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ON ZEROS OF SOME ENTIRE FUNCTIONS

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ABSTRACT. We study the distribution of zeros z_k for entire functions of the form $\sin z+\int_0^1 f(t)e^{iz(1-2t)}\,dt$ with f belonging to a space $X \hookrightarrow L_1(0,1)$. For a large class $\mathscr{X}$ of spaces X (including, e.g., the spaces $L_p(0,1)$ for all $p\in[1,\infty]$) we show that $z_k=\pi k+\zeta_k$, where $(\zeta_k)_{k\in\mathbb{Z}}$ is the sequence of Fourier coefficients for some function g in X, and study properties of the induced mapping $g\mapsto f$.

1. INTRODUCTION

The aim of this paper is to study the distribution of zeros for a class of entire functions of the form

$$F(z)=F_f(z):=\sin z+\int_0^1 f(t)e^{iz(1-2t)}\,dt, \tag{1.1}$$

where f is a function integrable over $(0,1)$. Such a class consists of functions of the so-called sine type [12, Lect. 22]. Moreover, to this form also reduce the functions

$$\tilde F(z)=m_-e^{-iz}+m_+e^{iz}+\int_{-1}^1\tilde f(t)e^{izt}\,dt, \qquad m_-m_+\neq 0,$$

which are the Fourier–Stieltjes transforms [7, Ch. 12.5] of the measures

$$m_-\delta_{-1}+m_+\delta_1+\tilde f(t)\,dt$$

on $[-1,1]$, with δ_x being the Dirac measure at the point x and $\tilde f\in L_1(-1,1)$. Indeed, taking $\alpha:=\frac{\pi}{2}+\frac{1}{2i}\log(m_-/m_+)$ and $c:=-2\sqrt{m_-m_+}$, we get by direct calculations the equality $\tilde F(z+\alpha)=cF_f(z)$ for a suitable $f\in L_1(0,1)$.

Asymptotic distribution of zeros for Fourier–Stieltjes transforms is of key significance in many areas of function theory, harmonic analysis, functional analysis, etc. and has been studied for many particular situations; see, e.g., [10, 11, 19]. Our interest in this topic stems from the spectral theory of Sturm–Liouville and Dirac operators, since their characteristic functions usually are of the above type; e.g., the case $f\in W_1^1(0,1)$ appears in [15], $f\in W_2^1(0,1)$ in [16, 17], $f\in W_2^s(0,1)$ with $s\in[0,1]$ in [9], $f\in L_p(0,1)$ with $p\in[1,\infty)$ in [3, 4], and $f\in BV[0,1]$ in [20].

By a refined version of the Riemann–Lebesgue lemma (cf. [16, Lemma 1.3.1]), the integral term in (1.1) is $o(e^{|\operatorname{Im} z|})$ as $|z|\to\infty$, and Rouché's theorem shows that the functions F_f and $\sin z$ have the same number of zeros in the discs $|z|\le\pi(l+\frac12)$ for all sufficiently large $l\in\mathbb{N}$ and that the zeros of F_f approach those of $\sin z$ for

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large $|z|$. Therefore the zeros of F_f can (and always will) be enumerated by z_n, $n \in \mathbb{Z}$, in such a way that $z_n = \pi n + \zeta_n$ with $\zeta_n = \mathrm{o}(1)$ as $n \to \pm\infty$.

We would like to study dependence between the decay rate of the remainders ζ_n and the regularity of the function f. An example of such an interrelation can be found in the paper by Levin and Ostrovskiĭ [13], namely:

Theorem A. *Assume that $f \in L_p(0,1)$ with $p \in (1,2]$ and let $z_n = \pi n + \zeta_n$ be zeros of the function F_f in (1.1). Then the sequence $(\zeta_n)_{n\in\mathbb{Z}}$ belongs to ℓ_q with $q := p/(p-1)$.*

It is interesting to know whether the statement of Theorem A can be reverted, i.e., whether for any sequence $(c_n)_{n\in\mathbb{Z}}$ from ℓ_q the numbers $\pi n + c_n$ are zeros (of respective multiplicities) of a function F_f in (1.1) with some $f \in L_p(0,1)$. Further related questions are about the cases when $f \in L_p(0,1)$ with $p = 1$ or $p > 2$ and, more generally, when the function f belongs to other spaces embedded into $L_1(0,1)$, as in the above-mentioned examples from the spectral theory.

Our results describe explicitly zero sequences of functions F_f when f runs through $L_p(0,1)$, $p \in [1,\infty]$, or through some other function spaces and, in particular, show that the statement of Theorem A can be neither reverted nor extended to $p > 2$. In fact, we prove that the remainders $\zeta_n = z_n - \pi n$ of the zeros z_n of the function F_f with $f \in L_p(0,1)$ form a sequence of Fourier coefficients of some function $g \in L_p(0,1)$. And conversely, for every $g \in L_p(0,1)$ the sequence $\big(\pi n + \hat{g}(n)\big)_{n\in\mathbb{Z}}$, $\hat{g}(n)$ denoting the nth Fourier coefficient of g, is a zero sequence for a function F_f, with a unique $f \in L_p(0,1)$. Since by the Hausdorff–Young theorem [7, Sect. 13.5] the sequences of Fourier coefficients of functions in $L_p(0,1)$ form a proper subset of ℓ_q, $q := p/(p-1)$, if $p \in (1,2)$ and do not form a subset of ℓ_q if $p > 2$, the above claim about Theorem A follows.

The main results of the paper are formulated in terms of the properties of the induced mapping $\phi : g \mapsto f$. We first prove that ϕ maps $L_1(0,1)$ surjectively and analytically onto $L_1(0,1)$ and is a local analytic homeomorphism when restricted to an open dense subset Ω of $L_1(0,1)$ consisting of those g for which the function $F_{\phi(g)}$ has no multiple zeros. The same results are shown to hold for a rich class $\mathscr{X}$ of Banach spaces X that are continuously embedded into $L_1(0,1)$ (among them $L_p(0,1)$ for $p \in (1,\infty]$); see Corollary 2.5. Our approach consists of constructing an analytic function

$$\Phi : L_1(0,1) \times L_1(0,1) \to L_1(0,1)$$

such that the relation $\phi(g) = f$ is equivalent to $\Phi(f,g) = 0$ if g belongs to Ω. It turns out that the partial derivatives $\partial_f \Phi$ and $\partial_g \Phi$ are boundedly invertible if and only if all zeros of the function F_f are simple, i.e., if and only if $g \in \Omega$, and the implicit function theorem then gives the properties of ϕ.

The paper is organized as follows. In the next section some definitions are given and the main results (Theorems 2.2 and 2.4 and Corollary 2.5) are formulated. Some auxiliary statements are proved in Section 3. In Section 4 we introduce the mappings ϕ and Φ and establish some of their properties. The final two sections are devoted to the proofs of Theorems 2.2 and 2.4 and Corollary 2.5.

Notation. Throughout the paper we abbreviate $L_1(0,1)$ with L_1. For a Banach space X, we denote by $\mathscr{B}(X)$ the algebra of all linear continuous operators in X.

The space L_1 will be regarded as a Banach algebra with respect to the convolution

$$(f * g)(x) := \int_0^1 f(x-t)g(t)\,dt,$$

where f is continued onto $(-1,1)$ to a periodic function of period 1. We denote by e_n, $n \in \mathbb{Z}$, the functions $\mathrm{e}_n(t) = \mathrm{e}^{2\pi i n t}$, $t \in [0,1]$, and write $\hat{f}$ for the Fourier transform of a function $f \in L_1$, viz.

$$\hat{f}(n) := \int_0^1 f(t)\mathrm{e}_{-n}(t)\,dt.$$

Occasionally we shall also use functions defined over $(-1,1)$; if $h \in L_1(-1,1)$, then $\hat{h}$ will stand for its Fourier transform over $(-1,1)$, i.e.,

$$\hat{h}(n) = \int_{-1}^1 h(t)\mathrm{e}^{-\pi i n t} dt.$$

It will be clear from the context as to which interval the function under consideration is defined, so that no confusion should arise. We note that the classes of Fourier transforms of functions in L_1 and $L_1(-1,1)$ coincide; indeed, if $f \in L_1$ and $h \in L_1(-1,1)$ are related via

$$h(t) := \begin{cases} \frac{1}{2} f(\frac{t}{2}+1), & \text{if} \quad t < 0, \\ \frac{1}{2} f(\frac{t}{2}), & \text{if} \quad t \geq 0, \end{cases}$$

then f and h have the same Fourier transform.

For a sequence $\Lambda = (\lambda_k)_{n\in\mathbb{Z}}$ of complex numbers we denote by $\mathscr{E}_\Lambda$ the system of exponentials $(\mathrm{e}^{i\lambda_n t})_{n\in\mathbb{Z}}$; as usual, for every ω that is repeated $l > 1$ times among λ_n, we replace the repeated occurrences of $\mathrm{e}^{i\omega t}$ with the functions $t\mathrm{e}^{i\omega t}, \dots, t^{l-1}\mathrm{e}^{i\omega t}$.

Finally, V.p. indicates that the corresponding summation or multiplication over the index set $\mathbb{Z}$ is taken in the sense of the principal value.

2. The main results

A sequence $\Lambda := (\lambda_n)_{n\in\mathbb{Z}}$ is called the zero sequence of an entire function F if every λ_n is a zero of F and every zero λ of F of order $n(\lambda)$ is repeated in Λ exactly $n(\lambda)$ times.

Definition 2.1. For a function $g \in L_1$, we denote by Λ_g the sequence $(\lambda_n)_{n\in\mathbb{Z}}$ with $\lambda_n = \lambda_n(g) := \pi n + \hat{g}(n)$. We also denote by Ω the subset of L_1 consisting of all $g \in L_1$, for which the sequence Λ_g has no repeating elements.

For $g \in L_1$, we denote by G_g the entire function given by the product

$$G_g(z) := \big(\lambda_0(g) - z\big) \lim_{n\to\infty} \prod_{0<|k|\leq n} \frac{z - \lambda_k(g)}{\pi k}. \tag{2.1}$$

The above product converges uniformly on compact sets (see Lemma 3.1 below) and thus determines an entire function. It turns out that this entire function has the form (1.1) for some $f \in L_1$, and, moreover, a function f with this property is unique. In this way we get the mapping ϕ such that

$$L_1 \ni g \mapsto \phi(g) := f \in L_1.$$

We observe that ϕ cannot be univalent; indeed, different enumerations of zeros of F_f lead to different g.

One of the main results of the paper is the following theorem.

Theorem 2.2. *The mapping $\phi : L_1 \to L_1$ is analytic and surjective, and its restriction $\phi_{|\Omega}$ onto the set Ω is a local analytic homeomorphism on Ω.*

We recall that a mapping $\psi : U \to V$ between subsets U and V of some Banach spaces is a local analytic homeomorphism at a point u if there is a neighbourhood U' of u and a neighbourhood V' of $\psi(u)$ such that $\psi_{|_{U'}} : U' \to V'$ is bijective, analytic, and homeomorphic on U'. This mapping ψ is a local analytic homeomorphism on U if it is such at every point $u \in U$. We refer the reader to [8] for the basic properties of analytic mappings between Banach spaces.

Theorem 2.2 shows that, for any $f \in L_1$, the remainders $\zeta_n := z_n - \pi n$ of the zeros z_n of the function F_f are Fourier coefficients of some function $g \in L_1$ and that, conversely, the sequence $(\pi n + \hat{g}(n))_{n\in\mathbb{Z}}$ for any $g \in L_1$ is the zero sequence for a function F_f with some $f \in L_1$. Next we define the class $\mathscr{X}$ of Banach spaces having similar properties.

Definition 2.3. We denote by $\mathscr{X}$ the set of all Banach spaces X that are continuously embedded into L_1 and satisfy the properties

(i) $\phi(X) = X$, and the mapping $\phi_{|X} : X \to X$ is analytic;
(ii) the mapping $\phi_{|X}$ is a local analytic homeomorphism on $\Omega \cap X$.

It turns out that the class $\mathscr{X}$ contains many classical function spaces. They can be constructed from a given $X \in \mathscr{X}$ using the following statement.

Theorem 2.4. *Assume that $X \in \mathscr{X}$ and that the operator M of multiplication by $i(1-2x)$ is continuous in X. Let Y be a Banach space that is continuously embedded into X; then Y belongs to $\mathscr{X}$ if the following conditions* (A1) *and one of* (A2) *and* (A2$'$) *hold:*

(A1) *the mapping $(f,g) \mapsto f * g$ is continuous from $X \times Y$ into Y and, moreover, for a fixed $f_0 \in X$ and a fixed $g_0 \in Y$ the linear operators*

$$Y \ni g \mapsto f_0 * g \in Y, \qquad X \ni f \mapsto f * g_0 \in Y$$

are compact;

(A2) *for some $m \in \mathbb{N}$, the m-fold convolution*

$$X^{\langle m\rangle} := \{f_1 * f_2 * \cdots * f_m \mid f_1, f_2, \dots, f_m \in X\}$$

of X is contained in Y;

(A2$'$) *the system $\{\mathrm{e}_n\}_{n\in\mathbb{Z}} \cap X$ is complete in X, and the system $\{\mathrm{e}_n\}_{n\in\mathbb{Z}} \cap Y$ is complete in Y.*

Some sufficient conditions under which (A1) holds are stated in Lemma 6.1.

Corollary 2.5. *The following spaces belong to the class $\mathscr{X}$:*

(i) $L_p(0,1)$ *for $p \in [1,\infty]$;*
(ii) *the space $C[0,1]$ of continuous functions over $[0,1]$;*
(iii) *the Hölder spaces $C^{\alpha}_{\mathrm{per}}[0,1]$, $\alpha \geq 0$, of periodic functions;*
(iv) *the Sobolev spaces $W^s_p(0,1)$, $p \in [1,\infty)$, $s \geq 0$;*

(v) *the space $BV(0,1)$ of functions of bounded total variation over* $[0,1]$*;*
(vi) *the closed linear span C_B in $C[0,1]$ of the set $\{\mathrm{e}_n \mid n \in B\}$, for every subset $B \subset \mathbb{Z}$.*

The above examples demonstrate to some extent the diversity of the function spaces in $\mathscr{X}$.

3. Preliminaries

For an arbitrary $g \in L_1$, we construct the infinite product G_g of (2.1) and would like to prove that $G_g = F_f$ for some $f \in L_1$. We show in this section that if such a function f exists, then it is unique, so that the above relation determines a mapping $\phi:\, g \mapsto f$. Put as before $\zeta_n := \hat{g}(n)$ and $z_n = \pi n + \zeta_n$, $n \in \mathbb{Z}$.

Lemma 3.1. *The product*

$$(z - z_0)\mathrm{V.p.}\prod_{n=-\infty}^{\infty}{}' \frac{z_n - z}{\pi n} = G_g(z),$$

the prime denoting that the factor corresponding to $n = 0$ is omitted, converges uniformly on compact sets and satisfies the relation

$$\lim_{y \to \pm\infty} \frac{G_g(iy)}{\sin iy} = 1.$$

Proof. That the product converges uniformly on compact sets follows from the same convergence of the series $\mathrm{V.p.}\sum_{n\in\mathbb{Z}}'(\zeta_n - z)/(\pi n)$; see [7, 10.1.5]. Using the canonical product for $\sin z$, we find that

$$\frac{G_g(z)}{\sin z} = \Big(1 - \frac{z_0}{z}\Big)\mathrm{V.p.}\prod_{n=-\infty}^{\infty}{}' \Big(1 + \frac{\zeta_n}{\pi n - z}\Big),$$

and it remains to prove that the function

$$\widetilde{G}(iy) := \mathrm{V.p.}\prod_{n=-\infty}^{\infty}{}' \Big(1 + \frac{\zeta_n}{\pi n - iy}\Big)$$

satisfies the relation $\lim_{y\to\pm\infty} \widetilde{G}(iy) = 1$.

To this end it suffices to show that the sum of the series

$$\mathrm{V.p.}\sum_{n=-\infty}^{\infty} \frac{\zeta_n}{\pi n - iy}$$

vanishes as $y \in \mathbb{R}$ tends to $\pm\infty$. We observe that $1/(\pi n - iy)$ for $y \neq 0$ is the nth Fourier coefficient of the function $u_y(t)$ that equals $i\mathrm{e}^{y(1-2t)}/\sinh y$ on $[0,1)$ and is periodically continued onto $\mathbb{R}$. Therefore $\zeta_n/(\pi n - iy)$ is the nth Fourier coefficient of the convolution $w_y := u_y * g$. The function u_y is of bounded variation over $[0,1]$, whence by [7, 3.1.5] such is w_y. It follows [7, 10.1.4] that the Fourier series of w_y converges pointwise to $\frac{1}{2}[w_y(\cdot - 0) + w_y(\cdot + 0)]$; in particular, we have

$$\mathrm{V.p.}\sum_{n=-\infty}^{\infty} \frac{\zeta_n}{\pi n - iy} = \tfrac{1}{2}[w_y(0) + w_y(1)] = \int_0^1 u_y(1-t)g(t)\,dt. \tag{3.1}$$

Since $|u_y(1-t)| \le \mathrm{e}^{|y|}/\sinh|y| \le 4$ for all $t \in (0,1)$ and all $y \in \mathbb{R}\setminus(-1,1)$ and since $u_y(t)$ go to zero as $|y| \to \infty$ for all $t \in (0,1)$, the Lebesgue dominated convergence theorem shows that the integral in (3.1) vanishes as $|y| \to \infty$. The proof is complete. □

We next show that F_f must coincide with G_g if both have the same zeros.

Lemma 3.2. *Assume that a sequence $\bigl(\pi n + \hat g(n)\bigr)_{n\in\mathbb{Z}}$ is the zero sequence for a function F_f with $f \in L_1$. Then $F_f = G_g$.*

Proof. We assume that none of $z_n := \pi n + \hat g(n)$ vanishes; the changes to be made otherwise are straightforward. Being of exponential type 1, the function F_f admits a representation as the Hadamard canonical product, i.e.,

$$F_f(z) = \mathrm{e}^{az+b}\,\mathrm{V.p.}\prod_{n=-\infty}^{\infty}\Bigl(1-\frac{z}{z_n}\Bigr)\mathrm{e}^{z/z_n}$$

with some constants a and b. Observe that the series

$$\mathrm{V.p.}\sum_{n=-\infty}^{\infty}\frac{1}{z_n} = \frac{1}{z_0} + \sum_{n=1}^{\infty}\frac{\hat g(n)+\hat g(-n)}{z_n z_{-n}}$$

converges; hence the factors e^{z/z_n} can be incorporated into e^{az+b} by modifying a accordingly.

Since the integral term in (1.1) is $\mathrm{o}\bigl(\mathrm{e}^{|\mathrm{Im}\, z|}\bigr)$ as $|z| \to \infty$ by [16, Lemma 1.3.1], we have

$$\lim_{y\to\pm\infty}\frac{F_f(iy)}{\sin iy} = 1. \tag{3.2}$$

On the other hand, for $z \in \mathbb{C}\setminus\pi\mathbb{Z}$,

$$\frac{F_f(z)}{\sin z} = \mathrm{e}^{az+b}\Bigl(\frac{1}{z}-\frac{1}{z_0}\Bigr)\mathrm{V.p.}\prod_{n=-\infty}^{\infty}{}'\Bigl(1+\frac{\hat g(n)}{\pi n - z}\Bigr)\mathrm{V.p.}\prod_{n=-\infty}^{\infty}{}'\Bigl(1-\frac{\hat g(n)}{z_n}\Bigr).$$

Both products above converge due to the convergence of the series $\mathrm{V.p.}\sum_{-\infty}^{\prime\,\infty}\hat g(n)/n$; see [7, 10.1.5]. Recalling (3.2) and the proof of Lemma 3.1, we conclude that $a = 0$ and that

$$\mathrm{e}^b = -z_0\mathrm{V.p.}\prod_{n=-\infty}^{\infty}{}'\frac{z_n}{\pi n}.$$

Substituting for e^b in the canonical product for F_f, we get $F_f = G_g$ as stated. □

We next show that there is at most one $f \in L_1$ such that $F_f = G_g$ for a given $g \in L_1$; see Corollary 3.4. This uniqueness statement follows from Levinson's theorems on closure $L_p(-1,1)$ of a system of exponentials [14, Chap. I] for the case where $f, g \in L_p$ with $p > 1$; however, these theorems are not applicable in the case $p = 1$. We recall that a system $\mathscr{S}$ of functions is called closed $L_p(-1,1)$ if the only function $h \in L_p(-1,1)$ such that

$$\int_{-1}^{1} h(t)f(t)\,dt = 0$$

for every $f \in \mathscr{S}$ is the identically zero function.

Lemma 3.3. *Assume that $r \in L_1(-1,1)$ and put $z_n := \pi n + \hat r(n)$ and $\Lambda := (z_n)_{n\in\mathbb{Z}}$. Then the system of exponentials $\mathscr{E}_\Lambda$ is closed $L_1(-1,1)$.*

Proof. Without loss of generality we may assume that none of z_k vanishes. Indeed, by [14, Theorem VI] finitely many of z_k may be arbitrarily changed without affecting the property of the system $\mathscr{E}_\Lambda$ to be closed $L_1(-1,1)$.

Assume that the system $\mathscr{E}_\Lambda$ is not closed $L_1(-1,1)$. Then there is $q \in L_1(-1,1)$ that is not identically zero and such that the function

$$Q(z) := \int_{-1}^{1} q(t)\mathrm{e}^{izt}\,dt$$

has zeros z_n, $n \in \mathbb{Z}$, counting multiplicities. Observe that the numbers z_n then give all the zeros of the entire function Q. Indeed, we clearly have

$$|Q(r\mathrm{e}^{i\theta})| \le \max_{-1\le t\le 1} |\exp\{ir\mathrm{e}^{i\theta}t\}| \int_{-1}^{1} |q(t)|\,dt = \mathrm{e}^{r|\sin\theta|}\int_{-1}^{1}|q(t)|\,dt,$$

and Jensen's theorem leads to the inequality

$$\begin{aligned}\int_0^r \frac{n(t)}{t}\,dt &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|Q(r\mathrm{e}^{i\theta})|\,d\theta - \log|Q(0)| \\ &\le \frac{r}{2\pi}\int_{-\pi}^{\pi}|\sin\theta|\,d\theta + C_1 = \frac{2r}{\pi} + C_1,\end{aligned} \tag{3.3}$$

where $n(r)$ is the number of zeros of Q in the disc $\{z \in \mathbb{C} \mid |z| \le r\}$ and C_1 is some constant. If, however, Q had at least one additional zero, then for any $\varepsilon > 0$ and all sufficiently large k (say, for $k \ge m$) we would have the estimate

$$n(\pi k + \pi\varepsilon) \ge 2k + 2.$$

Setting $t_k := \pi(k+\varepsilon)$ and using Stirling's approximation of the Euler gamma-function [1, Ch. 6], we then find that

$$\begin{aligned}\int_{t_m}^{t_{k+1}} \frac{n(t)}{t}\,dt &\ge \sum_{l=m}^{k}(2l+2)\log\frac{t_{l+1}}{t_l} \\ &= (2k+2)\log t_{k+1} - 2\sum_{l=m}^{k}\log t_l - 2m\log t_m \\ &= (2k+2)\log(t_{k+1}/\pi) - 2\log\Gamma(t_{k+1}/\pi) + C_2 \\ &\ge \frac{2t_{k+1}}{\pi} + (1-2\varepsilon)\log t_{k+1} + C_3\end{aligned}$$

for some real C_2 and C_3 independent of k, which contradicts (3.3). (Alternatively, we could then apply an analogue of Theorem III from [14] for exponentials $\mathrm{e}^{i\lambda_k t}$ with complex λ_k to contradict the assumption that q is not identically zero.)

Repeating the arguments of the proof of Lemma 3.2, we represent Q in the form

$$Q(z) = \mathrm{e}^{az+b}\,\mathrm{V.p.}\prod_{n=-\infty}^{\infty}\Bigl(1 - \frac{z}{z_n}\Bigr)$$

with some constants a and b and then show that

$$\lim_{y\to\pm\infty}\frac{Q(iy)\mathrm{e}^{-iay-b}}{\sin(iy)} = -\frac{1}{z_0}\mathrm{V.p.}\prod_{n=-\infty}^{\infty}{}'\Bigl(1 - \frac{\hat{r}(n)}{z_n}\Bigr) \ne 0.$$

On the other hand, the integral representation of Q implies the estimate $|Q(z)| = \mathrm{o}\big(\mathrm{e}^{|\operatorname{Im} z|}\big)$ as $|z| \to \infty$, which is inconsistent with at least one of the above limits. The contradiction derived shows that $q = 0$ and that the system $\mathscr{E}_\Lambda$ is closed $L_1(-1, 1)$. □

Corollary 3.4. *Assume that $g \in L_1$; then there exists at most one $f \in L_1$ such that Λ_g is the zero sequence of the entire function F_f of* (1.1).

Proof. Assume that there are $f, \tilde f \in L_1$ such that the functions F_f and $F_{\tilde f}$ vanish at all the points $z_n := \pi n + \hat g(n)$, $n \in \mathbb{Z}$, together with their derivatives of respective orders at the multiple points of Λ_g. We put $h := f - \tilde f$ and $H(z) := \int_0^1 h(t)\mathrm{e}^{iz(1-2t)}\,dt$ and observe that Λ_g is the zero sequence of H. Writing $H(z)$ as $\int_{-1}^1 q(t)\mathrm{e}^{izt}\,dt$ with $q(t) := \frac12 h(\frac{1-t}{2})$ and applying Lemma 3.3, we conclude that $q = 0$, i.e., that $f = \tilde f$. □

4. Properties of the mappings ϕ and Φ

The results of the previous section show that the mapping $\phi : L_1 \to L_1$ given by the relation $\phi(g) = f$ if $F_f = G_g$ on the domain

$$\operatorname{dom}\phi = \{g \in L_1 \mid \exists\, f \in L_1, F_f = G_g\}$$

is well defined. To derive its properties, we shall show that there exists an analytic map $\Phi : L_1 \times L_1 \to L_1$ such that the following holds:

(a) for every f and g in L_1, the equality $\phi(g) = f$ implies that $\Phi(f, g) = 0$;

(b) for every $f \in L_1$ and every $g \in \Omega$, the equality $\Phi(f, g) = 0$ yields $\phi(g) = f$.

Assume that $f, g \in L_1$ and that $\phi(g) = f$, i.e.,

$$F_f(\pi n + \zeta_n) = 0, \qquad \forall n \in \mathbb{Z}, \tag{4.1}$$

where $\zeta_n := \hat g(n)$. Taking into account (1.1) and the equality

$$\mathrm{e}^{i\zeta_n(1-2t)} = \sum_{k=0}^\infty \frac{[i\zeta_n(1-2t)]^k}{k!},$$

relation (4.1) can be recast as

$$\sin\zeta_n + \sum_{k=0}^\infty \frac{\zeta_n^k}{k!}\int_0^1 [i(1-2t)]^k f(t)\mathrm{e}^{-2\pi i n t}\,dt = 0. \tag{4.2}$$

We denote by $g^{\langle k\rangle}$ the k-fold convolution of g with itself, with $g^{\langle 0\rangle}$ being equal to the unity δ_0 of the convolution algebra L_1 and $g^{\langle 1\rangle} = g$, and put

$$s(g) := \sum_{k=0}^\infty \frac{(-1)^k g^{\langle 2k+1\rangle}}{(2k+1)!}.$$

Well-known properties of the convolution yield the relations

$$\sin\zeta_n = \widehat{s(g)}(n), \qquad \zeta_n^k = \widehat{g^{\langle k\rangle}}(n), \qquad n \in \mathbb{Z},$$

so that the inverse Fourier transform takes (4.2) into the relation

$$s(g) + f + \sum_{k=1}^\infty \frac{(M^k f) * g^{\langle k\rangle}}{k!} = 0,$$

where $(Mf)(x) := i(1-2x)f(x)$ is the operator of multiplication by $i(1-2x)$.

Assume now that the above equality holds for some $f \in L_1$ and $g \in \Omega$. Reverting the arguments, we get then that, for every $n \in \mathbb{Z}$,

$$F_f\big(\pi n + \zeta_n\big) = 0.$$

Since $g \in \Omega$, all the numbers $\lambda_n(g) = \pi n + \zeta_n$ are pairwise distinct. As in the proofs of Lemmas 3.2 and 3.3 it can be shown that F_f has no other zeros except $\lambda_n(g)$, i.e., that Λ_g is the zero sequence for the function F_f, so that $F_f = G_g$.

We have thus shown that the mapping $\Phi : L_1 \times L_1 \to L_1$ given by the formula

$$\Phi(f,g) := s(g) + f + \sum_{k=1}^{\infty} \frac{(M^k f) * g^{\langle k \rangle}}{k!} \tag{4.3}$$

satisfies properties (a) and (b). It remains to establish analyticity of Φ. It is easy to see that the Fréchet derivative $d\Phi(f,g)$, which is a linear continuous operator from $L_1 \times L_1$ into L_1 [8, Ch. 2], is equal to

$$d\Phi(f,g)(h_1,h_2) = \partial_f \Phi(f,g)(h_1) + \partial_g \Phi(f,g)(h_2),$$

where the partial derivatives $\partial_f \Phi(f,g)$ and $\partial_g \Phi(f,g)$ are linear continuous operators in L_1 and are given by the formulae

$$\partial_f \Phi(f,g)(h_1) = h_1 + \sum_{k=1}^{\infty} \frac{(M^k h_1) * g^{\langle k \rangle}}{k!}, \tag{4.4}$$

$$\partial_g \Phi(f,g)(h_2) = \Big(c(g) + \sum_{k=1}^{\infty} \frac{(M^k f) * g^{\langle k-1 \rangle}}{(k-1)!}\Big) * h_2, \tag{4.5}$$

with

$$c(g) := \sum_{k=0}^{\infty} \frac{(-1)^k g^{\langle 2k \rangle}}{(2k)!}.$$

Continuity of the convolution implies that the mappings

$$L_1 \times L_1 \ni (f,g) \mapsto \partial_f \Phi(f,g) \in \mathscr{B}(L_1),$$
$$L_1 \times L_1 \ni (f,g) \mapsto \partial_g \Phi(f,g) \in \mathscr{B}(L_1)$$

are continuous, and hence such is the mapping

$$L_1 \times L_1 \ni (f,g) \mapsto d\Phi(f,g) \in \mathscr{B}(L_1 \times L_1, L_1).$$

It follows [8, Ch. 2] that the function Φ is analytic.

Having in mind the use of the implicit function theorem, we have to find out when the partial derivatives of Φ are invertible. Since, for an arbitrary $h \in L_1$, the operator

$$f \mapsto f * h$$

is compact in L_1, we derive from (4.4) and (4.5) that the operators

$$\partial_f \Phi(f,g) - I, \qquad \partial_g \Phi(f,g) - I$$

are also compact. Henceforth the operators $\partial_f \Phi(f,g)$ and $\partial_g \Phi(f,g)$ are not invertible in L_1 if and only if their null-spaces are nontrivial [18, Theorem 4.25].

Assume that some $h_1 \in L_1 \setminus \{0\}$ verifies the relation $\partial_f \Phi(f,g)(h_1) = 0$. Taking the nth Fourier coefficient of (4.4), we find that

$$0 = \int_0^1 h_1(t) \mathrm{e}^{i\lambda_n(g)(1-2t)}\, dt = \frac{1}{2} \int_{-1}^1 h_1\Big(\frac{1-s}{2}\Big) \mathrm{e}^{i\lambda_n(g)s}\, ds. \tag{4.6}$$

Such a relation can hold for every $n \in \mathbb{Z}$ if and only if $g \in L_1 \setminus \Omega$. Indeed, for $g \in \Omega$ all $\lambda_n(g)$ are pairwise distinct, and since the system $\mathscr{E}_{\Lambda_g}$ is closed $L_1(-1,1)$ by Lemma 3.3, we get $h_1 = 0$. Conversely, $g \in L_1 \setminus \Omega$ means that at least two among $\lambda_n(g)$ coincide, say $\lambda_n(g) = \lambda_m(g) = \omega$. Since by [5, Prop. II.4.3] the system $\mathscr{E}_{\Lambda_g}$ forms a Riesz basis of $L_2(0,1)$, the system $\mathscr{E}_{\Lambda_g} \setminus \{t\mathrm{e}^{i\omega t}\}$ is not closed $L_1(-1,1)$, and thus there exists a nonzero h_1 satisfying (4.6) for all $n \in \mathbb{Z}$. Therefore we have shown that $\Phi(f,g) = 0$ implies the equivalence

$$\partial_f \Phi(f,g) \quad \text{is invertible} \iff g \in \Omega.$$

Assume now that f and g in L_1 are such that $\Phi(f,g) = 0$ and $\partial_g \Phi(f,g)$ is noninvertible, i.e., that $\partial_g \Phi(f,g)(h_2) = 0$ for some $h_2 \in L_1 \setminus \{0\}$. It follows that $F_f\bigl(\lambda_n(g)\bigr) = 0$ for every $n \in \mathbb{Z}$. Using formula (4.5) and equating to zero the nth Fourier coefficient of the function $\partial_g \Phi(f,g)(h_2)$, we arrive at the relation

$$F_f'\bigl(\lambda_n(g)\bigr)\hat{h}_2(n) = 0.$$

Since at least one Fourier coefficient of h_2 does not vanish, we conclude that at least one of $\lambda_n(g)$ is a multiple zero of F_f, i.e., that $g \in L_1 \setminus \Omega$. Conversely, if $\lambda_n(g)$ is a multiple zero of F_f, then $\partial_g \Phi(f,g)(h_2) = 0$ for $h_2(t) := \mathrm{e}^{2\pi i n t}$.

Therefore if some f and g in L_1 verify the equality $\Phi(f,g) = 0$, then the partial derivatives $\partial_f \Phi(f,g)$ and $\partial_g \Phi(f,g)$ are invertible if and only if $g \in \Omega$. The implicit function theorem now yields the following statement.

Proposition 4.1. *Assume that $f_0 \in L_1$, $g_0 \in \Omega$, and $\Phi(f_0, g_0) = 0$. Then there exist an open neighbourhood U of the point f_0, an open neighbourhood W of the point g_0, and a homeomorphism $\phi_W : W \to U$ such that $\Phi(\phi_W(g), g) = 0$ for every $g \in W$, and the mappings $\phi_W : W \to U$ and $\phi_W^{-1} : U \to W$ are analytic.*

It is clear that the mapping ϕ is well defined in W and coincides therein with ϕ_W. Since $0 \in \Omega$ and $\Phi(0,0) = 0$, we conclude from Proposition 4.1 that there is $\delta \in (0,1)$ such that ϕ is well defined in the ball

$$B_\delta = \{g \in L_1 \mid \|g\|_{L_1} < \delta\}$$

of radius δ. Next we shall show that $\operatorname{dom} \phi = L_1$.

We fix $l \in \mathbb{N}$ and $\mathbf{a} = (a_{-l}, \dots, a_l) \in \mathbb{C}^{2l+1}$ and show that ϕ is well defined and analytic in the δ-neighbourhood of the trigonometric polynomial

$$P_{\mathbf{a}}(t) := \sum_{|k| \le l} a_k \mathrm{e}^{2\pi i k t}.$$

Lemma 4.2. *For any $l \in \mathbb{N}$ and any $\mathbf{a} \in \mathbb{C}^{2l+1}$ the set $P_{\mathbf{a}} + B_\delta$ belongs to the domain of ϕ and there exist analytic functions*

$$\mathbb{C}^{2l+1} \ni \boldsymbol{\lambda} \mapsto h_l(\boldsymbol{\lambda}) \in L_1, \qquad \mathbb{C}^{2l+1} \ni \boldsymbol{\lambda} \mapsto A_l(\boldsymbol{\lambda}) \in \mathscr{B}(L_1)$$

such that

$$\phi(g + P_{\mathbf{a}}) = \phi(g) + h_l(\boldsymbol{\lambda}(g)) + A_l(\boldsymbol{\lambda}(g))\phi(g)$$

for all $g \in B_\delta$, with $\boldsymbol{\lambda}(g) := \bigl(\lambda_k(g)\bigr)_{k=-l}^{l} \in \mathbb{C}^{2l+1}$.

Proof. We shall start with the following remark. Fix $b \in \mathbb{C}$ and let $f \in L_1$ and $\zeta \in \mathbb{C}$ be such that $F_f(\zeta) = 0$. As in [11, p. 535] and [14, p. 10] it can be shown that there is $\tilde{f} \in L_1$ such that

$$F_{\tilde{f}}(z) = \frac{z - \zeta - b}{z - \zeta} F_f(z), \qquad z \in \mathbb{C}.$$

The mapping $f \mapsto \tilde{f}$ is affine and depends only on ζ; moreover, this dependence is analytic. In fact, a straightforward calculation gives that

$$\tilde{f} = f - b\bigl(h(\zeta) + A(\zeta)f\bigr),$$

where

$$[h(\zeta)](t) := \mathrm{e}^{i\zeta r(t)}, \qquad r(t) := \operatorname{sgn}(1-2t) - (1-2t), \qquad t \in [0,1],$$

and

$$[A(\zeta)f](t) = 2i\int_{a(t)}^{t} f(s)\mathrm{e}^{2i\zeta(t-s)}\,ds,$$

with $a(t) = 0$ if $t < \frac{1}{2}$ and $a(t) = 1$ otherwise. It is clear that the mappings

$$\mathbb{C} \ni \zeta \mapsto h(\zeta) \in L_1, \qquad \mathbb{C} \ni \zeta \mapsto A(\zeta) \in \mathscr{B}(L_1)$$

are entire.

Assume now that $f \in L_1$ and $\boldsymbol{\lambda} = (\lambda_k)_{k=-l}^{l} \in \mathbb{C}^{2l+1}$ are such that the numbers λ_k are pairwise distinct and $F_f(\lambda_k) = 0$ for every $k = -l, \dots, l$. Fixing $\mathbf{a} = (a_{-l}, \dots, a_l) \in \mathbb{C}^{2l+1}$ and applying the above transformation $2l+1$ times, we find that there is $\check{f}$ such that

$$F_{\check{f}}(z) = \Bigl(\prod_{|k|\le l} \frac{z-\lambda_k-a_k}{z-\lambda_k}\Bigr) F_f(z), \qquad z \in \mathbb{C}.$$

It is easy to see that $f \mapsto \check{f}$, being the composition of $2l+1$ affine mappings, is affine itself and has the form

$$\check{f} = f + h_l(\boldsymbol{\lambda}) + A_l(\boldsymbol{\lambda})f,$$

with analytic functions h_l and A_l from $\mathbb{C}^{2l+1}$ into L_1 and $\mathscr{B}(L_1)$, respectively.

Now let $g \in B_\delta$, $f = \phi(g)$, and $\boldsymbol{\lambda}(g) = \bigl(\lambda_k(g)\bigr)_{k=-l}^{l}$. As δ is less than 1, the numbers $\lambda_k(g)$ are pairwise distinct. The above arguments show that

$$\phi(g + P_{\mathbf{a}}) = \phi(g) + h_l(\boldsymbol{\lambda}(g)) + A_l(\boldsymbol{\lambda}(g))\phi(g),$$

and the lemma is proved. □

Since the set of all trigonometric polynomials is dense in L_1, it follows from Lemma 4.2 that $\operatorname{dom}\phi = L_1$ and that ϕ is an analytic mapping on the whole space L_1.

5. Proof of Theorem 2.2

In view of the results of the previous section it suffices to prove that ϕ is surjective.

We fix an arbitrary $f \in L_1$ and denote by $\Lambda = (z_n)_{n\in\mathbb{Z}}$ the sequence of zeros of the function F_f numbered so that

$$z_n = \pi n + \mathrm{o}(1), \qquad |n| \to \infty.$$

Then the system of exponentials $\mathscr{E}_\Lambda$ forms a Riesz basis of $L_2(-1,1)$ [5, Prop. II.4.3]. Let $(q_k)_{k\in\mathbb{Z}}$ be a biorthogonal basis. According to Proposition 4.1 there are $\varepsilon > 0$ and $\delta \in (0,1)$ such that $B_\varepsilon \subset \phi(B_\delta)$. Since $L_2(-1,1)$ is dense in $L_1(-1,1)$, there exist $n_0 \in \mathbb{N}$ and $q \in L_2(-1,1)$ of the form

$$q = \sum_{|k|\le n_0} c_k q_k$$

such that

$$\int_{-1}^{1}\Big|\frac{1}{2}f\Big(\frac{1-t}{2}\Big)-\overline{q}(t)\Big|\,dt<\varepsilon. \tag{5.1}$$

We can assume without loss of generality that

$$|z_k-\pi k|<1 \quad \text{if} \quad |k|>n_0, \tag{5.2}$$

enlarging n_0 if necessary. Set $f_0(t):=f(t)-2\overline{q}(1-2t)$, $t\in(0,1)$. In view of (5.1) we have $f_0\in B_\varepsilon$, so that there is $g_0\in B_\delta$ such that $f_0=\phi(g_0)$. In particular, it follows that the sequence Λ_{g_0} is the sequence of zeros of the function F_{f_0}, i.e., that

$$F_{f_0}(\lambda_k(g_0))=0, \qquad k\in\mathbb{Z}.$$

We also notice that

$$|\lambda_k(g_0)-\pi k|\le\|g_0\|_{L_1}\le\delta<1, \qquad k\in\mathbb{Z}. \tag{5.3}$$

It is clear that

$$\begin{aligned}F_{f_0}(z_k)&=F_{f_0}(z_k)-F_f(z_k)=-2\int_0^1\overline{q}(1-2t)\mathrm{e}^{iz_k(1-2t)}\,dt\\&=-\sum_{|l|\le n_0}\overline{c_l}\int_{-1}^{1}\mathrm{e}^{iz_ks}\overline{q_l}(s)\,ds.\end{aligned}$$

Since the sequence $(q_k)_{k\in\mathbb{Z}}$ is biorthogonal to $\mathscr{E}_\Lambda$, we conclude that $F_{f_0}(z_k)=0$ for all $k\in\mathbb{Z}$ such that $|k|>n_0$; it follows from (5.2) and (5.3) that $z_k=\lambda_k(g_0)$ for such k. Now we set

$$g:=g_0+\sum_{|k|\le n_0}\big(z_k-\lambda_k(g_0)\big)\mathrm{e}_k$$

and conclude that $\lambda_k(g)=z_k$ for all $k\in\mathbb{Z}$, so that $F_f=G_g$, and ϕ is surjective. The proof of Theorem 2.2 is complete.

6. Proof of Theorem 2.4 and Corollary 2.5

Proof of Theorem 2.4. We shall first prove that $\phi(Y)\subset Y$ and that the restriction $\phi_{|Y}$ of the mapping ϕ onto Y is analytic in Y. As $X\in\mathscr{X}$, ϕ is analytic in X, and continuity of the embedding $Y\hookrightarrow X$ implies that ϕ is also analytic as a mapping from Y into X.

Next, since $\Phi(\phi(g),g)=0$ for all $g\in Y$, by (4.3) we get

$$\phi(g)=-s(g)-\sum_{k=1}^{\infty}\frac{(M^k\phi(g))*g^{\langle k\rangle}}{k!}. \tag{6.1}$$

Assumption (A1) implies that $g\mapsto g^{\langle k\rangle}$ is a continuous k-homogeneous operator in Y, so that $g\mapsto s(g)$ is an analytic mapping in Y. Also analytic is the mapping $\Psi: X\times Y\to Y$ given by

$$\Psi(f,g):=\sum_{k=1}^{\infty}\frac{(M^kf)*g^{\langle k\rangle}}{k!},$$

and thus such is the composition of Ψ and ϕ, viz.

$$Y\ni g\mapsto\Psi(\phi(g),g)\in Y.$$

In view of (6.1) this establishes analyticity of the mapping $\phi_{|Y}$.

Next we prove that $\phi_{|_Y}$ is a local analytic homeomorphism on $Y\cap\Omega$. The above reasonings show that the mapping Φ sends $Y\times Y$ into Y, and we denote by $\Phi_{|_Y}$ its restriction onto $Y\times Y$. The Fréchet derivative and Fréchet partial derivatives of $\Phi_{|_Y}$ in Y are given by the same formulae as for L_1, but considered on elements of Y. Assumption (A1) implies that the operators

$$\partial_f\Phi_{|_Y}(f,g)-I,\qquad \partial_g\Phi_{|_Y}(f,g)-I$$

are compact, and the operators $\partial_f\Phi_{|_Y}(f,g)$ and $\partial_g\Phi_{|_Y}(f,g)$ have trivial null-spaces as soon as $g\in\Omega\cap Y$ by the reasoning of Section 4. Thus these partial derivatives are boundedly invertible in the algebra $\mathscr{B}(Y)$, which by the implicit function theorem yields the claim about $\phi_{|_Y}$.

It remains to show that $\phi(Y)=Y$ if either (A2) or (A2$'$) holds. Assume first that condition (A2) is satisfied, i.e., that $X^{\langle m\rangle}\subset Y$. We take an arbitrary $f\in Y$; since $Y\subset X$ and $X\in\mathscr{X}$, there is $g\in X$ such that $f=\phi(g)$. In other words, such a g satisfies the relation $\Phi(f,g)=0$, i.e.,

$$g=-\sum_{k=1}^{\infty}\frac{(-1)^k g^{\langle 2k+1\rangle}}{(2k+1)!}+f+\sum_{k=1}^{\infty}\frac{(M^kf)*g^{\langle k\rangle}}{k!}. \tag{6.2}$$

We set

$$Z_l:=X^{\langle l\rangle}+X^{\langle l+1\rangle}+\cdots+X^{\langle m-1\rangle}+Y$$

for $l=1,\dots,m-1$ and $Z_m:=Y$; then $g\in X\subset Z_1$. Assume that we have already shown that $g\in Z_l$ with $l<m$. Then the right-hand side of relation (6.2) belongs to Z_{l+1}, i.e., $g\in Z_{l+1}$, and, by induction, $g\in Z_m=Y$.

Assume now that (A2$'$) holds, take an arbitrary $f\in Y$, and choose trigonometric polynomials P_0 and P_1 in e_n such that $P_0\in Y$, $P_1\in X$, and

$$\|f-P_0\|_Y<\varepsilon_0,\qquad \|Mf-P_1\|_X<\varepsilon_1,$$

with ε_0 and ε_1 to be specified later. Consider the mapping $\psi:Y\to Y$ given by

$$\psi(h):=h-s(h)-(f-P_0)-(Mf-P_1)*h-\sum_{k=2}^{\infty}\frac{(M^kf)*h^{\langle k\rangle}}{k!}. \tag{6.3}$$

It is clear that ψ is analytic in Y and that $\psi(0)=-(f-P_0)$ and $d\,\psi(0)(h_1)=-(Mf-P_1)*h_1$. By Assumption (A1) there is $\varepsilon_1>0$ such that $\|d\psi(0)\|_{\mathscr{B}(Y)}\le\frac14$, whence for some $\varepsilon>0$ we have $\|d\psi(h)\|_{\mathscr{B}(Y)}\le\frac12$ for all h in the closed ball $B_{\varepsilon,Y}:=\{h\in Y\mid \|h\|_Y\le\varepsilon\}$. We now set $\varepsilon_0=\varepsilon/2$ and observe that ψ is a contraction of the ball $B_{\varepsilon,Y}$. Thus the equation $h=\psi(h)$ has a unique solution in $B_{\varepsilon,Y}$, which we denote by g_0.

Taking into account relations (4.3) and (6.3), we conclude that

$$\Phi(f,g_0)=g_0-\psi(g_0)+P_0+P_1*g_0=P_0+P_1*g_0=:P$$

is a trigonometric polynomial that belongs to the space Y. It is straightforward to verify that

$$F_f(\lambda_k(g_0))=(-1)^k\widehat{P}(k),$$

so that all but at most finitely many of the $\lambda_k(g_0)$ are zeros of the entire function F_f; moreover, $\widehat{P}(k)\ne0$ yields $\mathrm{e}_k\in Y$. Without loss of generality we can assume ε to be so small that $\|g_0\|_{L_1}<1$, and then the numbers $\lambda_k(g_0)$, $k\in\mathbb{Z}$, are pairwise distinct. Denote by $\Lambda=(z_k)_{k\in\mathbb{Z}}$ the zero sequence of the function F_f. The numbers

z_k can be labelled so that $z_k = \lambda_k(g_0)$ if the latter is a zero of F_f (i.e., if $\widehat{P}(k) = 0$). We now set

$$g := g_0 + \sum_{k:\, \widehat{P}(k)\neq 0} \big(z_k - \lambda_k(g_0)\big)\mathrm{e}_k.$$

Then $g \in Y$ and $\lambda_n(g) = z_n$ for all $n \in \mathbb{Z}$, which yields the relation $F_f = G_g$, i.e., the relation $\phi(g) = f$. Thus $\phi_{|_Y}$ is surjective under (A2′), and the proof of Theorem 2.4 is complete. □

Before proving Corollary 2.5, we give some sufficient conditions for assumption (A1) of Theorem 2.4 to be satisfied.

Lemma 6.1. *Assume that X, Y, and Z are Banach spaces continuously embedded into L_1 and that $X * Y \subset Z$. Then the following holds:*

(a) *the mapping $(f, g) \mapsto f * g$ is continuous from $X \times Y$ to Z;*

(b) *if the set $\mathscr{P} \cap X$ of all trigonometric polynomials belonging to X is dense in X, then the operator $R(f) : g \mapsto f * g$ is compact from Y to Z for every $f \in X$;*

(c) *if the set $\mathscr{P} \cap Y$ of all trigonometric polynomials belonging to Y is dense in Y, then the operator $L(g) : f \mapsto f * g$ is compact from X to Z for every $g \in Y$;*

(d) *if $X * Y$ is compactly embedded into Z, i.e., if the set*

$$B_{1,X} * B_{1,Y} := \{f * g \mid f \in B_{1,X},\, g \in B_{1,Y}\}$$

($B_{1,X}$ and $B_{1,Y}$ denoting the unit balls of X and Y, respectively) is precompact in Z, then the operators $R(f)$ and $L(g)$ are compact for every $f \in X$ and every $g \in Y$.

Proof. (a) Since the convolution operation is continuous in L_1 [7, Sect. 3.1.6], the operators $R(f)$ and $L(g)$ are closed [18, Sect. 13.1] for every $f \in X$ and every $g \in Y$ and hence, by the closed graph theorem [18, Thm. 2.15], they are continuous. The inequality

$$\|f * g\|_Z \le \|R(f)\|_{Y\to Z}\|g\|_Y$$

implies that, for every $f \in X$, the orbit $\{L(g)f \mid g \in B_{1,Y}\}$ is bounded in Z, and hence by the uniform boundedness principle [18, Thm. 2.6] we get that

$$c := \sup_{\|g\|_Y\le 1} \|L(g)\|_{X\to Z} < \infty.$$

Therefore $\|f*g\|_Z \le c\|f\|_X\|g\|_Y$ for all $f \in X$ and $g \in Y$, which yields the required continuity and the estimates $\|R(f)\|_{Y\to Z} \le c\|f\|_X$ and $\|L(g)\|_{X\to Z} \le c\|g\|_X$.

(b) Let $(p_n) \in \mathscr{P} \cap X$ be a sequence converging to f in X; then by the above

$$\|R(f) - R(p_n)\|_{Y\to Z} = \|R(f - p_n)\|_{Y\to Z} \le c\|f - p_n\|_X \to 0$$

as $n \to \infty$. Therefore $R(f)$ is the limit in the uniform operator topology of the operators $R(p_n)$; since the latter are of finite rank, the claim follows [18, Thm. 4.18]. The proof of item (c) is completely analogous.

(d) immediately follows from the definition of compact operators [18, Def. 4.16]. □

Proof of Corollary 2.5. For each of cases (i)–(vi) listed, we establish properties (A1) and one of (A2) and (A2′) of Theorem 2.4 for a suitable $X \in \mathscr{X}$, in which the multiplication operator M is continuous.

Case (i) *with* $p<\infty$. The spaces $X:=L_1\in\mathscr{X}$ and $Y:=L_p$ satisfy (A2$'$) and $L_1*L_p\subset L_p$ by [7, Sect. 3.1.6]. Thus by Lemma 6.1 condition (A1) is satisfied, and $L_p\in\mathscr{X}$ by Theorem 2.4.

Case (ii) *and Case* (i) *with* $p=\infty$. Since by [7, Sect. 3.1.4]

$$L_2*L_2\subset C[0,1]\subset L_\infty,$$

the operator $T(h):\,L_2\to C[0,1]$ of convolution with an element $h\in L_2$ is compact by Lemma 6.1. Now take $X=L_2\in\mathscr{X}$ and $Y=C[0,1]$ or $Y=L_\infty$. Then (A2) holds with $m=2$ by the above formula. The operator $R(f):\,g\mapsto f*g$ is compact in Y for every $f\in L_2$ as a composition of the continuous embedding of Y into X and the mapping $T(f)$, and $L(g):\,f\mapsto f*g$ acts compactly from X into Y for every $g\in L_\infty$ because it coincides with $T(g)$. Thus (A1) is verified, and $Y\in\mathscr{X}$.

Case (iii). By definition, the space $C^\alpha_{\rm per}[0,1]$, $\alpha=n+s$, with $n\in\mathbb{Z}_+$ and $s\in[0,1)$, is the completion of the set $\mathscr{P}$ by the norm

$$\|p\|_{C^\alpha_{\rm per}}:=\max_{0\le j\le n}\max_{x\in\mathbb{R}}|p^{(j)}(x)|+\sup_{x,y\in\mathbb{R},\ x\ne y}\frac{|p^{(n)}(x)-p^{(n)}(y)|}{|x-y|^s}.$$

If $s=0$, the second term in the above definition can be omitted. Set $X:=L_1$ and $Y:=C^\alpha_{\rm per}[0,1]$; then (A2$'$) holds. Moreover, for integer α, $X*Y\subset Y$ and $(f*g)^{(j)}=f*g^{(j)}$ for every $f\in X$, every $g\in Y$, and $j=1,\dots,\alpha$ by [7, Sect. 3.1.5]. For $\alpha=n+s$ we insert the above formula for $(f*g)^{(j)}$ in the definition of the norm to conclude that $f*g\in C^\alpha_{\rm per}[0,1]$. Now (A1) follows from Lemma 6.1, thus yielding $Y\in\mathscr{X}$.

Case (iv). We refer the reader to [2, Ch. 7] for the definition and basic properties of the Sobolev spaces $W^s_p:=W^s_p(0,1)$ with $p\ge1$ and $s\ge0$.

Direct verification shows that, for $n\in\mathbb{Z}_+$ and $s,t\in\{0;1\}$, we have

$$W^{n+s}_p*W^{n+t}_p\subset W^{n+s+t}_p; \tag{6.4}$$

therefore the multilinear interpolation [6, Ch. 4.4] justifies (6.4) for all $s,t\in[0,1]$. By the Arzéla-Ascoli theorem the space W^{n+1}_p is compactly embedded into W^n_p for every $n\in\mathbb{Z}_+$, and by the compactness theorem [6, Theorem 3.8] the embedding of W^s_p into W^t_p is compact as soon as $s>t\ge0$.

Let $s\in(0,1/p)$ and set $X:=L_p=W^0_p$ and $Y:=W^s_p$. Since the set $\mathscr{P}$ of all trigonometric polynomials is dense in the space $C_{\rm per}[0,1]$ of continuous 1-periodic functions and the latter is dense in Y by [2, Ch. 7], we conclude that (A2$'$) holds. By (6.4) and Lemma 6.1 condition (A1) is also verified, and thus $Y\in\mathscr{X}$.

Now take $X:=W^r_p$ with $r\in(0,1/p)$ and $Y=W^s_p$ with $s\in(r,1]$. By (6.4), $X^{\langle m\rangle}\subset Y$ if $m\ge1/r$, thus giving (A2). Since the embedding $X*Y\subset Y$ is compact by the above, (A1) holds by Lemma 6.1, and $Y\in\mathscr{X}$.

Assume that $X:=W^n_p$, $n\in\mathbb{N}$, is in $\mathscr{X}$ (the case $n=1$ being established above) and set $Y:=W^{n+t}_p$ with $t\in(0,1]$. Observe that $W^n_p*W^{n+t}_p\subset W^{n+1+t}_p$ for natural n and $t\in[0,1]$ (the cases $t=0$ and $t=1$ can be verified directly, and the case $t\in(0,1)$ follows by interpolation). Then $X*X\subset W^{n+1}_p\subset Y$, so that (A2) is satisfied with $m=2$, and the embedding of $X*Y\subset W^{n+1+t}_p$ into Y is compact. Therefore (A1) holds by Lemma 6.1 and $Y\in\mathscr{X}$; in particular, $W^{n+1}_p\in\mathscr{X}$. Using induction on $n\in\mathbb{N}$, we conclude that $W^s_p\in\mathscr{X}$ for all $s\ge0$, thus completing the proof of Case (iv).

Case (v). If $g \in BV[0,1]$ and $V(g)$ is the total variation of g, then $|n\hat{g}(n)| \le V(g)$ for all $n \in \mathbb{Z}$ by [7, Sect. 2.3.6], and thus

$$\sum_{n\in\mathbb{Z}} (1+n^2)^{1/3} |\hat{g}(n)|^2 < \infty.$$

It follows [2, Ch. 7] that $BV[0,1] \subset W_2^{1/3}$. Set $X := W_2^{1/3} \in \mathscr{X}$ and $Y := BV[0,1]$; then by (6.4) we have $X^{\langle 3\rangle} \subset W_2^1 \subset Y$, and (A2) holds. Take $f \in X$ and $g \in Y$; then the Fourier coefficients $\hat{h}(n) = \hat{f}(n)\hat{g}(n)$ of the convolution $h := f * g$ satisfy

$$\sum_{n\in\mathbb{Z}} (1+n^2)^{4/3} |\hat{h}(n)|^2 \le 2\big(|g(0)| + V(g)\big)^2 \sum_{n\in\mathbb{Z}} (1+n^2)^{1/3} |\hat{f}(n)|^2 < \infty.$$

It follows [2, Ch. 7] that $X * Y$ is contained into $W_2^{4/3}$ and thus is compactly embedded into W_2^1 and Y. Thus (A1) is verified by Lemma 6.1, and $Y \in \mathscr{X}$.

Case (vi). The spaces $X := L_1$ and $Y := C_B$ verify condition (A2′); moreover, the relation $f * \mathrm{e}_n = \mathrm{e}_n \int_0^1 f \mathrm{e}_{-n}$ for every $f \in X$ yields $X * Y \subset Y$. Thus (A1) holds by Lemma 6.1, and the proof of (vi) and of the corollary is complete. □

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