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REDUCIBLE AND ∂-REDUCIBLE HANDLE ADDITIONS

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Abstract. Let M be a simple 3-manifold with F a component of ∂M of genus at least two. For a slope α on F, we denote by $M(\alpha)$ the manifold obtained by attaching a 2-handle to M along a regular neighborhood of α on F. Suppose that α and β are two separating slopes on F such that $M(\alpha)$ and $M(\beta)$ are reducible. Then the distance between α and β is at most 2. As a corollary, if $g(F) = 2$, then there is at most one separating slope γ on F such that $M(\gamma)$ is either reducible or ∂-reducible.

1. Introduction

Let M be a compact 3-manifold. For a component F of ∂M, a slope γ on F is an isotopy class of essential simple closed curves on F. The distance between two slopes α and β on F, denoted by $\Delta(\alpha, \beta)$, is the minimal geometric intersection number among all the curves representing the slopes. For a slope γ on F, we denote by $M(\gamma)$ the manifold obtained by attaching a 2-handle to M along a regular neighborhood of γ on F, then capping off a possible 2-sphere component of the resulting manifold by a 3-ball. Note that if F is a torus, then $M(\gamma)$ is the Dehn filling along γ.

A compact, orientable 3-manifold M is said to be simple if it is irreducible, ∂-irreducible, anannular and atoroidal. By Thurston's theorem, a Haken 3-manifold M is hyperbolic if and only if M is simple. Two interesting problems on handle additions are the following:

Question 1. Suppose that M is a hyperbolic 3-manifold with F a component of ∂M. How many slopes γ are there on F such that $M(\gamma)$ is not hyperbolic?

Question 2. Suppose that M is a hyperbolic 3-manifold with F a component of ∂M and that M contains no essential closed surface of genus g. How many slopes γ are there on F such that $M(\gamma)$ contains an essential closed surface of genus g?

Let F be a torus. A. Hatcher has shown that there are only finitely many slopes γ such that $M(\gamma)$ contains an essential closed surface of genus g. See [4]. An idea for solving Question 1 is to estimate the upper bound of $\Delta(\alpha, \beta)$ when $M(\alpha)$ and $M(\beta)$ are non-hyperbolic. Now almost all the sharp upper bounds are given when $M(\alpha)$ and $M(\beta)$ are in distinct non-hyperbolic cases. The methods used are the labeled graph method developed by Gordon and Luecke and the representations of fundamental groups of 3-manifolds developed by Culler and Shalen. See [2].

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Now suppose that $g(F) \geq 2$. Now if $M(\gamma)$ is non-hyperbolic, then γ is called a degenerating slope. Scharlemann and Wu have shown that there are only finitely many basic degenerating slopes on F. In this case, the condition "basic degenerating" is necessary. As a corollary, if $g = 0$ or 1, then there are at most finitely many separating slopes γ on F such that $M(\gamma)$ contains an essential closed surface of genus g. See [10]. Qiu and Wang proved that there is a simple, small 3-manifold such that, for any integer $g \geq 2$, there are infinitely many separating slopes γ on F such that $M(\gamma)$ contains an essential closed surface of genus g. See [8] and [9].

An open problem on handle additions is Conjecture 2 in [10]. A basic case of the conjecture is the following:

Conjecture 1. *Let M be a simple 3-manifold, and let F be a component of ∂M of genus at least two. Suppose that α and β are two separating slopes on F such that each of $M(\alpha)$ and $M(\beta)$ is either reducible or ∂-reducible. Then $\Delta(\alpha, \beta) = 0$.*

In this paper, we shall study reducible and ∂-reducible handle additions. The main results are the following:

Theorem 1. *Suppose that M is a simple 3-manifold and that F is a component of ∂M of genus at least 2. If α and β are two separating slopes on F such that $M(\alpha)$ and $M(\beta)$ are reducible, then $\Delta(\alpha, \beta) \leq 2$.*

Theorem 2. *Suppose that M is a simple 3-manifold and that F is a component of ∂M of genus 2. Then there is at most one separating slope γ on F such that $M(\gamma)$ is either reducible or ∂-reducible.*

Proof of Theorem 2 under Theorem 1. Since $g(F) = 2$, so if $\alpha \neq \beta$ are two separating slopes on F, then $\Delta(\alpha, \beta) \geq 4$. By Theorem 1 in [6], there is at most one separating slope γ on F such that $M(\gamma)$ is ∂-reducible. Hence Theorem 2 follows immediately from Theorem 4.2 in [10] and Theorem 1. □

Theorem 2 means that Conjecture 1 is true when $g(F) = 2$.

The method for proving Theorem 1 is an extension of the labeled graph to handle additions. For details, we shall extend the technologies in [3], [5], [7] and [12] to study reducible handle additions.

2. Preliminaries

Suppose M is a simple manifold with F a component of ∂M of genus at least two, and α and β are two separating slopes on F. To prove Theorem 1, we assume that both $M(\alpha)$ and $M(\beta)$ are reducible. By Theorem 4.2 in [10], if one of $M(\alpha)$ and $M(\beta)$ is ∂-reducible, then $\Delta(\alpha, \beta) = 0$ and hence Theorem 1 holds. So in the following argument we always suppose that both $M(\alpha)$ and $M(\beta)$ are ∂-irreducible. We denote by H_α (resp. H_β) the 2-handle attached to M to obtain $M(\alpha)$ (resp. $M(\beta)$). Let $\hat{P}$ (resp. $\hat{Q}$) be an essential 2-sphere in $M(\alpha)$ (resp. $M(\beta)$) such that $|\hat{P} \cap H_\alpha|$ (resp. $|\hat{Q} \cap H_\beta|$) is minimal among all the essential 2-spheres in $M(\alpha)$ (resp. $M(\beta)$). Let $P = \hat{P} \cap M$ and $Q = \hat{Q} \cap M$. By Theorem 1 in [11], $\Delta(\alpha, \beta) \leq 4$. So we may assume that $\Delta(\alpha, \beta) = 4$ to obtain a contradiction.

Lemma 2.1. *P (resp. Q) is an incompressible and ∂-incompressible planar surface in M with all boundary components having the same slope α (resp. β).*

Proof. By assumption, $M(\alpha)$ and $M(\beta)$ are ∂-irreducible. Hence this lemma is immediate from the proof of Lemma 2.1 in [11]. □

We may assume that $|P \cap Q|$ is minimal. Then each component of $P \cap Q$ is either an essential arc or an essential simple closed curve on both P and Q. Let Γ_P be the graph in the 2-sphere $\hat{P}$ obtained by taking the arc components of $P \cap Q$ as edges and taking the boundary components of P as fat vertices. Similarly, we can define Γ_Q in the sphere $\hat{Q}$.

In this paper, the definitions of a cycle, the length of a cycle, a disk face and parallel edges are standard; see [3] and [10].

Lemma 2.2. (1) *There are no* 1*-sided disk faces in both* Γ_P *and* Γ_Q*.*
(2) Γ_P *contains no* $2q$ *parallel edges.*

Proof. The proofs follow from Lemma 2.1 and Lemma 5.2 in [11]. □

Number the components of ∂P as $\partial_1 P, \partial_2 P, \cdots, \partial_u P, \cdots, \partial_p P$ consecutively on F. This means that $\partial_u P$ and $\partial_{u+1} P$ bound an annulus in F with interior disjoint from P. Similarly, number the components of ∂Q as $\partial_1 Q, \partial_2 Q, \cdots, \partial_i Q, \cdots, \partial_q Q$. These give the corresponding labels of the vertices of Γ_P and Γ_Q.

For an endpoint x of an arc component of $P \cap Q$, if it belongs to $\partial_u P \cap \partial_i Q$, then we label it as (u,i) or i (resp. u) in Γ_P (resp. Γ_Q) for short when u (resp. i) is specified. In this case, i is called the Type A label of x in Γ_P. Furthermore, we give a sign $g(x)$ on x in [11], where $g(x) =$ "+" or "−", such that the signed labels $+1$, $+2, \cdots$, $+q, -q, \cdots, -1$ appear in the same direction around all the vertices of Γ_P. The signed label $g(x)i$ is called the Type B label of x in Γ_P. For more details about Type B labels, see [11].

Assumption 2.3. Without loss of generality, we assume that the labels $+1$, $+2, \cdots$, $+q$, $-q, \cdots, -1$ appear in the clockwise direction on each vertex of Γ_P.

Now each edge of Γ_P has a label pair of its two endpoints. For example, let e be an edge of Γ_P with its two endpoints x and y labeled with (u, i) and (v, j). Then (i, j) is called the Type A label pair of e, and $(g(x)i,\ g(y)j)$ is called the Type B label pair of e.

Then we have a weak parity rule as follows:

Lemma 2.4. *Each edge of* Γ_P *has different Type B labels at its two endpoints.*

Proof. See Lemma 3.3 in [11]. □

Lemma 2.5. *Suppose* $S = \{e_i \mid i = 1, 2, \cdots, n\}$ *is a set of parallel edges in* Γ_P*. If one of the edges, say* e_k*, has opposite Type B (or has the same Type A) labels at its two endpoints, then each edge in* S *has opposite Type B labels at its two endpoints.*

Proof. The lemma follows immediately from Assumption 2.3. □

Let x be a Type B label in $\{+1, +2, \cdots, +q,\ -q, \cdots, -2, -1\}$. An x-edge is an edge in Γ_P with label x at one of its two endpoints. We denote by B_P^x the subgraph of Γ_P consisting of all the vertices of Γ_P and all the x-edges.

A cycle of B_P^x which bounds a disk face in Γ_P is called a virtual Scharlemann cycle. By Assumption 2.3, each edge of a virtual Scharlemann cycle has the same label pair for either Type A or Type B. In this case, the label pair of the edges is said to be the label pair of the virtual Scharlemann cycle.

A virtual Scharlemann cycle with Type A label pair (i, j) is called a Scharlemann cycle if $i \neq j$.

A Scharlemann cycle with Type A label pair $(t, t+1)$ is said to be good if $2 \leq t \leq q-2$.

A cycle C of B_P^x is called an extended Scharlemann cycle if C immediately surrounds a good Scharlemann cycle C', that is, each edge of C is immediately parallel to an edge of C'.

For virtual Scharlemann cycles, we have the following lemmas:

Lemma 2.6. (1) *Γ_P contains no two Scharlemann cycles with distinct Type A label pairs.*

(2) *Γ_P contains no extended Scharlemann cycle.*

Proof. For (1), see the proof of Theorem 2.4 of [3]. For (2), see the proof of Lemma 2.3 in [12]. □

Lemma 2.7. *If Γ_P contains a Scharlemann cycle, then Q is separating, and $\hat{Q}$ bounds a punctured lens space in $M(\alpha)$.*

Proof. See the proof of Lemma 2.1 in [3]. □

Lemma 2.8. *Let $x \in \{+1, +2, \cdots, +q, -q, \cdots, -1\}$, and let D be a disk face of B_P^x. Then there is a virtual Scharlemann cycle lying in D.*

Proof. Relabel $-q, -(q-1), \cdots, -1$ as $+(q+1), +(q+2), \cdots, +(2q)$ respectively. Then, by Assumption 2.3, the labels $1, 2, 3, \cdots, q, (q+1), (q+2), \cdots, 2q$ appear in the clockwise direction on each vertex of Γ_p. (Repeat $\Delta(\alpha, \beta)/2$ times.) Hence we get Type C labels. In this case, by Lemma 2.4, each edge in Γ_P has different labels at its two endpoints. So the weak parity rule defined in [5] holds. By Proposition 5.1 in [5], an x-face contains a Scharlemann cycle with Type C labels. It is easy to see that it is a virtual Scharlemann cycle under Type B labels. □

Lemma 2.9. *Suppose $S = \{e_i \mid i = 1, 2, \cdots, n\}$ is a set of parallel edges of Γ_P. If $n > q$, then there is a virtual Scharlemann cycle in S.*

Proof. Suppose e_i is labeled with Type B pair (x_i, y_i) for each $1 \leq i \leq n$. If $n > q$, then $x_i = y_j = x$ for some $1 \leq i \leq n$ and some $1 \leq j \leq n$. This means that e_i and e_j bound an x-face. By Lemma 2.8, there is a virtual Scharlemann cycle in S. □

Lemma 2.10. (1) *B_P^x contains at least $p+2$ disk faces for each $x \in \{+1, +2, \cdots, +q, -q, \cdots, -1\}$.*

(2) *B_P^x has at least one 2-sided or 3-sided disk face.*

Proof. Since $\Delta(\alpha, \beta) = 4$, each vertex of B_P^x has valency at least 2. We denote by V, E and F the numbers of the vertices, edges and disk faces of B_P^x. Then $V = p$. Since $\Delta(\alpha, \beta) = 4$, by Lemma 2.4, $E = 2p$. By the Euler formula, $V - E + F \geq 2$. Hence $F \geq E - V + 2 = p + 2$. Thus (1) holds.

For (2), suppose that B_P^x contains no 2-sided or 3-sided disk faces. Then $2E \geq 4F$. Hence $V - E + F \leq p - 2p + p = 0 < 2$, a contradiction. □

By Lemma 2.10 (1) and Lemma 2.8, we have:

Corollary 2.11. *Γ_P contains at least $p+2$ virtual Scharlemann cycles.*

Assumption 2.12. Let D be a 2-sided disk face of B_P^x.

Suppose $\partial D = e_1 \cup e_n$. Then there is a set $S = \{e_k \mid k = 1, 2, \cdots, n\}$ of parallel edges of Γ_P in D. We assume that e_k is labeled with Type B label pair (x_k, y_k) for each $1 \leq k \leq n$. See Figure 1.

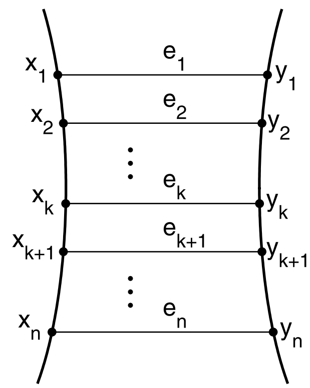


FIGURE 1

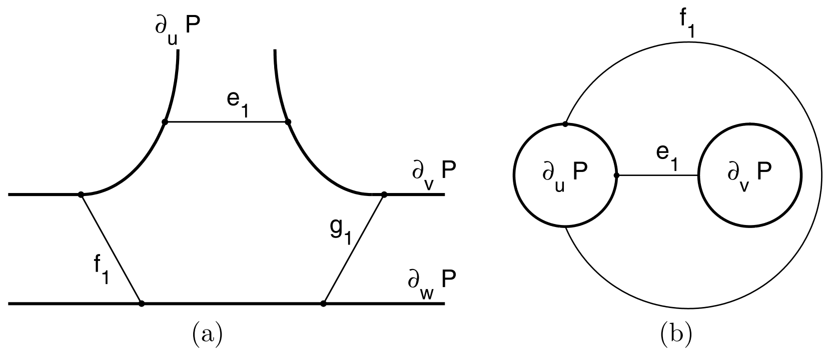


FIGURE 2

Assumption 2.13. Let D be a 3-sided disk face of B_P^x.

Now D is as in one of Figure 2(a) and (b). Since each vertex in B_P^x has valence at least 2, D is not as in Figure 2(b).

Now D is as in Figure 2(a). We denote by e_1, f_1 and g_1 the three edges of ∂D. We assume that $S_1 = \{e_k \mid k = 1, 2, \cdots, l\}$ is the set of the edges of Γ_P parallel to e_1 in D, $S_2 = \{f_k \mid k = 1, 2, \cdots, m\}$ is the set of the edges of Γ_P parallel to f_1 in D, and $S_3 = \{g_k \mid k = 1, 2, \cdots, n\}$ is the set of the edges of Γ_P parallel to g_1 in D. See Figure 3. Furthermore we assume the labels of the edges in S_1, S_2 and S_3 are as in Figure 3.

Since M is anannular, so $p, q > 2$.

The proof of Theorem 1 will be divided into three parts:

(1) Γ_P contains no Scharlemann cycle.

(2) Γ_P contains a Scharlemann cycle with Type A label pair $(1, 2)$ or $(q-1, q)$.

(3) Γ_P contains a good Scharlemann cycle.

3. Γ_P CONTAINS NO SCHARLEMANN CYCLE

In this section, we assume that Γ_P contains no Scharlemann cycle.

Lemma 3.1. *Γ_P contains an edge labeled with Type B label pair $(+i, -i)$ for each $1 \le i \le q$.*

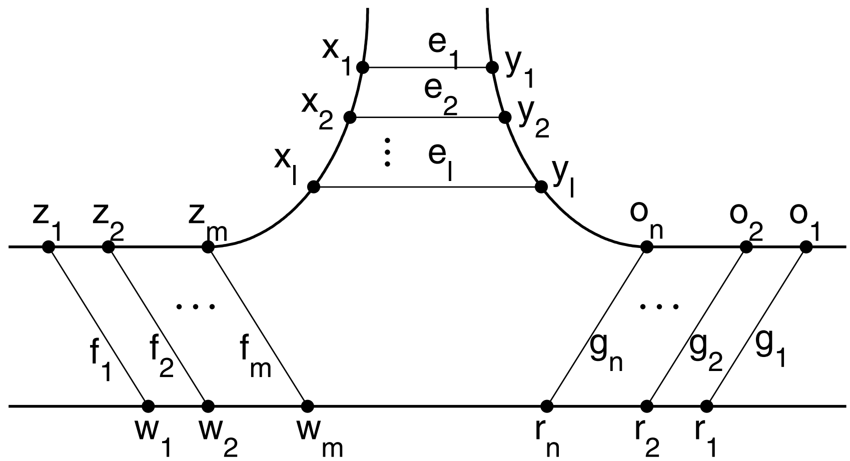


Figure 3

Proof. By Lemma 2.10 (2), B_P^{+i} contains a 2-sided or 3-sided disk face D. By Lemma 2.8, D contains a virtual Scharlemann cycle C. By the assumption of this section, C is labeled with Type B pair $(+1, -1)$ or $(+q, -q)$.

There are two cases:

Case 1. D is a 2-sided disk face of B_P^{+i}.

By Assumption 2.12, there is a family of parallel edges S in D, and the labels of the edges are as in Figure 1. Suppose e_k is an edge in C. Then the Type B label pair of e_k is either $(+1, -1)$ or $(+q, -q)$. By Lemma 2.5, each edge in S has opposite Type B labels at its two endpoints. Hence e_1 is labeled with $(+i, -i)$.

Case 2. D is a 3-sided disk face of B_P^{+i}.

By Assumption 2.13, there are three families of parallel edges S_1, S_2 and S_3 in D, and the labels of the edges are as in Figure 3.

Without loss of generality, we may assume $e_k \in C$ for some $1 \leq k \leq l$. Now the Type B label pair of e_k is either $(+1, -1)$ or $(+q, -q)$. By Lemma 2.5, each edge in S_1 has opposite Type B labels at its two endpoints. Hence e_1 is labeled with $(+i, -i)$. □

Lemma 3.2. *Γ_P contains no edges with Type B label pair $(+i, -i)$ for each $1 \leq i \leq q$ simultaneously.*

Proof. Suppose that for each $1 \leq i \leq q$, there is an edge e_i with Type B label pair $(+i, -i)$. This means that, for each vertex $\partial_i Q$ of Γ_Q, there is an edge with its two endpoints incident to $\partial_i Q$. Hence Γ_Q contains a 1-sided disk face, contradicting Lemma 2.2 (1). □

Proposition 3.3. *Theorem 1 is true for the case: Γ_P contains no Scharlemann cycle.*

Proof. This follows immediately from Lemma 3.1 and Lemma 3.2. □

4. Γ_P contains a Scharlemann cycle with Type A label pair (1,2) or (q,$q-1$)

In this section, we shall assume that Γ_P contains a Scharlemann cycle with Type A label pair $(1,2)$.

Lemma 4.1. *There is an edge in Γ_P labeled with Type B label pair $(+q,-q)$.*

Proof. Suppose that there is no edge in Γ_P with Type B label pair $(+q,-q)$. Then there is no virtual Scharlemann cycle with Type B label pair $(+q,-q)$.

Let D be a disk face of B_P^{+q}. By the assumption of this section and Lemma 2.6 (1), the virtual Scharlemann cycle in D has Type B label pair as one of $(+1,-1)$, $(+1,+2)$ and $(-1,-2)$.

Case 1. D contains a virtual Scharlemann cycle C with label pair $(+1,-1)$.

Since C contains at least two edges, there are at least two edges with label pair $(+1,-1)$. Since one endpoint of each edge in ∂D is labeled with $+q$, each edge in C does not lie in ∂D.

Case 2. D contains a virtual Scharlemann cycle C with label pair $(+1,+2)$.

Now there are at least two edges with label pair $(+1,+2)$. Since $q>2$, each edge in C does not lie in ∂D.

Case 3. D contains a virtual Scharlemann cycle C with label pair $(-1,-2)$.

Now each edge in C is not in ∂D, and there are at least two edges with label pair $(-1,-2)$, say b_1 and b_2. Adjacent to b_1 and b_2, there are two edges e_1 and e_2 with label pair $(+1,*)$. See Figure 4. If $e_k \subset \partial D$ for $k=1$ or 2, then the label pair of e_k is $(+1,+q)$. Since $q \geq 3$, there is at most one Scharlemann cycle with label pair $(-1,-2)$ adjacent to e_k if $e_k \subset \partial D$.

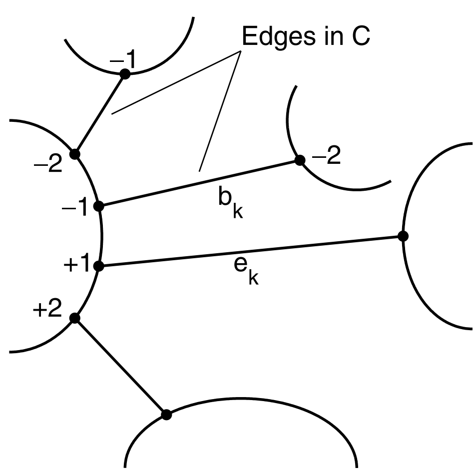


FIGURE 4

By Lemma 2.10 (1), B_P^{+q} contains at least $p+2$ disk faces. By the argument in Cases 1, 2 and 3, in each disk face, there are at least two edges with label pair $(+1,*)$. Furthermore, each such edge is counted only once. Hence there are at least $2p+4$ edges with Type B label pair $(+1,*)$ in Γ_P. Since $\Delta(\alpha,\beta)=4$, Γ_P contains exactly $2p$ edges with Type B label pair $(+1,*)$, a contradiction. $\square$

Suppose s is the smallest number such that Γ_P contains an edge e_k with Type B label pair $(+k, -k)$ for each $s < k \leq q$. By Lemma 3.2 and Lemma 4.1, $1 \leq s \leq q-1$. By the definition of s, Γ_P contains no edge with label pair $(+s, -s)$. Now e_k has both its two endpoints incident to $\partial_k Q$ on Γ_Q for each $k > s$.

Lemma 4.2. *For each $k > s$, $\partial_1 Q, \partial_2 Q, \cdots, \partial_s Q$ lie in the same component of $Q - e_k$.*

Proof. If $s = 1$, then this lemma is obviously true. Now we assume that $s \geq 2$.

By Lemma 2.10 (2), B_P^{+s} contains a 2-sided or 3-sided disk face D.

By Lemma 2.8, there is a virtual Scharlemann cycle C in D. By the proof of Lemma 3.1, the Type A label pair of C is neither $(1, 1)$ nor (q, q); otherwise, Γ_P contains an edge with Type B label pair $(s, -s)$. By the assumption of this section and Lemma 2.6 (1), the Type A label pair of C is $(1, 2)$. Hence $\partial_1 Q$ and $\partial_2 Q$ are connected by the edges in C. That means that if $s = 2$, then this lemma holds.

From now on, we assume that $s \geq 3$.

Case 1. D is a 2-sided disk face.

By Assumption 2.12, there is a family of parallel edges S in D, and the labels of the edges are as in Figure 1. Now the virtual Scharlemann cycle C is a length two cycle. Without loss of generality, we may assume that $e_k \subset C$. By Assumption 2.3, there are four subcases for the labels of x_k and y_k.

Case 1.1. $x_k = +1$, $y_k = +2$. See Figure 5(a).

Now we have:

$x_{k-1} = -1$, $y_{k-1} = +3$;

$x_{k-2} = -2$, $y_{k-2} = +4$;

$\vdots$

$x_{k-(s-2)} = -(s-2)$, $y_{k-(s-2)} = +s$.

Hence the edge incident to $y_{k-(s-2)}$ is an edge of B_P^{+s}. This means that e_1 is the edge incident to $y_{k-(s-2)}$, and $y_1 = y_{k-(s-2)}$. Hence $k = s-1$, and the labels of $e_1, e_2, \cdots, e_k$ are as in Figure 5(a). It is easy to see that $\partial_1 Q, \partial_2 Q, \cdots, \partial_s Q$ on Γ_Q are connected by the edges $e_1, e_2, \cdots, e_k$.

Case 1.2. $x_k = +2$, $y_k = +1$. See Figure 5(b).

Since $s \geq 3$, $k > 1$. Hence $x_{k-1} = +1$, and $y_{k-1} = +2$. By taking the place of e_k with e_{k-1} in the proof of Case 1.1, this lemma is true.

Case 1.3. $x_k = -2$, $y_k = -1$. See Figure 6(a).

$x_{k-1} = -3$, $y_{k-1} = +1$;

$x_{k-2} = -4$, $y_{k-2} = +2$;

$\vdots$

$x_{k-(s-2)} = -s$, $y_{k-(s-2)} = +(s-2)$;

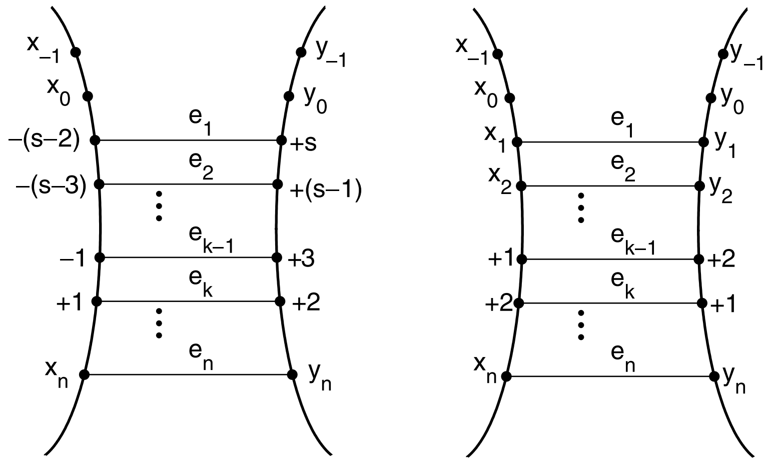
$x_{k-(s-1)} = *$, $y_{k-(s-1)} = +(s-1)$;

$x_{k-s} = *$, $y_{k-s} = +s$.

Hence the edges $e_k, e_{k-1}, \cdots, e_{k-(s-2)} \in S$. The labels of these edges are as in Figure 6(a). It is easy to see that $\partial_1 Q, \partial_2 Q, \cdots, \partial_s Q$ on Γ_Q are connected by the edges $e_k, e_{k-1}, \cdots, e_{k-(s-2)} \in S$.

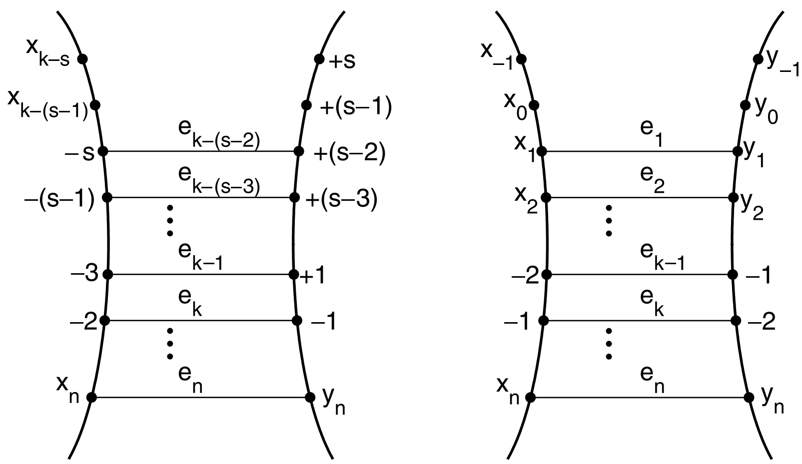
Case 1.4. $x_k = -1$, $y_k = -2$. See Figure 6(b).

Since $s \geq 3$, $k > 1$. Hence $x_{k-1} = -2$, and $y_{k-1} = -1$. By taking the place of e_k with e_{k-1} in the proof of Case 1.3, this lemma is true.



(a) (b)

FIGURE 5



(a) (b)

FIGURE 6

Case 2. D is a 3-sided disk face.

By Assumption 2.13, there are three families of parallel edges S_1, S_2 and S_3 in D, and the labels of the edges are as in Figure 3. By the assumption of this lemma, there is a virtual Scharlemann cycle C in D. By the proof of Lemma 3.1, the Type B label pair of C is neither $(+1,-1)$ nor $(+q,-q)$; otherwise, there is an edge with Type B label pair $(+s,-s)$, contradicting the definition of s. Hence the virtual Scharlemann cycle is labeled with Type A pair $(1,2)$. Without loss of generality, we may assume that $e_{k'} \in C$ for some $1 \leq k' \leq l$. By taking the place of S with S_1 in the argument of Case 1, this lemma holds. □

Proposition 4.3. *Theorem 1 is true for the case: Γ_P contains a Scharlemann cycle with Type A label pair $(1,2)$ or $(q,q-1)$.*

Proof. We firstly suppose that Γ_P contains a Scharlemann cycle with Type A label pair $(1,2)$. By the definition of s, for each $s < k \leq q$, there is an edge e_k with Type B label pair $(+k, -k)$. Hence e_k is a length one cycle incident to $\partial_k Q$ on Γ_Q. By Lemma 4.2, $\partial_1 Q, \partial_2 Q, \cdots, \partial_s Q$ lie in the same component of $Q - e_k$ for each $s < k \leq q$. Hence the innermost one of $S = \{e_k \mid s < k \leq q\}$ is a trivial loop in Γ_Q, contradicting Lemma 2.2 (1).

The proof of the condition that Γ_P contains a Scharlemann cycle with Type A label pair $(q, q-1)$ follows from the symmetry of the labels of the vertices of Γ_P. □

5. Γ_P contains a good Scharlemann cycle

In this section, we assume that Γ_P contains a good Scharlemann cycle with Type A label pair $(t, t+1)$, where $2 \leq t \leq q-2$. Hence $q \geq 4$. By Lemma 2.6 (1), the Type A label pair of each virtual Scharlemann cycle in Γ_P is one of $(1,1)$, (q,q) and $(t, t+1)$.

Lemma 5.1. *Let D be a 2-sided disk face of B_P^{+j}, where $j \notin \{t, t+1\}$. Then D contains no Scharlemann cycle with Type A pair $(t, t+1)$.*

Proof. By Assumption 2.12, there is a family of parallel edges S in D, and the labels of the edges are as in Figure 1. Suppose that there is a good Scharlemann cycle $C = e_k \cup e_{k+1}$ in D. Then there are two possibilities for the labels of e_k and e_{k+1} as in Figure 7 (a) and 7 (b).

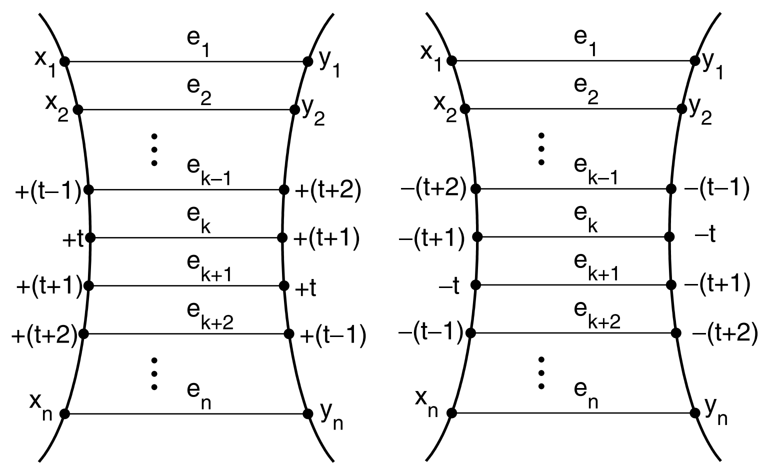


Figure 7

Case 1. $x_k = +t$, $y_k = +(t+1)$; $x_{k+1} = +(t+1)$, $y_{k+1} = +t$ as in Figure 7 (a). It is easy to see:
$x_{k-1} = +(t-1)$, $y_{k-1} = +(t+2)$; and
$x_{k+2} = +(t+2)$, $y_{k+2} = +(t-1)$.
Since $j \notin \{t, t+1\}$, the edges e_{k-1} and e_{k+2} belong to S. Hence e_{k-1} and e_{k+2} form an extended Scharlemann cycle, contradicting Lemma 2.6 (2).

Case 2. $x_k = -(t+1)$, $y_k = -t$; $x_{k+1} = -t$, $y_{k+1} = -(t+1)$ as in Figure 7 (b).

Then:
$x_{k-1} = -(t+2)$, $y_{k-1} = -(t-1)$; and
$x_{k+2} = -(t-1)$, $y_{k+2} = -(t+2)$.
Hence e_{k-1} and e_{k+2} form an extended Scharlemann cycle, contradicting Lemma 2.6 (2). □

Lemma 5.2. *Let D be a 3-sided disk face of Γ_P^{+j}. If $j \notin \{t, t+1\}$, then D contains no length 3 Scharlemann cycle with Type A pair $(t, t+1)$.*

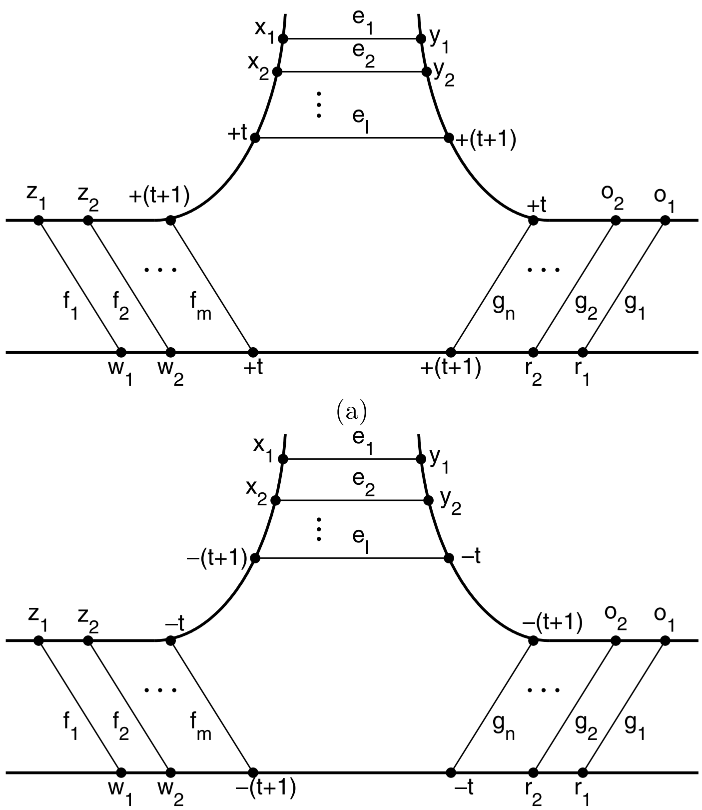


FIGURE 8

Proof. By Assumption 2.13, there are three families of parallel edges S_1, S_2 and S_3 in D, and the labels of the edges are as in Figure 3. Suppose that C is a length 3 Scharlemann cycle with Type A label pair $(t, t+1)$ in D. Then $C = e_l \cup f_m \cup g_n$. Hence the labels of e_l, f_m and g_n are as in one of Figures 8(a) and 8(b). By the same argument as in the proof of Lemma 5.1, e_{l-1}, f_{m-1} and g_{n-1} form an extended Scharlemann cycle, contradicting Lemma 2.6 (2). □

Lemma 5.3. *Let D be a 3-sided disk face of Γ_P^{+j}. If D contains a length 2 Scharlemann cycle C with Type B label pair $(+t, +(t+1))$, where $j \notin \{t, t+1\}$, then there is an edge with Type B label pair $(+j, -j)$.*

Proof. By Assumption 2.13, there are three families of parallel edges S_1, S_2 and S_3 in D, and the labels of the edges are as in Figure 3.

Without loss of generality, we may assume that $j < t$ and $C \subset S_1$. Then $C = e_k \cup e_{k+1}$. Since $j < t$, e_k does not lie in ∂D. Hence $k > 1$. If $l > k+1$, then e_{k-1} and e_{k+2} form an extended Scharlemann cycle, contradicting Lemma 2.6 (2). Hence $k + 1 = l$.

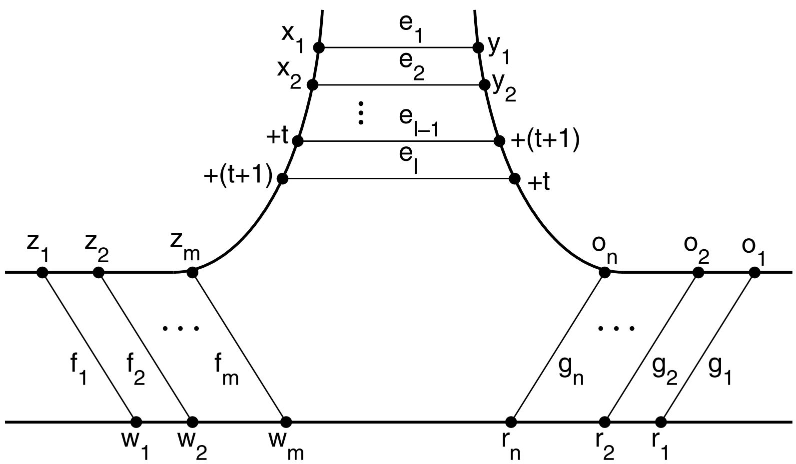


FIGURE 9

In this case, $x_l = +(t+1)$, $y_l = +t$, $x_{l-1} = +t$ and $y_{l-1} = +(t+1)$ as in Figure 9. It is easy to see that:

$x_{l-2} = +(t-1)$,

$\vdots$

$x_{l-(t-j+1)} = +j$.

Since $t - j + 1 \leq t < q$, each of $y_{l-2}, y_{l-3}, \cdots, y_{l-(t-j+1)}$ is either positive label greater than $+t$, or negative. Since $j < t$, $x_1 = +j$ and $e_1 = e_{l-(t-j+1)}$, hence $l = t - j + 2$.

Claim 1. If S_2 or S_3 contains a virtual Scharlemann cycle, then Lemma 5.3 holds.

Proof. Suppose that C' is a virtual Scharlemann cycle formed by the edges f_i and f_{i+1} in S_2.

We first assume that C' is labeled with Type A label pair $(t, t+1)$. Since f_1 is labeled with Type A label pair $(j, *)$ and $j < t$, $f_i \neq f_1$. Since $z_m = +(t+2)$, $f_{i+1} \neq f_m$. Hence $1 < i < i+2 \leq m$. This means that f_{i-1} and f_{i+2} form an extended Scharlemann cycle, contradicting Lemma 2.6 (2).

Assume now that C' is labeled with type A label pair $(1, 1)$ or (q, q). By Lemma 2.5, each edge in S_2 has opposite Type B labels at its two endpoints. Hence f_1 is labeled with $(+j, -j)$, and Lemma 5.3 holds.

By the same argument as above, if S_3 contains a virtual Scharlemann cycle, then Lemma 5.3 also holds. □

Claim 2. If $z_1 = +j$, then Lemma 5.3 holds.

Proof. By the above argument, $x_1 = +j$ and $l = t - j + 2$. Since $z_1 = +j$, $l + m = 2q + 1$, and $l = t - j + 2 \leq t + 1 \leq q$. Hence $m > q$. By Lemma 2.9, there is a virtual Scharlemann cycle in S_2. By Claim 1, Lemma 5.3 holds. □

Now by Claim 2, we assume that $w_1 = +j$.

Claim 3. If $r_1 = +j$, then Lemma 5.3 holds.

Proof. Since $r_1 = +j$ and $w_1 = +j$, $m + n = 2q + 1$. By the proof of Claim 2, Lemma 5.3 holds. □

By Claims 1–3, we may assume that $x_1 = +j$, $w_1 = +j$ and $o_1 = +j$. Since $o_n = +(t-1)$, $n = t - j$. Hence the labels of the edges in $S_1 \cup S_2 \cup S_3$ are as in Figure 10.

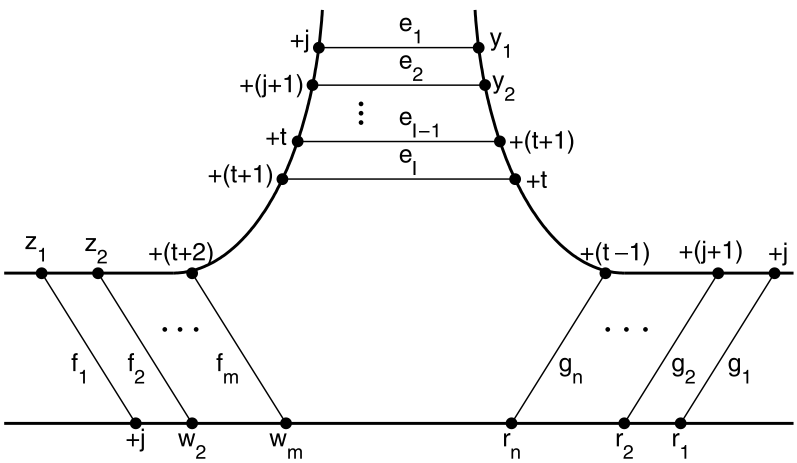


FIGURE 10

Claim 4. $m \notin \{n, n+1, n+2\}$.

Proof. Let A_1 be the annulus bounded by $\partial_{t-1}Q$ and $\partial_t Q$, A_2 be the annulus bounded by $\partial_{t+1}Q$ and $\partial_{t+2}Q$, and A_3 be the annulus bounded by $\partial_t Q$ and $\partial_{t+1}Q$ on ∂M. By Lemma 2.7, $\hat{Q}$ separates $M(\beta)$ into two parts, M_1 and M_2, one of which, say M_1, is a punctured lens space. It is easy to see that $A_1, A_2 \subset M_2$ and $A_3 \subset M_1$.

Since $w_1 = o_1 = +j$, by Assumption 2.3, we have $w_i = o_i$ for $1 \leq i \leq \min\{n, m\}$.

Case 1. $m = n$.

In this case, the labels of e_{l-2}, e_{l-1}, e_l, f_m and g_n are as in Figure 11.

Let D_1 be the disk face of Γ_P bounded by e_{l-2} and e_{l-1}(with subarcs of ∂P), D_2 be the disk face of Γ_P bounded by e_l, f_m and g_n (with subarcs of ∂P). See Figure 11.

We denote by Q' the surface obtained by doing a surgery on $Q \cup A_1 \cup A_2$ along D_1 and D_2. Then Q' is also a planar surface in M with all boundary components parallel to β. Denote by $\hat{Q}'$ the surface obtained by capping off all the components of $\partial Q'$ in $M(\beta)$.

Since $\hat{Q}$ is separating, $\hat{Q}'$ is also separating in $M(\beta)$. Since $A_1, A_2 \subset M_2$, so does $\hat{Q}' \subset M_2$. Suppose $M(\beta) = N_1 \cup_{\hat{Q}'} N_2$. Then $N_1 \supset M_1$ and $N_2 \subset M_2$. Since M_1 is

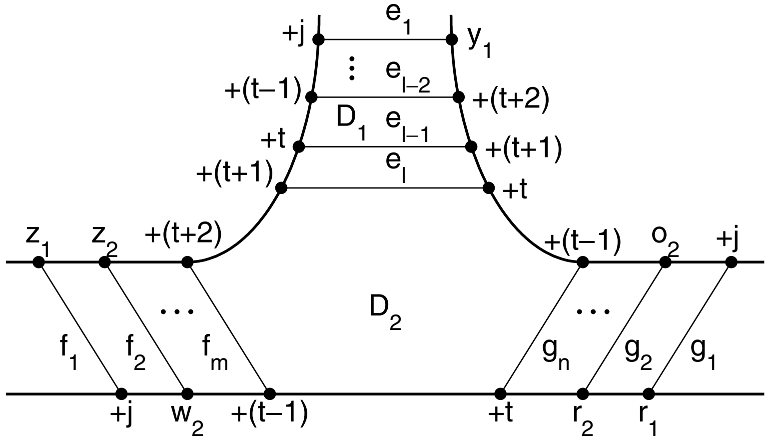


FIGURE 11

a once-punctured lens space, N_1 is not a 3-ball. Since $\partial M(\beta) \subset N_2$, N_2 is also not a 3-ball. Hence $\hat{Q}'$ is also a reducible 2-sphere in $M(\beta)$, and $|\partial Q'| < |\partial Q|$, which contradicts the minimality of $|\partial Q|$.

Case 2. $m = n + 1$.

In this case, the labels of e_{l-1}, e_l, f_m and g_n are as in Figure 12. Let D be the disk bounded by e_l, f_m and g_n and some arcs of ∂P, where $b_1 \subset A_1$, $b_2 \subset A_2$ and $b_3 \subset A_3$. This means that $\hat{Q}$ is non-separating, contradicting Lemma 2.7.

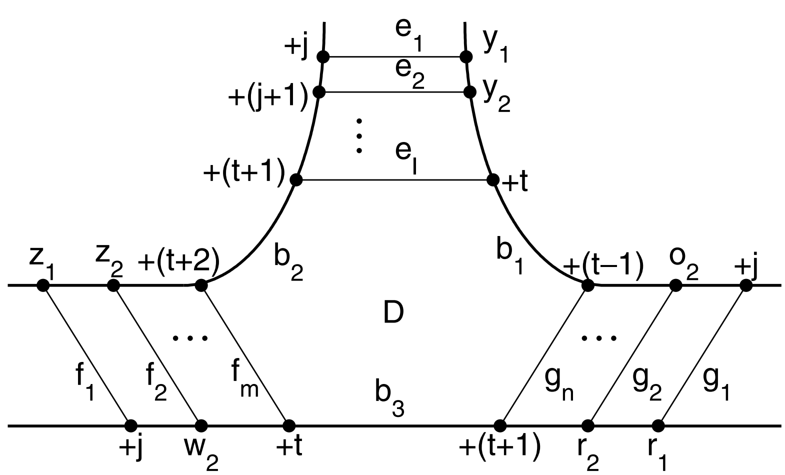


FIGURE 12

Case 3. $m = n + 2$.

In this case, the labels of e_{l-2}, e_{l-1}, e_l, f_m and g_n are shown in Figure 13. The argument is the same as the one of Case 1. □

By Claim 4, either $m < n$ or $m > n + 2$. Note that $z_m = +(t + 2)$ and $o_n = +(t - 1)$.

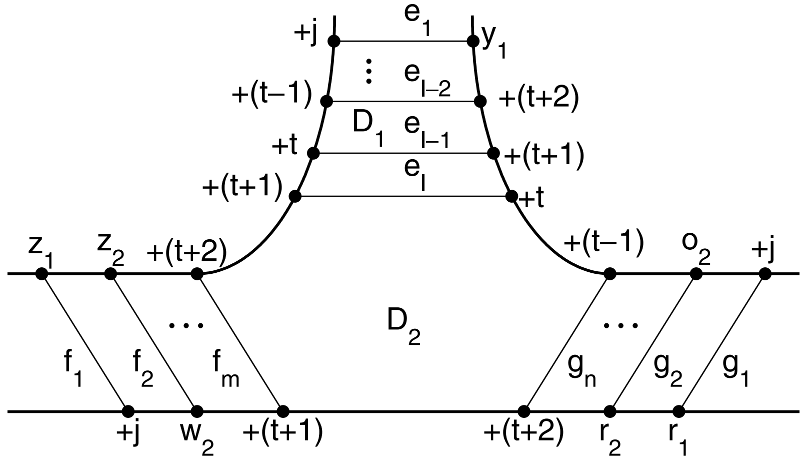


FIGURE 13

We first suppose $m < n$. See Figure 10. Since $w_i = o_i$ for $1 \leq i \leq m$ and $o_n = +(t-1)$, we have $+(t-1) \in \{r_i | \ i = 1, 2, \cdots, n\}$. Suppose $r_k = +(t-1)$. By Lemma 2.4, $k \neq n$. Hence g_n and g_k bound an x-face, where $x = +(t-1)$. By Lemma 2.8, there is a virtual Scharlemann cycle in S_3. By Claim 1, Lemma 5.3 holds.

Now suppose $m > n+2$. Since $w_i = o_i$ for $1 \leq i \leq n$, $w_{n+3} = +(t+2)$. Hence there is a virtual Scharlemann cycle in S_2. By Claim 1, Lemma 5.3 holds. □

Lemma 5.4. *Let D be a 3-sided disk face of Γ_P^{+j}. If D contains a length 2 Scharlemann cycle C with Type B label pair $(-t, -(t+1))$, where $j \notin \{t, t+1\}$, then there is an edge with Type B label pair $(+j, -j)$.*

Proof. By Assumption 2.13, there are three families of parallel edges S_1, S_2 and S_3 in D, and the labels of the edges are as in Figure 3.

Without loss of generality, we assume that $C \subset S_1$. By the proof of Lemma 5.3, $C = e_l \cup e_{l-1}$; otherwise, S_1 contains an extended Scharlemann cycle. See Figure 14. Hence $o_n = -(t+2)$ and $z_m = -(t-1)$.

Claim 5. If S_2 or S_3 contains a virtual Scharlemann cycle, then Lemma 5.4 holds.

Proof. See the proof of Claim 1 in Lemma 5.3. □

Claim 6. $w_m \notin \{-(t+2), -(t+1), -t\}$.

Proof. Suppose, otherwise, that $w_m \in \{-(t+2), -(t+1), -t\}$. Then the labels of e_{l-2}, e_{l-1}, e_l, f_m and g_n are as in one of Figures 15(a), (b) and (c). By the proof of Claim 4 in Lemma 5.3, this is impossible. □

Claim 7. If $w_1 = r_1 = +j$, then Lemma 5.4 holds.

Proof. Since $w_1 = r_1 = +j$, $m + n = 2q + 1$. Hence one of m and n, say $m > q$. By Lemma 2.9, there is a virtual Scharlemann cycle in S_2. By Claim 5, this claim holds. □

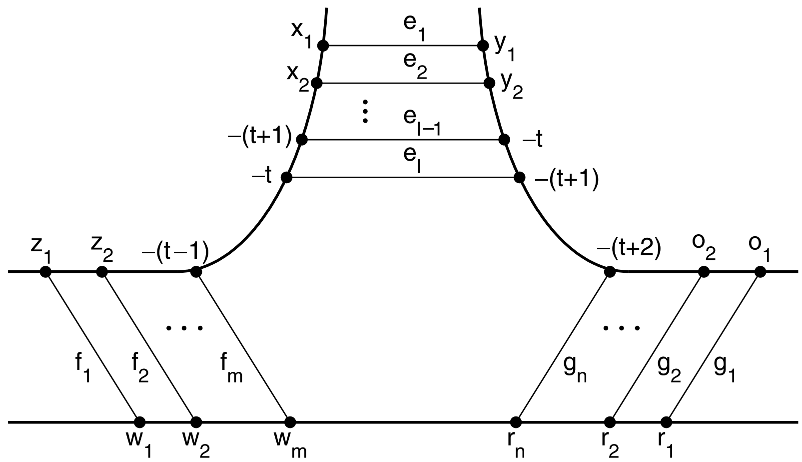


FIGURE 14

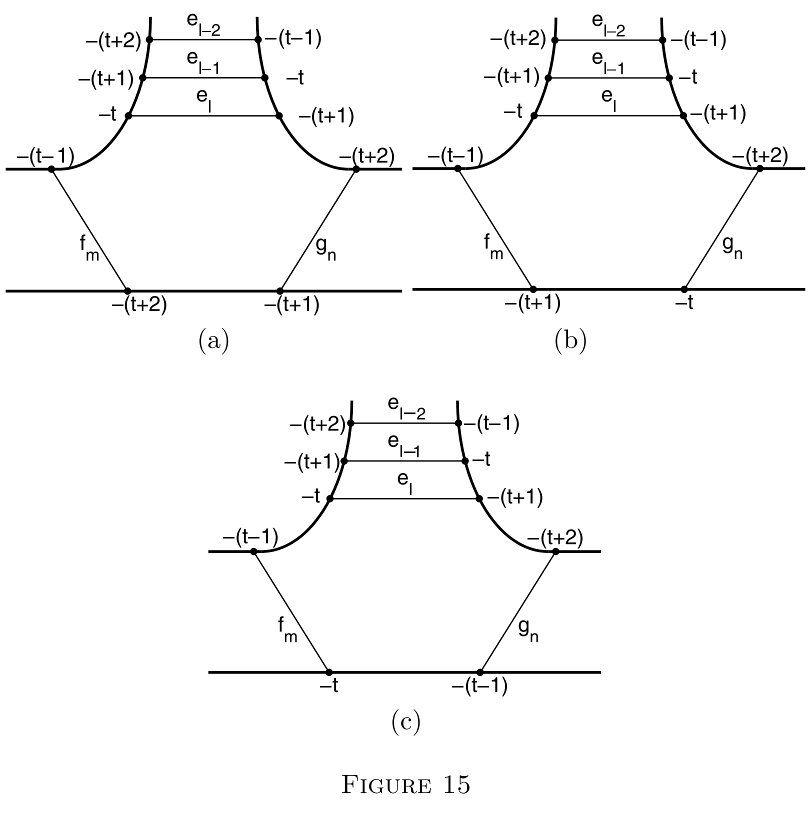


FIGURE 15

By Claim 7, either $z_1 = +j$ or $o_1 = +j$. Without loss of generality, we assume that $o_1 = +j$. Then $o_n = -(t+2), o_{n-1} = -(t+3), \cdots$, and $o_1 = +j$. See Figure 14. Since $n < 2q$, $\{o_n, o_{n-1}, \cdots, o_1\} = \{-(t+2), -(t+3), \cdots, -q, +q, +(q-1), \cdots, +j\}$. Hence $n = 2q - t - j$. Note that either $z_1 = +j$ or $w_1 = +j$.

Claim 8. If $z_1 = +j$, then Lemma 5.4 holds.

Proof. Now $z_m = -(t-1), z_{m-1} = -(t-2), \ldots, z_1 = +j$. Since $m < 2q$, $m = t + j - 1$. Hence $m + n = (t + j - 1) + (2q - t - j) = 2q - 1$. □

Since $z_1 = +j$ and $o_1 = +j$, $+j \notin \{w_1, w_2, \cdots, w_m, r_1, r_2, \cdots, r_n\}$. Hence $\{w_1, w_2, \cdots, w_m, r_1, r_2, \cdots, r_n\} = \{-1, -2, \cdots, -q, +1, +2, \cdots, +(j-1), +(j+1), \cdots, +q\}$.

If $r_k = -(t+2)$ for some $1 \leq k < n$, by Lemma 2.4, $k \neq n$. Hence g_n and g_k bound an x-face in Γ_P, where $x = -(t+2)$. See Figure 14. By Lemma 2.8, there is a virtual Scharlemann cycle in S_3. By Claim 5, Lemma 5.4 holds. Hence we may assume that $-(t+2) \in \{w_i \mid i = 1, 2, \cdots, m\}$.

Similarly, since $z_m = -(t-1)$, we may assume that $-(t-1) \in \{r_i \mid i = 1, 2, \cdots, n\}$.

Now $-(t+2) \leq w_m < -(t-1)$, contradicting Claim 6. □

Claim 9. If $w_1 = +j$, then Lemma 5.4 holds.

Proof. Since $o_1 = +j$ and $w_1 = +j$, by Assumption 2.3, $w_i = o_i$ for $1 \leq i \leq \min\{n, m\}$.

We first suppose $m < n$. Since $m < n$ and $o_n = -(t+2)$, $-(t+2) \in \{r_i | \ i = 1, 2, \cdots, n\}$. Hence there is a virtual Scharlemann cycle in S_3. By Claim 5, Lemma 5.4 holds.

Now suppose $m > n + 2$. Since $w_n = o_n = -(t+2)$, $w_{n+3} = -(t-1)$. Hence there is a virtual Scharlemann cycle in S_2. By Claim 5, Lemma 5.4 holds.

Suppose that $m \in \{n, n+1, n+2\}$. Then w_m is one of $-(t+2)$, $-(t+1)$ and $-t$. This contradicts Claim 6. □

By Claim 8 and Claim 9, Lemma 5.4 holds. □

Lemma 5.5. *If Γ_P has a good Scharlemann cycle labeled with type A label pair $(t, t+1)$, then Γ_P contains an edge labeled with type B label pair $(+j, -j)$ for each $j \notin \{t, t+1\}$.*

Proof. For each $j \notin \{t, t+1\}$, by Lemma 2.10 (2), B_P^{+j} contains a 2-sided or 3-sided disk face D.

If D is a 2-sided disk face, then, by Lemma 2.8 and Lemma 5.1, D contains a virtual Scharlemann cycle with Type A pair $(1, 1)$ or (q, q). By the proof of Lemma 3.1, there is an edge with type B label pair $(+j, -j)$.

If D is a 3-sided disk face, then, by Lemma 5.2, Lemma 5.3 and Lemma 5.4, we may assume that D contains a virtual Scharlemann cycle with type pair $(1, 1)$ or (q, q). By the proof of Lemma 3.1, there is an edge with type B label pair $(+j, -j)$. □

Proposition 5.6. *Theorem 1 is true for the case: Γ_P contains a good Scharlemann.*

Proof. Suppose that Γ_P contains a good Scharlemann cycle C with Type A label pair $\{t, t+1\}$. By Lemma 5.5, Γ_P contains an edge e^j with Type B label pair $\{+j, -j\}$ for each $j \notin \{t, t+1\}$. Hence e^j is a length one cycle in Γ_Q incident to the vertex $\partial_j Q$. Note that the edges in C connect $\partial_t Q$ to $\partial_{t+1} Q$ on Γ_Q. Hence e^j is a trivial loop on Γ_Q for some $j \notin \{t, t+1\}$, a contradiction. □

The proof of Theorem 1. Theorem 1 follows immediately from Proposition 3.3, Proposition 4.3 and Proposition 5.6. □

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