TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 361, Number 4, April 2009, Pages 1949–1961
S 0002-9947(08)04648-5
Article electronically published on October 22, 2008

ON THE NONEXISTENCE OF NONTRIVIAL INVOLUTIVE n-HOMOMORPHISMS OF $C^\star$-ALGEBRAS

EFTON PARK AND JODY TROUT

ABSTRACT. An n-homomorphism between algebras is a linear map $\phi : A \to B$ such that $\phi(a_1 \cdots a_n) = \phi(a_1) \cdots \phi(a_n)$ for all elements $a_1, \ldots, a_n \in A$. Every homomorphism is an n-homomorphism for all $n \geq 2$, but the converse is false, in general. Hejazian *et al.* (2005) ask: Is every $*$-preserving n-homomorphism between $C^\star$-algebras continuous? We answer their question in the affirmative, but the even and odd n arguments are surprisingly disjoint. We then use these results to prove stronger ones: If $n > 2$ is even, then ϕ is just an ordinary $*$-homomorphism. If $n \geq 3$ is odd, then ϕ is a difference of two orthogonal $*$-homomorphisms. Thus, there are no *nontrivial* $*$-linear n-homomorphisms between $C^\star$-algebras.

1. Introduction

Let A and B be algebras and $n \geq 2$ an integer. A linear map $\phi : A \to B$ is an *n-homomorphism* if for all $a_1, a_2, \ldots, a_n \in A$,

$$\phi(a_1 a_2 \cdots a_n) = \phi(a_1)\phi(a_2) \cdots \phi(a_n).$$

A 2-homomorphism is then just a homomorphism, in the usual sense, between algebras. Furthermore, every homomorphism is clearly also an n-homomorphism for all $n \geq 2$, but the converse is false, in general. The concept of n-homomorphism was studied for complex algebras by Hejazian, Mirzavaziri, and Moslehian [7]. This concept also makes sense for rings and (semi)groups. For example, an *AE_n-ring* is a ring R such that every additive endomorphism $\phi : R \to R$ is an n-homomorphism; Feigelstock [4, 5] classified all unital AE_n-rings.

In [7], Hejazian *et al.* ask: Is every $*$-preserving n-homomorphism between $C^\star$-algebras continuous? We answer in the affirmative by proving that every involutive n-homomorphism $\phi : A \to B$ between $C^\star$-algebras is in fact norm contractive: $\|\phi\| \leq 1$. Surprisingly, the arguments for the even and odd n cases are disjoint and, thus, are discussed in different sections. When $n = 3$, automatic continuity is reported by Bračič and Moslehian [2], but note that the proof of their Theorem 2.1 does not extend to the nonunital case since the unitization of a 3-homomorphism is not a 3-homomorphism, in general.

Received by the editors April 6, 2007.
2000 *Mathematics Subject Classification.* Primary 46L05; Secondary 47B99, 47L30.

©2008 American Mathematical Society
Reverts to public domain 28 years from publication

Using these automatic continuity results, we prove the following stronger results: If $n > 2$ is even, every $*$-linear n-homomorphism $\phi : A \to B$ between $C^\star$-algebras is in fact a $*$-homomorphism. If $n \geq 3$ is odd, every $*$-linear n-homomorphism $\phi : A \to B$ is a difference $\phi(a) = \psi_1(a) - \psi_2(a)$ of two orthogonal $*$-homomorphisms $\psi_1 \perp \psi_2$. Regardless, for all integers $n \geq 3$, every *positive* linear n-homomorphism is a $*$-homomorphism. Note that if ψ is a $*$-homomorphism, then $-\psi = 0 - \psi$ is a norm contractive $*$-preserving 3-homomorphism that is not positive linear.

There is also a dichotomy between the unital and nonunital cases. When the domain algebra A is unital, there is a simple representation of an n-homomorphism as a certain n-potent multiple of a homomorphism (discussed in the Appendix). The nonunital case is more subtle. For example, if A and B are nonunital (Banach) algebras such that $A^n = B^n = \{0\}$, then *every* linear map $L : A \to B$ (bounded or unbounded) is, trivially, an n-homomorphism (see Examples 2.5 and 4.3 of [7]).

The outline of the paper is as follows: In Section 2, we prove automatic continuity for the even case and in Section 3 for the odd case. In Section 4, we prove our nonexistence results. A key fact in many of our proofs is the Cohen Factorization Theorem [3] of $C^\star$-algebras. (See Proposition 2.33 in [8] for an elementary proof of this important result.) Finally, in Appendix A, we collect some facts about n-potents that we need.

The authors would like to thank Dana Williams and Tom Shemanske for their helpful comments and suggestions.

2. Automatic continuity: The even case

In this section, we prove that when $n > 2$ is even, every involutive (*i.e.,* $*$-linear) n-homomorphism between $C^\star$-algebras is completely positive and norm contractive, which generalizes the well-known result for $*$-homomorphisms ($n = 2$). Recall that a linear map $\theta : A \to B$ between $C^\star$-algebras is *positive* if $a \geq 0$ implies $\theta(a) \geq 0$ or, equivalently, for every $a \in A$ there is a $b \in B$ such that $\theta(a^*a) = b^*b$. We say that θ is *completely positive* if, for all $k \geq 1$, the induced map $\theta_k : M_k(A) \to M_k(B)$, $\theta_k((a_{ij})) = (\theta(a_{ij}))$, on $k \times k$ matrices is positive.

Theorem 2.1. *Let $\mathcal{H}$ be a Hilbert space. If $n \geq 2$ is even, then every involutive n-homomorphism from a $C^\star$-algebra A into $\mathcal{B}(\mathcal{H})$ is completely positive.*

Proof. Let $\phi : A \to \mathcal{B}(\mathcal{H})$ be an involutive n-homomorphism. We may assume $n = 2k > 2$. Let $\langle \cdot, \cdot \rangle$ denote the inner product on $\mathcal{H}$. By Stinespring's Theorem [9] (see Prop. II.6.6 in [1]), ϕ is completely positive if and only if for any $m > 1$ and elements $a_1, \ldots, a_m \in A$ and vectors $v_1, \ldots, v_m \in \mathcal{H}$ we have

$$\sum_{i,j=1}^{m} \langle \phi(a_i^* a_j) v_j, v_i \rangle \geq 0.$$

We proceed as follows: For each $1 \leq i \leq m$ use the Cohen Factorization Theorem [3] to factor $a_i = a_{i1} \cdots a_{ik}$ into a product of k elements. Thus, their adjoints factor as $a_i^* = a_{ik}^* \cdots a_{i1}^*$. Since $n = 2k$, we compute

$$\begin{aligned}\sum_{i,j=1}^m \langle \phi(a_i^* a_j)v_j, v_i\rangle &= \sum_{i,j=1}^m \langle \phi(a_{ik}^* \cdots a_{i1}^* a_{j1} \cdots a_{jk})v_j, v_i\rangle \\ &= \sum_{i,j=1}^m \langle \phi(a_{ik})^* \cdots \phi(a_{i1})^* \phi(a_{j1}) \cdots \phi(a_{jk})v_j, v_i\rangle \\ &= \langle \sum_{j=1}^m \phi(a_{j1}) \cdots \phi(a_{jk})v_j, \sum_{i=1}^m \phi(a_{i1}) \cdots \phi(a_{ik})v_j\rangle \\ &= \langle x, x\rangle \geq 0,\end{aligned}$$

where $x = \sum_{i=1}^m \phi(a_{i1}) \cdots \phi(a_{ik})v_i \in \mathcal{H}$. The result now follows. $\square$

Even though the previous result is a corollary of the more general theorem below, we have included it because the proof technique is different.

Lemma 2.2. *Let $\phi : A \to B$ be an n-homomorphism. Then, for all $k \geq 1$, the induced maps $\phi_k : M_k(A) \to M_k(B)$ on $k \times k$ matrices are n-homomorphisms. Moreover, if ϕ is involutive ($\phi(a^*) = \phi(a)^*$), then each ϕ_k is also involutive.*

Proof. Given n matrices $a^1 = (a_{ij}^1), \ldots, a^n = (a_{ij}^n)$ in $M_k(A)$, we can express their product $a^1 a^2 \cdots a^n = (a_{ij})$, where the (i,j)-th entry a_{ij} is given by the formula

$$a_{ij} = \sum_{m_1,\cdots,m_{n-1}=1}^k a_{im_1}^1 a_{m_1 m_2}^2 \cdots a_{m_{n-1}j}^n.$$

Since $\phi_k(a^1 a^2 \cdots a^n) = (\phi(a_{ij}))$ by definition and

$$\begin{aligned}\phi(a_{ij}) &= \sum_{m_1,\cdots,m_{n-1}=1}^k \phi(a_{im_1}^1 a_{m_1 m_2}^2 \cdots a_{m_{n-1}j}^n) \\ &= \sum_{m_1,\cdots,m_{n-1}=1}^k \phi(a_{im_1}^1)\phi(a_{m_1 m_2}^2) \cdots \phi(a_{m_{n-1}j}^n) \\ &= [\phi_k(a^1)\phi_k(a^2) \cdots \phi_k(a^n)]_{ij},\end{aligned}$$

it follows that $\phi_k : M_k(A) \to M_k(B)$ is an n-homomorphism. Now suppose that ϕ is involutive. We compute for all $a = (a_{ij}) \in M_k(A)$:

$$\phi_k(a^*) = \phi_k((a_{ji}^*)) = (\phi(a_{ji}^*)) = (\phi(a_{ji})^*) = \phi_k(a)^*$$

and hence each $\phi_k : M_k(A) \to M_k(B)$ is involutive. $\square$

Theorem 2.3. *Let $\phi : A \to B$ be an involutive n-homomorphism between $C^\star$-algebras. If $n \geq 2$ is even, then ϕ is completely positive. Thus, ϕ is bounded.*

Proof. We may assume $n = 2k > 2$. Since ϕ is linear, we want to show that for every $a \in A$ we have $\phi(a^* a) \geq 0$. By the Cohen Factorization Theorem, for any $a \in A$ we can find $a_1, ..., a_k \in A$ such that the factorization $a = a_1 \cdots a_k$ holds. Thus, the adjoint factors as $a^* = a_k^* \cdots a_1^*$. Since $n = 2k$ and ϕ is n-multiplicative and $*$-preserving,

$$\begin{aligned}\phi(a^*a) &= \phi(a_k^* \cdots a_1^* a_1 \cdots a_k)\\ &= \phi(a_k)^* \cdots \phi(a_1)^* \phi(a_1) \cdots \phi(a_k)\\ &= (\phi(a_1) \cdots \phi(a_k))^* (\phi(a_1) \cdots \phi(a_k))\\ &= b^* b \geq 0,\end{aligned}$$

where $b = \phi(a_1) \cdots \phi(a_k) \in B$. Thus, ϕ is a positive linear map. By the previous lemma, all of the induced maps $\phi_k : M_k(A) \to M_k(B)$ on $k \times k$ matrices are involutive n-homomorphisms and are positive. Hence, ϕ is completely positive and therefore bounded [1]. $\square$

We now wish to show that if $n \geq 2$ is even, then an involutive n-homomorphism is actually norm-contractive. First, we will need generalizations of the familiar $C^\star$-identity appropriate for n-homomorphisms.

Lemma 2.4. *Let A be a $C^\star$-algebra. For all $k \geq 1$, we have that*

$$\begin{cases} \|x\|^{2k} = \|(x^*x)^k\|, \\ \|x\|^{2k+1} = \|x(x^*x)^k\| \end{cases}$$

for all $x \in A$.

Proof. In the even case, we easily have that

$$\|x\|^{2k} = (\|x\|^2)^k = \|x^*x\|^k = \|(x^*x)^k\|$$

by the functional calculus since $x^*x \geq 0$. In the odd case, we compute again using the $C^\star$-identity and functional calculus:

$$\begin{aligned}\|x(x^*x)^k\|^2 &= \|(x(x^*x)^k)^*(x(x^*x)^k)\|\\ &= \|(x^*x)^k x^*x (x^*x)^k\|\\ &= \|(x^*x)^{2k+1}\| = \|(x^*x)\|^{2k+1}\\ &= (\|x\|^2)^{2k+1} = (\|x\|^{2k+1})^2;\end{aligned}$$

the result follows by taking square roots. $\square$

Theorem 2.5. *Let $\phi : A \to B$ be an involutive n-homomorphism of $C^\star$-algebras. If ϕ is bounded, then ϕ is norm-contractive ($\|\phi\| \leq 1$).*

Proof. Suppose $n = 2k$ is even. Then for all $x \in A$ we have

$$\phi\big((x^*x)^k\big) = \phi(x^*x \cdots x^*x) = \big(\phi(x^*)\phi(x)\big)^k = \big(\phi(x)^*\phi(x)\big)^k.$$

Thus by the previous lemma,

$$\begin{aligned}\|\phi(x)\|^n &= \|\phi(x)\|^{2k}\\ &= \|(\phi(x)^*\phi(x))^k\| = \|\phi((x^*x)^k)\|\\ &\leq \|\phi\|\|(x^*x)^k\| = \|\phi\|\|x\|^{2k} = \|\phi\|\|x\|^n,\end{aligned}$$

which implies that $\|\phi\| \leq 1$ by taking n-th roots.

The proof for the odd case $n = 2k + 1$ is similar. $\square$

3. Automatic continuity: The odd case

The positivity methods above do not work when n is odd, since the negation of a $*$-homomorphism defines an involutive 3-homomorphism that is (completely) bounded, but **not** positive. We need the following slight generalization of Lemma 3.5 of Harris [6].

Lemma 3.1. *Let A be a $C^\star$-algebra and let $\lambda \neq 0$ and $k \geq 1$. If $a \in A$, then $\lambda \in \sigma((a^*a)^k)$ if and only if there does not exist an element $c \in A$ with*

$$c\,(\lambda - (a^*a)^k) = a. \tag{1}$$

Proof. If $\lambda \notin \sigma((a^*a)^k)$, then $c = a(\lambda - (a^*a)^k)^{-1} \in A$ satisfies

$$c\,(\lambda - (a^*a)^k) = a(\lambda - (a^*a)^k)^{-1}(\lambda - (a^*a)^k) = a,$$

and so (1) holds.

On the other hand, if $\lambda \in \sigma((a^*a)^k)$ then, by the commutative functional calculus, there is a sequence $\{b_m\}_1^\infty$ in the unitization A^+ with $b_m \not\to 0$ but $d_m =_{\text{def}} (\lambda - (a^*a)^k)b_m \to 0$. Since $\lambda \neq 0$ we must have

$$a^*(aa^*)^{k-1}(ab_m) = (a^*a)^k b_m = \lambda b_m - d_m \not\to 0,$$

which implies $ab_m \not\to 0$. Hence, there does not exist an element $c \in A$ that can satisfy equation (1), since this would imply that

$$ab_m = c\,(\lambda - (a^*a)^k)b_m \to 0,$$

which is a contradiction. This proves the lemma. $\square$

We now prove automatic continuity for involutive n-homomorphisms of $C^\star$-algebras for all odd values of n. Note that we do not assume that A is unital, nor do we appeal to the *unitization* $\phi^+ : A^+ \to B^+$ of ϕ, which is **not** an n-homomorphism, in general.

Theorem 3.2. *Let $\phi : A \to B$ be an involutive n-homomorphism between $C^\star$-algebras. If $n \geq 3$ is odd, then $\|\phi\| \leq 1$, i.e., ϕ is norm-contractive.*

Proof. Let $n = 2k + 1$ where $k \geq 1$. Given any $a \in A$ and $\lambda > 0$ such that $\lambda \notin \sigma((a^*a)^k)$, there is, by the previous lemma, an element $c \in A$ such that

$$a = c\,(\lambda - (a^*a)^k) = (\lambda c - c(a^*a)^k).$$

Noting that $c(a^*a)^k$ is a product of $2k + 1 = n$ elements in A, and ϕ is a $*$-linear n-homomorphism, we compute:

$$\begin{aligned}\phi(a) &= \phi(\lambda c - c(a^*a)^k) = \lambda\phi(c) - \phi(c(a^*a)^k)\\ &= \lambda\phi(c) - \phi(c)(\phi(a)^*\phi(a))^k = \phi(c)(\lambda - (\phi(a)^*\phi(a))^k),\end{aligned}$$

which yields that there is an element $\phi(c) \in B$ with:

$$\phi(c)(\lambda - (\phi(a)^*\phi(a))^k) = \phi(a).$$

By the previous lemma, we conclude that $\lambda \notin \sigma((\phi(a)^*\phi(a))^k)$. Thus, we have shown the following inclusion of spectra:

$$\sigma((\phi(a)^*\phi(a))^k) \subseteq \sigma((a^*a)^k) \cup \{0\}.$$

Therefore, by the spectral radius formula [1, II.1.6.3] and the generalization of the $C^\star$-identity in Lemma 2.4, we must deduce that:

$$\begin{aligned}\|\phi(a)\|^{2k} &= \|(\phi(a)^*\phi(a))^k\| \\ &= r((\phi(a)^*\phi(a))^k) \leq r((a^*a)^k) \\ &= \|(a^*a)^k\| = \|a\|^{2k},\end{aligned}$$

which implies that $\|\phi(a)\| \leq \|a\|$ for all $a \in A$, as desired. $\square$

Note that the argument in the previous proof does *not* work for $n = 2k$ even, since we would need to employ $(a^*a)^{k-1}a$ which is a product of $2k-1 = n-1$ elements as needed, but not self-adjoint, in general. Thus, we could not appeal to the spectral radius formula for self-adjoint elements and Lemma 3.1 would not apply. Hence, the even and odd n arguments are essentially disjoint.

4. Nonexistence of nontrival involutive n-homomorphisms of $C^\star$-algebras

Our first main result is the nonexistence of nontrivial n-homomorphisms on unital $C^\star$-algebras for all $n \geq 3$. We do the unital case first since it is much simpler to prove and helps to frame the argument for the nonunital case.

Theorem 4.1. *Let $\phi : A \to B$ be an involutive n-homomorphism between the $C^\star$-algebras A and B, where A is unital. If $n \geq 2$ is even, then ϕ is a $*$-homomorphism. If $n \geq 3$ is odd, then ϕ is the difference $\phi(a) = \psi_1(a) - \psi_2(a)$ of two orthogonal $*$-homomorphisms $\psi_1 \perp \psi_2 : A \to B$.*

Proof. In either case, by Proposition A.1, the element $e = \phi(1) \in B$ is an n-potent ($e^n = e$) and is self-adjoint because

$$e = \phi(1) = \phi(1^*) = \phi(1)^* = e^*.$$

Also, there is an associated algebra homomorphism $\psi : A \to B$ defined for all $a \in A$ by the formula

$$\psi(a) = e^{n-2}\phi(a) = \phi(a)e^{n-2}$$

such that $\phi(a) = e\psi(a) = \psi(a)e$. In either case, ψ is $*$-linear since ϕ is $*$-linear and e is self-adjoint and commutes with the range of ϕ:

$$\psi(a^*) = e^{n-2}\phi(a^*) = e^{n-2}\phi(a)^* = \left(e^{n-2}\phi(a)\right)^* = \psi(a)^*.$$

Now, if $n = 2k$ is even, $e = e^n = (e^k)^*e^k \geq 0$ and so $e = p$ is a projection. Thus, $\phi(a) = p\psi(a) = \psi(a)p = p\psi(a)p$ is a $*$-homomorphism. If $n \geq 3$ is odd, then by Lemma A.8, e is the difference of two orthogonal projections $e = p_1 - p_2$ which must commute with both ψ and ϕ by the functional calculus. Define $\psi_1, \psi_2 : A \to B$ by $\psi_i(a) = p_i\psi(a)p_i$ for all $a \in A$ and $i = 1, 2$. Then $\psi_2 \perp \psi_2$ are orthogonal $*$-homomorphisms, and

$$\psi_1(a) - \psi_2(a) = p_1\psi(a) - p_2\psi(a) = e\psi(a) = \phi(a)$$

for all $a \in A$, from which the desired result follows. $\square$

Corollary 4.2. *Let $\phi : A \to B$ be a linear map between $C^\star$-algebras. If A is unital, the following are equivalent for all integers $n \geq 2$:*

(a) *ϕ is a $*$-homomorphism.*
(b) *ϕ is a positive n-homomorphism.*
(c) *ϕ is an involutive n-homomorphism and $\phi(1) \geq 0$.*

Proof. Clearly (a) $\implies$ (b) $\implies$ (c). If $n \geq 2$ is even, then (c) $\implies$ (a) by the previous result. If $n \geq 3$ is odd, then by the previous result, we only need to show that ϕ is positive. Let $n = 2k+1$. Given any $a \in A$, by the Cohen Factorization Theorem, we can write $a = a_1 \cdots a_k$. Since $\phi(1) \geq 0$, by hypothesis, and $n = 2k+1$, we compute:

$$\begin{aligned}\phi(a^*a) = \phi(a^*1a) &= \phi(a_k^* \cdots a_1^* 1 a_1 \cdots a_k)\\ &= \phi(a_k)^* \cdots \phi(a_1)^* \phi(1)\phi(a_1)\cdots\phi(a_k)\\ &= \big(\phi(a_1)\cdots\phi(a_k)\big)^*\phi(1)\big(\phi(a_1)\cdots\phi(a_k)\big)\\ &= b^*\phi(1)b \geq 0,\end{aligned}$$

where $b = \phi(a_1)\cdots\phi(a_k) \in B$. Thus, ϕ is positive linear and therefore a $*$-homomorphism. □

Next, we extend our nonexistence results to the nonunital case, by appealing to approximate unit arguments (which require continuity!) and the following important factorization property of $*$-preserving n-homomorphisms.

Lemma 4.3 (Coherent Factorization Lemma)**.** *Let $\phi : A \to B$ be an involutive n-homomorphism of $C^\star$-algebras. For any $1 \leq k \leq n$ and any $a \in A$, if $a = a_1 \cdots a_k = b_1 \cdots b_k$ in A, then*

$$\phi(a_1)\cdots\phi(a_k) = \phi(b_1)\cdots\phi(b_k) \in B.$$

Note that, in general, $\phi(a) \neq \phi(a_1)\cdots\phi(a_k)$ when $1 < k < n$.

Proof. Clearly, we may assume $1 < k < n$. Since ϕ is $*$-linear, the range $\phi(A) \subset B$ is a self-adjoint linear subspace of B (but not necessarily a subalgebra, in general). Given any $d = \phi(c) \in \phi(A)$, using the Cohen Factorization Theorem, write $d = d_1 \cdots d_n = \phi(c_1)\cdots\phi(c_n)$ where $d_i = \phi(c_i)$ for $1 \leq i \leq n$. Consider the following computation:

$$\begin{aligned}\phi(a_1)\cdots\phi(a_k)d &= \phi(a_1)\cdots\phi(a_k)\phi(c_1)\cdots\phi(c_n)\\ &= \phi(a_1\cdots a_k c_1 \cdots c_{n-k})\phi(c_{n-k+1})\cdots\phi(c_n)\\ &= \phi(b_1\cdots b_k c_1 \cdots c_{n-k})\phi(c_{n-k+1})\cdots\phi(c_n)\\ &= \phi(b_1)\cdots\phi(b_k)\phi(c_1)\cdots\phi(c_n)\\ &= \phi(b_1)\cdots\phi(b_k)d.\end{aligned}$$

Let $f = \phi(a_1)\cdots\phi(a_k) - \phi(b_1)\cdots\phi(b_k)$. Then $fd = 0$ for all $d \in \phi(A) \subset B$, and thus $fd = 0$ for all d in the $*$-subalgebra A_ϕ of B generated by $\phi(A)$. In particular, for the element

$$d_a = \phi(a_k^*)\cdots\phi(a_1^*) - \phi(b_k^*)\cdots\phi(b_1^*) = f^* \in A_\phi.$$

Hence, $ff^* = fd_a = 0$ and so $\|f\|^2 = \|ff^*\| = 0$ by the $C^\star$-identity. Therefore,

$$\phi(a_1)\cdots\phi(a_k) - \phi(b_1)\cdots\phi(b_k) = f = 0,$$

and the result is proven. □

Definition 4.4. An approximate unit for a (nonunital) $C^\star$-algebra A is a net $\{e_\lambda\}_{\lambda\in\Lambda}$ of elements in A indexed by a directed set Λ such that

(a) $0 \le e_\lambda$ and $\|e_\lambda\| \le 1$ for all $\lambda \in \Lambda$.
(b) $e_\lambda \le e_\mu$ if $\lambda \le \mu$ in Λ.
(c) For all $a \in A$,

$$\lim_{\lambda\to\infty} \|ae_\lambda - a\| = \lim_{\lambda\to\infty} \|e_\lambda a - a\| = 0.$$

Every $C^\star$-algebra has an approximate unit, which is countable ($\Lambda = \mathbb{N}$) if A is separable (see Section II.4 of Blackadar [1]).

Theorem 4.5. *Suppose $\phi : A \to B$ is an involutive n-homomorphism of $C^\star$-algebras, where A is nonunital. Then, for all $a \in A$, the limit*

$$\psi(a) = \lim_{\lambda\to\infty} \phi(e_\lambda)^{n-2}\phi(a) = \lim_{\lambda\to\infty} \phi(a)\phi(e_\lambda)^{n-2}$$

exists, independently of the choice of the approximate unit $\{e_\lambda\}$ of A, and defines a $*$-homomorphism $\psi : A \to B$ such that

$$\phi(a) = \lim_{\lambda\to\infty} \phi(e_\lambda)\psi(a)$$

for all $a \in A$.

Proof. We may assume $n \ge 3$. Given $a \in A$, use the Cohen Factorization Theorem to factor $a = a_1a_2\cdots a_n$. Define a map $\psi : A \to B$ by

$$\psi(a) = \phi(a_1a_2)\phi(a_3)\cdots\phi(a_n) = \phi(a_1)\cdots\phi(a_{n-2})\phi(a_{n-1}a_n),$$

which is *well-defined* by the Coherent Factorization Lemma. The continuity of ϕ implies that

$$\begin{aligned}\lim_{\lambda\to\infty} \phi(e_\lambda)^{n-2}\phi(a) &= \lim_{\lambda\to\infty} \phi(e_\lambda)^{n-2}\phi(a_1)\cdots\phi(a_n)\\ &= \lim_{\lambda\to\infty} \phi(e_\lambda^{n-2}a_1a_2)\phi(a_3)\cdots\phi(a_n)\\ &= \phi(a_1a_2)\phi(a_3)\cdots\phi(a_n) = \psi(a) \in B.\end{aligned}$$

It follows that we can write:

$$\psi(a) = \lim_{\lambda\to\infty} \phi(e_\lambda)^{n-2}\phi(a) = \lim_{\lambda\to\infty} \phi(a)\phi(e_\lambda)^{n-2},$$

and so $\psi : A \to B$ is linear since ϕ is linear. Moreover, since ϕ is $*$-linear, it follows that ψ is also $*$-linear:

$$\begin{aligned}\psi(a)^* &= \big(\phi(a_1a_2)\phi(a_3)\cdots\phi(a_n)\big)^*\\ &= \phi(a_n)^*\cdots\phi(a_3)^*\phi(a_1a_2)^*\\ &= \phi(a_n^*)\cdots\phi(a_3^*)\phi(a_2^*a_1^*)\\ &= \phi(a_{n1}^*a_{n2}^*)\phi(a_{n-1}^*)\cdots\phi(a_{12}^*)\\ &= \psi((a_{n1}^*a_{n2}^*)(a_{n-1}^*)\cdots(a_{12}^*))\\ &= \psi(a_n^*\cdots a_1^*) = \psi(a^*).\end{aligned}$$

In the computation above, we factored $a_n = a_{n2}a_{n1}$ and set $a_{12} = a_1a_2$ to obtain the factorization $a^* = a_n^*\cdots a_1^* = (a_{n1}^*a_{n2}^*)a_{n-1}^*\cdots a_{12}^*$ into n elements. Given

$a, b \in A$ with factorizations $a = a_1 \cdots a_n$ and $b = b_1 \cdots b_n$, the fact that ϕ is an n-homomorphism implies:

$$\begin{aligned}\psi(a)\psi(b) &= \big(\phi(a_1a_2)\phi(a_3)\cdots\phi(a_n)\big)\big(\phi(b_1b_2)\phi(b_3)\cdots\phi(b_n)\big)\\ &= \phi((a_1a_2)a_3\cdots a_n(b_1b_2))\phi(b_3)\cdots\phi(b_n)\\ &= \phi((ab_1)b_2)\phi(b_3)\cdots\phi(b_n)\\ &= \psi(ab);\end{aligned}$$

note that $ab = (ab_1)b_2b_3\cdots b_n$ is a factorization of ab into n elements. A second proof of multiplicativity goes as follows:

$$\begin{aligned}\psi(ab) &= \lim_{\lambda\to\infty}\phi(e_\lambda)^{n-2}\phi(ab) = \lim_{\lambda\to\infty}\phi(e_\lambda)^{n-2}\phi(\lim_{\mu\to\infty} ae_\mu^{n-2}b)\\ &= \lim_{\lambda\to\infty}\phi(e_\lambda)^{n-2}\lim_{\mu\to\infty}\phi(ae_\mu^{n-2}b)\\ &= \lim_{\lambda\to\infty}\phi(e_\lambda)^{n-2}\lim_{\mu\to\infty}\phi(a)\phi(e_\mu)^{n-2}\phi(b)\\ &= \lim_{\lambda\to\infty}\phi(e_\lambda)^{n-2}\phi(a)\lim_{\mu\to\infty}\phi(e_\mu)^{n-2}\phi(b)\\ &= \psi(a)\psi(b).\end{aligned}$$

Thus, ψ is a well-defined $*$-homomorphism. Finally, we compute:

$$\begin{aligned}\lim_{\lambda\to\infty}\phi(e_\lambda)\psi(a) &= \lim_{\lambda\to\infty}\phi(e_\lambda)\phi(a_1a_2)\phi(a_3)\cdots\phi(a_n)\\ &= \lim_{\lambda\to\infty}\phi(e_\lambda(a_1a_2)a_3\cdots a_n) = \lim_{\lambda\to\infty}\phi(e_\lambda a)\\ &= \phi(a).\end{aligned}$$

□

Using similar factorizations, the fact that $\{e_\lambda^n\}$ is also an approximate unit for A, and the fact that the strict completion of the $C^\star$-algebra $C^\star(\phi(A))$ generated by the range $\phi(A)$ is the multiplier algebra $M(C^\star(\psi(A)))$, we obtain the nonunital version of Proposition A.1.

Corollary 4.6. *Suppose that A and B are $C^\star$-algebras with A nonunital, and let $\phi : A \to B$ be an involutive n-homomorphism with associated $*$-homomorphism $\psi : A \to B$. Then there is a self-adjoint n-potent $e = e^* = e^n \in M(C^*(\phi(A)))$ such that $\phi(e_\lambda) \to e$ strictly for any approximate unit $\{e_\lambda\}$ of A, and with the property that*

$$\phi(a) = e\psi(a) = \psi(a)e,$$
$$\psi(a) = e^{n-2}\phi(a)$$

for all $a \in A$.

Proof. By the previous proof, we can define $e \in M(C^\star(\phi(A)))$ on generators $\phi(a)$ by

$$e\phi(a) = \lim_{\lambda\to\infty}\phi(e_\lambda)\phi(a) = \phi(a_1a_2\cdots a_{n-1})\phi(a_n) \in C^\star(\phi(A))$$

for any $a = a_1\cdots a_n \in A$. It follows that:

$$\begin{aligned}e^n\phi(a) &= \lim_{\lambda\to\infty}\phi(e_\lambda)^n\phi(a)\\ &= \lim_{\lambda\to\infty}\phi(e_\lambda^n)\phi(a_1)\phi(a_2)\cdots\phi(a_n)\\ &= \lim_{\lambda\to\infty}\phi((e_\lambda^n)a_1a_2\cdots a_{n-1})\phi(a_n)\\ &= \phi(a_1\cdots a_{n-1})\phi(a_n) = e\phi(a),\end{aligned}$$

which implies $e \in M(C^\star(\phi(A)))$ is n-potent. The fact that $e = e^*$ follows from $\phi(e_\lambda)^* = \phi(e_\lambda^*) = \phi(e_\lambda)$. The other statements follow from the previous proof. $\square$

The dichotomy between the unital and nonunital cases is now clear. If A is unital, then $C^\star(\phi(A)) \subset B$ is a unital $C^\star$-subalgebra of B with unit $\psi(1) = \phi(1)^{n-1} \in B$ (which is a projection!) and so

$$M(C^\star(\psi(A))) = C^\star(\phi(A)) \subset B.$$

However, for A nonunital, we cannot identify the multiplier algebra $M(C^\star(\phi(A)))$ as a subalgebra of B, or even $M(B)$, unless ϕ is surjective. In general, we only have inclusions $\psi(A) \subset C^\star(\phi(A)) \subset B$.

Now that we know, as in the unital case, every involutive n-homomorphism is an n-potent multiple of a $*$-homomorphism, we can prove the following general version of Theorem 4.1 and its corollary in a similar manner using Lemma A.8.

Theorem 4.7. *Let $\phi : A \to B$ be an involutive n-homomorphism of $C^\star$-algebras. If $n \geq 2$ is even, then ϕ is a $*$-homomorphism. If $n \geq 3$ is odd, then ϕ is the difference $\phi(a) = \psi_1(a) - \psi_2(a)$ of two orthogonal $*$-homomorphisms $\psi_1 \perp \psi_2 : A \to B$.*

Corollary 4.8. *For all $n \geq 2$ and $C^\star$-algebras A and B, $\phi : A \to B$ is a positive n-homomorphism if and only if ϕ is a $*$-homomorphism.*

Appendix A. On n-homomorphisms and n-potents

An element $x \in A$ is called an *n-potent* if $x^n = x$. Note that if $\phi : A \to B$ is an n-homomorphism, then $\phi(x) = \phi(x^n) = \phi(x)^n \in B$ is also an n-potent. The following important result is Proposition 2.2 in [7], whose proof is included for completeness.

Proposition A.1. *If A is a unital algebra (or ring) and $\phi : A \to B$ is an n-homomorphism, then there is a homomorphism $\psi : A \to B$ and an n-potent $e = e^n \in B$ such that $\phi(a) = e\psi(a) = \psi(a)e$ for all $a \in A$. Also, e commutes with the range*[1] *of ϕ, i.e., $e\phi(a) = \phi(a)e$ for all $a \in A$.*

Proof. Note that $e = \phi(1) = \phi(1^n) = \phi(1)^n = e^n \in B$ is an n-potent. Define a linear map $\psi : A \to B$ by $\psi(a) = e^{n-1}\phi(a)$ for all $a \in A$. For all $a, b \in A$,

$$\begin{aligned}\psi(ab) &= e^{n-2}\phi(ab) = e^{n-2}\phi(a1^{n-2}b) \\ &= \big(e^{n-2}\phi(a)\big)\big(\phi(1)^{n-2}\phi(b)\big) \\ &= \big(e^{n-2}\phi(a)\big)\big(e^{n-2}\phi(b)\big) = \psi(a)\psi(b),\end{aligned}$$

and so ψ is an algebra homomorphism. Furthermore,

$$e\psi(a) = \phi(1)(\phi(1)^{n-2}\phi(a)) = \phi(1)^{n-1}\phi(a) = \phi(1^{n-1}a) = \phi(a).$$

Similarly, $\psi(a)e = \phi(a)$ for all $a \in A$. The final statement is a consequence of the fact that for all $a \in A$,

$$e\phi(a) = \phi(1)\phi(a1^{n-1}) = \big(\phi(1)\phi(a)\phi(1)^{n-2}\big)\phi(1) = \phi(1a1^{n-2})e = \phi(a)e.$$

$\square$

The following computation will be more significant when we consider the nonunital case (see the proof of Theorem 4.5).

[1] Note that the range $\phi(A)$ is not a subalgebra of B in general.

Corollary A.2. *Let ϕ and ψ be as in Proposition* A.1 *and $n \geq 3$. Then for all $a \in A$, if $a = a_1 a_2 \cdots a_n$ with $a_1, \ldots, a_n \in A$,*

$$\psi(a) = \phi(a_1 a_2)\phi(a_3) \cdots \phi(a_n).$$

Proof. We compute as follows:

$$\begin{aligned}\psi(a) =_{\mathrm{def}} e^{n-2}\phi(a) &= \phi(1)^{n-2}\phi(a_1 \cdots a_n) \\ &= \phi(1)^{n-2}\phi(a_1) \cdots \phi(a_n) \\ &= \big(\phi(1)^{n-2}\phi(a_1)\phi(a_2)\big)\phi(a_3) \cdots \phi(a_n) \\ &= \phi(1^{n-2}a_1 a_2)\phi(a_3) \cdots \phi(a_n) \\ &= \phi(a_1 a_2)\phi(a_3) \cdots \phi(a_n).\end{aligned}$$

□

Definition A.3. Let A be a unital algebra. An *n-partition of unity* is an ordered n-tuple $(e_0, e_1, \ldots, e_{n-1})$ of idempotents ($e_k^2 = e_k$) that sum to the identity $e_0 + e_1 + \cdots + e_{n-1} = 1$ and are pairwise mutually orthogonal, *i.e.*, $e_j e_k = \delta_{jk} 1$ for all $0 \leq j, k \leq n-1$, where δ_{jk} is the Kronecker delta.

Note that $e_0 = 1 - (e_1 + \cdots + e_{n-1})$ is completely determined by $e_1, e_2, \ldots, e_{n-1}$ and is thus redundant in the notation for an n-partition of unity.

Definition A.4. Let $\omega_0 = 0$ and $\omega_k = e^{2\pi i(k-1)/(n-1)}$ for $1 \leq k \leq n-1$. Note that $\omega_1 = 1$ and $\omega_1, \ldots, \omega_{n-1}$ are the $(n-1)$-th roots of unity and $\Sigma_n = \{\omega_0, \omega_1, \ldots, \omega_{n-1}\}$ are the n roots of the polynomial equation $x^n - x = x(x^{n-1} - 1) = 0$.

If A is a complex algebra, we let $\tilde{A}$ denote A, if A is unital, or the unitization $A^+ = A \oplus \mathbb{C}$, if A is nonunital.

Theorem A.5. *Let A be a complex algebra. If $e \in A$ is an n-potent, there is a unique n-partition of unity $(e_0, e_1, \ldots, e_{n-1})$ in $\tilde{A}$ such that*

$$e = \sum_{k=1}^{n-1} \omega_k e_k.$$

If A is nonunital, then $e_1, \ldots, e_{n-1} \in A$.

Proof. Define the n polynomials $p_0, p_1, \ldots, p_{n-1}$ by

$$p_k(x) = \frac{\prod_{j \neq k}(x - \omega_j)}{\prod_{j \neq k}(\omega_k - \omega_j)}.$$

In particular, $p_0(x) = 1 - x^{n-1}$. Each polynomial p_k has degree $n-1$ and satisfies $p_k(\omega_k) = 1$ and $p_k(\omega_j) = 0$ for all $j \neq k$. It follows that $p_j(x)p_k(x) = 0$ for all $x \in \Sigma_n$. We also claim for all $x \in \mathbb{C}$ that

$$\sum_{k=0}^{n-1} p_k(x) = p_0(x) + \cdots + p_{n-1}(x) = 1. \tag{2}$$

$$x = \sum_{k=0}^{n-1} \omega_k p_k(x). \tag{3}$$

Indeed, these identities follow from the fact that these polynomial equations have degree $n-1$ but are satisfied by the n *distinct* points in Σ_n.

Now, given any $x^n = x$ in $\mathbb{C}$ it follows that $p_k(x)^2 = p_k(x)$. Hence, for any n-potent $e \in A$, if we define $e_k = p_k(e)$, then $(e_0, e_1, \ldots, e_{n-1})$ consists of idempotents $e_k^2 = p_k(e)^2 = p_k(e) = e_k$ and satisfy, by (2),

$$\sum_{k=0}^{n-1} e_k = \sum_{k=0}^{n-1} p_k(e) = 1_{\tilde{A}}.$$

They are pairwise orthogonal because $e_j e_k = p_j(e)p_k(e) = 0$ for $j \neq k$. Moreover,

$$e = \sum_{k=1}^{n-1} \omega_k p_k(e) = \sum_{k=1}^{n-1} \omega_k e_k$$

by Equation (3). For $1 \leq k \leq n-1$, note that $p_k(x) = xq_k(x)$ for some polynomial $q_k(x)$. Hence, if A is nonunital and $1 \leq k \leq n-1$, we have $e_k = p_k(e) = eq_k(e) \in A$, since A is an ideal in $\tilde{A}$. $\square$

The following result is the n-homomorphism version of the previous n-potent result. Recall say that two linear maps $\psi_i, \psi_j : A \to B$ are *orthogonal* ($\psi_i \perp \psi_j$) if

$$\psi_i(a)\psi_j(b) = \psi_j(b)\psi_i(a) = 0$$

for all $a, b \in A$.[2]

Proposition A.6. *Let A and B be complex algebras. If A is unital, then a linear map $\phi : A \to B$ is an n-homomorphism if and only if there exist $n-1$ mutually orthogonal homomorphisms $\psi_1, \ldots, \psi_{n-1} : A \to B$ such that for all $a \in A$,*

$$\phi(a) = \sum_{k=1}^{n-1} \omega_k \psi_k(a).$$

Proof. ($\Rightarrow$) Let $\phi : A \to B$ be an n-homomorphism. By Proposition A.1, there is an n-potent $e \in B$ and a homomorphism $\psi : A \to B$ such that $\phi(a) = e\psi(a) = \psi(a)e$. Using the previous result, write $e = \sum_{k=1}^{n-1} \omega_k e_k$, where $(e_0, e_1, \ldots, e_{n-1})$ is the associated n-partition of unity in $\tilde{A}$ defined by the polynomials p_k. Since $e_k = p_k(e)$, we have that $e_k\psi(a) = \psi(a)e_k$ for $1 \leq k \leq n-1$. Define $\psi_k : A \to B$ by

$$\psi_k(a) =_{\mathrm{def}} e_k\psi(a) = e_k^2\psi(a) = e_k\psi(a)e_k.$$

Then $\psi_1, \ldots, \psi_{n-1}$ are orthogonal homomorphisms and, for all $a \in A$,

$$\phi(a) = e\psi(a) = \sum_{k=1}^{n-1} \omega_k e_k \psi(a) = \sum_{k=1}^{n-1} \omega_k \psi_k(a).$$

($\Leftarrow$) Follows from the fact that $\omega_k^n = \omega_k$ for all $k = 1, \ldots, n-1$. $\square$

Remark A.7. If A is nonunital, the above result does not hold. One reason is that the unitization $\phi^+ : A^+ \to B^+$ of an n-homomorphism is not, in general, an n-homomorphism. Also, if $A^n = B^n = \{0\}$, then every linear map $L : A \to B$ is an n-homomorphism. (See Examples 2.5 and 4.3 of Hejazian *et. al.* [7].)

Let Σ_n be the n roots of the polynomial equation $x = x^n$ from Definition A.4. If A is a $C^\star$-algebra, it follows that a normal n-potent $e = e^n$ must have spectrum $\sigma(e) \subseteq \Sigma_n$. Recall that a projection is an element $p = p^* = p^2 \in A$. Two projections p_1 and p_2 are orthogonal if $p_1 p_2 = 0$. A tripotent is a 3-potent element $e^3 = e \in A$.

[2]Note that the zero homomorphism is orthogonal to every homomorphism.

The following characterization of self-adjoint n-potents in $C^\star$-algebras is important for our nonexistence results on n-homomorphisms.

Lemma A.8. *Let A be a $C^\star$-algebra.*

(a) *If $n \geq 2$ is an even integer, the following are equivalent:*
   (i) *e is a projection.*
   (ii) *e is a positive n-potent.*
   (iii) *e is a self-adjoint n-potent.*

(b) *If $n \geq 3$ is an odd integer, the following are equivalent:*
   (i) *e is a self-adjoint tripotent.*
   (ii) *$e = p_1 - p_2$ is a difference of two orthogonal projections.*
   (iii) *e is a self-adjoint n-potent.*

Proof. In both the even and odd cases, (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) (See Theorem A.5.) Suppose (iii) holds. If $n = 2k$ is even,

$$e = e^* = e^n = e^{2k} = (e^k)^*(e^k) \geq 0,$$

and so the spectrum of e satisfies $\sigma(e) \subset \Sigma_n \cap [0, \infty] = \{0, 1\}$. Thus, e is a projection. If $n \geq 3$ is odd, then since $e = e^*$ we must have $\sigma(e) \subset \Sigma_n \cap \mathbb{R} = \{-1, 0, 1\}$. Thus, $\lambda = \lambda^3$ for all $\lambda \in \sigma(e)$, which implies $e = e^3$ is tripotent. □

References

[1] B. Blackadar, *Theory of C^*-algebras and von Neumann algebras*, Encyclopaedia of Mathematical Sciences, 122. Operator Algebras and Non-commutative Geometry, III. Springer-Verlag, Berlin, 2006. MR2188261 (2006k:46082)

[2] J. Bračič and S. Moslehian, *On Automatic Continuity of 3-Homomorphisms on Banach Algebras*, to appear in Bull. Malays. Math. Sci. Soc. arXiv: math.FA/0611287.

[3] P. Cohen, *Factorization in group algebras*, Duke Math. J. 26 (1959) 199–205. MR0104982 (21:3729)

[4] S. Feigelstock, *Rings whose additive endomorphisms are N-multiplicative*, Bull. Austral. Math. Soc. 39 (1989), no. 1, 11–14. MR976254 (89j:16043)

[5] S. Feigelstock, *Rings whose additive endomorphisms are N-multiplicative. II*, Period. Math. Hungar. 25 (1992), no. 1, 21–26. MR1200838 (93j:16018)

[6] L. Harris, *A Generalization of $C^\star$-algebras*, Proc. London Math. Soc. 42 (1981), no. 3, 331–361. MR607306 (82e:46089)

[7] M. Hejazian, M. Mirzavaziri, and M.S. Moslehian, *n-homomorphisms*, Bull. Iranian Math. Soc. 31 (2005), no. 1, 13-23. MR2228453 (2007b:47091)

[8] I. Raeburn and D. P. Williams, *Morita Equivalence and Continuous-Trace $C^\star$-Algebras*, Mathematical Surveys and Monographs, vol. 60, American Mathematical Society, 1998. MR1634408 (2000c:46108)

[9] W. Stinespring, *Positive functions on $C^\star$-algebras*, Proc. Amer. Math. Soc. 6 (1955), 211–216. MR0069403 (16:1033b)

Department of Mathematics, Texas Christian University, Box 298900, Fort Worth, Texas 76129

E-mail address: `e.park@tcu.edu`

Department of Mathematics, Dartmouth College, 6188 Kemeny Hall, Hanover, New Hampshire 03755

E-mail address: `jody.trout@dartmouth.edu`