1}(a) = e^{\varpi_6}(a) = -1$ and $e^{\varpi_j}(a) = 1$ for any $j \in \{2, 3, 4, 5\}$, and when G is of type $\mathbf{E}_7$,

$$S_1 = SL_2(\mathbb{R}) \times Spin(4, 5).$$

The group $\tilde{U} = U/\ker(\sigma)$ is a Heisenberg group and S_1 is a subgroup of the group of automorphisms of $\tilde{U}$ which fix $\mathcal{Z}(\tilde{U})$ pointwise. This means that S_1 acts on $\tilde{U}/\mathcal{Z}(\tilde{U})$ as a subgroup of $Sp(\tilde{U}/\mathcal{Z}(\tilde{U}))$. In other words, there exists a Lie group homomorphism

$$S_1 \mapsto Sp(\tilde{U}/\mathcal{Z}(\tilde{U})). \tag{5.15}$$

Let $Mp(\tilde{U}/\mathcal{Z}(\tilde{U}))$ denote the metaplectic cover of $Sp(\tilde{U}/\mathcal{Z}(\tilde{U}))$. The extension $\hat{\sigma}$ is obtained as a restriction of the oscillator representation of $Mp(\tilde{U}/\mathcal{Z}(\tilde{U})) \ltimes \tilde{U}$. However, the oscillator representation is a genuine representation of $Mp(\tilde{U}/\mathcal{Z}(\tilde{U}))$, and therefore the group homomorphism in (5.15) is not enough to prove the existence of $\hat{\sigma}$. Next we show that the homomorphism given in equation (5.15) is actually equal to the composition $\varphi_2 \circ \varphi_1$ of two group homomorphisms φ_1 and φ_2, where

$$\varphi_1 : S_1 \mapsto Mp(\tilde{U}/\mathcal{Z}(\tilde{U}))$$

and

$$\varphi_2 : Mp(\tilde{U}/\mathcal{Z}(\tilde{U})) \mapsto Sp(\tilde{U}/\mathcal{Z}(\tilde{U})).$$

Here φ_2 is the standard covering map. The key idea in the proof is to use the explicit formulas of the oscillator representation on the Siegel parabolic subgroup (see [Ho, §1]).

When G is of type $\mathbf{E}_6$ the action of $\mathbb{R}^\times \ltimes Spin(3,4)$ leaves $\mathfrak{X}$ and $\mathfrak{Y}$ invariant. In fact, if $GL(\mathfrak{X})$ denotes the Levi factor of the Siegel parabolic subgroup of $Sp(\tilde{U}/\mathcal{Z}(\tilde{U}))$, then $\mathbb{R}^\times \ltimes Spin(3,4)$ is a subgroup of the connected component of the identity in $GL(\mathfrak{X})$. The existence of the group homomorphism φ_1 follows from formulas in [Ho, §1] for the oscillator representation on the Siegel parabolic subgroup.

When G is of type $\mathbf{E}_7$ the situation is only slightly more complicated. Recall that

$$S_1 = SL_2(\mathbb{R}) \times Spin(4,5).$$

Similar to when G is of type $\mathbf{E}_6$, $Spin(4,5)$ leaves the vector spaces $\mathfrak{X}$ and $\mathfrak{Y}$ invariant. Therefore $Spin(4,5)$ can be considered as a subgroup of the connected component of the identity in $GL(\mathfrak{X})$. Again formulas of the oscillator representation on the Siegel parabolic subgroup imply that the group homomorphism

$$Spin(4,5) \mapsto Sp(\tilde{U}/\mathcal{Z}(\tilde{U}))$$

breaks into the composition of two group homomorphisms

$$Spin(4,5) \mapsto Mp(\tilde{U}/\mathcal{Z}(\tilde{U})) \mapsto Sp(\tilde{U}/\mathcal{Z}(\tilde{U})). \tag{5.16}$$

The action of $SL_2(\mathbb{R})$ does not preserve the polarization $\mathfrak{X} \oplus \mathfrak{Y}$ of $\tilde{U}$. However, one can choose a different polarization which is preserved by the action of $SL_2(\mathbb{R})$. This polarization can be described as follows. Let Ω be the set of roots $\alpha \in \Delta^+$ such that $\mathfrak{g}_\alpha \subset \mathfrak{X} \oplus \mathfrak{Y}$. Then Ω can be partitioned into a disjoint union

$$\Omega = \bigsqcup_{t=1}^{8} \Omega_t$$

where for any t, Ω_t has four elements and moreover, elements of each Ω_t can be ordered such that we have

$$\Omega_t = \{\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \alpha^{(4)}\}, \tag{5.17}$$

where the following two conditions hold:

a. For some $j \in \{1,2\}$ depending on t, we have

$$\alpha^{(2)} = \alpha^{(1)} + \alpha_7, \alpha^{(3)} = \beta_j - \alpha^{(2)} \text{ and } \alpha^{(4)} = \alpha^{(3)} + \alpha_7.$$

b. $\mathfrak{g}_{\alpha^{(1)}} \oplus \mathfrak{g}_{\alpha^{(3)}} \subseteq \mathfrak{X}$ and $\mathfrak{g}_{\alpha^{(2)}} \oplus \mathfrak{g}_{\alpha^{(4)}} \subseteq \mathfrak{Y}$.

A polarization preserved by $SL_2(\mathbb{R})$ has the form $\mathfrak{X}_1 \oplus \mathfrak{Y}_1$, where for every $1 \leq t \leq 8$, if Ω_t is sorted as in (5.17), then $\mathfrak{X}_1$ contains the direct sum $\mathfrak{g}_{\alpha^{(1)}} \oplus \mathfrak{g}_{\alpha^{(2)}}$ and $\mathfrak{Y}_1$ contains the direct sum $\mathfrak{g}_{\alpha^{(3)}} \oplus \mathfrak{g}_{\alpha^{(4)}}$. For the reader's convenience, we give one such polarization explicitly in section 10.

Since $SL_2(\mathbb{R})$ preserves a polarization, we can argue as above and see that the group homomorphism

$$SL_2(\mathbb{R}) \mapsto Sp(\tilde{U}/\mathcal{Z}(\tilde{U}))$$

breaks into the composition of two group homomorphisms

$$SL_2(\mathbb{R}) \mapsto Mp(\tilde{U}/\mathcal{Z}(\tilde{U})) \mapsto Sp(\tilde{U}/\mathcal{Z}(\tilde{U})). \tag{5.18}$$

Next we show that the group homomorphism

$$SL_2(\mathbb{R}) \times Spin(4,5) \mapsto Sp(\tilde{U}/\mathcal{Z}(\tilde{U}))$$

breaks into the composition of two group homomorphisms

$$SL_2(\mathbb{R}) \times Spin(4,5) \mapsto Mp(\tilde{U}/\mathcal{Z}(\tilde{U})) \mapsto Sp(\tilde{U}/\mathcal{Z}(\tilde{U})).$$

Let ϕ_1 and ϕ_2 be the maps from $Spin(4,5)$ and $SL_2(\mathbb{R})$ into $Mp(\tilde{U}/\mathcal{Z}(\tilde{U}))$ given in (5.16) and (5.18), respectively. Consider the map

$$\Phi : Spin(4,5) \times SL_2(\mathbb{R}) \mapsto Mp(\tilde{U}/\mathcal{Z}(\tilde{U}))$$

given by $\Phi(a \times b) = \phi_1(a)\phi_2(b)$. We show that the map Φ is the appropriate group homomorphism from S_1 to $Mp(\tilde{U}/\mathcal{Z}(\tilde{U}))$. Continuity of Φ is obvious. To show that Φ is a group homomorphism, it suffices to show that $\phi_1(a)$ and $\phi_2(b)$ commute. But the images of $\phi_1(a)$ and $\phi_2(b)$ inside $Sp(\tilde{U}/\mathcal{Z}(\tilde{U}))$ commute with each other, and since $Spin(4,5) \times SL_2(\mathbb{R})$ is connected, the commutator of $\phi_1(a)$ and $\phi_2(b)$ should be a constant function. Checking for when a and b are the identity elements implies that this commutator is equal to the identity element of $Mp(\tilde{U}/\mathcal{Z}(\tilde{U}))$.

The uniqueness of $\hat{\sigma}$ follows from the fact that the group $SL_2(\mathbb{R}) \times Spin(4,5)$ is perfect. $\square$

6. A class of unitary representations of rank two

For any irreducible unitary representation σ of a nilpotent simply connected Lie group, let $\mathcal{O}_\sigma$ be the coadjoint orbit associated to σ (see [Ki1], [Ki2]). Recall the following elementary results from Kirillov's orbit method.

Proposition 6.1. *Let $N^1 \subset N^2$ be nilpotent simply connected Lie groups and assume N^1 is a Lie subgroup of N^2 of codimension n.*

a. *If σ^1 is an irreducible unitary representation of N^1, then the support of $\mathrm{Ind}_{N^1}^{N^2}\sigma^1$ lies inside irreducible unitary representations σ of N^2 for which*

$$\dim(\mathcal{O}_\sigma) \leq 2n + \dim(\mathcal{O}_{\sigma^1}).$$

b. *If σ is an irreducible unitary representation of N^2, then the support of $\mathrm{Res}_{N^1}^{N^2}\sigma$ lies inside irreducible unitary representations σ^1 of N^1 for which*

$$\dim(\mathcal{O}_{\sigma^1}) \leq \dim(\mathcal{O}_\sigma).$$

Proposition 6.2. *The degenerate principal series representations π_χ of G are of rank two.*

Proof. Recall that $N_B = [B,B]$. From the Bruhat decomposition it follows that in the double coset space $N_B \backslash G/\overline{P}$, $N_B\overline{P}$ has full measure. Therefore by Mackey's subgroup theorem,

$$\mathrm{Res}_{N_B}^G \pi_\chi = \mathrm{Res}_{N_B}^G \mathrm{Ind}_{\overline{P}}^G \chi = \mathrm{Ind}_{N_B \cap \overline{P}}^{N_B} \chi.$$

Let $U_1 = N_B \cap \overline{P}$. Since $U_1 \subset [\overline{P}, \overline{P}]$, the restriction of χ to U_1 is the trivial representation. Therefore

$$\mathrm{Res}_{N_B}^G \pi_\chi = \mathrm{Ind}_{U_1}^{N_B} 1.$$

Recall that β_1 is the highest root of $\mathfrak{g}$. Let G_{β_1} be the one-parameter unipotent subgroup of G which corresponds to $\mathfrak{g}_{\beta_1}$. One can see that $U_2 = U_1 G_{\beta_1}$ is actually a Lie subgroup of N_B and $U_2 \approx U_1 \times G_{\beta_1}$. If e denotes the identity element of G, then

$$\mathrm{Ind}_{U_1}^{U_2} 1 = \mathrm{Ind}_{U_1}^{U_1} \hat{\otimes} \mathrm{Ind}_{\{e\}}^{G_{\beta_1}} 1 = 1 \hat{\otimes} L^2(G_{\beta_1}, dg_{\beta_1}),$$

where $L^2(G_{\beta_1}, dg_{\beta_1})$ is the left regular representation of G_{β_1}. Consequently, $\mathrm{Ind}_{U_1}^{U_2} 1$ is actually a direct integral of one-dimensional unitary representations of U_2. Any one-dimensional unitary representation is associated to a coadjoint orbit of dimension zero. Proposition 6.1 and the relation

$$\mathrm{Ind}_{U_1}^{N_B} 1 = \mathrm{Ind}_{U_2}^{N_B} \mathrm{Ind}_{U_1}^{U_2} 1$$

imply that $\mathrm{Ind}_{U_1}^{N_B} 1$ (respectively $\mathrm{Res}_{N_\Gamma}^{G} \pi_\chi = \mathrm{Res}_{N_\Gamma}^{N_B} \mathrm{Ind}_{U_1}^{N_B} 1$) is supported on irreducible unitary representations of N_B (respectively N_Γ) whose coadjoint orbits have dimensions at most twice the codimension of U_2 in N_B. A simple calculation shows that the codimension of U_2 in N_B is equal to 15 (respectively 26) when G is of type $\mathbf{E}_6$ (respectively when G is of type $\mathbf{E}_7$). However, by [Sa1, Cor. 4.2.3], the dimension of the coadjoint orbit associated to a rankable representation of N_Γ of rank 3 is 32 when G is of type $\mathbf{E}_6$ and 56 when G is of type $\mathbf{E}_7$. But $2 \times 15 < 32$ and $2 \times 26 < 56$(!), which imply that π_χ should be of rank at most two. A comparison of K-types and the uniqueness of the minimal representation [Sa2, Prop 14] justify that π_χ is not of rank one. $\square$

Proposition 6.2 can actually be extended to include the complementary series representations π_s and the representation π° which appear when G is of type $\mathbf{E}_7$. Note that the proof of Proposition 6.2 is not applicable to the case of complementary series.

Proposition 6.3. *π° and π_s (for $0 \leq s < 1$) are of rank two.*

Proof. We use a construction of these representations given in [BSZ]. First consider a complementary series representation π_s. Using [BSZ, Corollary 8.7] one can describe the $\overline{P}$-action of π_s on $L^2(\mathfrak{n}, \nabla(X)^{-s} d_s X)$, where ∇ is a cubic L-semi-invariant polynomial and $d_s X$ is a normalization of the Lebesgue measure. The Fourier transform gives an isometry

$$\widehat{\ } : L^2(\mathfrak{n}, d_s X) \mapsto L^2(\overline{\mathfrak{n}}, d_s \overline{X}),$$

where $d_s \overline{X}$ is an appropriate normalization of the Lebesgue measure on $\overline{\mathfrak{n}}$. The isometry $f \mapsto \widehat{f \nabla^{-\frac{s}{2}}}$ from $L^2(\mathfrak{n}, \nabla(X)^{-s} d_s X)$ to $L^2(\overline{\mathfrak{n}}, d_s \overline{X})$ is an intertwining operator between the action of $\overline{P}$ on $L^2(\mathfrak{n}, \nabla(X)^{-s} d_s X)$ given in [BSZ, (8.5)] and its action on $L^2(\overline{\mathfrak{n}}, d_s \overline{X})$ given as follows:

$$e^{\overline{Y}} \cdot f(\overline{X}) = f(\overline{X} + \overline{Y}) \qquad \text{for every } \overline{X}, \overline{Y} \in \overline{\mathfrak{n}}, \tag{6.4}$$

$$l \cdot f(\overline{X}) = \delta_P(l)^{\frac{1}{2}} f(l^{-1} \cdot \overline{X}) \qquad \text{for every } l \in L, \overline{X} \in \overline{\mathfrak{n}}. \tag{6.5}$$

Note that these formulas are independent of s. From the description of the unitary principal series in the "noncompact" picture in [Kn2, Chapter VII, (7.3b)] it follows that for any unitary character χ, $\pi_{\chi|\overline{P}}$ and $\pi_{s|\overline{P}}$ are isomorphic. It follows that

$$\mathrm{Res}_P^G \pi_\chi = \mathrm{Res}_P^G \pi_s.$$

Since $N_\Gamma \subset P$, the rank of π_s is equal to the rank of π_χ.

Next we prove that π° is of rank two. Our proof actually shows something slightly stronger: we prove that the (spectrum of the) restriction of π° to $U \cap N$ is multiplicity-free. Note that this is false for any unitary representation of G of rank three, as it already fails for a rankable representation of N_Γ of rank three.

We can conjugate $N \cap U$ by the longest Weyl element and obtain its "opposite" group, $\overline{N} \cap \overline{U}$. It suffices to prove that the $\overline{N} \cap \overline{U}$-spectrum of π° is multiplicity-free. We use [BSZ, Theorem 8.11].

The Lie algebra of $N \cap U$ is equal to $\mathfrak{Y} \oplus \mathcal{Z}(\mathfrak{u})$, with $\mathfrak{Y}$ as in (5.8), and we have

$$\mathfrak{n} = \mathfrak{Y} \oplus \mathcal{Z}(\mathfrak{u}) \oplus \mathfrak{g}_{\alpha_7}.$$

If we think of $\mathfrak{n}$ as the Jordan algebra $\mathrm{Herm}(3, \mathbb{O}_{\mathrm{split}})$, then from results in [BSZ, §2] it follows that $\mathfrak{Y}$, $\mathcal{Z}(\mathfrak{u})$ and $\mathfrak{g}_{\alpha_7}$ correspond to matrices of the forms

$$\begin{bmatrix} 0 & v & w \\ v^* & 0 & 0 \\ w^* & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} t_1 & u & 0 \\ u^* & t_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t_3 \end{bmatrix}, \tag{6.6}$$

respectively.

Let $\mathcal{O}_2$ be the L-orbit of

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

inside $\mathfrak{n}$. From [BSZ, Table 2] and [BSZ, Lemma 2.10] it follows that there exists a cubic L-semi-invariant polynomial ∇ on $\mathfrak{n}$. By [BSZ, Theorem 8.11], the restriction of π° to $\overline{P}$ can be realized on $L^2(\mathcal{O}_2, d\nu_2)$, where $d\nu_2$ is the L-quasi-invariant measure of $\mathcal{O}_2$. The closure of $\mathcal{O}_2$ in $\mathfrak{n}$ is equal to the vanishing set of ∇ in $\mathfrak{n}$.

Write $\mathrm{Herm}(3, \mathbb{O}_{\mathrm{split}})$ as

$$\mathrm{Herm}(3, \mathbb{O}_{\mathrm{split}}) = \mathrm{Herm}(2, \mathbb{O}_{\mathrm{split}}) \oplus \mathbb{O}_{\mathrm{split}} \oplus \mathbb{O}_{\mathrm{split}} \oplus \mathbb{R} \tag{6.7}$$

in a fashion compatible with the matrices shown in (6.6). More precisely, the summands $\mathrm{Herm}(2, \mathbb{O}_{\mathrm{split}})$, $\mathbb{O}_{\mathrm{split}} \oplus \mathbb{O}_{\mathrm{split}}$, and $\mathbb{R}$ and correspond to $\mathcal{Z}(\mathfrak{u}), \mathfrak{Y}$, and $\mathfrak{g}_{\alpha_7}$, respectively. The decomposition (6.7) allows us to represent points of the left hand side of (6.7) by quadruples $\mathbf{u} = (u_1, u_2, u_3, u_4)$, where:

$$u_1 \in \mathrm{Herm}(2, \mathbb{O}_{\mathrm{split}}), u_2, u_3 \in \mathbb{O}_{\mathrm{split}}, \text{ and } u_4 \in \mathbb{R}.$$

From the description of the action of $\overline{N}$ on $L^2(\mathcal{O}_2, d\nu_2)$ in [BSZ, Lemma 8.10] it follows that $\overline{N} \cap \overline{U}$ acts on a space of functions on $\mathcal{O}_2$ by pointwise multiplication by characters, and two distinct points $\mathbf{u}$ and $\mathbf{u}'$ are separated by these characters unless we have $u_i = u_i'$ for any $i \in \{1, 2, 3\}$, but $u_4 \neq u_4'$. If we can show that for a set $\mathcal{S} \subseteq \mathcal{O}_2$ of full measure, any two distinct points differ in at least one of the first three coordinates, then it follows that the action of $\overline{N} \cap \overline{U}$ separates points of $\mathcal{S}$, and therefore the (spectrum of the) action of $\overline{N} \cap \overline{U}$ on $L^2(\mathcal{O}_2, d\nu_2)$ is multiplicity-free. Our next aim is to prove the existence of the set $\mathcal{S}$.

For an element $\mathbf{u} = (u_1, u_2, u_3, u_4)$, we have

$$\nabla(\mathbf{u}) = u_4 \nabla_1(u_1) + F(u_1, u_2, u_3),$$

where ∇_1 is the "determinant" of the Jordan algebra $\mathrm{Herm}(2, \mathbb{O}_{\mathrm{split}})$ and $F(u_1, u_2, u_3)$ is a cubic polynomial in 26 variables obtained by real coordinates of u_1, u_2, u_3 (see [SV, §5, (5.11)]). Obviously, if $\mathbf{u} \in \mathcal{O}_2$ is such that $\nabla_1(u_1) \neq 0$, then the equation $\nabla(\mathbf{u}) = 0$ uniquely determines u_4 in terms of u_1, u_2, u_3. Therefore we can choose $\mathcal{S}$ to be the set of all $\mathbf{u} \in \mathfrak{n}$ for which $\nabla(\mathbf{u}) = 0$ but $\nabla_1(u_1) \neq 0$. It remains to show that this set has full measure in $\mathcal{O}_2$. To this end, we show that the complement of $\mathcal{S}$ in $\mathcal{O}_2$ is a submanifold of $\mathcal{O}_2$ of positive codimension.

In fact we can work with the complexifications. Let $\overline{\mathcal{O}_2}$ denote the Zariski closure of $\mathcal{O}_2$ in $\mathrm{Herm}(3, \mathbb{O}_{\mathrm{split}}) \otimes \mathbb{C}$. Recall that the closure of $\mathcal{O}_2$ in $\mathfrak{n}$ is equal to the set of $\mathbb{R}$-rational points of the vanishing set of ∇ in $\mathrm{Herm}(\mathbb{O}_{\mathrm{split}}) \otimes \mathbb{C}$. The latter vanishing set is the closure of an orbit of the action of the complexification of L on a 27-dimensional complex affine space, and therefore it is an irreducible algebraic variety. Moreover, there exist elements $\mathbf{u}$ of $\mathcal{O}_2$ for which $\nabla_1(u_1) \neq 0$. Therefore the set of all $\mathbf{u}$ in $\overline{\mathcal{O}_2}$ for which $\nabla_1(u_1) = 0$ is an algebraic set of positive codimension in $\overline{\mathcal{O}_2}$. Since the complex dimension of $\overline{\mathcal{O}_2}$ is the same as the real dimension of $\mathcal{O}_2$, the complement of $\mathcal{S}$ in $\mathcal{O}_2$ is a submanifold of positive codimension. □

7. Proof of Theorem 2.2.A

In this section we study the correspondence (5.11) for unitary representations π_χ when G is of type $\mathbf{E}_7$. Our main tool is standard Mackey theory. Recall that $\overline{P}$ is the parabolic opposite to P, and $\pi_\chi = \mathrm{Ind}_{\overline{P}}^G \chi$ where χ is a unitary character of $\overline{P}$. The parabolic subgroup R can be expressed as

$$R = (\mathbb{R}^\times \ltimes (SL_2(\mathbb{R}) \times Spin(5,5))) \ltimes U \tag{7.1}$$

where $\mathbb{R}^\times$ is an appropriate subgroup of A. (Conjugation by $-1 \in \{\pm 1\} \subset \mathbb{R}^\times$ induces a nontrivial automorphism of $Spin(5,5)$.) Let B_{SL_2} denote the Borel subgroup of $SL_2(\mathbb{R})$ which contains $A \cap SL_2(\mathbb{R})$ and the unipotent subgroup corresponding to $-\alpha_7$. Let $N_{SL_2} = [B_{SL_2}, B_{SL_2}]$. Observe that the vector space $\mathfrak{X}$ is in fact a commutative Lie subalgebra of $\mathfrak{g}$ which lies inside the Lie algebra of N_B. Therefore $\mathfrak{X}$ is the Lie algebra of a Lie subgroup of G. We abuse our notation slightly to denote this Lie subgroup by $\mathfrak{X}$ as well.

Bearing in mind that $\mathbb{R}^\times$ represents a specific subgroup of A as in (7.1), we consider the following subgroups of R:

$$R_2 = (\mathbb{R}^\times \ltimes (B_{SL_2} \times Spin(5,5))) \ltimes \mathfrak{X}, \tag{7.2}$$
$$R_3 = (\mathbb{R}^\times \ltimes (B_{SL_2} \times Spin(5,5))) \ltimes U. \tag{7.3}$$

Note that indeed $R_2 = R \cap \overline{P}$.

By the Bruhat decomposition, $R\overline{P}$ is an open double coset of full measure in $R \backslash G / \overline{P}$. Therefore Mackey's subgroup theorem implies that

$$\mathrm{Res}_R^G \pi_\chi = \mathrm{Ind}_{R_2}^R \chi. \tag{7.4}$$

Next observe that the isomorphism

$$R_3/((N_{SL_2} \times Spin(5,5)) \ltimes U) \approx R_2/((N_{SL_2} \times Spin(5,5)) \ltimes \mathfrak{X})$$

implies that χ extends to a character $\hat{\chi}$ of R_3. By the projection formula,

$$\mathrm{Ind}_{R_2}^R \chi = \mathrm{Ind}_{R_3}^R \mathrm{Ind}_{R_2}^{R_3} \chi = \mathrm{Ind}_{R_3}^R (\hat{\chi} \otimes \mathrm{Ind}_{R_2}^{R_3} 1). \tag{7.5}$$

Let $\eta = \mathrm{Ind}_{R_2}^{R_3} 1$. Let σ be the big representation of U introduced in the statement of Proposition 5.13 and let $\hat{\sigma}$ be the extension of σ to R_1. We will now prove the following lemma.

Lemma 7.6. *There exists a unitary character $\hat{\kappa}$ of R_3 such that*

$$\eta = \hat{\kappa} \otimes (\mathrm{Res}_{R_3}^R \mathrm{Ind}_{R_1}^R \hat{\sigma}). \tag{7.7}$$

Proof. To prove Lemma 7.6 note that $UR_2 = R_3$ and therefore by Mackey's subgroup theorem,

$$\mathrm{Res}_U^{R_3}\eta = \mathrm{Res}_U^{R_3}\mathrm{Ind}_{R_2}^{R_3}1 = \mathrm{Ind}_{\mathfrak{X}}^U 1.$$

The right hand side, which is equal to the U-spectrum of η, is a multiplicity-free direct integral of big representations of U. The fact that the action of $\mathbb{R}^\times \ltimes Spin(5,5) \subset R_3$ on the U-spectrum of η is transitive implies that η is an irreducible unitary representation of R_3. Next we apply standard Mackey theory to η. The stabilizer of σ in R_3 is $R_3 \cap R_1$, and therefore by Mackey theory we can write

$$\eta = \mathrm{Ind}_{R_1\cap R_3}^{R_3}(\kappa \otimes \hat{\sigma}), \tag{7.8}$$

where κ is an irreducible unitary representation of $B_{SL_2} \times Spin(4,5)$ which is extended (trivially on U) to $R_1 \cap R_3$. We can use Mackey's subgroup theorem again and see that the U-spectrum of the right hand side of (7.8) is multiplicity-free only if κ is a one-dimensional unitary representation. Therefore κ should be a unitary character of $B_{SL_2} \times Spin(4,5)$ and hence it is trivial on $N_{SL_2} \times Spin(4,5)$. Obviously κ extends to a unitary character $\hat{\kappa}$ of R_3 which factors through a character of

$$R_3/((N_{SL_2} \times Spin(5,5)) \ltimes U).$$

From (7.8), the projection formula, Mackey's subgroup theorem, and $R_3R_1 = R$, we have

$$\eta = \hat{\kappa} \otimes \mathrm{Ind}_{R_1\cap R_3}^{R_3}\hat{\sigma} = \hat{\kappa} \otimes \mathrm{Res}_{R_3}^R\mathrm{Ind}_{R_1}^R\hat{\sigma}. \qquad \square$$

Let $\hat{\eta} = \mathrm{Ind}_{R_1}^R\hat{\sigma}$. From (7.7) it follows that

$$\hat{\kappa}^{-1} \otimes \eta = \mathrm{Res}_{R_3}^R\hat{\eta}. \tag{7.9}$$

Next we continue with the right hand side of (7.5). Using (7.9) and the projection formula we have

$$\mathrm{Ind}_{R_3}^R(\hat{\chi} \otimes \eta) = \mathrm{Ind}_{R_3}^R(\hat{\chi} \otimes \hat{\kappa} \otimes \hat{\kappa}^{-1} \otimes \eta) = \hat{\eta} \otimes \mathrm{Ind}_{R_3}^R(\hat{\kappa} \otimes \hat{\chi}) = \hat{\eta} \otimes \hat{\zeta}, \tag{7.10}$$

where $\hat{\zeta} = \mathrm{Ind}_{R_3}^R(\hat{\kappa} \otimes \hat{\chi})$. By Mackey's subgroup theorem,

$$\mathrm{Res}_{SL_2(\mathbb{R})}^R\hat{\zeta} = \mathrm{Res}_{SL_2(\mathbb{R})}^R\mathrm{Ind}_{R_3}^R(\hat{\kappa} \otimes \hat{\chi}) = \mathrm{Ind}_{B_{SL_2}}^{SL_2(\mathbb{R})}(\hat{\kappa} \otimes \hat{\chi}).$$

Let $\zeta = \mathrm{Ind}_{B_{SL_2}}^{SL_2(\mathbb{R})}(\hat{\kappa} \otimes \hat{\chi})$. Clearly ζ is a unitary principal series representation of $SL_2(\mathbb{R})$. Since the only one-dimensional unitary representation of $Spin(4,5)$ is the trivial representation, it follows immediately that

$$\mathrm{Res}_{SL_2(\mathbb{R})\times Spin(4,5)}^R\hat{\zeta} = \zeta\hat{\otimes}1. \tag{7.11}$$

By Mackey's subgroup theorem we see that the unitary representation

$$\mathrm{Res}_U^R\hat{\zeta} = \mathrm{Res}_U^R\mathrm{Ind}_{R_3}^R(\hat{\kappa} \otimes \hat{\chi})$$

is a direct integral of unitary representations of the form $\mathrm{Ind}_U^U 1$; i.e., $\hat{\zeta}$ acts trivially when restricted to U, or in other words, $\hat{\zeta}$ factors through a unitary representation of R/U.

Let $\tilde{\zeta} = \mathrm{Res}_{R_1}^R\hat{\zeta}$. Then by (7.4), (7.5), (7.10), and the projection formula we have

$$\mathrm{Res}_R^G\pi_\chi = \mathrm{Ind}_{R_3}^R(\hat{\chi} \otimes \eta) = \hat{\eta} \otimes \hat{\zeta} = (\mathrm{Ind}_{R_1}^R\hat{\sigma}) \otimes \hat{\zeta} = \mathrm{Ind}_{R_1}^R(\hat{\sigma} \otimes \tilde{\zeta}).$$

Since $\tilde{\zeta}$ comes from a unitary representation of R_1/U, it follows from standard Mackey theory and (7.11) that $\Psi(\pi_\chi) = \zeta\hat{\otimes}1$.

8. Proof of Theorem 2.2.B

In this section we study the correspondence of (5.11) for representations π_χ when G is of type $\mathbf{E}_6$. The argument is very similar to the case when G is of type $\mathbf{E}_7$.

Recall that $\overline{P}$ is the parabolic subgroup opposite to P and χ is a unitary multiplicative character of $\overline{P}$ such that $\pi_\chi = \mathrm{Ind}_{\overline{P}}^G \chi$.

The vector space $\mathfrak{X}$ is a Lie algebra and corresponds to a Lie subgroup of N_B. We abuse our notation to let $\mathfrak{X}$ denote this Lie subgroup of N_B as well. Let $R_2 = R\cap\overline{P}$. Then

$$R_2 = ((\mathbb{R}^\times \times \mathbb{R}^\times) \ltimes \mathit{Spin}(4,4)) \ltimes \mathfrak{X}.$$

Since the restriction of χ to $\mathit{Spin}(4,4) \ltimes \mathfrak{X}$ is trivial, χ extends to a character $\hat{\chi}$ of R. By the projection formula,

$$\mathrm{Res}_R^G \mathrm{Ind}_{\overline{P}}^G \chi = \mathrm{Ind}_{R\cap\overline{P}}^R \chi = \hat{\chi} \otimes \mathrm{Ind}_{R\cap\overline{P}}^R 1.$$

Lemma 8.1. *Fix an extension $\hat{\sigma}$ of σ to R_1. (See the statement of Proposition* 5.13.*) Then for some unitary character $\tilde{\chi}$ of R_1 whose restriction to U is trivial, we have*

$$\mathrm{Ind}_{R\cap\overline{P}}^R 1 = \mathrm{Ind}_{R_1}^R (\tilde{\chi} \otimes \hat{\sigma}). \tag{8.2}$$

Proof. By Mackey's subgroup theorem, we have

$$\mathrm{Res}_U^R \mathrm{Ind}_{R\cap\overline{P}}^R 1 = \mathrm{Ind}_{\mathfrak{X}}^U 1,$$

which implies that the restriction of the left hand side of (8.2) to U is a multiplicity-free direct integral of big representations of U. Transitivity of the action of the Levi subgroup of R on this U-spectrum implies that the left hand side of (8.2) is an irreducible unitary representation of R. It follows that if we use standard Mackey theory to write the left hand side of (8.2) as $\mathrm{Ind}_{R_1}^R(\tau \otimes \hat{\sigma})$, then τ has to be a one-dimensional unitary representation of R_1/U. Therefore $\tau = \tilde{\chi}$, for some unitary character $\tilde{\chi}$. $\square$

Using Lemma 8.1, we have

$$\mathrm{Res}_R^G \pi_\chi = \hat{\chi} \otimes \mathrm{Ind}_{R\cap\overline{P}}^R 1 = \hat{\chi} \otimes \mathrm{Ind}_{R_1}^R (\tilde{\chi} \otimes \hat{\sigma}) = \mathrm{Ind}_{R_1}^R (\hat{\chi} \otimes \tilde{\chi} \otimes \hat{\sigma}),$$

which implies that $\Psi(\pi_\chi) = \tilde{\chi} \otimes \hat{\chi}$.

9. The image of π° under Ψ

From the K-type structure of the unitary representation π° it follows that its Gelfand-Kirillov dimension is strictly larger than the minimal representation but strictly smaller than a generic degenerate principal series representation (see [Sah]). This suggests that in the correspondence (5.11), the image of π° should be a unitary representation of $SL_2(\mathbb{R}) \times \mathit{Spin}(4,5)$ which is "smaller than" the image of π_χ. It turns out that the only possibility is the trivial representation of $SL_2(\mathbb{R}) \times \mathit{Spin}(4,5)$. A rigorous proof of this statement can be given using the property of π° which was shown in the proof of Proposition 6.3: since the $N \cap U$-spectrum of π° is multiplicity-free, Mackey's subgroup theorem applied to

$$\mathrm{Res}_{N\cap U}^R \mathrm{Ind}_{R_1}^R (\Psi(\pi^\circ) \otimes \hat{\sigma})$$

implies that $\Psi(\pi^\circ)$ should be one-dimensional. But the only one-dimensional unitary representation of $SL_2(\mathbb{R}) \times Spin(4,5)$ is the trivial representation. In other words, $\Psi(\pi^\circ)$ is the trivial representation.

10. Tables

In this section we prove Lemma 5.9 and describe explicitly the polarization $\mathfrak{X}_1 \oplus \mathfrak{Y}_1$ used in the proof of Proposition 5.13. Recall the labelling of simple roots shown in section 2. For any root α, we can write

$$\alpha = c_1(\alpha)\alpha_1 + \cdots + c_n(\alpha)\alpha_n$$

for integers $c_i(\alpha)$. We represent a root by putting these integers in their corresponding locations on the nodes of the Dynkin diagram. When describing a set of roots, a "$*$" in a location of the Dynkin diagram means that the corresponding coefficient c_i can assume any possible value, given the fixed coefficients, to make the entire labelling represent a root.

Parts a, b, c and e of Lemma 5.9 follow easily from the given tables. Part d requires a simple calculation of weights of the standard basis of $\mathbb{R}^{n-2,n-2}$ under the action of the Cartan subgroup of $Spin(n-2,n-2)$.

Let G be of type $\mathbf{E}_6$. The tables below suffice for the proof of Lemma 5.9.

$\mathfrak{W}$	$\mathfrak{W}^*$	$\mathfrak{X}$	$\mathfrak{Y}$	$\mathcal{Z}(\mathfrak{u})$
$0****$ 1	$1****$ 1	$0***1$	$1***0$	$1***1$

e_1	e_2	e_3	e_4
12321 2	12321 1	12221 1	12211 1
e_{-1}	e_{-2}	e_{-3}	e_{-4}
11111 0	11111 1	11211 1	11221 1

f_2	f_3	f_4	f_{-2}	f_{-3}	f_{-4}
00000 1	00100 1	00110 1	01210 1	01110 1	01100 1

Now let G be of type $\mathbf{E}_7$. The tables below suffice for the proof of Lemma 5.9.

$\mathfrak{W}$	$\mathfrak{W}^*$	$\mathfrak{X}$	$\mathfrak{Y}$	$\mathcal{Z}(\mathfrak{u})$
$0****1$ $*$	$1****1$ $*$	$01****$ $*$	$11****$ $*$	$12****$ $*$

e_1	e_2	e_3	e_4	e_5
123432 2	123431 2	123421 2	123321 2	123321 1
e_{-1}	e_{-2}	e_{-3}	e_{-4}	e_{-5}
122210 1	122211 1	122221 1	122321 1	122321 2

f_2	f_3	f_4	f_5
000001 0	000011 0	000111 0	000111 1
f_{-2}	f_{-3}	f_{-4}	f_{-5}
001221 1	001211 1	001111 1	001111 0

Finally, when G is of type $\mathbf{E}_7$, the polarization $\mathfrak{X}_1 \oplus \mathfrak{Y}_1$ which appears in the proof of Proposition 5.13 can be described as follows:

$\mathfrak{X}_1$	$\mathfrak{Y}_1$
$*1**00$ $*$	$*1**10$ $*$
$*1**11$ $*$	$*1**21$ $*$

.

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