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ON THE DYNAMICS OF HOMOLOGY-PRESERVING HOMEOMORPHISMS OF THE ANNULUS

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Abstract. We consider the homeomorphisms of the compact annulus $\mathbb{A} = \mathbb{S}^1 \times [-1, 1]$ isotopic to the symmetry $S_{\mathbb{A}}$ which interchanges the two boundary components. We prove that if such a homeomorphism is, in some sense, conservative and twisted, then it possesses a periodic orbit of period exactly two. This can be regarded as a counterpart of the Poincaré-Birkhoff theorem in the isotopy class of $S_{\mathbb{A}}$.

1. Introduction and motivation

A celebrated result in topological dynamics is the following, whose first complete proof is generally attributed to G.D. Birkhoff ([2], see also [4]).

Theorem 1.1 (Poincaré-Birkhoff). *Let h be a homeomorphism of the compact annulus $\mathbb{A}$ isotopic to the identity map $\mathrm{Id}_{\mathbb{A}}$. If h preserves the area and satisfies the boundary twist condition*[1]*, then it has at least two fixed points.*

There are numerous variants and generalizations of this theorem. An important idea, which already appears in the paper [19] by Poincaré, roughly says that the area-preserving assumption can be replaced with the next one: there is no essential subannulus of $\mathbb{A}$ containing its image as a proper subset. This is the point of view adopted for example in [5, 9, 12, 13, 16, 21], and we will also think of a conservative homeomorphism of the annulus by following this line. Suitable generalized twist properties can be found in [7, 8, 12, 21]. An interesting feature of these properties is that they deal with the behaviour of h inside the annulus and not only on its boundary. As we will see below, this is naturally adapted to our purpose. Let us quote finally [1, 7, 14] for analogous statements in the open annulus.

The aim of the present paper is to show that there exist close results in the isotopy class of the symmetry $S_{\mathbb{A}}$ which interchanges the boundary components of $\mathbb{A}$. The reasons for such an investigation are the following. First of all, the homeomorphisms h of $\mathbb{A}$ isotopic to either $S_{\mathbb{A}}$ or $\mathrm{Id}_{\mathbb{A}}$ share the following property, which is actually one of the mostly basic facts for proving the various versions of the Poincaré-Birkhoff theorem: any lift H of h to the universal cover $\tilde{\mathbb{A}}$ commutes with the deck transformations. In other words, they induce the identity map at the

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[1]The boundary twist assumption intuitively says that h rotates the two boundary components of $\mathbb{A}$ in opposite directions. See e.g. [7] or [12] for a precise statement.

level of homology. Secondly, it is well known that the Poincaré-Birkhoff theorem can be deduced from (the proof of) another classical result in dynamics, namely the Brouwer plane translation theorem for fixed point free and orientation-preserving planar homeomorphisms. This idea can be traced back to Kerékjártó ([16]) and also plays a central role in some modern papers such as [1], [12]-[14], [21]. Moreover the Brouwer plane translation theorem has a counterpart in the framework of orientation-reversing homeomorphisms: roughly, for such a homeomorphism of the sphere $\mathbb{S}^2$, any kind of recurrent behaviour in the complement of the fixed point set implies the existence of a 2-periodic orbit (see [3]). It is then natural to expect for a result saying vaguely that "a conservative homeomorphism h of the annulus $\mathbb{A}$ isotopic to $S_{\mathbb{A}}$ and without a 2-periodic orbit (but possibly with fixed points) cannot twist the annulus". Producing nontrivial results in this direction requires describing what is an untwisted homeomorphism h by considering its behaviour in the whole annulus; indeed, if h^2 has no fixed point on the boundary of $\mathbb{A}$, then it moves points in the same direction on the two boundary components. Following L. Guillou in [12], this is carried out by looking at the way h moves the arcs joining the two boundary components of $\mathbb{A}$. Precisely, our main result is the following.

Theorem 1.2. *Let h be a homeomorphism of the compact annulus $\mathbb{A} = \mathbb{S}^1 \times [-1,1]$ isotopic to the symmetry $S_{\mathbb{A}} : \mathbb{A} \to \mathbb{A}$, $(z,r) \mapsto (z,-r)$; equivalently, h reverses the orientation and interchanges the two boundary components of $\mathbb{A}$. We assume that h has only finitely many fixed points. If h has no 2-periodic point, i.e. if $\mathrm{Fix}(h) = \mathrm{Fix}(h^2)$, then at least one of the two following assertions holds.*

(1) *There exists an essential Jordan curve $J \subset \mathbb{A}$ such that*
   - $J \cap h^2(J) = J \cap \mathrm{Fix}(h)$,
   - *the Jordan curves J and $h^2(J)$ have no point of transverse intersection; equivalently, $h^2(J)$ meets only one connected component of $\mathbb{A} \setminus J$.*

(2) *There exists an arc α joining the two boundary components of $\mathbb{A}$ such that*
   - $\alpha \cap h(\alpha) = \alpha \cap h^2(\alpha) = \alpha \cap \mathrm{Fix}(h)$,
   - *the set $h(\alpha) \cup h^2(\alpha)$ does not meet the two (local) sides of α; as a consequence, $h^2(\alpha)$ does not meet the two sides of $h(\alpha)$,*
   - *the three arcs $h^i(\alpha)$, $i \in \{0,1,2\}$, are either in the (circular) order $\alpha, h(\alpha), h^2(\alpha)$ or $h^2(\alpha), h(\alpha), \alpha$.*

A typical homeomorphism illustrating the untwisted situation (2) is of course $h = S_{\mathbb{A}} \circ R_\theta$, where R_θ is the rotation by angle $\theta \notin \{0, \pi\}$. Such an example satisfies $\mathrm{Fix}(h) = \mathrm{Fix}(h^2) = \emptyset$ but it can be easily modified, by slowing down the rotation near $\mathbb{S}^1 \times \{0\}$, in order to create finitely many fixed points on this circle. As a remark, let us observe that a Jordan curve J as in the dissipative situation (1) is actually contained in the interior of $\mathbb{A}$, due to the fact that $\mathrm{Fix}(h^2) = \mathrm{Fix}(h)$ is disjoint from the boundary of $\mathbb{A}$. It should also be pointed out that we do not know anything about the relative positions of the Jordan curves $J, h(J)$ and $h^2(J)$. Finally, the finiteness of the fixed point set $\mathrm{Fix}(h)$ is a technical assumption appearing as the price to pay in order to interpret the conservative and twist properties in terms of curves such as J and α.

Another approach for considering twist properties inside the annulus, due to J. Franks ([7, 8]), is by means of the rotation set. From this point of view, a homeomorphism of the annulus $\mathbb{A}$ is untwisted if it moves asymptotically all the points of $\mathbb{A}$ in the same direction, clockwise or counterclockwise. Observing that the notions of rotation number and rotation set also make sense for a homeomorphism of $\mathbb{A}$ isotopic to $S_{\mathbb{A}}$, we explain in an Appendix how some results of Franks extend to this framework. This leads to another version of our result for homeomorphisms without wandering point where we drop the assumption concerning $\mathrm{Fix}(h)$ (Theorem 4.4).

2. Background

2.1. **Notation and vocabulary.** We think of the compact annulus as $\mathbb{A} = \mathbb{S}^1 \times [-1,1]$ and of the open annulus as $\mathbb{A}' = \mathbb{S}^1 \times \mathbb{R}$. We use the same letter Π for the two universal covering maps $\tilde{\mathbb{A}} = \mathbb{R} \times [-1,1] \to \mathbb{A}$ and $\widetilde{\mathbb{A}'} = \mathbb{R}^2 \to \mathbb{A}'$ both defined by $(\theta, r) \mapsto (e^{2i\pi\theta}, r)$. The deck transformations are then the iterates τ^n of the translation $\tau : (x,y) \mapsto (x+1, y)$ defined either on $\tilde{\mathbb{A}}$ or on $\mathbb{R}^2$. We write $\mathrm{Bd}^{\pm}(\mathbb{A})$ for the two boundary components $\mathbb{S}^1 \times \{\pm 1\}$ of $\mathbb{A}$ and $\mathrm{Bd}(\mathbb{A}) = \mathrm{Bd}^-(\mathbb{A}) \cup \mathrm{Bd}^+(\mathbb{A})$. We have similarly $\mathrm{Bd}(\tilde{\mathbb{A}}) = \mathrm{Bd}^-(\tilde{\mathbb{A}}) \cup \mathrm{Bd}^+(\tilde{\mathbb{A}})$ for the boundary of the strip $\tilde{\mathbb{A}}$. Each boundary component of $\mathrm{Bd}(\tilde{\mathbb{A}})$ is naturally ordered from the left to the right, i.e. for a given $\sigma = \pm 1$, we write $(a, \sigma) \leq (a', \sigma)$ if and only if $a \leq a'$. In a general way, $\mathrm{Bd}(M)$ denotes the boundary of a manifold M which is always a surface or a 1-dimensional manifold in this paper. Since it does not seem to be entirely standard, let us point out that a 1-dimensional submanifold N of a surface S with boundary is always assumed to satisfy $\mathrm{Bd}(N) = \mathrm{Bd}(S) \cap N$. In other words, we define $N \subset S$ to be a 1-dimensional submanifold of S if the pair (S, N) is locally homeomorphic to either $(\mathbb{R}^2, \mathbb{R} \times \{0\})$ or $\big(\{(x,y) \in \mathbb{R}^2 \,|\, x \leq 0\}, \{(x,y) \in \mathbb{R}^2 \,|\, x \leq 0,\ y = 0\}\big)$.

If Y is a topological space and $X \subset Y$ we write generally $\mathrm{Cl}_Y(X)$, $\mathrm{Int}_Y(X)$ and $\partial_Y X$ for respectively the closure, interior and frontier of X with respect to Y. Nevertheless, most of our constructions are made in the strip $\tilde{\mathbb{A}}$, so we denote simply $\overline{X}$, $\mathrm{Int}(X)$ and ∂X instead of respectively $\mathrm{Cl}_{\tilde{\mathbb{A}}}(X)$, $\mathrm{Int}_{\tilde{\mathbb{A}}}(X)$, $\partial_{\tilde{\mathbb{A}}} X$ for any set $X \subset \tilde{\mathbb{A}}$.

A subset X of a surface S is said to be an *arc*, a *Jordan curve*, a *topological closed disc*, a *line*, a *half-line* if it is homeomorphic to respectively $[0,1]$, $\mathbb{S}^1$, the closed unit disc of $\mathbb{R}^2$, $\mathbb{R}$, $[0, +\infty)$. A line or a half-line $X \subset S$ is said to be *properly embedded* in S if it is a closed subset of S. If $\alpha \subset S$ is an arc with a given orientation and a, b two points of α which are met in this order on α, then $[a,b]_\alpha$ denotes the subarc of α from a to b. We say briefly that an arc $\alpha \subset \mathbb{A}$ *crosses* $\mathbb{A}$ if it joins the two boundary components of $\mathbb{A}$, i.e. if it is contained in $\mathbb{A} \setminus \mathrm{Bd}(\mathbb{A})$ except for one endpoint on each boundary component of $\mathbb{A}$. Similarly for an arc $\tilde{\alpha} \subset \tilde{\mathbb{A}}$. It follows from the Jordan curve theorem that if $\tilde{\alpha}$ is an arc crossing $\tilde{\mathbb{A}}$, then $\tilde{\mathbb{A}} \setminus \tilde{\alpha}$ has exactly two connected components W, W' and that $\partial W = \partial W' = \tilde{\alpha}$. Only one of these two sets W, W' is unbounded on the right (resp. on the left), which means that it contains points (x,y) (resp $(-x,y)$) of $\tilde{\mathbb{A}}$ with arbitrarily large $x > 0$; it is named the *domain on the right* (resp. *on the left*) of $\tilde{\alpha}$.

For any map $f : X \to X$, a point $x \in X$ is said to be *k-periodic* if k is the *smallest* positive integer such that $f^k(x) = x$. The integer $k \geq 1$ is named the *f-period* or simply the *period* of x. We also say that k is the period of the orbit $\mathcal{O} = \{x, f(x), \dots, f^{k-1}(x)\}$ since any point in this orbit is k-periodic. The fixed point set is $\mathrm{Fix}(f) = \{x \in X \,|\, f(x) = x\}$.

2.2. **Homology-preserving homeomorphisms of the annulus.** There are precisely four isotopy classes for the homeomorphims of the annulus $\mathbb{A}$. We are mainly interested in this paper in the one of the symmetry $S_{\mathbb{A}} : \mathbb{A} \to \mathbb{A}$ defined by the formula $S_{\mathbb{A}}(z,r) = (z,-r)$. A homeomorphism in this class lifts to homeomorphisms of $\tilde{\mathbb{A}}$ commuting with τ, exactly as those isotopic to $\mathrm{Id}_{\mathbb{A}}$. This is not true for a homeomorphism of $\mathbb{A}$ which is in one of the two remaining isotopy classes. This commutation property can be rephrased in a more topological way by saying that h induces the identity map on the homology group $H_1(\mathbb{A},\mathbb{Z})$. Indeed fix $x_0 \in \mathbb{A}$ and let $\omega : [0,1] \to \mathbb{A}$ be a path from $\omega(0) = x_0$ to $\omega(1) = h(x_0)$. If $\alpha : [0,1] \to \mathbb{A}$ is a loop based at x_0 whose lifts $\tilde{\alpha} : [0,1] \to \tilde{\mathbb{A}}$ satisfy $\tilde{\alpha}(1) = \tau(\tilde{\alpha}(0))$, then its homotopy class $[\alpha]$ generates the fundamental group $\pi_1(\mathbb{A}, x_0) \cong \mathbb{Z}$ and for any lift H of h to $\tilde{\mathbb{A}}$ we have

$$\tau \circ H = H \circ \tau \iff [\alpha] = [\omega * h \circ \alpha * \omega^{-1}],$$

where $*$ is the concatenation of paths. Since the Hurewicz isomorphism from $\pi_1(\mathbb{A}, x_0)$ to $H_1(\mathbb{A},\mathbb{Z})$ maps the homotopy class $[\beta]$ of a loop β to the homology class $\{\beta\}$ of the 1-cycle β (see e.g. [11]) we obtain

$$\tau \circ H = H \circ \tau \iff \{\alpha\} = \{\omega * h \circ \alpha * \omega^{-1}\} = \{h \circ \alpha\} = h_{*,1}(\{\alpha\}) \iff h_{*,1} = \mathrm{Id},$$

where $h_{*,1}$ is the endomorphism induced by h on $H_1(\mathbb{A},\mathbb{Z})$. In this situation, we say that h is a *homology-preserving homeomorphism*. Remark that all this remains valid if one replaces $\mathbb{A}$ with $\mathbb{A}'$.

Property 2.1. We let $\mathbb{B}$ be either $\mathbb{A}$ or $\mathbb{A}'$. We suppose that h is a homology-preserving homeomorphism of $\mathbb{B}$ and that $H : \tilde{\mathbb{B}} \to \tilde{\mathbb{B}}$ is a lift of h to the universal cover.

(1) If $\tilde{z} \in \tilde{\mathbb{B}}$ is a 2-periodic point of H, then $z = \Pi(\tilde{z})$ is a 2-periodic point of h.
(2) If H has no 2-periodic point, then it extends to a homeomorphism (again denoted H) of the whole sphere $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$ without a 2-periodic point.

Proof. 1) We have $\tilde{z} = H^2(\tilde{z})$, so $z = h^2(z)$. If $z = h(z)$, then $H(\tilde{z}) = \tau^n(\tilde{z})$ for an integer $n \in \mathbb{Z}$ and consequently $\tilde{z} = H^2(\tilde{z}) = \tau^{2n}(\tilde{z})$. So we get $n = 0$ and then $H(\tilde{z}) = \tilde{z}$, a contradiction.

2) If $\mathbb{B} = \mathbb{A}'$, then, of course, we just let $H(\infty) = \infty$. If $\mathbb{B} = \mathbb{A}$, let us write τ_b for the vertical translation by the vector $(0,b) \in \mathbb{R}^2$. We obtain the required extension by letting $H(\infty) = \infty$ and for $(x,y) \in \mathbb{R}^2$, $|y| \geq 1$:

$$H(x,y) = \begin{cases} \tau^{-1}_{y/|y|-y} \circ H \circ \tau_{y/|y|-y}(x,y) & \text{if } h \text{ is isotopic to } \mathrm{Id}_{\mathbb{A}}, \\ \tau_{y/|y|-y} \circ H \circ \tau_{y/|y|-y}(x,y) & \text{if } h \text{ is isotopic to } S_{\mathbb{A}}. \end{cases}$$

□

2.2.1. *Rotation number and rotation set.* This section will be useful in the Appendix. We recall some definitions and results which are very classical for homeomorphisms of the annulus $\mathbb{A}$ isotopic to $\mathrm{Id}_{\mathbb{A}}$, and the reader is just asked to observe that they extend immediately to any homology-preserving homeomorphism h of $\mathbb{A}$. The projection on the first coordinate $\tilde{\mathbb{A}} \to \mathbb{R}$, $(x,y) \mapsto x$ is denoted by p_1. We consider a lift $H : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ of h.

• We say that $\tilde{m} \in \tilde{\mathbb{A}}$ has *rotation number* $\rho_H(\tilde{m}) \in \mathbb{R}$ if $\lim_{+\infty} \frac{p_1 \circ H^n(\tilde{m}) - p_1(\tilde{m})}{n} = \rho_H(\tilde{m})$. In this case, all the points $\tau^k(\tilde{m})$, $k \in \mathbb{Z}$, have the same rotation number as $\tilde{m}$ and one can define the rotation number of $m = \Pi(\tilde{m}) \in \mathbb{A}$ by letting $\rho_H(m) =$

$\rho_H(\tilde{m})$. In particular if $m \in \mathbb{A}$ is a q-periodic point of h, then there exists $p \in \mathbb{Z}$ such that $H^q(\tilde{m}) = \tau^p(\tilde{m})$ for any $\tilde{m} \in \Pi^{-1}(\{m\})$ and we have $\rho_H(m) = p/q$.

- The rotation set $\rho(H)$ may be defined by

$$\rho(H) = \left\{ \int_{\mathbb{A}} \phi \, d\mu \,|\, \mu \text{ is an } h\text{-invariant Borel probability measure on } \mathbb{A} \right\},$$

where $\phi : \mathbb{A} \to \mathbb{R}$ is well defined by $\phi(m) = p_1 \circ H(\tilde{m}) - p_1(\tilde{m})$ for any $m \in \mathbb{A}$ and $\tilde{m} \in \Pi^{-1}(\{m\})$. The rotation set is a compact interval $[a, b]$ (possibly $a = b$) whose endpoints are realized by ergodic probability measures. By using the Birkhoff ergodic theorem one deduces that a, b are the rotation numbers of some points; i.e., there exist $\tilde{m}, \tilde{m}' \in \tilde{\mathbb{A}}$ such that $a = \rho_H(\tilde{m})$ and $b = \rho_H(\tilde{m}')$. We also have

$$\forall p, q \in \mathbb{Z} \quad \rho(\tau^{-p} \circ H^q) = q\rho(H) - p.$$

2.3. **A recurrence property.** The next result is Lemma 5.4 of [3]. It will give some important properties for the brick decompositions used farther in the paper (see Section 2.4).

Lemma 2.2. *Let $H : \mathbb{S}^2 \to \mathbb{S}^2$ be an orientation-reversing homeomorphism. Assume that there exists a sequence $D_1, \ldots, D_n$ of topological closed discs in $\mathbb{S}^2$ satisfying*

(i) $\forall i, j \in \{1, \ldots, n\} \quad D_i = D_j$ *or* $\mathrm{Int}_{\mathbb{S}^2}(D_i) \cap \mathrm{Int}_{\mathbb{S}^2}(D_j) = \emptyset$,
(ii) $\forall i \in \{1, \ldots, n\} \quad H(D_i) \cap D_i = \emptyset = H^2(D_i) \cap D_i$,
(iii) $\forall i, j \in \{1, \ldots, n\} \quad D_j$ *meets at most one of the two sets* $H^{-1}(D_i)$ *or* $H(D_i)$*; equivalently:* $H(D_i) \cap D_j \neq \emptyset \Rightarrow H(D_j) \cap D_i = \emptyset$,
(iv) $\forall i \in \{1, \ldots, n-1\} \, \exists k_i \geq 1$ *such that* $H^{k_i}(D_i) \cap D_{i+1} \neq \emptyset$ *and* $\exists k_n \geq 1$ *such that* $H^{k_n}(D_n) \cap D_1 \neq \emptyset$.

Then H has a 2-periodic point.

2.4. **Brick decompositions.** This notion was introduced by P. Le Calvez and A. Sauzet in [17] and [20].

2.4.1. *Definition and a topological property.* Let S be a surface. A *brick decomposition* $\mathcal{D}$ of S is a collection $\{B_i\}_{i \in I}$ of topological closed discs such that

- $\bigcup_{i \in I} B_i = S$,
- if $i \neq j$, then $B_i \cap B_j$ is either empty or an arc contained in $\partial_S B_i \cap \partial_S B_j$,
- for every $z \in S$, the set $I(z) = \{i \in I \,|\, z \in B_i\}$ contains at most three elements and $\bigcup_{i \in I(z)} B_i$ is a neighbourhood of z in S,
- if S has a boundary, then for every B_i the set $B_i \cap \mathrm{Bd}(S)$ is either empty or an arc. In particular there are at most two B_i's containing a given point $z \in \mathrm{Bd}(S)$.

A brick decomposition can be readily constructed from a triangulation of S. The B_i's are called the *bricks* of the decomposition. An elementary but important fact is the following. The reader is asked to keep in mind our convention concerning 1-submanifolds of bordered surfaces (Section 2.1).

Property 2.3. For any nonempty set $J \subsetneq I$, the set $\bigcup_{i \in J} B_i$ is closed in S and its frontier $\partial_S\big(\bigcup_{i \in J} B_i\big)$ is a 1-dimensional submanifold of S.

Observe that the second assertion in the above property requires the fourth item in the definition of a brick decomposition. Further details can be found for example in [3, Section 5.2] in the case of a surface S which is an open subset of the sphere $\mathbb{S}^2$. The adaptation to our slightly more general setting is straightforward.

2.4.2. *A Lemma of Guillou.* Lemma 2.4 below is implicit in [13]. For the convenience of the reader, we write it in a form adapted to our purpose and we give a proof.

Lemma 2.4. *Let F be a closed subset of $\tilde{\mathbb{A}}$ disjoint from $\mathrm{Bd}(\tilde{\mathbb{A}})$ and containing only isolated points (maybe $F = \emptyset$). Let $\tilde{\mathcal{D}} = \{B_i\}_{i\in I}$ be a brick decomposition of the surface $S = \tilde{\mathbb{A}} \setminus F$. Suppose that $X \subsetneq S$ is a nonempty union of bricks; i.e., $X = \bigcup_{i\in J} B_i$ for some nonempty $J \subsetneq I$, and has the following properties:*

(i) *X is connected,*

(ii) *X is unbounded as well as any connected component of $S \setminus X = \tilde{\mathbb{A}} \setminus (F \cup X)$.*

Then $\partial\overline{X}$ is a 1-dimensional submanifold of $\tilde{\mathbb{A}}$ such that $\partial_S X \subset \partial\overline{X} \subset F \cup \partial_S X$. Furthermore one of the two next assertions holds for any connected component Δ of $\partial\overline{X}$.

(1) *The set Δ is a line, or a half-line, properly embedded in $\tilde{\mathbb{A}}$.*

(2) *The set Δ is an arc crossing $\tilde{\mathbb{A}}$.*

Proof. The set X is closed in $S = \tilde{\mathbb{A}} \setminus F$ (Property 2.3), so $\overline{X} \subset X \cup F$. Moreover S is an open subset of $\tilde{\mathbb{A}}$; hence easily $\partial_S X \subset \partial\overline{X} \subset F \cup \partial_S X$. The set $\partial_S X$ is already known to be a 1-dimensional submanifold of S, and so of $\tilde{\mathbb{A}}$. Thus, in order to prove that $\partial\overline{X}$ is a 1-dimensional submanifold of $\tilde{\mathbb{A}}$, we just have to check that any given $z \in \partial\overline{X} \cap F \subset \tilde{\mathbb{A}} \setminus \mathrm{Bd}(\tilde{\mathbb{A}})$ possesses a neighbourhood in $\partial\overline{X}$ which is homeomorphic to $\mathbb{R}$. Consider any connected component δ of $\partial_S X$. Because of (ii) the set δ cannot be a Jordan curve and if δ is a line, then $\overline{\delta} \setminus \delta \subset F$ is different from $\{z\}$, i.e. $\delta \cup \{z\}$ is not a Jordan curve. Let V be an open disc in $\tilde{\mathbb{A}}$ containing z so small that $\overline{V} \cap F = \{z\}$ and $\overline{V} \cap \mathrm{Bd}(\tilde{\mathbb{A}}) = \emptyset$. We have $\emptyset \neq X \cap V = X \cap (V \setminus \{z\})$ and $\overline{V \setminus \{z\}} = \overline{V} \not\subset \overline{X}$; hence the connected set $V \setminus \{z\}$ meets $\partial_S X$. According to the above remarks we have

$$\forall\, \delta \in \pi_0(\partial_S X) \quad \delta \cap V \neq \emptyset \Rightarrow \delta \cap \partial V \neq \emptyset.$$

Since a brick decomposition of S is locally finite and $\partial_S X \subset \bigcup_{i\in J} \partial_S B_i$, it follows from the compactness of $\partial V \subset S$ that there are only finitely many δ meeting V. Moreover at least one of them satisfies $z \in \overline{\delta}$ since otherwise one could find a smaller V such that $V \cap \partial_S X = \emptyset$. Thus the set $\mathcal{E} = \{\delta \in \pi_0(\partial_S X) \,|\, z \in \overline{\delta}\}$ is nonempty and has finite cardinality. It is now enough to check that it contains precisely two elements. Clearly $z \in \tilde{\mathbb{A}} \setminus \mathrm{Bd}(\tilde{\mathbb{A}})$ implies that $\mathcal{E}$ has cardinality at least two. It cannot contain three distinct δ_i $(i = 1, 2, 3)$ since otherwise one could construct a Jordan curve $J \subset X$ as follows. For $i = 1, 2$, pick a point $a_i \in \delta_i$ and write β_i for the subarc of $\overline{\delta_i}$ joining a_i to z. Since X is a connected union of bricks there is an arc $\gamma \subset X$ joining a_1, a_2 and meeting $\partial_S X$ only in its two endpoints a_1, a_2. Thus $J = \beta_1 \cup \gamma \cup \beta_2$ is a Jordan curve disjoint from δ_3. Possibly after renaming the δ_i's, the arc γ can be chosen in such a way that δ_3 is contained in a bounded connected component of $\tilde{\mathbb{A}} \setminus J$, contradicting (ii). Finally, let Δ be a connected component of $\partial\overline{X}$. It is obviously closed in $\tilde{\mathbb{A}}$ and, if it is an arc, then its two endpoints cannot be on the same connected component of $\mathrm{Bd}(\tilde{\mathbb{A}})$ because of (ii). $\square$

2.4.3. *Attractors and repellers.* Suppose now that h is a homeomorphism of a surface S with a brick decomposition $\mathcal{D} = \{B_i\}_{i\in I}$. For any brick $B_{i_0} \in \mathcal{D}$ define

$$I_0 = \{i_0\}, \quad \mathcal{A}_0 = \mathcal{R}_0 = \bigcup_{i\in I_0} B_i = B_{i_0}$$

and inductively, for $n \in \mathbb{N}$,

$$I_{n+1} = \{i \in I \,|\, h(\mathcal{A}_n) \cap B_i \neq \emptyset\}, \quad \mathcal{A}_{n+1} = \bigcup_{i\in I_{n+1}} B_i,$$

$$I_{-n-1} = \{i \in I \,|\, h^{-1}(\mathcal{R}_{-n}) \cap B_i \neq \emptyset\}, \quad \mathcal{R}_{-n-1} = \bigcup_{i\in I_{-n-1}} B_i.$$

Following Le Calvez and Sauzet, we consider the sets $\mathcal{A} = \bigcup_{n\geq 1} \mathcal{A}_n$ and $\mathcal{R} = \bigcup_{n\geq 1} \mathcal{R}_{-n}$, which are called respectively the *attractor* and *repeller* associated to B_{i_0} and h. If necessary we specify the brick B_{i_0} by writing respectively $\mathcal{A}_n(B_{i_0})$ and $\mathcal{A}(B_{i_0})$ in place of $\mathcal{A}_n$ and $\mathcal{A}$ (the homeomorphism h is always clear from the context) and similarly for the sets $\mathcal{R}_n$ and $\mathcal{R}$. The next key result is Lemma 5.9 of [3] although it is stated there for a surface S which is an open subset of $\mathbb{S}^2$. As in [3], it is a consequence of Lemma 2.2.

Lemma 2.5. *Let $H : \mathbb{S}^2 \to \mathbb{S}^2$ be an orientation-reversing homeomorphism without a 2-periodic point and let $\mathcal{D} = \{B_i\}_{i\in I}$ be a brick decomposition of a surface $S \subset \mathbb{S}^2$ such that $H(S) = S$. Assume furthermore that for any two bricks B_i, B_j in $\mathcal{D}$ we have:*

- *$H(B_i) \cap B_i = \emptyset = H^2(B_i) \cap B_i$,*
- *at most one of the two sets $H^{-1}(B_i) \cap B_j$ or $H(B_i) \cap B_j$ is nonempty.*

Then the attractor $\mathcal{A}$ and the repeller $\mathcal{R}$ associated to any brick $B_{i_0} \in \mathcal{D}$ are such that

(i) $\mathrm{Int}_S(B_{i_0}) \cap \mathcal{A} = \emptyset$,
(ii) $\mathcal{A} \cap \mathrm{Int}_S(\mathcal{R}) = \emptyset$.

In other words, (i) B_{i_0} is not a brick of $\mathcal{A}$ and (ii) $\mathcal{A}$ and $\mathcal{R}$ do not have any common brick.

2.5. **Two versions of the Poincaré-Birkhoff theorem.** The next result is due to Guillou.

Theorem 2.6 ([12]). *Let h be a fixed point free homeomorphism of the annulus $\mathbb{A}$ isotopic to $\mathrm{Id}_{\mathbb{A}}$. Then there is an essential Jordan curve $J \subset \mathbb{A}$ such that $h(J) \cap J = \emptyset$ or there is an arc α crossing $\mathbb{A}$ such that $h(\alpha) \cap \alpha = \emptyset$.*

As explained in the introduction, this theorem was helpful for finding a reasonable notion of (un)twisted homeomorphism in our framework. From a more technical point of view, some of our constructions were also inspired by [12, 13]. In particular Lemma 3.7 below is an adaptation of some arguments in [12]. Nevertheless, no familiarity with these papers is needed for the reading of the present work. On the other hand, the following close result of H.E. Winkelnkemper is an ingredient for proving Theorem 1.2.

Theorem 2.7 ([21]). *Let h be a homeomorphism of the annulus $\mathbb{A}$ isotopic to $\mathrm{Id}_{\mathbb{A}}$ and let $H : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ be a lift of h. If H is not conjugate to τ, then either H has a fixed point or there exists an essential Jordan curve $J \subset \mathbb{A}$ such that $h(J) \cap J = \emptyset$.*

3. Proof of the main result (Theorem 1.2)

Let h be as in Theorem 1.2 without any 2-periodic point. Let $H_0 : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ be a lift of h. Since h^2 has no fixed point on $\mathrm{Bd}(\mathbb{A})$ there exists $k \in \mathbb{Z}$ such that $\tilde{z} < \tau^k \circ H_0^2(\tilde{z}) < \tau(\tilde{z})$ for every $\tilde{z} \in \mathrm{Bd}^-(\tilde{\mathbb{A}})$ and one deduces from $H_0 \circ \tau = \tau \circ H_0$ that these inequalities also hold for $\tilde{z} \in \mathrm{Bd}^+(\tilde{\mathbb{A}})$. If k is even (resp. odd) let us define $H = \tau^{k/2} \circ H_0$ (resp. $H = \tau^{(1-k)/2} \circ H_0^{-1}$) and afterwards $G = \tau \circ H^{-2}$. Thus H and G are lifts of h and h^{-2} (resp. of h^{-1} and h^2) and one can check from the above inequalities that

$$\forall \tilde{z} \in \mathrm{Bd}(\tilde{\mathbb{A}}) \quad \tilde{z} < H^2(\tilde{z}) < \tau(\tilde{z}) \text{ and } \tilde{z} < G(\tilde{z}) < \tau(\tilde{z}).$$

We denote from now on $F = \mathrm{Fix}(H) = \mathrm{Fix}(H^2)$ (see Property 2.1) and $S = \tilde{\mathbb{A}} \setminus F$. We have $S = \tau(S) = H(S)$. We recall that H can be thought of as the restriction of a homeomorphism of the whole sphere $\mathbb{S}^2$ without any 2-periodic orbit. For later use we make the following remark.

Claim. The homeomorphism G is fixed point free.

Indeed $\tilde{z} = G(\tilde{z})$ implies $\Pi(\tilde{z}) \in \mathrm{Fix}(h^2) = \mathrm{Fix}(h)$; hence $H(\tilde{z}) = \tau^n(\tilde{z})$ for some $n \in \mathbb{Z}$ and consequently $\tau^{1-2n}(\tilde{z}) = G(\tilde{z}) = \tilde{z}$, which is absurd.

To avoid repetitions we introduce some vocabulary: for any map $f : X \to X$ we say that a family $\mathcal{E}$ of subsets of X is *2-aperiodic with respect to f* if we have

$$\forall Y, Z \in \mathcal{E} \quad \begin{cases} Y \cap f(Y) = \emptyset = Y \cap f^2(Y), \\ f(Y) \cap Z = \emptyset \text{ or } f(Z) \cap Y = \emptyset. \end{cases}$$

Thus Lemma 2.2 and Lemma 2.5 can be rephrased by demanding that respectively $\{D_1, \ldots, D_n\}$ and the brick decomposition $\mathcal{D}$ are 2-aperiodic w.r.t. H.

We now construct a brick decomposition of $S = \tilde{\mathbb{A}} \setminus F$ which, in addition to the general features recalled in Section 2.4, captures some part of the behaviour of H^2 on $\mathrm{Bd}(\tilde{\mathbb{A}})$.

Lemma 3.1. *There exist an $\epsilon \in (0,1)$ and a brick decomposition $\tilde{\mathcal{D}}_H = \{B_i\}_{i \in I}$ of the surface $S = \tilde{\mathbb{A}} \setminus F$ satisfying the following properties:*

1. *$\tilde{\mathcal{D}}_H$ is τ-equivariant, which means $\{\tau(B_i)\}_{i \in I} = \tilde{\mathcal{D}}_H$.*
2. *$\tilde{\mathcal{D}}_H$ is 2-aperiodic w.r.t. H.*
3. *The bricks meeting $\mathrm{Bd}^-(\tilde{\mathbb{A}})$ (resp. $\mathrm{Bd}^+(\tilde{\mathbb{A}})$) are rectangles $B_n^- = [a_n, a_{n+1}] \times [-1, -1+\epsilon]$ (resp. $B_n^+ = [b_n, b_{n+1}] \times [1-\epsilon, 1]$), where $(a_n)_{n \in \mathbb{Z}}$ and $(b_n)_{n \in \mathbb{Z}}$ are two strictly increasing sequences of reals numbers such that $\lim_{\pm\infty} a_n = \lim_{\pm\infty} b_n = \pm\infty$.*
4. *For every $n \in \mathbb{Z}$ we have either $H^2(B_n^\pm) \cap B_{n+1}^\pm \neq \emptyset$ or there exists an integer $m = m(n)$ such that $H^{-1}(B_m^\mp) \cap B_n^\pm \neq \emptyset \neq H(B_m^\mp) \cap B_{n+1}^\pm$. Consequently*

$$B_{n+1}^\pm \subset \mathcal{A}_2(B_n^\pm) \subset \mathcal{A}(B_n^\pm) \text{ and } B_{n-1}^\pm \subset \mathcal{R}_{-2}(B_n^\pm) \subset \mathcal{R}(B_n^\pm).$$

Proof. We first construct the trace of $\tilde{\mathcal{D}}_H$ on $\mathrm{Bd}(\tilde{\mathbb{A}})$ in order to get (4) and afterwards we extend it to obtain the whole brick decomposition. Let us call an *interval decomposition* of $\mathrm{Bd}(\mathbb{A})$ a set $\mathcal{I}$ of compact intervals of $\mathrm{Bd}(\mathbb{A})$ satisfying $\bigcup_{\alpha \in \mathcal{I}} \alpha = \mathrm{Bd}(\mathbb{A})$ and such that, for $\alpha \neq \alpha'$ in $\mathcal{I}$, we have

$$z \in \alpha \cap \alpha' \Rightarrow z \text{ is a common endpoint of } \alpha \text{ and } \alpha'.$$

We define in the same way an interval decomposition of $\mathrm{Bd}(\tilde{\mathbb{A}})$. Since h^2 is fixed point free on $\mathrm{Bd}(\mathbb{A})$, one can easily construct a finite interval decomposition $\mathcal{I}_0$ which is 2-aperiodic w.r.t. h. Remark that $\mathcal{I}_0$ has at least three intervals on each boundary component of $\mathbb{A}$. Let $\widetilde{\mathcal{I}_0}$ be the set obtained by "lifting $\mathcal{I}_0$", that is, $\widetilde{\mathcal{I}_0}$ contains the connected components of all the sets $\Pi^{-1}(\alpha), \alpha \in \mathcal{I}_0$. This provides an interval decomposition of $\mathrm{Bd}(\tilde{\mathbb{A}})$ which is both τ-equivariant and 2-aperiodic w.r.t. H. Now if $\tilde{\mathcal{I}}, \tilde{\mathcal{I}}'$ are two interval decompositions of $\mathrm{Bd}(\tilde{\mathbb{A}})$, we write $\tilde{\mathcal{I}} \preceq \tilde{\mathcal{I}}'$ if $\tilde{\mathcal{I}}$ is finer than $\tilde{\mathcal{I}}'$, that is, if every interval of $\tilde{\mathcal{I}}$ is contained in an interval of $\tilde{\mathcal{I}}'$. This yields a partial ordering on the set of all the interval decompositions of $\mathrm{Bd}(\tilde{\mathbb{A}})$. There exists $\tilde{\mathcal{I}} \succeq \widetilde{\mathcal{I}_0}$ which is maximal among the interval decompositions of $\mathrm{Bd}(\tilde{\mathbb{A}})$ which are both τ-equivariant and 2-aperiodic w.r.t. H. Consider two intervals $\tilde{\alpha}, \tilde{\alpha}' \in \tilde{\mathcal{I}}$ with a common endpoint, say $\tilde{\alpha} = [a, a'] \times \{\sigma\}, \tilde{\alpha}' = [a', a''] \times \{\sigma\}$ $(a < a' < a'', \sigma = \pm 1)$. We have $\tilde{\alpha}' \neq \tau(\tilde{\alpha})$ since otherwise we would get

$$(a, \sigma) < H^2(a, \sigma) < \tau(a, \sigma) = (a', \sigma)$$

on the boundary component of $\tilde{\mathbb{A}}$ containing these three points, which contradicts $\tilde{\alpha} \cap H^2(\tilde{\alpha}) = \emptyset$. Consequently we get a new τ-equivariant interval decomposition $\tilde{\mathcal{I}}' \succeq \tilde{\mathcal{I}}$ by replacing in $\tilde{\mathcal{I}}$ all the translates of $\tilde{\alpha}$ and $\tilde{\alpha}'$ by those of $\tilde{\alpha} \cup \tilde{\alpha}'$; precisely we define

$$\tilde{\mathcal{I}}' = \big(\tilde{\mathcal{I}} \setminus \{\tau^n(\tilde{\alpha}), \tau^n(\tilde{\alpha}') \,|\, n \in \mathbb{Z}\}\big) \cup \{\tau^n(\tilde{\alpha} \cup \tilde{\alpha}') \,|\, n \in \mathbb{Z}\}.$$

This interval decomposition $\tilde{\mathcal{I}}'$ cannot be 2-aperiodic w.r.t. H because of the maximality of $\tilde{\mathcal{I}}$. Since $H \circ \tau = \tau \circ H$, it follows that either we have

$$\emptyset \neq H^2(\tilde{\alpha} \cup \tilde{\alpha}') \cap (\tilde{\alpha} \cup \tilde{\alpha}') = \big(H^2(\tilde{\alpha}) \cap \tilde{\alpha}'\big) \cup \big(H^2(\tilde{\alpha}') \cap \tilde{\alpha}\big) \tag{1}$$

or there exists $\tilde{\alpha}'' = [b, b'] \times \{-\sigma\} \in \tilde{\mathcal{I}}$ such that

$$H^{-1}(\tilde{\alpha}'') \cap (\tilde{\alpha} \cup \tilde{\alpha}') \neq \emptyset \neq H(\tilde{\alpha}'') \cap (\tilde{\alpha} \cup \tilde{\alpha}') \tag{2}$$

and the latter can be restated as

$$\big(H^{-1}(\tilde{\alpha}'') \cap \tilde{\alpha} \neq \emptyset \neq H(\tilde{\alpha}'') \cap \tilde{\alpha}'\big) \text{ or } \big(H^{-1}(\tilde{\alpha}'') \cap \tilde{\alpha}' \neq \emptyset \neq H(\tilde{\alpha}'') \cap \tilde{\alpha}\big). \tag{3}$$

Because of the behaviour of H^2 on $\mathrm{Bd}(\tilde{\mathbb{A}})$, Equation (1) gives $H^2(\tilde{\alpha}) \cap \tilde{\alpha}' \neq \emptyset$. Moreover Equation (3) implies $H^{-1}(\tilde{\alpha}'') \cap \tilde{\alpha} \neq \emptyset \neq H(\tilde{\alpha}'') \cap \tilde{\alpha}'$. Indeed observe that $H^{-1}(\tilde{\alpha}'') \cap \tilde{\alpha}' \neq \emptyset \neq H(\tilde{\alpha}'') \cap \tilde{\alpha}$ would imply that $H(\tilde{\alpha}'')$ meets both $H^2(\tilde{\alpha}')$ and $\tilde{\alpha}$ which are respectively on the right and on the left of $\tilde{\alpha}'$ in the connected component of $\mathrm{Bd}(\tilde{\mathbb{A}})$ which contains them. Consequently we would get $\tilde{\alpha}' \subset H(\tilde{\alpha}'')$ and then

$$\emptyset \neq \tilde{\alpha}'' \cap H(\tilde{\alpha}') \subset \tilde{\alpha}'' \cap H^2(\tilde{\alpha}'') = \emptyset,$$

which is absurd. For a given $\epsilon \in (0,1)$ one can associate to each $\tilde{\alpha} = [a, a'] \times \{-1\} \in \tilde{\mathcal{I}}$ the rectangle $\tilde{R}_{\tilde{\alpha}} = [a, a'] \times [-1, -1+\epsilon] \subset \tilde{\mathbb{A}}$ and similarly for $\tilde{\alpha} = [b, b'] \times \{1\} \in \tilde{\mathcal{I}}$. The family $\{\tilde{R}_{\tilde{\alpha}} \,|\, \tilde{\alpha} \in \tilde{\mathcal{I}}\}$ is τ-equivariant and, for a small enough $\epsilon > 0$, it is also 2-aperiodic w.r.t. H. It remains to complete this family to get a brick decomposition of S as required. By the construction, each rectangle $\tilde{R}_{\tilde{\alpha}}$ is mapped homeomorphically by the covering map Π onto $R_\alpha = \Pi(\tilde{R}_{\tilde{\alpha}}) \subset \mathbb{A} \setminus \Pi(F)$. Because of the finiteness of $\mathcal{I}$ there are only finitely many different R_α's, call them $R_1, \ldots, R_k$, and it is not difficult to construct a brick decomposition $\mathcal{D}_0$ of $\mathbb{A} \setminus \Pi(F)$ possessing all the R_i's as bricks (one can proceed as for Lemma 5.10 of [3]). Since every brick $B \in \mathcal{D}_0$ is topologically a disc, so are the connected components of $\Pi^{-1}(B)$ and we get a τ-equivariant brick decomposition $\widetilde{\mathcal{D}_0}$ of S by considering all the connected components of the sets $\Pi^{-1}(B)$, $B \in \mathcal{D}_0$. Subdividing suitably some bricks of $\widetilde{\mathcal{D}_0} \setminus \{\tilde{R}_{\tilde{\alpha}} \,|\, \tilde{\alpha} \in \tilde{\mathcal{I}}\}$ if necessary, one gets a τ-equivariant brick decomposition $\tilde{\mathcal{D}}$ of S which is also 2-aperiodic w.r.t. H. The reader can for example adapt the proof of Lemma 5.10 of [3]; the only difference is that if a brick $B \in \widetilde{\mathcal{D}_0} \setminus \{\tilde{R}_{\tilde{\alpha}} \,|\, \tilde{\alpha} \in \tilde{\mathcal{I}}\}$ is subdivided, then the translated subdivision must be performed on each $\tau^k(B)$, $k \in \mathbb{Z}$, in order to keep the τ-equivariance. $\square$

Let $\tilde{\mathcal{D}}_H$ be a brick decomposition of S as given by Lemma 3.1. We keep the notation $B_n^{\pm}, a_n, b_n$ from this lemma. Choose any brick meeting $\mathrm{Bd}(\tilde{\mathbb{A}})$, say $B_0^- = [a_0, a_1] \times [-1, -1+\epsilon]$, and consider the attractor $\mathcal{A} = \mathcal{A}(B_0^-)$ and the repeller $\mathcal{R} = \mathcal{R}(B_0^-)$ associated to B_0^- and H. According to Property (4) of Lemma 3.1 we have

$$\bigcup_{n\geq 1} B_n^- \subset \bigcup_{n\geq 1} \mathcal{A}_{2n} \subset \mathcal{A} \text{ and } \bigcup_{n\leq -1} B_n^- \subset \bigcup_{n\geq 1} \mathcal{R}_{-2n} \subset \mathcal{R}.$$

We let

$$\mathcal{A}_e = \bigcup_{n\geq 1} \mathcal{A}_{2n},\ \mathcal{R}_e = \bigcup_{n\geq 1} \mathcal{R}_{-2n} \text{ and } \mathcal{A}_o = \bigcup_{n\geq 0} \mathcal{A}_{2n+1},\ \mathcal{R}_o = \bigcup_{n\geq 0} \mathcal{R}_{-2n-1},$$

where the subscripts e and o stand for respectively even and odd. It follows from $B_1^- \subset \mathcal{A}_2$ and $B_0^- \cap B_1^- \neq \emptyset$ that $\mathcal{A}_e$ and $\mathcal{A}_o$ are connected; hence $\mathcal{A} = \mathcal{A}_e \cup \mathcal{A}_o$ has at most two connected components. We clearly have $H(\mathcal{A}_e) \subset \mathcal{A}_o$ and $H(\mathcal{A}_o) \subset \mathcal{A}_e$. Since the union of the bricks containing a given point $\tilde{z} \in S$ is a neighbourhood of $\tilde{z}$, we can be more precise and observe that

$$H(\mathcal{A}_e) \subset \mathrm{Int}(\mathcal{A}_o) \quad \text{and} \quad H(\mathcal{A}_o) \subset \mathrm{Int}(\mathcal{A}_e).$$

Remark that the set $\mathcal{A}_o \cap \mathrm{Bd}^+(\tilde{\mathbb{A}})$ contains $H([a_1, +\infty) \times \{-1\})$, so it is unbounded on the right. Similarly $B_{-1}^- \subset \mathcal{R}_{-2}$ and $B_{-1}^- \cap B_0^- \neq \emptyset$ implies that $\mathcal{R}_e$ and $\mathcal{R}_o$ are connected, so $\mathcal{R} = \mathcal{R}_e \cup \mathcal{R}_o$ has at most two connected components. Moreover we have

$$H^{-1}(\mathcal{R}_e) \subset \mathrm{Int}(\mathcal{R}_o) \quad \text{and} \quad H^{-1}(\mathcal{R}_o) \subset \mathrm{Int}(\mathcal{R}_e).$$

The set $\mathcal{R}_o \cap \mathrm{Bd}^+(\tilde{\mathbb{A}})$ is unbounded on the left since it contains $H^{-1}((-\infty, a_0] \times \{-1\})$. It is now enough to prove the two next propositions.

Proposition 3.2. *If either $\mathcal{A}$ or $\mathcal{R}$ is not connected, then there exists a Jordan curve J as announced in Theorem* 1.2.

Proposition 3.3. *If $\mathcal{A}$ and $\mathcal{R}$ are connected, then the conclusion of Theorem* 1.2 *holds.*

Proof of Proposition 3.2. We first suppose that $\mathcal{A}$ is not connected, i.e. $\mathcal{A}_e \cap \mathcal{A}_o = \emptyset$. Then we necessarily have $\mathcal{A}_e \cap \mathrm{Bd}^+(\tilde{\mathbb{A}}) = \emptyset$. Otherwise $\mathcal{A}_e$ contains a brick B_i^+ and Property (4) of Lemma 3.1 implies $\bigcup_{n\geq i} B_n^+ \subset \mathcal{A}_e$. Consequently $\mathrm{Bd}^+(\tilde{\mathbb{A}}) \cap \mathcal{A}_o$ would be bounded on the right, a contradiction. Let $X_0 = \bigcup_{n\in\mathbb{Z}} \tau^n(\mathcal{A}_e)$ and let X be the union of X_0 with all the bounded connected components of $S \setminus X_0$. Because of the τ-equivariance of $\tilde{\mathcal{D}}_H$, each set $\tau^n(\mathcal{A}_e)$ $(n \in \mathbb{Z})$ is a union of bricks and then so are X_0 and X. Since each set $\tau^n(\mathcal{A}_e)$ is connected and $\tau^n(B_1^-) \subset \tau^n(\mathcal{A}_e) \cap \tau^m(\mathcal{A}_e)$ for $n \geq m$ in $\mathbb{Z}$ we obtain that X is connected and then satisfies the assumptions (i)-(ii) of Lemma 2.4. Since $\overline{X} \subset X \cup F$ is disjoint from $\mathrm{Bd}^+(\tilde{\mathbb{A}})$ and

$$\mathrm{Bd}^-(\tilde{\mathbb{A}}) \subset \mathrm{Int}\big(\bigcup_{n\in\mathbb{Z}} B_n^-\big) \subset \mathrm{Int}(X)$$

there exists a connected component $\tilde{J}$ of $\partial\overline{X}$ which is a line properly embedded in $\tilde{\mathbb{A}}$, disjoint from $\mathrm{Bd}(\tilde{\mathbb{A}})$, which separates $\mathrm{Bd}^+(\tilde{\mathbb{A}})$ and $\mathrm{Bd}^-(\tilde{\mathbb{A}})$ in $\tilde{\mathbb{A}}$. We have $\tau(X) = X$; hence $\tau(\partial\overline{X}) = \partial\overline{X}$ and consequently $\tau(\tilde{J}) = \tilde{J}$ since otherwise $\tilde{J}$ and $\tau(\tilde{J}) \subset \partial\overline{X}$ would be two disjoint and properly embedded lines in $\tilde{\mathbb{A}}$ separating $\mathrm{Bd}^-(\tilde{\mathbb{A}})$ and $\mathrm{Bd}^+(\tilde{\mathbb{A}})$. This contradicts the connectedness of $\overline{X}$ because $\mathrm{Bd}^-(\tilde{\mathbb{A}}) \subset \mathrm{Int}(\overline{X})$. Thus $\tilde{J}$ projects onto an essential Jordan curve $J = \Pi(\tilde{J}) \subset \mathbb{A} \setminus \mathrm{Bd}(\mathbb{A})$ and it is enough to show that $H^2(\tilde{J} \cap S)$ is contained in one of the two connected components of $\tilde{\mathbb{A}} \setminus \tilde{J}$. We know that

$$\tilde{J} \subset \partial\overline{X} \subset \partial_S X \cup F \text{ and also } \partial_S X \subset \partial_S X_0 \subset \bigcup_{n\in\mathbb{Z}} \partial_S(\tau^n(\mathcal{A}_e)),$$

the last inclusion following from the fact that $X_0 = \bigcup_{n\in\mathbb{Z}} \tau^n(\mathcal{A}_e)$ is closed in S (Property 2.3). Thus any point $\tilde{z} \in \tilde{J} \cap S$ belongs to $\partial_S(\tau^n(\mathcal{A}_e)) = \tau^n(\partial_S \mathcal{A}_e)$ for some $n \in \mathbb{Z}$; hence

$$H^2(\tilde{z}) \in H^2(\tau^n(\partial_S\mathcal{A}_e)) = \tau^n(H^2(\partial_S\mathcal{A}_e)) \subset \tau^n(\mathrm{Int}(\mathcal{A}_e)) \subset \mathrm{Int}(X).$$

The set $\mathrm{Int}(X)$ being connected (as the interior of a connected union of bricks) and disjoint from $\tilde{J}$, this proves the proposition when $\mathcal{A}$ is not connected. If $\mathcal{R}$ is not connected, the proof is the same after replacing $\mathcal{A}_e$ and H with respectively $\mathcal{R}_e$ and H^{-1}. □

Proof of Proposition 3.3. This is broken into the following four lemmas.

Lemma 3.4. *If $\mathcal{A}$ and $\mathcal{R}$ are connected, then $\mathcal{A}$ is bounded on the left while $\mathcal{R}$ is bounded on the right.*

Proof. Since $\mathcal{A}$ is a connected union of bricks and meets the two boundary components of $\tilde{\mathbb{A}}$, one can find an arc $\tilde{\gamma}$ crossing $\tilde{\mathbb{A}}$ from a point $\tilde{z}_0 \in \mathrm{Bd}^-(\tilde{\mathbb{A}}) \cap \mathcal{A}$ to a point $\tilde{z}_1 \in \mathrm{Bd}^+(\tilde{\mathbb{A}}) \cap \mathcal{A}$ and lying entirely in $\mathrm{Int}(\mathcal{A})$. We know from Lemma 2.5 that $\mathcal{R} \subset \tilde{\mathbb{A}} \setminus \mathrm{Int}(\mathcal{A}) \subset \tilde{\mathbb{A}} \setminus \tilde{\gamma}$ and furthermore $\bigcup_{n\leq -1} B_n^- \subset \mathcal{R}$ is unbounded on the left. Consequently $\mathcal{R}$ is contained in the domain on the left of $\tilde{\gamma}$ and is then bounded on the right. One gets the assertion concerning $\mathcal{A}$ by reversing the roles of $\mathcal{A}$ and $\mathcal{R}$. □

Lemma 3.5. *If $\mathcal{A}$ and $\mathcal{R}$ are connected, then at least one of the following two assertions is true.*

(1) *There exists an essential Jordan curve $J \subset \mathbb{A}$ such that $J \cap h^2(J) = \emptyset$.*
(2) *There exists an arc $\tilde{\beta}$ crossing $\tilde{\mathbb{A}}$ such that, writing W_r for the domain on the right of $\tilde{\beta}$, we have*
   - $\tau(W_r) \subset H^2(W_r) \subset H(W_r) \subset W_r$*;*
     equivalently, $G(W_r) \subset W_r \subset H^{-1}(W_r) \subset H^{-2}(W_r)$.
   - $H(\tilde{\beta}) \cap \tilde{\beta} = H^2(\tilde{\beta}) \cap \tilde{\beta} = \tilde{\beta} \cap F$.

Proof. Let X be the union of $\mathcal{A}$ with all the bounded components of $S \setminus \mathcal{A}$, so that X is a union of bricks of $\tilde{\mathcal{D}}_H$ satisfying (i)-(ii) of Lemma 2.4 and $\partial_S X \subset \partial_S \mathcal{A}$. The set $\mathcal{A}$ meets the two boundary components of $\tilde{\mathbb{A}}$ and is bounded on the left (Lemma

3.4), so the same is true for $\overline{X}$. It follows that there is a connected component $\tilde{\alpha}$ of $\partial\overline{X} \subset \partial_S X \cup F$ which is an arc crossing $\tilde{\mathbb{A}}$. Write U_l (resp. U_r) for the domain on the left (resp. on the right) of $\tilde{\alpha}$. We have $\mathrm{Int}(X) \subset U_r$ because $\mathrm{Int}(X)$ is connected, disjoint from $\tilde{\alpha} \subset \partial\overline{X}$ and unbounded on the right. We also observe that $H(U_r) \subset U_r$. Indeed H fixes the two ends of $\tilde{\mathbb{A}}$, so it is enough to check $\partial H(U_r) \subset \overline{U_r}$ and this turns out to be true because

$$\partial H(U_r) = H(\tilde{\alpha}) = H(\tilde{\alpha} \cap S) \cup (\tilde{\alpha} \cap F)$$

and

$$H(\tilde{\alpha} \cap S) \subset H(\partial\overline{X} \cap S) \subset H(\mathcal{A}) \subset \mathrm{Int}(\mathcal{A}) \subset \mathrm{Int}(X) \subset U_r.$$

The homeomorphism $G = \tau \circ H^{-2}$ being fixed point free, Theorem 2.7 tells us that either G is conjugate to the translation τ or there exists an essential Jordan curve $J \subset \mathbb{A}$ such that $J \cap h^{\pm 2}(J) = \emptyset$. So we can suppose that G is conjugate to τ. Defining $V = \bigcup_{n\in\mathbb{N}} G^n(U_r)$ and keeping in mind that $G(\tilde{z}) > \tilde{z}$ for any $\tilde{z} \in \mathrm{Bd}(\tilde{\mathbb{A}})$, one obtains since G is conjugate to τ that $V = \bigcup_{0\le n\le m} G^n(U_r)$ for some integer $m \geq 0$ and consequently $\overline{V} = \bigcup_{0\le n\le m} G^n(\overline{U_r})$. In particular this shows that $\tilde{\mathbb{A}} \setminus \overline{V}$ has only one unbounded (on the left) connected component which we call W_l. A classical result of Kerékjártó asserts that if U_1, U_2 are two Jordan domains[2] of $\mathbb{R}^2$, then any connected component of $U_1 \cap U_2$ is also a Jordan domain whose frontier is contained in $\partial_{\mathbb{R}^2} U_1 \cup \partial_{\mathbb{R}^2} U_2$ (see [15]). One deduces that ∂W_l is an arc crossing $\tilde{\mathbb{A}}$ and contained in $\bigcup_{0\le n\le m} G^n(\tilde{\alpha})$. We let $\tilde{\beta} = \partial W_l$ and $W_r = \tilde{\mathbb{A}} \setminus \overline{W_l}$. Thus W_l, W_r are the domains respectively on the left and on the right of $\tilde{\beta}$ and $V \subset W_r$. To prove the first point of (2), first remark that

$$G(V) \subset V \Rightarrow W_l \subset G(W_l) \iff G(W_r) \subset W_r \iff \tau(W_r) \subset H^2(W_r).$$

Furthermore we know that $H(U_r) \subset U_r$; hence by using $H \circ G = G \circ H$ we obtain that $H(V) \subset V$. It follows that $W_l \subset H(W_l)$ and afterwards $H^2(W_r) \subset H(W_r) \subset W_r$. To prove the second point of (2), remark that $G(S) = S$ gives

$$\tilde{\beta} \cap S \subset \Big(\bigcup_{0\le n\le m} G^n(\tilde{\alpha}) \Big) \cap S \subset \bigcup_{0\le n\le m} G^n(\partial_S \mathcal{A}).$$

Thus we get for $i \in \{1, 2\}$,

$$H^i(\tilde{\beta} \cap S) \subset \bigcup_{0\le n\le m} G^n(H^i(\partial_S\mathcal{A})) \subset \bigcup_{0\le n\le m} G^n(\mathrm{Int}(\mathcal{A})) \subset \bigcup_{0\le n\le m} G^n(U_r) = V \subset W_r$$

and consequently $H^i(\tilde{\beta}) \cap \tilde{\beta} = H^i(\tilde{\beta} \cap F) \cap \tilde{\beta} = \tilde{\beta} \cap F$. □

Lemma 3.6. *Let $\tilde{\beta}$ and W_r be as in Lemma* 3.5. *We suppose that $\tau(\overline{W_r}) \not\subset W_r$, i.e. that $\tau(\tilde{\beta}) \cap \tilde{\beta} \neq \emptyset$. Then there exists a Jordan curve J as announced in Theorem* 1.2.

Proof. We adopt the convention that any arc crossing $\tilde{\mathbb{A}}$ is oriented from its endpoint on $\mathrm{Bd}^-(\tilde{\mathbb{A}})$ to its endpoint on $\mathrm{Bd}^+(\tilde{\mathbb{A}})$ and we keep the notation W_l for the domain on the left of $\tilde{\beta}$. Let $\tilde{z}_0, \tilde{z}_1$ be the endpoints of $\tilde{\beta}$ on respectively $\mathrm{Bd}^-(\tilde{\mathbb{A}})$ and $\mathrm{Bd}^+(\tilde{\mathbb{A}})$. Let $\tilde{z}$ be the first point of $\tilde{\beta}$ such that $\tau(\tilde{z}) \in \tilde{\beta}$. Clearly $\tilde{z} \notin \{\tilde{z}_0, \tilde{z}_1\}$, so we have two arcs $[\tilde{z}_0, \tau(\tilde{z})]_{\tilde{\beta}}$ and $\tau([\tilde{z}_0, \tilde{z}]_{\tilde{\beta}}) = [\tau(\tilde{z}_0), \tau(\tilde{z})]_{\tau(\tilde{\beta})}$ contained in $\tilde{\mathbb{A}} \setminus \mathrm{Bd}(\tilde{\mathbb{A}})$ except for their origins $\tilde{z}_0, \tau(\tilde{z}_0)$ and meeting only in their common endpoint $\tau(\tilde{z})$.

[2] A *Jordan domain* of $\mathbb{R}^2$ is the bounded complementary domain of a Jordan curve $J \subset \mathbb{R}^2$.

Since $\tau(W_r) \subset W_r$, two points of $\tilde\beta \cap \tau(\tilde\beta)$ are met in the same order on $\tilde\beta$ and on $\tau(\tilde\beta)$, so we get more precisely

$$\{\tau(\tilde z)\} = [\tilde z_0, \tau(\tilde z)]_{\tilde\beta} \cap \tau([\tilde z_0, \tilde z]_{\tilde\beta}) = [\tilde z_0, \tau(\tilde z)]_{\tilde\beta} \cap \tau(\tilde\beta).$$

Letting $\tilde\gamma = [\tilde z_0, \tau(\tilde z)]_{\tilde\beta} \cup \tau([\tilde z_0, \tilde z]_{\tilde\beta})$, it follows that $\tilde\gamma = \partial\Omega$ for a connected component Ω of $\tilde{\mathbb{A}} \setminus (\tilde\beta \cup \tau(\tilde\beta))$. Since any point $\tilde m \in \mathrm{Bd}^-(\tilde{\mathbb{A}})$ such that $\tilde z_0 < \tilde m < \tau(\tilde z_0)$ belongs to $W_r \cap \tau(W_l)$, we deduce that Ω is a connected component of $W_r \cap \tau(W_l)$. Moreover we have $\tilde\beta \cap \tau(\tilde\beta) \subset \tilde\beta \cap H(\tilde\beta) \subset F$ by item (2) of Lemma 3.5, so $\tau^k(\tilde z) \in F$ for every $k \in \mathbb{Z}$.

FIRST CASE: The points $\tilde z, \tau(\tilde z)$ are met in this order on $\tilde\beta$.

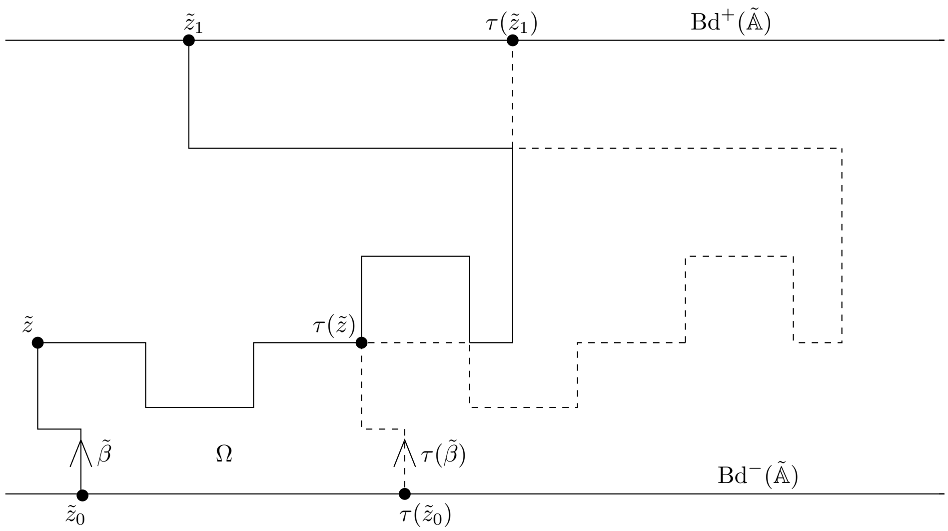

FIGURE 1. The arcs $\tilde\beta$ and $\tau(\tilde\beta)$ when $\tilde z$ precedes $\tau(\tilde z)$ on $\tilde\beta$

Let $\tilde J = [\tilde z, \tau(\tilde z)]_{\tilde\beta}$, so that $\tilde\gamma = [\tilde z_0, \tilde z]_{\tilde\beta} \cup \tilde J \cup \tau([\tilde z_0, \tilde z]_{\tilde\beta})$. The arc $\tilde J$ projects onto an essential loop $J = \Pi(\tilde J) \subset \mathbb{A} \setminus \mathrm{Bd}(\mathbb{A})$, and we have

$$\{\tilde z, \tau(\tilde z)\} \subset \tilde J \cap H^2(\tilde J) \subset \tilde\beta \cap H^2(\tilde\beta) \subset F.$$

To prove that J is actually a Jordan curve as required, it is enough to show the following properties:

(i) $\tilde J \cup H^2(\tilde J) \subset \Omega \cup \tilde J \cup [\tau(\tilde z_0), \tau(\tilde z)]_{\tau(\tilde\beta)} = \overline{\Omega} \setminus \left([\tilde z_0, \tilde z]_{\tilde\beta} \setminus \{\tilde z\}\right)$,

(ii) the covering map Π induces a one-to-one map from $\overline{\Omega} \setminus [\tilde z_0, \tilde z]_{\tilde\beta}$ onto $\Pi(\overline{\Omega})$.

Because of $H^2(W_r) \subset W_r$, two points of $\tilde\beta \cap H^2(\tilde\beta)$ are met in the same order on $\tilde\beta$ and on $H^2(\tilde\beta)$; hence $H^2(\tilde J) \cap [\tilde z_0, \tilde z]_{\tilde\beta} = \{\tilde z\}$. So we just have to check that $H^2(\tilde J) \subset \overline{\Omega}$ for proving (i). First observe that Ω is the only connected component of $W_r \cap \tau(W_l)$ containing $\tilde z$ in its frontier, due to the fact that if $\Omega' \neq \Omega$ is another connected component of $W_r \cap \tau(W_l)$, then $\partial\Omega' \cap \partial\Omega \subset \tilde\beta \cap \tau(\tilde\beta) \subset [\tau(\tilde z), \tilde z_1]_{\tilde\beta}$. Consider now an arbitrary neighbourhood V of $\tilde z$ in $\tilde{\mathbb{A}}$. We have $\tilde z \in \overline{W_r} \setminus \tau(\tilde\beta) \subset$

$\tau(W_l)$ and $H^2(\tilde{z}) = \tilde{z}$, so any point $\tilde{m} \in \tilde{J} \cap S \subset \tilde{\beta} \cap S$ close enough to $\tilde{z}$ satisfies $H^2(\tilde{m}) \in V \cap \tau(W_l) \cap W_r$ and consequently $H^2(\tilde{J}) \cap \Omega \neq \emptyset$. To see that $H^2(\tilde{J})$ is entirely contained in $\overline{\Omega}$ remark that otherwise $H^2(\tilde{J} \setminus \{\tilde{z}, \tau(\tilde{z})\}) \subset H^2(\tilde{\beta}) \subset \overline{W_r} \cap \tau(\overline{W_l})$ would contain a point of $\partial\Omega \cap \tilde{\beta} \cap \tau(\tilde{\beta}) = \{\tau(\tilde{z})\}$, which is not possible since $\tau(\tilde{z})$ is a fixed point of H^2. Property (ii) can be rephrased by saying that

$$\forall n \in \mathbb{Z} \setminus \{0\} \quad \tau^n\big(\overline{\Omega} \setminus [\tilde{z}_0, \tilde{z}]_{\tilde{\beta}}\big) \cap \big(\overline{\Omega} \setminus [\tilde{z}_0, \tilde{z}]_{\tilde{\beta}}\big) = \emptyset.$$

This is true for $n = \pm 1$ because $\overline{\Omega} \setminus [\tilde{z}_0, \tilde{z}]_{\tilde{\beta}} \subset W_r \cap \tau(\overline{W_l}) = W_r \setminus \tau(W_r)$. Moreover we have $\overline{\Omega} \cap \tau(\overline{\Omega}) = \tau([\tilde{z}_0, \tilde{z}]_{\tilde{\beta}}) \subset \partial\Omega$ and classical arguments from Brouwer's theory of fixed point free orientation-preserving planar homeomorphisms[3] then ensure $\overline{\Omega} \cap \tau^n(\overline{\Omega}) = \emptyset$ for $|n| \geq 2$.

SECOND CASE: The points $\tau(\tilde{z}), \tilde{z}$ are in this order on $\tilde{\beta}$.

Remark that $\tau(\tilde{z})$ is the first point on $\tilde{\beta}$ whose image by τ^{-1} belongs to $\tilde{\beta}$. Hence we reduce to the first case by considering $\tau(\tilde{z}), \tilde{z}, \tau^{-1}, H^{-1}$ in place of respectively $\tilde{z}, \tau(\tilde{z}), \tau, H$ and by interchanging the roles of W_l, W_r. ☐

Lemma 3.7. *Let $\tilde{\beta}$ and W_r be as in Lemma* 3.5*. We suppose that $\tau(\overline{W_r}) \subset W_r$, i.e. that $\tau(\tilde{\beta}) \cap \tilde{\beta} = \emptyset$. Then there exists an arc α as announced in Theorem* 1.2*.*

Proof (adapted from [12]*).* For every integer $n \geq 1$, let us define $X_n = \bigcap_{i=0}^{n} G^i(\tilde{\beta})$ and $X = \bigcap_{i\in\mathbb{N}} G^i(\tilde{\beta}) = \bigcap_{n\geq 1} X_n$. We remark that $X = \emptyset$. Otherwise, since $X \subset G(X)$, one can consider the map $f = (G^{-1})|_X : X \to X$ which preserves the order induced by $\tilde{\beta}$ on X because of $G(W_r) \subset W_r$. We get a contradiction because every increasing map from a nonempty closed subset of the interval $[0,1]$ into itself has a fixed point. Hence there exists a least integer $N \geq 1$ such that $X_N = \emptyset$. If $N = 1$, i.e. if $\tilde{\beta} \cap G(\tilde{\beta}) = \emptyset$, then we have $H^2(\tilde{\beta}) \cap \tau(\tilde{\beta}) = \emptyset$ and we simply take $\alpha = \Pi(\tilde{\beta})$. We complete the proof by showing the following induction step:

If $N \geq 2$ there exists an arc $\tilde{\gamma}$ crossing $\tilde{\mathbb{A}}$ such that, writing W'_r for the domain on the right of $\tilde{\gamma}$ and $X'_n = \bigcap_{i=0}^{n} G^i(\tilde{\gamma})$ $(n \geq 1)$, we have

(a) $\tau(W'_r) \subset H^2(W'_r) \subset H(W'_r) \subset W'_r$,
(b) $\tau(\overline{W'_r}) \subset W'_r$,
(c) $H(\tilde{\gamma}) \cap \tilde{\gamma} = H^2(\tilde{\gamma}) \cap \tilde{\gamma} = \tilde{\gamma} \cap F$,
(d) $X'_{N-1} = \emptyset$.

We first observe the following properties of X_{N-1}.

(i) $\tau(X_{N-1}) \subset H^2(W_r)$; in particular $\tau(X_{N-1}) \cap H^2(X_{N-1}) = \emptyset$.
Equivalently, $G(X_{N-1}) \subset W_r$ and then $G(X_{N-1}) \cap X_{N-1} = \emptyset$,
(ii) $\forall i = 1, 2,\ H^i(X_{N-1}) \subset W_r$; in particular $H^i(X_{N-1}) \cap X_{N-1} = \emptyset$,

[3]Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a fixed point free and orientation-preserving homeomorphism. It is well known that a topological closed disc $D \subset \mathbb{R}^2$ disjoint from its first iterate $f(D)$ is also disjoint from every $f^n(D)$, $n \neq 0$ (see e.g. [12, Proposition 3.5]). Suppose now that $D' \subset \mathbb{R}^2$ is a topological closed disc such that $D' \cap f(D') \subset \partial_{\mathbb{R}^2} D'$ and $D' \cap f^n(D') \neq \emptyset$ with $|n| \geq 2$. Pick $m \in D' \cap f^{-n}(D')$ and consider an arc γ from m to $f^n(m)$ lying entirely in $\mathrm{Int}_{\mathbb{R}^2}(D')$ except possibly its endpoints in $\partial_{\mathbb{R}^2} D'$. The points $m, f(m), f^n(m), f^{n+1}(m)$ are pairwise distinct because f has no periodic point and $n \notin \{0, \pm 1\}$; moreover $f(\mathrm{Int}_{\mathbb{R}^2}(D')) \cap \mathrm{Int}_{\mathbb{R}^2}(D') = \emptyset$, so $f(\gamma) \cap \gamma = \emptyset$. One can find a topological closed disc $D \subset \mathbb{R}^2$ containing γ and so thin that $D \cap f(D) = \emptyset$. We also have $f^n(m) \in D \cap f^n(D)$, which contradicts the above property.

(iii) $\tau(X_{N-1}) \subset W_r$; in particular $\tau(X_{N-1}) \cap X_{N-1} = \emptyset$,
(iv) $X_{N-1} \cap \mathrm{Bd}(\tilde{\mathbb{A}}) = \emptyset$.

Indeed we have

$$G(X_{N-1}) \subset G(\tilde\beta) \subset G(\overline{W_r}) \subset \overline{W_r} \quad \text{and} \quad G(X_{N-1}) \cap \tilde\beta = X_N = \emptyset,$$

which gives $G(X_{N-1}) \subset W_r$. To get (ii) remark that $H^i(X_{N-1}) \subset H^i(\tilde\beta) \subset \overline{W_r}$ and afterwards

$$\begin{aligned}\tilde\beta \cap H^i(X_{N-1}) \subset \tilde\beta \cap H^i(\tilde\beta \cap G(\tilde\beta)) &= \tilde\beta \cap H^i(\tilde\beta) \cap \big(\tau \circ H^{i-2}(\tilde\beta)\big)\\ &= \tilde\beta \cap F \cap \big(\tau \circ H^{i-2}(\tilde\beta)\big) \subset \tilde\beta \cap \tau(\tilde\beta) = \emptyset.\end{aligned}$$

Finally (iii)-(iv) are clear since $\tau(\overline{W_r}) \subset W_r$ and G has no fixed point. By the compactness of X_{N-1} there exists an open neighbourhood U of X_{N-1} in $\tilde{\mathbb{A}}$ such that all the properties appearing in (i)-(iv) are still true when one replaces X_{N-1} with U. Consider a finite covering $X_{N-1} \subset \bigcup_{i=1}^n \tilde\alpha_i$, where the $\tilde\alpha_i$'s are connected components of $U \cap \tilde\beta$. For each $i = 1, \ldots, n$ one can choose an arc $\tilde\beta_i \subset \tilde\alpha_i$ in such a way that $X_{N-1} \cap \tilde\alpha_i = X_{N-1} \cap \tilde\beta_i$ and there exists an arc $\tilde\gamma_i$ having the same endpoints as $\tilde\beta_i$, call them a_i, b_i, such that $\tilde\gamma_i \setminus \{a_i, b_i\} \subset W_l \cap U$. These $\tilde\gamma_i$'s can be supposed pairwise disjoint and we then define $\tilde\gamma$ to be the arc obtained by replacing in $\tilde\beta$ each $\tilde\beta_i$ with $\tilde\gamma_i$. Precisely we set

$$\tilde\gamma = \big(\tilde\beta \setminus \bigcup_{i=1}^n \tilde\beta_i\big) \cup \bigcup_{i=1}^n \tilde\gamma_i.$$

It remains to check properties (a)-(d). Remark that $W_r \subset W'_r$ by construction. The homeomorphisms τ, H fixing the ends of $\tilde{\mathbb{A}}$, Property (a) follows from

$$\tau(\tilde\gamma) \subset \tau(\tilde\beta \cup U) \subset H^2(\overline{W_r}) \subset H^2(\overline{W'_r}) \quad \text{and} \quad H(\tilde\gamma) \subset H(\tilde\beta \cup U) \subset \overline{W_r} \subset \overline{W'_r}.$$

One deduces (b) similarly from $\tau(\tilde\gamma) \subset \tau(\tilde\beta \cup U) \subset W_r \subset W'_r$. Remembering that $\tilde\gamma \setminus \tilde\beta \subset W_l \cap U$ one obtains for $i \in \{1, 2\}$,

$$H^i(\tilde\beta) \cap (\tilde\gamma \setminus \tilde\beta) = H^i(\tilde\gamma \setminus \tilde\beta) \cap \tilde\beta = H^i(\tilde\gamma \setminus \tilde\beta) \cap (\tilde\gamma \setminus \tilde\beta) = \emptyset$$

and consequently $H^i(\tilde\gamma) \cap \tilde\gamma \subset H^i(\tilde\beta) \cap \tilde\beta \subset F$, which proves (c). To prove (d) we first observe that $X'_n \subset X_n$ for every $n \geq 1$. Because of $X_{n+1} = X_n \cap G(X_n)$, and similarly for X'_{n+1}, it is enough to check $X'_1 \subset X_1$. This inclusion follows again from $\tilde\gamma \setminus \tilde\beta \subset W_l \cap U$, which implies

$$G(\tilde\beta) \cap (\tilde\gamma \setminus \tilde\beta) = G(\tilde\gamma \setminus \tilde\beta) \cap \tilde\beta = G(\tilde\gamma \setminus \tilde\beta) \cap (\tilde\gamma \setminus \tilde\beta) = \emptyset;$$

hence $\tilde\gamma \cap G(\tilde\gamma) \subset \tilde\beta \cap G(\tilde\beta)$, i.e. $X'_1 \subset X_1$. Consequently we get $X'_{N-1} \subset \tilde\gamma \cap X_{N-1}$ and the latter set is empty by the construction of $\tilde\gamma$. □

4. Appendix: A remark on a theorem of J. Franks

The following version of the Poincaré-Birkhoff theorem is essentially due to Franks.

Theorem 4.1. *Let h be a homeomorphism of the compact annulus $\mathbb{A}$ isotopic to $\mathrm{Id}_{\mathbb{A}}$ and with every point nonwandering. Let $H : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ be a lift of h.*

(1) *If $0 \in \rho(H)$, then H (and so h) has a fixed point.*
(2) *If 0 is an interior point of $\rho(H)$, then h has at least two fixed points.*

The assertion (1) is an easy consequence of [8, Corollary 2.3] or of Theorem 2.7 whereas (2) is contained in [7, Theorem 3.3]. As a remark, the conservative assumption in Theorem 4.1 (namely, h has no wandering point) is not the best one in order to get only one fixed point but is enough for our purpose. Our interest is actually in the techniques developed in [7] for finding the second fixed point of h. Our goal here is to point out that the same arguments lead to Theorem 4.4 below, provided one replaces Proposition 1.3 of [7] (which is a consequence of Brouwer's lemma on translation arcs) with its analogue from [3], that is, Lemma 2.2. Our original motivation was to find a statement in the same spirit as Theorem 1.2 but without any hypothesis on $\mathrm{Fix}(h)$. Nevertheless, observe that the conservative assumption is stronger in Theorem 4.4 than in Theorem 1.2. Indeed, if h has no wandering point, then so does h^2 (see for example Property 4.2) and this clearly implies that the situation (1) of Theorem 1.2 does not occur. Before proving Theorem 4.4, let us state a classical corollary of Theorem 4.1. It may be useful to first recall the next fact.

Property 4.2. Let X be a topological space and let $h : X \to X$ be a homeomorphism. We suppose that h has no wandering point. Then the same is true for any iterate h^q, $q \geq 2$.

Proof. It is enough to check that for a given nonempty open set $U \subset X$ there exist an integer $n \geq 1$ multiple of q and a set $V \subset U$ such that $V \cap h^n(V) \neq \emptyset$. Because h has no wandering point there exists an integer $n_1 \geq 1$ such that the open set $U_1 = U \cap h^{n_1}(U)$ is nonempty. For the same reason there exists $n_2 \geq 1$ such that $U_2 = U_1 \cap h^{n_2}(U_1) \neq \emptyset$ and so on. Thus we get a decreasing sequence of nonempty open sets $U_0 = U, U_1, \ldots, U_q$ and a sequence of positive integers $n_1, \ldots, n_q$ such that $U_{k+1} = U_k \cap h^{n_{k+1}}(U_k)$, $0 \leq k \leq q-1$. Consider the $q+1$ integers $p_0 = 0, p_1 = n_1, p_2 = n_1 + n_2, \ldots, p_q = \sum_{k=1}^{q} n_k$; at least two of them are equal modulo q, say $p_i = p_j \bmod q$, where $0 \leq i < j \leq q$, so that $p_j - p_i = \sum_{k=i+1}^{j} n_k \geq 1$ is a multiple of q. We conclude by observing that $U_j = U_{j-1} \cap h^{n_j}(U_{j-1}) \subset U_i \cap h^{n_j+\cdots+n_{i+1}}(U_i)$. ☐

We make the convention that a rational number p/q is always written with $q \in \mathbb{N} \setminus \{0\}$, $p \in \mathbb{Z}$ and p, q relatively prime. According to Property 4.2, one can apply Theorem 4.1 to h^q and its lift $\tau^{-p} \circ H^q$ to obtain the following result.

Corollary 4.3. *Let h be a homeomorphism of $\mathbb{A}$ isotopic to $\mathrm{Id}_{\mathbb{A}}$ and with every point nonwandering. For any rational number $p/q \in \rho(H)$, the homeomorphism h possesses a q-periodic point with rotation number p/q.*

The result to be proved in this Appendix is the following.

Theorem 4.4. *Let h be a homeomorphism of the compact annulus $\mathbb{A}$ isotopic to $S_{\mathbb{A}}$ and with every point nonwandering. Let $H : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ be a lift of h. If 0 is an interior point of the rotation set $\rho(H)$, then H (and so h) admits a 2-periodic point.*

Remark. It is easy to construct a homeomorphism of $\mathbb{A}$ showing that the conclusion of Theorem 4.4 does not hold if one supposes only $0 \in \rho(H)$.

Let us explain how to adapt the arguments in [7] in order to get Theorem 4.4. We first recall the

Definition 4.5 ([7])**.** Let h be a homeomorphism of the open annulus $\mathbb{A}'$ isotopic to $\mathrm{Id}_{\mathbb{A}'}$ and let H be a lift of h to the universal cover $\widetilde{\mathbb{A}'} = \mathbb{R}^2$. A *positively returning disc* for H is a topological closed disc $D \subset \mathbb{R}^2$ satisfying the following properties.

(i) The covering map Π induces a homeomorphism from D onto $\Pi(D)$; i.e., $\tau^l(D) \cap D = \emptyset$ for every integer $l \neq 0$,
(ii) $H^n(\mathrm{Int}_{\mathbb{R}^2}(D)) \cap \tau^k(\mathrm{Int}_{\mathbb{R}^2}(D)) \neq \emptyset$ for some integers $n \geq 1, k \geq 1$,
(iii) $H(D) \cap D = \emptyset$.

A *negatively returning disc* is defined similarly but with $k \leq -1$ in the second item. We chose to include (i) in our definition in view of [10] and in order to save words. One can also remark that the true definition used in [7] deals with *open* discs. This does not alter the validity of Theorem 4.6 below because if D is a positively returning disc as defined above, then $U = \mathrm{Int}_{\mathbb{R}^2}(D)$ is an (open) positively returning disc in the sense of [7] (conversely, if U is an open positively returning disc as in [7], then a large enough closed disc D inside U also satisfies (ii)-(iii) in Definition 4.5). A similar remark holds for negatively returning discs. Thus our choices in Definition 4.5 will allow us to use directly both results from [7, 10] and Lemma 2.2.

After minor modifications, which are explained below, the main result of [7] can be stated as follows.

Theorem 4.6 ([7, 10])**.** *Let h be a homeomorphism of $\mathbb{A}'$ isotopic to $\mathrm{Id}_{\mathbb{A}'}$ and satisfying the following conditions.*

(1) *Every point of $\mathbb{A}'$ is nonwandering under h.*
(2) *There is a lift H of h to the universal cover $\mathbb{R}^2$ which possesses both a positively returning disc and a negatively returning disc.*

Then H (and so h) has a fixed point.

Theorem 2.1 of [7] (see also [10]) contains an additional hypothesis, namely h is supposed to have at most finitely many fixed points, but gives a stronger conclusion: H, and so h, has a fixed point of positive index. The finiteness of $\mathrm{Fix}(h)$ ensures that $\mathbb{R}^2 \setminus \mathrm{Fix}(H)$ is connected, which is a crucial point in the proof of [7, Theorem 2.1]. To prove now Theorem 4.6, suppose that $\mathrm{Fix}(H) = \emptyset$; then, of course, $\mathbb{R}^2 = \mathbb{R}^2 \setminus \mathrm{Fix}(H)$ is connected so that verbatim the same proof as in [7] gives a fixed point for H, a contradiction. In fact, Franks' result could be rephrased in an even more precise way by saying that, for h and H as in Theorem 4.6, either $\mathrm{Fix}(H)$ separates $\mathbb{R}^2$ or there exists a Jordan curve $J \subset \mathbb{R}^2 \setminus \mathrm{Fix}(H)$ such that the index of J w.r.t. H is 1. It is easily seen that we cannot drop the first possibility.

We give now a suitable definition of a positively/negatively returning disc for a homeomorphism of $\mathbb{A}'$ isotopic to the symmetry $S_{\mathbb{A}'} : \mathbb{A}' \to \mathbb{A}'$, $(z, r) \mapsto (z, -r)$.

Definition 4.7. Let h be a homeomorphism of $\mathbb{A}'$ isotopic to $S_{\mathbb{A}'}$ and let H be a lift of h to the universal cover $\mathbb{R}^2$. A *positively returning disc* for H is a topological closed disc $D \subset \mathbb{R}^2$ satisfying the properties (i),(ii) of Definition 4.5 and

(iii′) $H^i(D) \cap D = \emptyset$ for $i = 1, 2$ and there is no integer l such that $H(D)$ meets both $\tau^l(D)$ and $\tau^{-l}(D)$.

The definition of a *negatively returning disc* is the same except that we demand $k \leq -1$ in (ii). Replacing (iii) with (iii′) when h is isotopic to $S_{\mathbb{A}'}$ is essential in order to use Lemma 2.2 instead of Proposition 1.3 of [7]. Then Theorem 4.6 becomes:

Theorem 4.8. *Let h be a homeomorphism of $\mathbb{A}'$ isotopic to $S_{\mathbb{A}'}$ and satisfying the following conditions.*

(1) *Every point of $\mathbb{A}'$ is nonwandering under h.*
(2) *There is a lift H of h to the universal cover $\mathbb{R}^2$ which possesses both a positively returning disc and a negatively returning disc.*

Then H (and so h) has a 2-periodic point.

Proof. • We first suppose that $\mathbb{R}^2 \setminus \mathrm{Fix}(H)$ is not connected. Then there is a connected component $\tilde{K}$ of $\mathrm{Fix}(H)$ such that $\mathbb{R}^2 \setminus \tilde{K}$ is not connected (see e.g. [18, Chapter V]). According to a result of Epstein ([6, Theorem 2.5]), if $f : S \to S$ is an orientation-reversing homeomorphism of a compact surface S, then any connected component K of $\mathrm{Fix}(f)$ is either a point, an arc or a Jordan curve; in the last two cases f interchanges the two (local) sides of K. Working in the sphere $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$ and combining with the Jordan curve theorem, we obtain that $\tilde{K}$ is a line properly embedded in $\mathbb{R}^2$; furthermore $\mathbb{R}^2 \setminus \tilde{K}$ has exactly two connected components $\widetilde{\mathbb{A}'_1}, \widetilde{\mathbb{A}'_2}$ which satisfy $\partial_{\mathbb{R}^2}\widetilde{\mathbb{A}'_i} = \tilde{K}$ $(i = 1, 2)$ and are interchanged by H. In particular it is enough to find a fixed point of H^2 in $\widetilde{\mathbb{A}'_1}$. This switch property also gives $\tilde{K} = \mathrm{Fix}(H)$; hence $\tilde{K}$ is τ-invariant as well as $\widetilde{\mathbb{A}'_i}$ $(i = 1, 2)$. Consequently $K = \Pi(\tilde{K})$ is an essential Jordan curve of $\mathbb{A}'$ and the sets $\mathbb{A}'_i = \Pi(\widetilde{\mathbb{A}'_i})$ $(i = 1, 2)$ are the two connected components of $\mathbb{A}' \setminus K$; they are homeomorphic to $\mathbb{A}'$ and they are interchanged by h. We reduce now to Theorem 4.6 by considering the homeomorphism $h^2|_{\mathbb{A}'_1} : \mathbb{A}'_1 \to \mathbb{A}'_1$. Let us give a few details. First of all, one knows from Property 4.2 that h^2 has no wandering point (alternatively, this also follows from the fact h has no wandering point and that a connected set $U \subset \mathbb{A}' \setminus K$ is disjoint from all its odd iterates $h^{2n+1}(U)$, $n \in \mathbb{Z}$). Remark now that if D is a positively (resp. negatively) returning disc for H, then obviously $D \cap \tilde{K} = \emptyset$ and $H(D)$ is also a positively (resp. negatively) returning disc for H. Consequently $\widetilde{\mathbb{A}'_1}$ contains both a positively and a negatively returning disc of H. If $D \subset \widetilde{\mathbb{A}'_1}$ is such a disc, then any integer $n \geq 1$ as in (ii) of Definition 4.7 is necessarily even, so D is also a positively (resp. negatively) returning disc for the lift H^2 of h^2 in the sense of Definition 4.5. Pick any homeomorphism f from $\mathbb{A}'_1$ onto $\mathbb{A}'$ mapping a loop of $\mathbb{A}'_1 \subset \mathbb{A}'$ homotopically to itself. Since $\widetilde{\mathbb{A}'_1}$ is simply connected, the map $\Pi_1 = \Pi|_{\widetilde{\mathbb{A}'_1}} : \widetilde{\mathbb{A}'_1} \to \mathbb{A}'_1$ is a universal covering map. As a consequence, there exists a homeomorphism $\Phi : \widetilde{\mathbb{A}'_1} \to \mathbb{R}^2$ such that $\Pi \circ \Phi = f \circ \Pi_1$. The group of all the deck transformations of $\Pi : \mathbb{R}^2 \to \mathbb{A}'$ (resp. of $\Pi_1 : \widetilde{\mathbb{A}'_1} \to \mathbb{A}'_1$) is $\mathcal{G} = \{\tau^k \mid k \in \mathbb{Z}\}$ (resp. $\mathcal{G}_1 = \{(\tau|_{\widetilde{\mathbb{A}'_1}})^k \mid k \in \mathbb{Z}\}$) and the map $\Phi_* : \mathcal{G} \to \mathcal{G}_1$ defined by $\Phi_*(t) = \Phi^{-1} \circ t \circ \Phi$ is an isomorphism of groups such that

$$\Phi^{-1} \circ \tau \circ \Phi = \Phi_*(\tau) = \tau|_{\widetilde{\mathbb{A}'_1}}, \text{ i.e. } \tau \circ \Phi = \Phi \circ \tau|_{\widetilde{\mathbb{A}'_1}}.$$

The homeomorphism $g = f \circ h^2|_{\mathbb{A}'_1} \circ f^{-1} : \mathbb{A}' \to \mathbb{A}'$ is isotopic to $\mathrm{Id}_{\mathbb{A}'}$ because $h^2|_{\mathbb{A}'_1}$ preserves the orientation and the two ends of the subannulus $\mathbb{A}'_1$. Furthermore g is conjugate to $h^2|_{\mathbb{A}'_1}$, so it has no wandering point. The homeomorphism $G =$

$\Phi \circ H^2|_{\widetilde{\mathbb{A}'_1}} \circ \Phi^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ is a lift of g to $\mathbb{R}^2$ and one deduces from $\tau \circ \Phi = \Phi \circ \tau|_{\widetilde{\mathbb{A}'_1}}$ that if $D \subset \widetilde{\mathbb{A}'_1}$ is a positively (resp. negatively) returning disc of H^2, then $\Phi(D)$ is a positively (resp. negatively) returning disc of G. Theorem 4.6 then gives a fixed point for G and therefore for $H^2|_{\widetilde{\mathbb{A}'_1}}$.

• We now deal with the case where $\mathbb{R}^2 \setminus \mathrm{Fix}(H)$ is connected by following closely [7]. Arguing by contradiction, we suppose that $\mathrm{Fix}(H) = \mathrm{Fix}(H^2)$. Let B_+ (resp. B_-) be the set of all the points of $\mathbb{R}^2$ contained in the interior of a positively (resp. negatively) returning disc of H. Every point $\tilde{m} \in \mathbb{R}^2 \setminus \mathrm{Fix}(H) = \mathbb{R}^2 \setminus \mathrm{Fix}(H^2)$ belongs to the interior of a disc $D \subset \mathbb{R}^2$ so small that it satisfies the properties (i) and (iii′) in Definition 4.7. Since $m = \Pi(\tilde{m}) \in \mathbb{A}'$ is a nonwandering point of h, there exists an integer $n \geq 1$ such that $h^n(\Pi(\mathrm{Int}_{\mathbb{R}^2}(D))) \cap \Pi(\mathrm{Int}_{\mathbb{R}^2}(D)) \neq \emptyset$ and consequently $H^n(\mathrm{Int}_{\mathbb{R}^2}(D)) \cap \tau^k(\mathrm{Int}_{\mathbb{R}^2}(D)) \neq \emptyset$ for some $k \in \mathbb{Z}$. Because of Lemma 2.2 we have $k \neq 0$, so D is either a positively returning disc or a negatively returning disc of H. This proves $\mathbb{R}^2 \setminus \mathrm{Fix}(H) = B_- \cup B_+$. Since $\mathbb{R}^2 \setminus \mathrm{Fix}(H)$ is connected and since $B_\pm$ are nonempty open subsets of $\mathbb{R}^2 \setminus \mathrm{Fix}(H)$ we obtain $B_- \cap B_+ \neq \emptyset$. Hence there exists $\tilde{m} \in \mathrm{Int}_{\mathbb{R}^2}(D) \cap \mathrm{Int}_{\mathbb{R}^2}(D')$ where D (resp. D') is a positively (resp. negatively) returning disc of H. Choose another disc D'' with $\tilde{m} \in \mathrm{Int}_{\mathbb{R}^2}(D'') \subset D \cap D'$. As above D'' is either a positively returning disc or a negatively returning disc. We only deal with the latter case, the first one being similar. Because of $D'' \subset D$ we obtain that D is both a positively and a negatively returning disc; there exist four positive integers m, n, k, l such that

$$H^n(D) \cap \tau^k(D) \neq \emptyset \quad \text{and} \quad H^m(D) \cap \tau^{-l}(D) \neq \emptyset.$$

Let us consider the following sequence of discs:

$$D, \tau^k(D), \tau^{2k}(D), \ldots, \tau^{lk}(D), \tau^{l(k-1)}(D), \tau^{l(k-2)}(D), \ldots, \tau^l(D).$$

Property (iii′) in Definition 4.7 implies that for any two integers $p, q \in \mathbb{Z}$ one has $H(\tau^p(D)) \cap \tau^p(D) = \emptyset = H^2(\tau^p(D)) \cap \tau^p(D)$ and furthermore at most one of the two sets $H(\tau^p(D)) \cap \tau^q(D)$ or $H(\tau^q(D)) \cap \tau^p(D)$ is nonempty. Moreover we have

$$\forall i = 0, \ldots, l-1 \quad H^n(\tau^{ik}(D)) \cap \tau^{(i+1)k}(D) = \tau^{ik}\big(H^n(D) \cap \tau^k(D)\big) \neq \emptyset,$$
$$\forall i = 1, \ldots, k \quad H^m(\tau^{li}(D)) \cap \tau^{l(i-1)}(D) = \tau^{li}\big(H^m(D) \cap \tau^{-l}(D)\big) \neq \emptyset,$$

so that Lemma 2.2 gives a 2-periodic point for H, a contradiction. □

For completeness, we now give the proof of Theorem 4.4. It is almost the same as the one of Theorem 3.3 in [7].

Proof of Theorem 4.4. Let us write $\rho(H) = [a, b]$ with $a < 0 < b$ and recall that a, b are the rotation numbers of some points of $\tilde{\mathbb{A}}$, say $b = \rho_H(\tilde{m})$ and $a = \rho_H(\tilde{m}')$. We also have $b = \lim_{+\infty} \frac{1}{2n}\big(p_1 \circ H^{2n}(\tilde{m}) - p_1(\tilde{m})\big)$ from which one deduces that the set

$$\big\{n \in \mathbb{N} \setminus \{0\} \,|\, p_1 \circ H^2 \circ H^{2n}(\tilde{m}) - p_1 \circ H^{2n}(\tilde{m}) > b\big\}$$

has infinite cardinality, so there exists a sequence $(n_k)_{k\in\mathbb{N}}$ in this latter set with $\lim_{+\infty} n_k = +\infty$. We let $m = \Pi(\tilde{m}) \in \mathbb{A}$. Considering a subsequence if necessary, one can suppose that $(h^{2n_k}(m))_{k\in\mathbb{N}}$ converges to a point $z \in \mathbb{A}$. Pick any point $\tilde{z} \in \Pi^{-1}(\{z\})$ and remark that it cannot be a fixed point of H^2 because of $p_1 \circ H^2 \circ H^{2n_k}(\tilde{m}) - p_1 \circ H^{2n_k}(\tilde{m}) > b > 0$. There exists a closed Euclidean disc $D \subset \mathbb{R}^2$

with center $\tilde{z}$ and so small that, by letting $\delta = D \cap \tilde{\mathbb{A}}$, we have

- Π induces a homeomorphism from δ onto $\Pi(\delta)$,
- $H^i(\delta) \cap \delta = \emptyset$ for $i = 1, 2$ and there is no integer l such that $H(\delta)$ meets both $\tau^{-l}(\delta)$ and $\tau^l(\delta)$.

For k large enough we have $h^{2n_k}(m) \in \Pi(\mathrm{Int}(\delta))$; hence $H^{2n_k}(\tilde{m}) \in \tau^{i_k}(\mathrm{Int}(\delta))$ for some $i_k \in \mathbb{Z}$. We have $\lim_{+\infty} i_k = +\infty$ because $\lim_{+\infty} p_1 \circ H^{2n}(\tilde{m}) = +\infty$, so there exist k, l such that $n_l > n_k$, $i_l > i_k$ and $H^{2n_l - 2n_k}(\mathrm{Int}(\delta)) \cap \tau^{i_l - i_k}(\mathrm{Int}(\delta)) \neq \emptyset$. If $\tilde{z} \notin \mathrm{Bd}(\tilde{\mathbb{A}})$, then one can choose $\delta = D \subset \tilde{\mathbb{A}} \setminus \mathrm{Bd}(\tilde{\mathbb{A}})$. If $\tilde{z} \in \mathrm{Bd}(\tilde{\mathbb{A}})$, then δ is a half-disc and there exists a topological closed disc $\delta' \subset \delta \cap (\tilde{\mathbb{A}} \setminus \mathrm{Bd}(\tilde{\mathbb{A}}))$ such that $H^{2n_l - 2n_k}(\mathrm{Int}(\delta')) \cap \tau^{i_l - i_k}(\mathrm{Int}(\delta')) \neq \emptyset$; it suffices to consider the intersection of δ with the substrip $\{(x, y) \in \tilde{\mathbb{A}} \mid -1 + \epsilon \leq y \leq 1 - \epsilon\}$ for a small enough $\epsilon > 0$. Thus we obtain a positively returning disc for H contained in $\tilde{\mathbb{A}} \setminus \mathrm{Bd}(\tilde{\mathbb{A}})$. We get in the same way a negatively returning disc of H in $\tilde{\mathbb{A}} \setminus \mathrm{Bd}(\tilde{\mathbb{A}})$. We conclude similarly as in the first part of the proof of Theorem 4.8, by using that $\Pi_1 = \Pi|_{\tilde{\mathbb{A}} \setminus \mathrm{Bd}(\tilde{\mathbb{A}})} : \tilde{\mathbb{A}} \setminus \mathrm{Bd}(\tilde{\mathbb{A}}) \to \mathbb{A} \setminus \mathrm{Bd}(\mathbb{A})$ is a universal covering map. Given a suitable homeomorphism $f : \mathbb{A} \setminus \mathrm{Bd}(\mathbb{A}) \to \mathbb{A}'$ there is a homeomorphism $\Phi : \tilde{\mathbb{A}} \setminus \mathrm{Bd}(\tilde{\mathbb{A}}) \to \mathbb{R}^2$ such that $\Pi \circ \Phi = f \circ \Pi_1$ and $\tau \circ \Phi = \Phi \circ \tau|_{\tilde{\mathbb{A}} \setminus \mathrm{Bd}(\tilde{\mathbb{A}})}$ (actually f and Φ could be made explicit here). One deduces that Theorem 4.8 applies to $g = f \circ h|_{\mathbb{A} \setminus \mathrm{Bd}(\mathbb{A})} \circ f^{-1} : \mathbb{A}' \to \mathbb{A}'$ lifted by $G = \Phi \circ H|_{\tilde{\mathbb{A}} \setminus \mathrm{Bd}(\tilde{\mathbb{A}})} \circ \Phi^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$. This gives a 2-periodic point for G and so for $H|_{\tilde{\mathbb{A}} \setminus \mathrm{Bd}(\tilde{\mathbb{A}})}$. $\square$

We end with a result similar to Corollary 4.3.

Corollary 4.9. *Let h be a homeomorphism of $\mathbb{A}$ isotopic to $S_{\mathbb{A}}$ and with every point nonwandering. We suppose that there exists a lift $H : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ of h such that $\rho(H) = [a, b]$, $a < b$. Then for any rational number $p/q \in (a, b)$ we have:*

- *if q is even, then h possesses a q-periodic point m with $\rho_H(m) = p/q$,*
- *if q is odd, then h possesses a $2q$-periodic point m with $\rho_H(m) = p/q$.*

Proof. The homeomorphism h^q is isotopic to $\mathrm{Id}_{\mathbb{A}}$ (resp. to $S_{\mathbb{A}}$) when q is even (resp. odd) and we know from Property 4.2 that it has no wandering point. Furthermore $\rho(\tau^{-p} \circ H^q) = [aq - p, bq - p]$ contains 0 in its interior; hence one can apply either Theorem 4.1 or Theorem 4.4 to h^q and its lift $\tau^{-p} \circ H^q$, depending on whether q is even or odd. In the first case one gets a point $\tilde{m} \in \tilde{\mathbb{A}}$ such that $H^q(\tilde{m}) = \tau^p(\tilde{m})$, so $h^q(m) = m$ for $m = \Pi(\tilde{m}) \in \mathbb{A}$. The integer q is the h-period of m because p/q is irreducible. In the second case one obtains $\tilde{m} \in \tilde{\mathbb{A}}$ such that $H^{2q}(\tilde{m}) = \tau^{2p}(\tilde{m})$ but $H^q(\tilde{m}) \neq \tau^p(\tilde{m})$. One deduces from the irreducibility of p/q that $2q$ is the h-period of $m = \Pi(\tilde{m})$. $\square$

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