TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 361, Number 4, April 2009, Pages 1791–1809
S 0002-9947(08)04632-1
Article electronically published on October 30, 2008

ON THE HOMOTOPY OF FINITE CW-COMPLEXES WITH POLYCYCLIC FUNDAMENTAL GROUP

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ABSTRACT. Let X be a finite connected CW-complex of dimension q. If its fundamental group $\pi_1(X)$ is polycyclic of Hirsch number $h > q$, we show that at least one homotopy group $\pi_i(X)$ is not finitely generated. If $h = q$ or $h = q - 1$ the same conclusion holds unless X is an Eilenberg-Mac Lane space $K(\pi_1(X), 1)$.

1. INTRODUCTION

Let X be a finite connected CW-complex of dimension q. Consider the homotopy groups $\pi_i(X) = [\mathbf{S}^i, X]$, for $i \geq 2$. If all these (Abelian) groups are finitely generated we say that the homotopy of X is finitely generated. For simply connected complexes, a celebrated theorem of Serre [31] asserts that

Theorem 1.1. *If $\pi_1(X) = 1$ the homotopy of X is finitely generated.*

On the other hand simple examples such as $X = \mathbf{S}^1 \vee \mathbf{S}^2$ show that the homotopy of X is not always finitely generated.

One can then ask if there is a general negative statement asserting that the homotopy of X is not finitely generated under some hypothesis on computable invariants of X, such as its fundamental group. This is the aim of the present paper. The statements we will prove have the following form:

> If the fundamental group of X satisfies the conditions (C) (which depend on the dimension q), then the homotopy of X is not finitely generated unless X is an Eilenberg-Mac Lane space $K(\pi_1(X))$.

If X is an Eilenberg-Mac Lane space, the cohomological dimension of $\pi_1(X)$ is less than or equal to $q = dim(X)$. So, we may add $cd(\pi_1(X)) > q$ to the hypothesis (C) in order to have the desired conclusion on the homotopy of X. For instance when $\pi_1(X)$ has non-trivial torsion elements the cohomological dimension $cd(\pi_1(X))$ is infinite and the above conclusion is valid.

1.1. **Statement of the results.** Before stating the main theorem we remind the reader of the following:

Definition. A group G is called the polycyclic if it admits a series

$$1 = G_0 < G_1 < \cdots < G_k = G$$

Received by the editors January 26, 2007.
2000 *Mathematics Subject Classification.* Primary 57R70, 55P15, 57Q10.
Key words and phrases. Novikov homology, closed 1-forms, CW-complexes, homotopy groups.

with cyclic factors G_{i+1}/G_i. When all the factor groups are infinite cyclic we call G a poly-$\mathbf{Z}$ group.

K. A. Hirsch proved in [15] that the number of infinite cyclic factors in such a series is an invariant of G. It is called the Hirsch number of G and it is denoted by $h(G)$.

Here are our main statements:

Theorem 1.2. *Suppose that X is a finite connected CW-complex of dimension q and that $\pi_1(X)$ is polycyclic. Then*

a) *If $h(\pi_1(X)) > q$, the homotopy of X is not finitely generated.*

b) *If $h(\pi_1(X)) \in \{q-1, q\}$, then the homotopy of X is not finitely generated unless X is a $K(\pi_1(X), 1)$. In particular, when $\pi_1(X)$ has torsion, then the conclusion of* a) *holds.*

Note that, by a result of J-P. Serre (see Theorem 4.2 below), when the homotopy of X is not finitely generated at least one of the groups $\pi_2(X), \pi_3(X), \ldots, \pi_q(X)$ is not finitely generated. When X is a manifold we can improve this result:

Theorem 1.3. *Let M^n be a closed manifold. Let $r = \max\left\{\left[\frac{n}{2}\right], 3\right\}$. Suppose that $\pi_1(M)$ is polycyclic. Then, if $h(\pi_1(M)) \geq n-1$, the group $\pi_i(M)$ is not finitely generated for some $i \leq r$ unless $h(\pi_1(M)) = n$ and M is a $K(\pi_1, 1)$.*

Moreover, if $n \geq 6$ we can replace the assertion "M is a $K(\pi_1, 1)$" by the stronger one "the universal cover of M is diffeomorphic to $\mathbf{R}^n$".

When $n = 3, 4$, we are able to prove the above statement for $r = 2$, namely:

Theorem 1.4. *Let M^n be a closed connected manifold of dimension $n = 3$ or $n = 4$ which has a polycyclic fundamental group. If $h(\pi_1(M)) \geq n-1$, then $\pi_2(M)$ is not finitely generated unless $h(\pi_1(M)) = n$ and M is a $K(\pi_1, 1)$.*

Remark. Conversely, for any torsion free polycyclic group G with $h(G) = n$, there is a closed manifold M with universal cover diffeomorphic to $\mathbf{R}^n$ and such that $\pi_1(M) = G$. This result was proved by L. Auslander and F. E. A. Johnson in [1]. Their result generalizes a previous one by C. T. C. Wall [37] (chapter 15B), which is valid for poly-$\mathbf{Z}$ groups.

In the next theorem we weaken the hypothesis on the fundamental group but we suppose in addition that $\chi(X) \neq 0$. Recall first the finiteness properties of a group, which were introduced by C. T. C. Wall in [36]:

Definition. Let $r \geq 1$ be an integer. A group G is of type $\mathcal{F}_r$ if there is an Eilenberg-Mac Lane space $K(G, 1)$ whose r-skeleton has a finite number of cells. Equivalently a group is of type $\mathcal{F}_r$ if it acts freely, properly, cellularly and cocompactly on an $(r-1)$-connected cell complex.

A group is of type $\mathcal{F}_\infty$ if it is of type $\mathcal{F}_r$ for any integer $r > 0$.

Remark. A group G is of type $\mathcal{F}_1$ if and only if it is finitely generated. G is of type $\mathcal{F}_2$ if and only if it is finitely presented.

Theorem 1.5. *Let X^q be a finite connected CW-complex of dimension q with fundamental group of type $\mathcal{F}_{q+1}$. Suppose that there is a non-vanishing morphism $u : \pi_1(X) \to \mathbf{Z}$ such that $Ker(u)$ is of type $\mathcal{F}_{q+1}$. Then, if $\chi(X) \neq 0$, the homotopy of X is not finitely generated.*

If X is a manifold with non-zero Euler characteristic, the same holds for $\pi_1(X)$ and $Ker(u)$ of type $\mathcal{F}_{[\frac{q}{2}]}$.

The hypothesis on u may be reformulated in terms of the Bieri-Renz invariants of $\pi_1(X)$ as follows:

$$(*) \qquad \Sigma^{q+1}(\pi_1) \cap -\Sigma^{q+1}(\pi_1) \neq \emptyset.$$

The Bieri-Renz invariants $\Sigma^i(G)$ of a group G are open subsets of the unit sphere of $\mathbf{R}^{rk(G)}$. We recall the definition and the properties of these invariants in Section 4.

In the next subsection we show that polycyclic fundamental groups satisfy the hypothesis of Theorem 1.5 and we give other examples of groups for which the condition $(*)$ is fulfilled.

1.2. **Comments on the results.** An Abelian group is obviously polycyclic of Hirsch number equal to its rank. It is also of type $\mathcal{F}_\infty$. For Abelian fundamental groups the results of the previous subsection were proved by the author in [10]. Here are other remarks about these statements.

Remarks. 1. The lower bound $q-1$ for $h(\pi_1)$ in the hypothesis of Theorem 1.2 is optimal. Indeed, the complex $X = \mathbf{T}^{q-2} \times \mathbf{S}^2$ fulfills the conditions of the hypothesis of Theorem 1.2 except for $h(\pi_1(X)) = q-2$. It is a consequence of Theorem 1.1 of J-P. Serre, quoted above, that X does not satisfy the conclusion of Theorem 1.2.

2. If G is a polycyclic group, then $[G,G]$ is obviously polycyclic and therefore for every morphism $u : G \to \mathbf{Z}$, $Ker(u)$ has the same property. This implies that the hypothesis on the fundamental group of Theorem 1.5 is fulfilled when $\pi_1(X)$ is polycyclic. Indeed we have:

Proposition 1.6. *Polycyclic groups are of type $\mathcal{F}_\infty$.*

Proof. The proposition is an immediate corollary of the following:

Lemma 1.7. *Consider an exact sequence of groups*

$$1 \to K \to G \to Q \to 1.$$

If K and H are of type $\mathcal{F}_\infty$, then so is G.

To prove the above lemma one may use the following results:

a) If G is finitely presented, then G is $\mathcal{F}_\infty$ iff $H_*(G,\cdot)$ commutes with direct products. This was proved by R. Bieri and B. Eckmann in [3].

b) The Hochschild-Serre spectral sequence [17] which satisfies

$$E^2_{pq} = H_p(Q, H_q(K,R))$$

and converges towards $H_*(G,R)$. (The complete statement (Theorem 4.6) is given in Section 4.)

Now K and Q are finitely presented ($\mathcal{F}_2$) and it is obvious that G is also finitely presented. Then, it is clear how a) and b) imply Lemma 1.7 thus proving Proposition 1.6. □

3. Other examples of groups satisfying the hypothesis of Theorem 1.5 were constructed by M. Bestvina and N. Brady in [2] and then in a more general statement by J. Meier, H. Meinert and L. Van Wyk in [21]. Let us describe them briefly.

Definition. A flag complex L is a finite simplicial complex with the property that any collection of $q+1$ mutually adjacent vertices span a q-simplex in L. The *right angled Artin group* associated to L is the group G_L spanned by the vertices $v_1, \ldots, v_s$ of L with relations $[v_i, v_j] = 1$ whenever v_i and v_j are adjacent in L.

Note that the group G_L admits a finite Eilenberg-Mac Lane space [2] and therefore it is of type $\mathcal{F}_\infty$. We have

Theorem 1.8 (Bestvina, Brady, Meier, Meinert, Van Wyk)**.** *Let L and G_L be as above and consider a morphism $u : G_L \to \mathbf{Z}$ such that $u(v_i) \neq 0$ for each generator v_i. Then, if L is $(r-1)$ connected, $Ker(u)$ is of type $\mathcal{F}_r$.*

Considering appropriate flag complexes L (remark that the barycentric subdivision of any complex is a flag complex) we may find many examples of groups which fulfill the hypothesis of Theorem 1.5.

1.3. **Idea of the proof.** The general idea of the proof of Theorem 1.3 is the following: we show first that a manifold M as in Theorem 1.3 whose dimension is greater than or equal to 6 and whose homotopy groups $\pi_i(M)$ are finitely generated for $i \leq r$ admits a fibration over the circle. This is the main difficulty of the proof. To overcome it, we will use Novikov homology theory and some of its applications on Morse functions $f : M \to \mathbf{S}^1$.

Remark that if the manifold F^{n-1} is a fiber, the inclusion $j : F \hookrightarrow M$ induces isomorphisms in π_i for $i \geq 2$. The lift of j induces a homotopy equivalence between the universal covers of F and M.

Now take a manifold M, as in Theorem 1.3, and consider the product $M \times \mathbf{S}^3$ in order to fulfill the condition on the dimension. Supposing that the homotopy groups $\pi_i(M)$ are finitely generated for $i \leq r$, apply the previous argument to this product and get a fiber F as above. Then check that F still satisfies the hypothesis of Theorem 1.3. If its dimension is still greater than or equal to 6 we apply the same argument to F. By succesive iterations we find a manifold F_0 of low dimension whose universal cover has the same homotopy type as the one of $M \times \mathbf{S}^3$. In particular the homology groups of $\widetilde{F_0}$ and $\widetilde{M} \times \mathbf{S}^3$ are isomorphic. By comparing them, we infer that the universal cover of M is acyclic, which means that M is Eilenberg-Mac Lane. So, either the homotopy of M is not finitely generated, or M is Eilenberg-Mac Lane, as in the statement of Theorem 1.3.

To prove Theorem 1.2 we embed X into a Euclidian space and thicken it to a manifold W, with boundary M. Then we apply the above argument to M.

In the hypothesis of Theorem 1.5, supposing that the homotopy of X is finitely generated, we are only able to prove that *the Novikov homology* $H_*(M, u)$ of the corresponding manifold M vanishes. But this implies that $\chi(M) = \chi(X) = 0$, yielding a contradiction.

The paper is organized as follows. In Section 2 we state the result from Theorem 2.1 on the fibration over the circle which was roughly sketched above. Supposing Theorem 2.1 is true, we show how it can be succesively applied in order to prove Theorems 1.3, 1.4 and 1.2. In Section 3 we recall the definition and some useful properties of the Novikov homology. We also recall some basic facts about Morse theory of circle-valued functions and point out the relation between the (vanishing of the) Novikov homology and the existence of a fibration over the circle. In Section 4 we prove Theorems 2.1 and 1.5. We will use the Bieri-Renz criterion which we recall in Subsection 4.3, devoted to the Bieri-Renz invariants.

2. Iterated fibrations

Our main result (Theorem 1.2) is a consequence of the following:

Theorem 2.1. *Let M^n be a closed manifold of dimension $n \geq 2$ with vanishing Whitehead group $Wh(\pi_1(M))$. Suppose that $\pi_1(M)$ is of type $\mathcal{F}_{\left[\frac{n}{2}\right]}$. Suppose also that $\pi_i(M)$ are finitely generated for $i \leq \left[\frac{n}{2}\right]$.*

a) *Suppose that $n \geq 6$ and that there is a non-zero cohomology class $u \in H^1(M;\mathbf{Z}) \approx Hom(\pi_1(M),\mathbf{Z})$ such that $Ker(u)$ is of type $\mathcal{F}_{\left[\frac{n}{2}\right]}$. Then there is a fibration $f : M \to \mathbf{S}^1$ such that $[f^* d\theta] = u$.*

b) *More generally, suppose that $Ker(u)$ is of type $\mathcal{F}_r$, where $r = \max\left\{\left[\frac{n}{2}\right], 2\right\}$. Then, for $p \geq \max\{6-n, 2\}$ there is a fibration $f : M \times \mathbf{S}^p \to \mathbf{S}^1$, such that $[f^* d\theta] = u \in H^1(M \times \mathbf{S}^p) \approx H^1(M)$. For p odd the hypothesis on $Wh(\pi_1(M))$ can be dropped.*

Remarks. 1. The Whitehead group is defined as follows:

$$Wh(\pi) = \frac{GL\mathbf{Z}[\pi]}{[GL\mathbf{Z}[\pi], GL\mathbf{Z}[\pi]], \{\pm g | g \in \pi\}},$$

where $GL\mathbf{Z}[\pi] = \lim_{\to} GL_m\mathbf{Z}[\pi]$.

2. T. Farrell and W. Hsiang proved in [12] that $Wh(\pi)$ vanishes when π is poly-$\mathbf{Z}$. The Whitehead group also vanishes when $\pi = G_L$ is one of the examples of Bestvina and Brady. This result was proved by B. Hu [18].

It is conjectured that $Wh(\pi) = 0$ for any torsion-free group π.

3. For $\pi_1 = \mathbf{Z}$, Theorem 2.1 was proved by W. Browder and J. Levine in [7].

The proof of Theorem 2.1 will be given in Section 4. Let us now show how this theorem implies our main results, Theorems 1.3, 1.4 and 1.2.

Proof of Theorem 2.1 $\Longrightarrow$ *Theorem* 1.3. Without restricting the generality of our statements, we may suppose that $n \geq 3$. We begin with the statement of the following result, due to K. A. Hirsch ([16], Theorem 2). □

Theorem 2.2. *Let G be a polycyclic group. There exists a normal subgroup N of finite index in G which is poly-$\mathbf{Z}$ and such that $h(N) = h(G)$.*

Now let $\pi_1^0 \leq \pi_1$ be a subgroup as in Theorem 2.2 and consider the associated finite cover $M_0 \to M$. Let

$$1 = G_0 < G_1 < \cdots < G_k = \pi_1^0$$

be a series with infinite cyclic factor groups: we have therefore $h(\pi_1) = k$, so by hypothesis $k \geq n-1$. Denote by u_1 the projection $G_k \to G_k/G_{k-1} \approx \mathbf{Z}$.

Suppose that $\pi_i(M)$ is finitely generated for $i \leq r = \max\left\{\left[\frac{n}{2}\right], 3\right\}$.

Consider first the case $n \geq 6$. Using Proposition 1.6 we find that u_1 fulfills the hypothesis of Theorem 2.1. We apply Theorem 2.1 and we get a fibration $M_0 \to \mathbf{S}^1$. Let F_1 be a fibre of this fibration. So F_1 satisfies:

1. $\pi_1(F_1) = Ker(u_1) = G_{k-1}$.
2. $\pi_i(F_1) \approx \pi_i(M)\ \forall\, i \geq 2$.

We wish to apply Theorem 2.1 to F_1. We use the cohomology class $u_2 : G_{k-1} \to G_{k-1}/G_{k-2}$. Its kernel is G_{k-2}, so, again using Proposition 1.6, $Ker(u_2)$ is of type $\mathcal{F}_{\left[\frac{n}{2}\right]}$ (actually $\mathcal{F}_\infty$, since it is polycyclic). The hypothesis on the higher homotopy groups of Theorem 2.1 is fulfilled by F_1 (because of condition 2 above), therefore we may apply Theorem 2.1 to F_1 if its dimension is no less than 6.

We thus get a closed connected manifold $F_2 \subset F_1$ whose higher homotopy groups are those of M and whose fundamental group is G_{k-2}. If its dimension is greater than or equal to 6 we may again apply Theorem 2.1 to the couple $(F_2, u_3 : G_{k-2} \to G_{k-2}/G_{k-3})$.

By iterating this argument we get a sequence

$$F_{n-5} \hookrightarrow F_{n-6} \hookrightarrow \cdots \hookrightarrow F_1 \hookrightarrow M_0 \tag{1}$$

such that for $j = 1, \ldots, n-1$:

1. $\pi_1(F_j) = G_{k-j}$.
2. $dim(F_j) = n - j$.
3. The inclusion $F_j \hookrightarrow M$ induces isomorphisms in π_i for $i \geq 2$.

The manifold F_{n-5} is of dimension 5, so it does not verify the dimension hypothesis of Theorem 2.1. Its fundamental group is G_{k-n+5}. In order to continue to apply Theorem 2.1, we consider the product $F = F_{n-5} \times \mathbf{S}^3$. As above, we have the cohomology class $u_{k-n+4} : G_{k-n+5} \to G_{k-n+5}/G_{k-n+4}$ which is non-zero and has a (poly-$\mathbf{Z}$) kernel of type $\mathcal{F}_{\left[\frac{n}{2}\right]}$. By 2.1.b), we get a fibration of F over $\mathbf{S}^1$. Its fiber K_0 is a closed connected manifold of dimension 7 with finitely generated π_i for $i = 1, \ldots, r = \max\left\{\left[\frac{n}{2}\right], 3\right\}$ (since $\mathbf{S}^3$ has the same property by Theorem 1.1). Its fundamental group is G_{k-n+4}. Since $[7/2] = 3$, we may apply Theorem 2.1 to K_0 and then again to its submanifold K_1 given by Theorem 2.1 to obtain a sequence as above, where the maps are inclusions of fibers of fibrations over the circle which therefore induce isomorphisms at the level of π_i for $i \geq 2$:

$$K_2 \hookrightarrow K_1 \hookrightarrow K_0 \hookrightarrow F.$$

It follows that the universal covers of K_2 and F are homotopically equivalent, in particular

$$H_*(\widetilde{K}_2) \approx H_*(\widetilde{F}_{n-5} \times \mathbf{S}^3). \tag{2}$$

Now K_2 is a closed connected 5-dimensional manifold whose fundamental group is $G = G_{k-n+2}$. Since $k \geq n-1$, G is an infinite group. It follows that $H_5(\widetilde{K}_2) = 0$ (see [6], p.346). We have therefore that $H_i(\widetilde{K}_2)$ vanishes for $i > 4$.

Using the Kunneth formula we infer from (2) that $H_i(\widetilde{F}_{n-5})$ vanishes for $i > 0$. Therefore $\widetilde{F}_{n-5}$ is contractible and, using the sequence (1), $\widetilde{M_0} = \widetilde{M}$ is contractible, too. So M is an Eilenberg-Mac Lane space $K(\pi_1(M), 1)$. The cohomological dimension of $\pi_1(M)$ is therefore equal to n. Now we use the following well known result (for a proof, see [13], Lemma 8, p. 154):

Proposition 2.3. *If G is a polycyclic group, then $cd(G)$ is finite if and only if G is torsion free. In this case $cd(G) = h(G)$.*

Applying Proposition 2.3 we get that M cannot be a $K(\pi_1, 1)$ unless $h(\pi_1(M)) = n$. If this relation is not valid, we get a contradiction and therefore the homotopy of M cannot be finitely generated, completing the proof of Theorem 1.3 for $n \geq 6$. If it is and if M is a $K(\pi_1, 1)$, let us show that its universal cover is $\mathbf{R}^n$. Since according to Theorem 2.1 M fibers over $\mathbf{S}^1$, we have

$$\widetilde{M} \approx \widetilde{F} \times \mathbf{R}.$$

To finish the proof we just have to apply a celebrated theorem of J. Stallings [30] which asserts:

Theorem 2.4. *If a manifold $P^{n\geq 5}$ is a product of two open non-trivial contractible manifolds, then P is diffeomorphic to $\mathbf{R}^n$.*

Proof of Theorem 2.1 $\Longrightarrow$ *Theorem* 1.4 (*cases* $n=3$ *and* $n=4$). Consider first the case $n = 4$. Suppose that $\pi_2(M)$ is finitely generated. As in the proof of Theorem 1.3, use Theorem 2.2 and consider a finite cover M_0 with poly-$\mathbf{Z}$ fundamental group π_1^0 of Hirsch number greater than 2. We first apply Theorem 2.1.b) to $M_0 \times \mathbf{S}^2$ and to the morphism $u_1 : G_k \to G_k/G_{k-1}$. We get a manifold F of dimension 5 which is a fiber of a fibration $M_0 \to \mathbf{S}^1$. Since $[dim(F)/2] = 2$ we may again apply Theorem 2.1.b) to the product $F \times \mathbf{S}^2$ and the cohomology class $u_2 : G_{k-1} \to G_{k-1}/G_{k-2}$; we get a 6-dimensional submanifold $K \hookrightarrow F \times \mathbf{S}^2$. We have a sequence

$$K \hookrightarrow F \times \mathbf{S}^2 \hookrightarrow M \times \mathbf{S}^2 \times \mathbf{S}^2,$$

which induces homotopy equivalences at the level of the universal covers. In particular,

$$H_*(\widetilde{K}) \sim H_*(\widetilde{M} \times \mathbf{S}^2 \times \mathbf{S}^2).$$

Now, as above, since $k \geq 3$, the fundamental group G_{k-2} of K is infinite, so the homology of $\widetilde{K}$ vanishes in degrees $i > 5$. This implies that the homology of $\widetilde{M}$ is zero, so M is a $K(\pi_1, 1)$. We conclude using Proposition 2.3.

Now let $n = 3$ and consider the 4-manifold $N = M \times \mathbf{S}^1$. We have $h(\pi_1(N)) \geq 3$ and $\pi_2(N) \approx \pi_2(M)$. Applying the above argument to N, we get that if $\pi_2(M)$ is finitely generated, then N has a contractible universal covering. The same is valid for M, and as above this is only possible when $h(\pi_1(M)) = 3$ (by Proposition 2.3). □

Proof of Theorem 2.1 $\Longrightarrow$ *Theorem* 1.3 *in the case* $n=5$. Suppose that $\pi_2(M)$ and $\pi_3(M)$ are finitely generated. We apply Theorem 2.1.b) 3 times to the 8-dimensional manifold $M \times \mathbf{S}^3$. We get a sequence:

$$K_3 \hookrightarrow K_2 \hookrightarrow K_1 \hookrightarrow M \times \mathbf{S}^3,$$

as above. The manifold K_3 is of dimension 5 and has an infinite fundamental group. So, $H_i(\widetilde{K}_3) = 0$ for $i \geq 5$. Since this homology is isomorphic to $H_*(\widetilde{M} \times \mathbf{S}^3)$, it follows that $\widetilde{M}$ is acyclic, and therefore that M is an Eilenberg-Mac Lane space. As above, we use Proposition 2.3 to finish the proof. □

Proof of Theorem 2.1 $\Longrightarrow$ *Theorem* 1.2. Assume that the homotopy of X is finitely generated.

a) Suppose that $h(\pi_1(X)) = k$ for some integer $k > q$. Using Theorem 2.2, consider a finite cover X_0 of X such that $\pi_1(X_0)$ is poly-$\mathbf{Z}$ and $h(\pi_1(X_0)) = k$.

Embed X_0 in a Euclidian space $\mathbf{R}^{2q+r+1}$ for some $r \geq 0$ which will be fixed later in the proof. Let W be a tubular neighbourhood of X_0 and denote by M^{2q+r} the smooth manifold ∂W. Since M is a deformation retract of $W \setminus X_0$, using a general position argument we get isomorphisms between $\pi_i(W)$ and $\pi_i(M)$ for $i \leq q+r-1$. In particular, for $r > 1$ the inclusion $M \hookrightarrow W$ induces an isomorphism at the level of fundamental groups and, since X_0 is a retract of W, the higher homotopy groups $\pi_i(M)$ are finitely generated for $i \leq q + r - 1$. If $r \geq 1$, then $q + r - 1 \geq \left[\frac{2q+r}{2}\right]$

and the hypothesis on the higher homotopy groups of Theorem 2.1 is satisfied by the manifold M. Let

$$1 = G_0 < G_1 < \cdots < G_k = \pi_1(X_0)$$

be a series with infinite cyclic factor groups. For r large enough ($r \geq k-2q+5$), we may apply Theorem 2.1 k times, as in the proof of Theorem 1.3, and get a sequence of closed connected manifolds

$$F_k \hookrightarrow F_{k-1} \hookrightarrow \cdots \hookrightarrow F_1 \hookrightarrow M \tag{3}$$

such that for $j = 0, \ldots, k$:

1. $\pi_1(F_j) = G_{k-j}$.
2. $dim(F_j) = 2q + r - j$.
3. The inclusion $F_j \hookrightarrow M$ induces isomorphisms in π_i for $i \geq 2$.

The constant $r \geq 1$ must be chosen large enough to have:

a) $dim(F_j) \geq 5\ \forall j:$ needed for the dimension hypothesis in Theorem 2.1 (this means $2q + r - k \geq 5$).

b) $r + q - 1 \geq \left[\frac{2q+r}{2}\right]$ (which is true for $r \geq 1$) to insure the hypothesis on the higher homotopy groups in Theorem 2.1, as we explained above.

c) $r > k - q$.

It follows that the first manifold F_k in the sequence (1) is closed and simply connected, of dimension $2q + r - k > q$. Using property 3 we find that the composition

$$F_k \hookrightarrow M \hookrightarrow W \to X_0 \tag{4}$$

induces isomorphisms at the level of the higher homotopy groups π_i for $i = 2, \ldots, q + r - 1$. By Whitehead's theorem, the induced application between the universal covers of F_k and X_0 is an isomorphism at the level of the i^{th} homology group for $i \leq q + r - 1$. In particular, since $q + r - 1 \geq 2q + r - k$ (since by hypothesis $k > q$), we have

$$H_{2q+r-k}(\widetilde{F}_k) \approx H_{2q+r-k}(\widetilde{X}_0). \tag{5}$$

But F_k is simply connected of dimension $2q + r - k$, so the left side of (5) is $\mathbf{Z}$. On the other hand $2q + r - k > q = dim(X_0)$, so the right side of (5) vanishes. This contradicts the initial assumption on the finite generation of the homotopy of X and statement a) of the theorem is proved.

b) Suppose now that $k = h(\pi_1(X)) \in \{q-1, q\}$. As above we get a closed simply connected manifold F_k of dimension $2q + r - k$ (i.e. $q + r + 1$ or $q + r$) such that the application $H_i(F_k) \to H_i(\widetilde{X_0})$ is an isomorphism for $i \leq q + r - 1$. In particular, the homology $H_i(F_k; \mathbf{Z})$ vanishes for $i = q + 1, \ldots, q + r - 1$.

By the universal coefficients theorem we obtain that the cohomology $H^i(F_k; \mathbf{Z})$ is zero for $i = q + 2, \ldots, q + r - 1$, so Poincaré duality implies that $H_i(F_k; \mathbf{Z})$ also vanishes for $i = q - k + 1, \ldots, q + r - k - 2$. Now $q - k + 1 \in \{1, 2\}$ and F_k is simply connected. We infer after choosing r sufficiently large that the integer homology of F_k vanishes in the degrees $i \leq q$ which means that $\widetilde{X}_0$ is contractible. Therefore X_0 and X are Eilenberg-Mac Lane spaces. In particular the cohomological dimension of $\pi_1(X)$ cannot exceed q, which implies that $\pi_1(X)$ is torsion free.

This completes the proof. $\square$

3. Novikov homology and fibrations over the circle

In the preceding section we showed that our main results (Theorems 1.2, 1.3 and 1.4) are consequences of Theorem 2.1. Let M be a closed connected manifold and $u \in H^1(M; \mathbf{Z}) \approx Hom(\pi_1, \mathbf{Z})$. The aim of the present section is to introduce the Novikov homology $H_*(M, u)$ and to describe the situations when the vanishing of $H_*(M, u)$ implies the existence of a fibration $f : M \to \mathbf{S}^1$, i.e. the conclusion of Theorem 2.1. Then in Section 4 we prove that the hypothesis of Theorem 2.1 implies the vanishing of the Novikov homology $H_*(M, u)$.

3.1. **Novikov homology.** Let $u \in H^1(M; \mathbf{R})$. Denote by Λ the ring $\mathbf{Z}[\pi_1(M)]$ and by $\hat{\Lambda}$ the Abelian group of formal series $\mathbf{Z}[[\pi_1(M)]]$. Consider a $\mathcal{C}^1$-triangulation of M which we lift to the universal cover $\widetilde{M}$. We get a Λ-free complex $C_\bullet(M)$ spanned by (fixed lifts of) the cells of the triangulation of M.

We define now *the completed ring* Λ_u :

$$\Lambda_u := \left\{ \lambda = \sum n_i g_i \in \hat{\Lambda} \mid g_i \in \pi_1(M),\ n_i \in \mathbf{Z},\ u(g_i) \to +\infty \right\}.$$

The convergence to $+\infty$ means here that for all $A > 0$, $u(g_i) < A$ only for a finite number of g_i which appear with a non-zero coefficient in the sum λ.

Remark. Let $\lambda = 1 + \sum n_i g_i$ where $u(g_i) > 0$ for all i. Then λ is invertible in Λ_u. Indeed, if we denote by $\lambda_0 = \sum n_i g_i$, then it is easy to check that $\sum_{k \geq 0} (-\lambda_0)^k$ is an element of Λ_u and it is obvious that it is the inverse of λ.

Definition. Let $C_\bullet(M, u)$ be the Λ_u-free complex $\Lambda_u \otimes_\Lambda C_\bullet(M)$. The Novikov homology $H_*(M, u)$ is the homology of the complex $C_\bullet(M, u)$.

A purely algebraic consequence of the previous definition is the following version of the universal coefficients theorem ([14], p. 102, Th. 5.5.1):

Theorem 3.1. *There is a spectral sequence E^r_{pq} which converges to $H_*(M, u)$ and such that*

$$E^2_{pq} = Tor^\Lambda_p(H_q(\widetilde{M}), \Lambda_u).$$

We will use this result in Section 4 to prove that in the hypothesis of Theorem 1.3 the Novikov homology associated to some class vanishes.

3.2. **Morse-Novikov theory.** We recall in this subsection the relation between Novikov homology and closed one forms. In dimension $n \geq 6$, when the Novikov homology vanishes, some hypothesis on π_1 stated below implies the existence of a nowhere vanishing closed one form on M. It is well known (see [35]) that the existence of such a form is equivalent to the existence of a fibration of M over $\mathbf{S}^1$.

Let α be a closed generic one form in the class u. Let ξ be the gradient of α with respect to some generic metric on M. For every critical point c of α we fix a point $\tilde{c}$ above c in the universal cover $\widetilde{M}$. We can then define a complex $C_\bullet(\alpha, \xi)$ spanned by the zeros of α: the incidence number $[d, c]$ for two zeros of consecutive indices is the (possibly infinite) sum $\sum n_i g_i$ where n_i is the algebraic number of flow lines which join c and d and which are covered by a path in $\widetilde{M}$ joining $g_i \tilde{c}$ and $\tilde{d}$. It turns out that this incidence number belongs to Λ_u, so $C_\bullet(\alpha, \xi)$ is actually a Λ_u-free complex.

The fundamental property of the Novikov homology is the following [23], [33]:

Theorem 3.2. *For any generic couple (α, ξ) the Novikov homology $H_*(M, [\alpha])$ is isomorphic to the homology of the complex $C_\bullet(\alpha, \xi)$.*

We have a straightforward corollary:

Corollary 3.3. *For any $q \geq 2$ and $u \in H^1(M)$ we have an isomorphism*

$$H_*(M \times \mathbf{S}^q, u) \approx H_*(M, u) \oplus H_{*+q}(M, u).$$

Proof of Corollary 3.3. Take a generic couple (α, ξ) on M, and a couple (df, η) on $\mathbf{S}^q$, where f is a function which has only two critical points. Then, obviously, we have the equality

$$C_\bullet(\alpha + df, \xi + \eta) = C_\bullet(\alpha, \xi) \oplus C_{\bullet+q}(\alpha, \xi).$$

The lemma is proved, using Theorem 3.2. □

Another consequence of Theorem 3.2 is obtained by comparing the complexes $C_\bullet(\alpha, \xi)$ and $C_\bullet(-\alpha, -\xi)$. We get the following duality property (see Prop. 2.8 in [9] and 2.30 in [19]):

Theorem 3.4. *Let M^n be a closed connected manifold, let $u \in H^1(M; \mathbf{R})$ and let l be an integer. If $H_i(M, -u) = 0$ for $i \leq l$, then $H_i(M, u) = 0$ for $i \geq n - l$.*

If the form α has no zeroes, then $C_\bullet(\alpha, \xi)$ vanishes and therefore we have $H_*(M, [\alpha]) = 0$. Conversely, one can ask if the vanishing of $H_*(M, u)$ implies the existence of a nowhere vanishing closed 1-form belonging to the class $u \in H^1(M)$. For $n \geq 6$ this problem was independently solved by F. Latour [19] and A. Pajitnov [24], [25]. The statement is ([19], Th.1′):

Theorem 3.5. *For $dim(M) \geq 6$ the following set of conditions is equivalent to the existence of a nowhere vanishing closed 1-form in $u \in H^1(M, \mathbf{Z})$:*

1. *Vanishing Novikov homology $H_*(M, u)$.*
2. *Vanishing Whitehead torsion $\tau(M, u) \in Wh(M, u)$.*
3. *Finitely presented $Ker(u) \subset \pi_1(M)$.*

Using Corollary 3.3 and Remark 1, below, one immediately infers:

Corollary 3.6. *If $q \geq 2$ is an integer such that $dim(M) + q \geq 6$, the same conditions 1-3 of Theorem 3.5 are equivalent to the existence of a nowhere vanishing closed one form on $M \times \mathbf{S}^q$ in the cohomology class*

$$u \in H^1(M, \mathbf{Z}) \approx H^1(M \times \mathbf{S}^q, \mathbf{Z}).$$

Moreover, if q is odd condition 2 is always fulfilled (see Remark 1 below), so 1 and 3 are sufficient for the existence of a nowhere vanishing one form in the class u.

Remarks. 1. The definitions of the generalized Whitehead group $Wh(M, u)$ and of the Whitehead torsion $\tau(M, u)$ are given in [19]. $Wh(M, u)$ is an Abelian group which only depends on $\pi_1(M)$ and on u. The Whitehead torsion is an element of $Wh(\pi_1(M), u)$ which is associated to the acyclic complex $C_\bullet(M, u)$. Whitehead torsion is additive:

$$\tau(C_\bullet \oplus D_\bullet) = \tau(C_\bullet) + \tau(D_\bullet),$$

and satisfies

$$\tau(C_{\bullet+1}) = -\tau(C_\bullet).$$

In particular, the equality of complexes in the proof of Corollary 3.3 above shows that

$$\tau(M \times \mathbf{S}^q, u) \;=\; \tau(M, u) + (-1)^q \tau(M, u),$$

therefore

$$\tau(M, u) = 0 \;\Rightarrow\; \tau(M \times \mathbf{S}^q, u) = 0$$

and $\tau(M \times \mathbf{S}^q, u) = 0$ for q odd.

2. In the statement of [24] the first two conditions are replaced by:

$1'$. $C_\bullet(M, u)$ is simply equivalent to zero.

Actually, one can show (see [20]) that $1'$ is equivalent to "1 and 2".

3. In earlier works on the subject as those of F. T. Farrell [11] and L. Siebenmann [32] the algebraic conditions which are equivalent to the existence of a nowhere vanishing closed 1-form in a rational cohomology class u were stated in the hypothesis that the infinite cyclic cover P associated to u is finitely dominated. ($P \to M$ is defined to be the pull-back of the universal covering $\mathbf{R} \to \mathbf{S}^1$ defined by a function $f : M \to \mathbf{S}^1$ such that $[f^* d\theta] = u$.) The relation between the finite domination of P and vanishing of the Novikov homology was first established by A. Ranicki in [28], [29]. A. Ranicki ([27], Chap.14) established a relation between the Whitehead "fibering obstruction" from [11] and [32] and condition 2 above. Then A. Pajitnov and A. Ranicki proved in [26] the following

Theorem 3.7. *If G is a group with $Wh(G) = 0$ and $u : G \to \mathbf{Z}$ is a morphism, then $Wh(G, u) = 0$.*

It follows that under the hypothesis of Theorem 2.1 condition 2 of Theorem 3.5 is always satisfied.

In order to prove Theorem 1.5 we use

Proposition 3.8. *If $H_*(M, u) = 0$ for some class u, then $\chi(M) = 0$.*

Proof. The complex $C_\bullet(M, u)$ is acyclic. As in [20] one can then show that this complex is simply equivalent to a complex of the form

$$0 \to \Lambda_u^p \xrightarrow{\partial} \Lambda_u^q \to 0.$$

This means that the second complex is isomorphic to the first after adding or cancelling a finite number of trivial summands

$$0 \to \Lambda_u \xrightarrow{Id} \Lambda_u \to 0.$$

Now the ring Λ_u satisfies an invariant basis property so $p = q$, and in particular

$$\chi(M) = \chi(C_\bullet(M, u)) = 0. \qquad \square$$

4. Novikov homology and finiteness properties of groups

In this section we will achieve the proof of our results Theorems 1.2 ... 1.5. Up to now we have proved that Theorems 1.2, 1.3 and 1.4 are implied by Theorem 2.1 (Section 2) and that the conclusion of Theorem 2.1 is valid if the three conditions of Theorem 3.5 are fulfilled. It remains to show that the hypothesis of Theorem 2.1 implies these three conditions. It is clearly the case for the third one, since in the hypothesis of Theorem 2.1, $Ker(u)$ is of type $\mathcal{F}_{[\frac{n}{2}]}$, in particular of type $\mathcal{F}_2$, i.e. finitely presented. The second condition is also satisfied, according to Theorem 3.7.

The main point of our proof is the vanishing of the Novikov homology $H_*(M,u)$ under the assumption of the hypothesis of Theorem 2.1.

Recall that, by Theorem 3.1, there is a spectral sequence E^r_{pq} which converges to $H_*(M,u)$ and whose term E^2_{pq} is equal to $Tor_p^{\Lambda}(H_q(\widetilde{M}),\Lambda_u)$. Recall also that, by the duality result of Theorem 3.4 we have the implication

$$H_i(M,\pm u)=0\ \forall i\le\left[\frac{n}{2}\right]\quad\Rightarrow\quad H_i(M,u)=0\,\forall i.$$

It suffices therefore to prove the following statement:

Proposition 4.1. *Let M^n be a closed manifold. Suppose that $\pi_1(M)$ is of type $\mathcal{F}_{\left[\frac{n}{2}\right]}$. Suppose also that $\pi_i(M)$ are finitely generated for $i\le\left[\frac{n}{2}\right]$.*

Suppose there is a non-zero cohomology class $u\in H^1(M;\mathbf{R})\approx Hom(\pi_1(M),\mathbf{R})$ such that $Ker(u)$ is of type $\mathcal{F}_{\left[\frac{n}{2}\right]}$. Then, for all integers $0\le p,q\le\left[\frac{n}{2}\right]$, we have

$$Tor_p^{\Lambda}(H_q(\widetilde{M}),\Lambda_{\pm u})\;=\;0.$$

Note that this result is purely algebraic. In order to prove it we use some facts about

4.1. **Hurewicz-type morphisms.** Recall that the classical Hurewicz theorem asserts that for $q\ge 2$ the canonical morphism $I_q:\pi_q(M)\to H_q(M)$ is an isomorphism provided that M is $(q-1)$-connected.

In [31] J-P. Serre generalized this theorem (see also [34], p. 504). For some "admissible" classes of groups $\mathcal{C}$, he showed that, if X is simply connected such that $\pi_i(X)\in\mathcal{C}$ for $i=1,\dots,q-1$, where $q\ge 2$, then I_q is an isomorphism mod $\mathcal{C}$: This means that $Ker(I_q)$ and $Coker(I_q)$ are in $\mathcal{C}$.

The class of finitely generated Abelian groups is such an admissible class. In particular we have:

Theorem 4.2. *Let X be a simply connected space. Then $\pi_i(X)$ is finitely generated for $i\le q$ iff $H_i(X)$ is finitely generated for $i\le q$.*

In particular any compact, simply connected CW-complex has finitely generated homotopy groups (which is Theorem 1.1*).*

By applying this theorem to $\widetilde{M}$ we may replace the hypothesis on $\pi_i(M)$ by the analogue hypothesis on $H_i(\widetilde{M})$. From now on we will suppose that for all $i\le\left[\frac{n}{2}\right]$, $H_i(\widetilde{M})=\mathbf{Z}^{r_i}\oplus T_i$, where T_i is a torsion finitely generated $\mathbf{Z}$-module. The proof of Proposition 4.1 relies on the following statement:

Proposition 4.3. *Let π be a group of type $\mathcal{F}_p$ for some positive integer p and $u:\pi\to\mathbf{Z}$ a morphism whose kernel is of type $\mathcal{F}_p$. Suppose that $\mathbf{Z}^r$ is a π-module. Denote by Λ the ring $\mathbf{Z}[\pi]$ and by Λ_u the completed ring, as above. Let $\pi_0\le\pi$ be a normal subgroup of finite index and Λ_0 the corresponding group ring $\mathbf{Z}[\pi_0]$. Then for $i\le p$ we have*

$$Tor_i^{\Lambda_0}(\mathbf{Z}^r,\Lambda_u)\;=\;0.$$

Remark. For $\pi_0=\pi$ and $r=1$ (and therefore for arbitrary r and trivial action of π on $\mathbf{Z}^r$) the statement above was proved by J-C. Sikorav in this thesis [33].

We postpone the proof of Proposition 4.3 to Subsection 4.4. We now show:

4.2. **Proof of Proposition 4.3 $\implies$ Proposition 4.1.** Fix $p, q \le \left[\frac{n}{2}\right]$. For $g \in \pi_1(M)$, denote by ϕ_g the automorphism of $H_q(\widetilde{M}) = \mathbf{Z}^{r_q} \oplus T_q$, given by the action of $\pi_1(M)$. We have:

$$\phi_g = \begin{pmatrix} a_g & 0 \\ b_g & c_g \end{pmatrix},$$

where $a_g : \mathbf{Z}^{r_q} \to \mathbf{Z}^{r_q}$, $b_g : \mathbf{Z}^{r_q} \to T_q$ and $c_g : T_q \to T_q$. Note that a_g and c_g are automorphisms (of inverses $a_{g^{-1}}$, resp. $c_{g^{-1}}$). Let

$$\pi_0 = \{ g \in \pi_1 \,|\, c_g = Id \}.$$

Since T_q is finite, there is only a finite number of automorphisms $c : T_q \to T_q$. Therefore π_0 is a normal subgroup of $\pi_1(M)$ of finite index. (It is the kernel of the morphism $\pi_1(M) \to Aut(T_q)$ defined by $g \mapsto c_g$.) We use the following:

Lemma 4.4. *Let G be a group of type $\mathcal{F}_p$ and $G_0 \le G$ a normal subgroup of finite index. Then G_0 is of type $\mathcal{F}_p$.*

The proof of Lemma 4.4 is obvious: if Q is a $K(G,1)$ with finite p-skeleton, then the finite cover of Q corresponding to $G_0 \le G$ will be a $K(G_0,1)$ with finite p-skeleton.

Now let $u : \pi_1(M) \to \mathbf{R}$, as in the statement of Proposition 4.1, and let $u_0 = u|_{\pi_0}$. Obviously, $Ker(u_0)$ has finite index in $Ker(u)$ so, using Lemma 4.4, both π_0 and $Ker(u_0)$ are of type $\mathcal{F}_p$.

Now consider the short exact sequence:

$$0 \to T_q \xrightarrow{(0,Id)} \mathbf{Z}^{r_q} \oplus T_q \xrightarrow{Id \oplus 0} \mathbf{Z}^{r_q} \to 0.$$

One immediately checks that this is an exact sequence of π_0-modules (where the action of π_0 on $\mathbf{Z}^{r_q}$ is $x \mapsto a_g(x)$). Now consider $\Lambda_0 = \mathbf{Z}[\pi_0]$, and view the completed ring Λ_u as a Λ_0-module. The tensor product of Λ_u and the exact sequence above yields a long exact sequence:

$$(1) \qquad \cdots \to Tor_p^{\Lambda_0}(T_q, \Lambda_u) \to Tor_p^{\Lambda_0}(\mathbf{Z}^{r_q} \oplus T_q, \Lambda_u) \to Tor_p^{\Lambda_0}(\mathbf{Z}^{r_q}, \Lambda_u) \to \cdots.$$

As a consequence of Proposition 4.3, the right term in the sequence above vanishes. In order to prove that the left term is zero, we consider an exact sequence of the form

$$0 \to \mathbf{Z}^m \to \mathbf{Z}^m \to T_q \to 0,$$

which is viewed as a sequence of π_0-modules with trivial actions (recall that, by construction, π_0 acts trivially on T_q). The corresponding long exact sequence given by the tensor product with Λ_u writes:

$$(2) \qquad \cdots \to Tor_p^{\Lambda_0}(\mathbf{Z}^m, \Lambda_u) \to Tor_p^{\Lambda_0}(T_q, \Lambda_u) \to Tor_{p-1}^{\Lambda_0}(\mathbf{Z}^m, \Lambda_u) \to \cdots.$$

Applying again Proposition 4.3 we infer $Tor_p^{\Lambda_0}(T_q, \Lambda_u) = 0$, therefore the middle term in (1) vanishes.

We have thus established that for each $p, q \le \left[\frac{n}{2}\right]$

$$(3) \qquad Tor_p^{\Lambda_0}(H_q(\widetilde{M}), \Lambda_u) = 0,$$

which is the assertion required in Proposition 4.1, with Λ_0 instead of Λ. To complete the proof we need the following results.

Proposition 4.5. *Let G be a group, let L be a $\mathbf{Z}[G]$ right module and let N be a $\mathbf{Z}[G]$ left module. Assume that N is $\mathbf{Z}$-torsion-free. Then*

$$Tor_*^{\mathbf{Z}[G]}(L,N) \approx H_*(G, L \otimes_{\mathbf{Z}} N),$$

where G acts diagonally on $L \otimes_{\mathbf{Z}} N$: $g(x \otimes y) = xg^{-1} \otimes gy$.

This proposition is proved in [8] (prop. 2.2, p. 61).

Theorem 4.6. *For any group extension*

$$1 \to K \to G \to Q \to 1$$

and any G-module R there is a spectral sequence of the form

$$E^2_{ij} = H_i(Q, H_j(K,R))$$

which converges to $H_{i+j}(G,R)$.

This theorem is due to G. Hochschild and J-P. Serre [17] (see also [8], p. 171).

We first apply Proposition 4.5 and infer from (3) that for each $p, q \leq \left[\frac{n}{2}\right]$ we have:

$$H_p(\pi_0, H_q(\widetilde{M}) \otimes_{\mathbf{Z}} \Lambda_u) = 0. \tag{4}$$

Note that the hypothesis of Proposition 4.5 is fulfilled by $N = \Lambda_u$.

We fix q and denote by R the $\pi_1(M)$-module $H_q(\widetilde{M}) \otimes_{\mathbf{Z}} \Lambda_u$ (for the diagonal action). Then we apply Theorem 4.6 to the extension

$$1 \to \pi_0 \to \pi_1(M) \to \pi_1(M)/\pi_0 \to 1$$

and to the module R. We find using (4) that $E^2_{ij} = 0$ for all $j \leq \left[\frac{n}{2}\right]$ and for all $i \in \mathbf{N}$. According to Theorem 4.6, this implies that

$$H_i(\pi_1(M), R) = 0 \ \ \forall i \leq \left[\frac{n}{2}\right]. \tag{5}$$

Finally, we once again apply Proposition 4.5 and we get that for all $i, q \leq \left[\frac{n}{2}\right]$ we have

$$Tor_i^{\Lambda}(H_q(\widetilde{M}), \Lambda_u) = 0.$$

The analogous relation for Λ_{-u} instead of Λ_u can be established in the same way, so Proposition 4.1 follows. □

We only have to prove Proposition 4.3 to complete the proof of Theorems 1.2, 1.3 and 1.4. The proof involves some facts about Bieri-Renz invariants. We recall the definition and some properties of these invariants in the subsection below.

4.3. **Bieri-Renz invariants. Proof of Proposition 4.3.** Let G be a group of type $\mathcal{F}_m$. We call two non-zero homomorphisms $u, v : G \to \mathbf{R}$ equivalent if $u = \lambda v$ for some positive $\lambda \in \mathbf{R}$. We denote by $S(G)$ the quotient $Hom(G, \mathbf{R})/\approx$ (which is an $(rk(G)-1)$-dimensional sphere). The Bieri-Renz invariants $\Sigma^i(G)$ and $\Sigma^i(G, \mathbf{Z})$, defined for $i = 1, \ldots, m$, are open subsets of $S(G)$. They were introduced by R. Bieri, W. Neumann and R. Strebel in [4] for $i = 1$ and by R. Bieri and B. Renz in [5] for $i \geq 2$.

These invariants are defined as follows. Let X be a $K(G,1)$ which is a complex with finite m-skeleton and let $u : G \to \mathbf{R}$ be a non-zero homomorphism. Then there exists an *equivariant height function* $f : \widetilde{X} \to \mathbf{R}$, i.e. a function which satisfies $f(gx) = f(x) + u(g)$. (If X is a manifold, then f is a primitive of the pullback of some 1-form in the class u.) It can be shown that the difference of two such functions is bounded. Consider the maximal subcomplex X_f of $\widetilde{X}$ whose image by f is contained in $[0,+\infty[$.

Definition. Let $u \in S(G)$ and $i \in \{1,\dots,m\}$. Then u belongs to $\Sigma^i(G)$ (resp. to $\Sigma^i(G,\mathbf{Z})$), if for some couple (X,f) as above the subcomplex X_f is $(i-1)$-connected (resp. $(i-1)$-acyclic).

Bieri and Renz show that the definition does not depend on (X,f). The following properties of the Bieri-Renz invariants are obvious from the definition:
a) $\Sigma^i(G) \subset \Sigma^i(G,\mathbf{Z})$.
b) $\Sigma^1(G) = \Sigma^1(G,\mathbf{Z})$.
c) For $i \geq 2$, $\Sigma^i(G) = \Sigma^i(G,\mathbf{Z}) \cap \Sigma^2(G)$.
The most striking application of these invariants is stated in the following:

Theorem 4.7. *Let $N \subset G$ be a normal subgroup with Abelian quotient G/N. Denote by $S(G,N)$ the subset of $S(G)$ defined by $\{u \in S(G) \,|\, u|_N = 0\}$. Then, for any $i = 1,\dots,m$, we have the equivalence:*
N is of type $\mathcal{F}_i$ iff $S(G,N) \subset \Sigma^i(G)$.

Note that $\{u,-u\} \subset S(G, Ker(u))$. When $Im(u)$ is cyclic it is easy to show that these two sets coincide. Note also that $S(G,[G,G]) = S(G)$. So, one immediately infers the following corollary:

Corollary 4.8. i) *Let $u \in S(G)$ and let i be an integer as above. If $Ker(u)$ is $\mathcal{F}_i$, then $\pm u \in \Sigma^i(G)$. The converse is valid if $Im(u)$ is cyclic.*
ii) *$\Sigma^i(G) = S(G)$ if and only if $[G,G]$ is of type $\mathcal{F}_i$.*

In the sequel we give an algebraic description of the invariants $\Sigma^i(G,\mathbf{Z})$ following [5], Section 4. Fix a non-zero homomorphism $u : G \to \mathbf{R}$. Let F be a finitely generated free $\mathbf{Z}[G]$-module, and $\{e_i\}_{i=1,\dots,k}$ a basis of F. We define an application $v : F \to \mathbf{R}$ as follows: v is defined arbitrarily on the elements e_i. Then for any $g \in G$ we put $v(ge_i) = v(e_i) + u(g)$. Finally, $\lambda = \sum_{i,j} n_{ij} g_j e_i$ for $\lambda \in F$ in the basis $\{e_i\}$, we define

$$v(\lambda) \,=\, inf\{v(g_j e_i) \,|\, n_{ij} \neq 0\},$$

and $v(0) = +\infty$. Following Bieri and Renz we call v a *valuation extending* u. Now suppose that G is of type $\mathcal{F}_m$ and let

$$P_m \overset{\partial_m}{\to} P_{m-1} \overset{\partial_{m-1}}{\to} \cdots \overset{\partial_1}{\to} P_0 \overset{\partial_0}{\to} \mathbf{Z} \to 0$$

be a $\mathbf{Z}[G]$-free, finitely generated resolution (which exists since there exists a $K(G,1)$ with finite m-skeleton). Define as above $v_i : P_i \to \mathbf{R}$ which are valuations extending u. We may suppose in addition that for any $i = 1,\dots,m$ and for any $x \in P_i$ we have

$$v_i(x) \leq v_{i-1}(\partial_i(x)). \qquad (1)$$

Indeed, one easily sees that it suffices to check the relation above on the basis $\{e_i\}$ of P_i. We can construct v inductively, by choosing $v(e_i)$ sufficiently negative in order to satisfy the inequality (1).

Bieri and Renz proved the following theorem ([5], theorem 4.1):

Theorem 4.9. *Let $P_\bullet \to \mathbf{Z}$ be a $\mathbf{Z}[G]$-free, finitely generated resolution of length m and let $v : P_\bullet \to \mathbf{R}$ be a valuation extending u satisfying* (1). *Then $u \in \Sigma^m(G,\mathbf{Z})$ if* *and only if there exists a chain endomorphism* $\Phi : P_\bullet \to P_\bullet$ *which lifts the identity of* $\mathbf{Z}$ *and which satisfies the property*

$$v(\Phi(x)) > v(x) \ \ \forall\, x \in P_\bullet. \tag{2}$$

We will reformulate this theorem as follows. It is obvious that one can construct the valuation $v : P_\bullet \to \mathbf{R}$ such that it satisfies an additional feature: for all $j = 1, \dots, m$, v is constant on the set $\{e_i^j\}$ of the basis elements of P_j. Denote this constant by ν_j. Then, the valuation $v_j : P_j \to \mathbf{R}$ is given by

$$v_j(\lambda) \, = \, \nu_j + inf\{u(g_k) \,|\, n_{ik}^j \neq 0\},$$

where $\lambda = \sum_{i,k} n_{ik}^j g_k e_i^j$, $g_k \in G$, $n_{ik}^j \in \mathbf{Z}$. Thus, for an endomorphism Φ given by Theorem 4.9, if

$$\Phi(e_i^j) \, = \, \sum_{i,k} n_{ik}^j g_{ik}^j e_k^j$$

(where $g_{ik}^j \in G$ and $n_{ik}^j \in \mathbf{Z}$), the inequality (2) implies that $u(g_{ik}^j) > 0$ for all elements of G appearing with non-zero coefficient n_{ik}^j.

We call an element $\lambda = \sum_i n_i g_i$ of $\mathbf{Z}[G]$ u-positive if $u(g_i) > 0$ for any g_i which has a non-zero coefficient n_i in the writing of λ. We call a matrix $A \in \mathcal{M}_k(\mathbf{Z}[G])$ u-positive if all its entries are u-positive. Taking into account the preceding remarks, Theorem 4.9 can be stated as follows:

Theorem 4.10. *Let* $P_\bullet \to \mathbf{Z}$ *be a resolution of length* m *which is* $\mathbf{Z}[G]$*-free and finitely generated. Fix a basis for each* P_j *for all* $j = 1, \dots, m$*. Then* $u \in \Sigma^m(G, \mathbf{Z})$ *if and only if there exists a chain endomorphism* $\Phi : P_\bullet \to P_\bullet$ *which lifts the identity of* $\mathbf{Z}$ *and such that for all* $j = 1, \dots, m$ *the matrices of* $\Phi_j : P_j \to P_j$ *in the fixed basis are* u*-positive.*

Now we are able to complete the

Proof of Proposition 4.3. Applying Proposition 4.5 we infer that $Tor_i^{\Lambda_0}(\mathbf{Z}^r, \Lambda_u)$ is isomorphic to $H_i(\pi_0, \mathbf{Z}^r \otimes_{\mathbf{Z}} \Lambda_u)$, where the action of π_0 on $\mathbf{Z}^r \otimes_{\mathbf{Z}} \Lambda_u$ is given by $x \otimes \lambda \mapsto xg^{-1} \otimes g\lambda$. It suffices therefore to prove that the latter vanishes for $i \leq p$.

Let

$$P_p \to P_{p-1} \to \cdots \to P_1 \to P_0 \to \mathbf{Z}$$

be a $\mathbf{Z}[\pi_0]$-free resolution which we may suppose is finitely generated since π_0 is $\mathcal{F}_p$. By definition we have

$$H_i(\pi_0, \mathbf{Z}^r \otimes_{\mathbf{Z}} \Lambda_u) \approx H_i(P_\bullet \otimes_{\mathbf{Z}[\pi_0]} (\mathbf{Z}^r \otimes_{\mathbf{Z}} \Lambda_u)). \tag{3}$$

We will prove that the right term of (3) vanishes for all $i \leq p$. Fix a basis $\{e_i^j\}_i$ for each module P_j. Since $Ker(u|_{\pi_0})$ is of type $\mathcal{F}_p$, it follows by Corollary 4.8.i) that $u \in \Sigma^p(\pi_0, \mathbf{Z})$, so, applying Theorem 4.10 we obtain an endomorphism $\Phi : P_\bullet \to P_\bullet$ such that for all $j = 1, \dots, m$ the matrix of Φ_j associated to the basis $\{e_i^j\}$ is u-positive.

Let $\Psi = \Phi \otimes Id : P_\bullet \otimes_{\mathbf{Z}[\pi_0]} (\mathbf{Z}^r \otimes_{\mathbf{Z}} \Lambda_u) \to P_\bullet \otimes_{\mathbf{Z}[\pi_0]} (\mathbf{Z}^r \otimes_{\mathbf{Z}} \Lambda_u)$. □

The proof of Proposition 4.3 will be complete if we prove the following:

Lemma 4.11. *The homomorphism* $Id - \Psi$ *is invertible and it induces zero in homology.*

Proof. Let us prove first that $Id - \Psi$ vanishes in homology. Since $\Phi : P_\bullet \to P_\bullet$ lifts the identity of $\mathbf{Z}$, and $P_\bullet$ is free, it is well known and easy to prove that there exists a homotopy $s : P_\bullet \to P_{\bullet+1}$ between Φ and Id. It follows that $s \otimes Id$ is a homotopy between Ψ and Id, so $Id - \Psi$ induces the zero morphism in homology.

Now let us prove the first assertion of the lemma. Let $\{f_1, f_2, \ldots, f_r\}$ be the canonical basis of $\mathbf{Z}^r$. We can see $\mathbf{Z}^r \otimes_{\mathbf{Z}} \Lambda_u$ as a right Λ_u-module endowed with the canonical structure $(x \otimes \lambda)\mu = x \otimes \lambda\mu$. It is a free module of rank r and $\{f_1 \otimes 1, f_2 \otimes 1, \ldots, f_r \otimes 1\}$ is a basis for this module.

Recall that we denoted by Λ_0 the ring $\mathbf{Z}[\pi_0]$. For any $j = 1, \ldots, m$, the product $P_j \otimes_{\Lambda_0} (\mathbf{Z}^r \otimes_{\mathbf{Z}} \Lambda_u)$ inherits the structure of the right Λ_u-module described above. On the other hand, since P_j is free we have

$$P_j \otimes_{\Lambda_0} (\mathbf{Z}^r \otimes_{\mathbf{Z}} \Lambda_u) \approx (\mathbf{Z}^r \otimes_{\mathbf{Z}} \Lambda_u)^{rk(P_j)} \approx (\Lambda_u)^{r \cdot rk(P_j)}. \tag{4}$$

If $\{e_i^j\}_{i=1,\ldots,rk(P_j)}$ is the given basis of P_j, then

$$\{e_i^j \otimes_{\Lambda_0} (f_s \otimes 1)\}_{i=1,\ldots,rk(P_j),\, s=1,\ldots,r} \tag{5}$$

will be a basis for $P_j \otimes_{\Lambda_0} (\mathbf{Z}^r \otimes_{\mathbf{Z}} \Lambda_u)$.

It is easy to check that the isomorphisms (4) preserve the right Λ_u-module structure. Moreover, the differential $\partial \otimes_{\Lambda_0} Id$ of the complex $P_\bullet \otimes_{\Lambda_0} (\mathbf{Z}^r \otimes_{\mathbf{Z}} \Lambda_u)$ respects this structure, therefore it is a complex of free right Λ_u-modules. □

We claim that the matrices of Ψ in the basis (5) are u-positive. Indeed, for fixed j, let $(\lambda_{ik})_{i,k=1,\ldots,rk(P_j)}$ be the matrix of Φ_j in the basis $\{e_i^j\}$. This matrix is known as u-positive. We dropped the index j from the coefficients of the matrix to simplify the notations. Denote by $\bar{\lambda}$ the image of an element $\lambda \in \Lambda_0$ under the endomorphism $\Lambda_0 \to \Lambda_0$ induced by the involution of π_0: $g \mapsto g^{-1}$. The right action of $\bar{\lambda}_{ik}$ on $\mathbf{Z}^r$, evaluated on the basis $\{f_s\}$, is:

$$f_s \bar{\lambda}_{ik} = \sum_{l=1}^{r} n_{iks}^l f_l \,,$$

for some integers n_{iks}^l.

We infer that

$$\begin{aligned} \Psi_j(e_i^j \otimes_{\Lambda_0} (f_s \otimes 1)) &= \sum_k e_k^j \lambda_{ik} \otimes_{\Lambda_0} (f_s \otimes 1) \\ = \sum_k e_k^j \otimes_{\Lambda_0} (f_s \bar{\lambda}_{ik} \otimes \lambda_{ik}) &= \sum_{k,l} e_k^j \otimes_{\Lambda_0} (n_{iks}^l f_l \otimes \lambda_{ik}) \\ = \sum_{k,l} e_k^j \otimes_{\Lambda_0} (f_l \otimes n_{iks}^l \lambda_{ik}) &= \sum_{k,l} [e_k^j \otimes_{\Lambda_0} (f_l \otimes 1)]\, n_{iks}^l \lambda_{ik}, \end{aligned}$$

so the matrix of Ψ_j is u-positive.

Denote this matrix by A_j and define another matrix B_j by $B_j = Id + \sum_{k=1}^{+\infty} A_j^k$. As A_j is u-positive it is easy to check that the matrix B_j belongs to $\mathcal{M}_{r \cdot rk(P_j)}(\Lambda_u)$. It therefore defines, using the basis (5), an endomorphism of $P_j \otimes_{\Lambda_0} (\mathbf{Z}^r \otimes_{\mathbf{Z}} \Lambda_u)$. It is actually an automorphism since obviously $(Id - A_j)B_j = Id$. Finally, as Ψ is a morphism of complexes, the morphism $Id + \Psi + \Psi^2 + \cdots$ induced by B_j will also commute with the differential. We finally have an automorphism of $P_\bullet \otimes_{\Lambda_0} (\mathbf{Z}^r \otimes_{\mathbf{Z}} \Lambda_u)$ whose inverse is $Id - \Psi$. This completes the proof of Lemma 4.11 and hence the proof of Proposition 4.3 which implies our main theorems 1.2, 1.3 and 1.4. □

4.4. **Proof of Theorem 1.5.** Assume that the homotopy of X is finitely generated. By embedding X in $\mathbf{R}^{2q+3}$ construct a manifold $M = \partial W$ of dimension $2q+2$ as in the proof of Theorem 1.2. We have $\chi(X) = \chi(W) = 2\chi(M)$. By general position $\pi_i(M) \approx \pi_i(X)$ for $i \leq q+1$. These groups are therefore finitely generated. The hypothesis of Proposition 4.1 is fulfilled and, by applying this result, we get $H_*(M, u) = 0$. But according to Proposition 3.8 this implies $\chi(M) = 0$, thus contradicting the hypothesis on $\chi(X)$.

If X is a manifold, the result follows directly from Propositions 4.1 and 3.8 and Theorem 3.1. The proof is finished. $\square$

Ackowledgements

The author is grateful to Thomas Delzant for useful discussions on the subject.

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