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ON ESTIMATES FOR THE RATIO OF ERRORS IN BEST RATIONAL APPROXIMATION OF ANALYTIC FUNCTIONS

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Abstract. Let E be an arbitrary compact subset of the extended complex plane $\overline{\mathbb{C}}$ with nonempty interior. For a function f continuous on E and analytic in the interior of E denote by $\rho_n(f;E)$ the least uniform deviation of f on E from the class of all rational functions of order at most n. In this paper we show that if f is not a rational function and if K is an arbitrary compact subset of the interior of E, then $\prod_{k=0}^{n}(\rho_k(f;K)/\rho_k(f;E))$, the ratio of the errors in best rational approximation, converges to zero geometrically as $n \to \infty$ and the rate of convergence is determined by the capacity of the condenser $(\partial E, K)$. In addition, we obtain results regarding meromorphic approximation and sharp estimates of the Hadamard type determinants.

1. Introduction

Let E be a compact subset of the extended complex plane $\overline{\mathbb{C}}$ and denote by $C(E)$ the space of continuous functions on E with the supremum norm

$$\|f\|_E = \sup_{z\in E} |f(z)|.$$

By $A(E)$ we mean the algebra of functions in $C(E)$ which are analytic on the interior of E. Also, for $f \in A(E)$ and each nonnegative integer n, let $\rho_n(f;E)$ denote the error in best rational approximation of f in the supremum norm on E by rational functions of order at most n; that is,

$$\rho_n(f;E) = \inf_{r\in\mathcal{R}_n} ||f-r||_E,$$

where $\mathcal{R}_n = \{r : r = p/q,\ \deg p \le n,\ \deg q \le n,\ q \not\equiv 0\}$ is the class of all rational functions of order at most n.

From now on we will always assume that E has a nonempty interior. In this paper, the main object of study is the ratio of errors in the best rational approximation of f on E and an arbitrary compact subset of its interior. More precisely, we investigate the asymptotic behaviors of the ratio $\rho_n(f;K)/\rho_n(f;E)$ and the product $\prod_{k=0}^{n}(\rho_k(f;K)/\rho_k(f;E))$ as $n \to \infty$, where K denotes a compact subset of the E's interior. We make two trivial observations regarding the ratio of the

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errors. First of all one has to exclude rational functions since in this case the error $\rho_n(f;E)$ would vanish for all but finitely many n. Secondly, since $K \subset E$, it follows directly from the definition that $\rho_n(f;K)/\rho_n(f;E) \leq 1$ for all $n \geq 0$. Our main result is Theorem A. Also note that ∂E stands for the boundary of the set E and by $C(F,K)$ we mean the *capacity* of the *condenser* (F,K) for a pair of disjoint compact subsets of $\overline{\mathbb{C}}$ (see, for example, [8] and [16] for more details and the exact definition).

Theorem A. *Let E be a compact subset of $\overline{\mathbb{C}}$ with nonempty interior and suppose that K is a compact subset of the interior of E. If $f \in A(E)$ and f is not a rational function, then*

$$\limsup_{n\to\infty}\left(\prod_{k=0}^{n}(\rho_k(f;K)/\rho_k(f;E))\right)^{1/n^2} \leq \exp(-1/C(\partial E,K)).$$

In [14], the second author proves the above inequality in the case where the complements of E and K are both connected. Therefore, Theorem A can be considered as the generalization of the result in [14] with no additional assumptions on the compact sets E and K. One immediate consequence of Theorem A is the following estimate for the lower limit of $(\rho_n(f;K)/\rho_n(f;E))^{1/n}$ as $n \to \infty$.

Corollary 1. *Under the assumptions of Theorem A, we have*

$$\liminf_{n\to\infty}\left(\frac{\rho_n(f;K)}{\rho_n(f;E)}\right)^{1/n} \leq \exp(-2/C(\partial E,K)).$$

As another application of Theorem A, we state the following result regarding the degree of rational approximation of analytic functions.

Corollary 2. *Suppose E and F are disjoint compact subsets of $\overline{\mathbb{C}}$. If f is analytic on $\overline{\mathbb{C}} \setminus F$, then*

$$\limsup_{n\to\infty}\left(\prod_{k=0}^{n}\rho_k(f;E)\right)^{1/n^2} \leq \exp(-1/C(E,F)); \tag{1.1}$$

$$\limsup_{n\to\infty}\rho_n(f;E)^{1/n} \leq \exp(-1/C(E,F)); \tag{1.2}$$

$$\liminf_{n\to\infty}\rho_n(f;E)^{1/n} \leq \exp(-2/C(E,F)). \tag{1.3}$$

We remark that (1.2) and (1.3) follow directly from (1.1). Inequality (1.2) is the well–known theorem of Walsh (see [19] and [2]). Estimate (1.3) is known as Gonchar's conjecture [7]. Parfenov [9] gives a proof of (1.1) and (1.3) for the case where E is a continuum with connected complement. In [12], the second author proves (1.1) and (1.3) for an arbitrary compact set E.

This paper is organized as follows. In Section 2 we present the needed notation and some facts about the theory of Hankel operators which includes the AAK theorem and its generalization. Section 3 contains Theorem 5 related to the estimates of the Hadamard type determinants. The second author (see [13]) has proved the corresponding result for domains bounded by finitely many closed analytic Jordan curves. Finally, in Section 4 we give the proof of Theorem A.

2. Notation and related topics from the theory of Hankel operators

We fix the following notation which will be used throughout this paper. Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces. For a compact operator $A : \mathcal{H} \to \mathcal{K}$, denote by $\{s_n(A)\}_{n\geq 0}$ the sequence of *singular numbers* (counted with multiplicities) of the operator A; that is, $\{s_n(A)\}_{n\geq 0}$ is the sequence of eigenvalues of the operator $(A^*A)^{1/2}$, where $A^* : \mathcal{K} \to \mathcal{H}$ denotes the adjoint of A. Furthermore, we shall always assume that the sequence $\{s_n(A)\}_{n\geq 0}$ is nonincreasing. Also, one can think of $s_n(A)$ as the minimum distance of A, in the operator norm, from the class of operators of rank at most n. More precisely,

$$s_n(A) = \inf ||A - L||, \tag{2.1}$$

where the infimum is taken over the class of all operators $L : \mathcal{H} \to \mathcal{K}$ of rank at most n, and $||\cdot||$ is the usual operator norm. In fact, the infimum in (2.1) is always achieved for some finite rank operator; that is, there exists an operator $M : \mathcal{H} \to \mathcal{K}$ of rank at most n for which $s_n(A) = ||A - M||$ (see [5] for more details and facts about the singular numbers).

Let Γ be the union of a finite number of rectifiable Jordan curves. Denote by $L_2(\Gamma)$ the Hilbert space of square integrable functions φ with respect to the Lebesgue measure on Γ, where the usual norm and inner product are given by

$$||\varphi||_2 = \left(\int_\Gamma |\varphi(\xi)|^2 \, |d\xi| \right)^{1/2}$$

and

$$\langle \varphi, \psi \rangle_{L_2(\Gamma)} = \int_\Gamma \varphi(\xi)\overline{\psi(\xi)} \, d\xi, \quad \varphi, \psi \in L_2(\Gamma).$$

We also will be concerned with $L_\infty(\Gamma)$, the space of essentially bounded functions φ on Γ with the norm

$$||\varphi||_\infty = \operatorname*{ess\,sup}_\Gamma |\varphi(\xi)|.$$

Next suppose that G is a bounded domain of the complex plane $\mathbb{C}$ such that G's boundary Γ consists of a finite number of closed analytic Jordan curves. Fix $1 \leq p < \infty$. An analytic function φ on G belongs to the *Smirnov class* $E_p(G)$ if there is a sequence of domains $G_1, G_2, \ldots$ with rectifiable boundaries $\partial G_1, \partial G_2, \ldots$ such that $G_k \subset G_{k+1}$, $\overline{G}_k \subset G$, $\bigcup_k G_k = G$, and

$$\sup_k \int_{\partial G_k} |\varphi(\xi)|^p \, |d\xi| < \infty.$$

It should be mentioned that for such domains G, the Smirnov class $E_p(G)$ coincides with the *Hardy space* $H_p(G)$ (see [3], [10], or [18] for more details). The Smirnov class $E_\infty(G)$ is always the same as $H_\infty(G)$ (the class of bounded analytic functions on G). Moreover, it follows that each function (or equivalent class functions) in $E_p(G)$, $1 \leq p \leq \infty$, can be identified with its boundary function in the sense of nontangential limit (see [3] and [10]); and, $E_p(G)$ can be considered as a closed subspace of $L_p(\Gamma)$. We will use this fact throughout without further notice.

For a domain G with the boundary Γ (described as above) and $f \in C(\Gamma)$, define the *Hankel operator* $A_{f,G}$ with symbol f by

$$A_{f,G} : E_2(G) \to E_2^\perp(G) = L_2(\Gamma) \ominus E_2(G)$$

and

$$A_{f,G}(\varphi) = \mathbf{P}_-(\varphi f) \quad \text{for all} \quad \varphi \in E_2(G),$$

where $\mathbf{P}_-$ is the orthogonal projection from $L_2(\Gamma)$ onto $E_2^\perp(G)$. From now on, whenever G is understood, we shall denote $A_{f,G}$ simply by A_f. It is not hard to see that A_f is a compact operator (see, for example, [11]).

Finally, let $\mathcal{M}_n(G) = \{h : h = p/q,\ p \in E_\infty(G),\ \deg q \le n,\ q \not\equiv 0\}$ be a class of meromorphic functions on G with at most n poles (counted with multiplicities), and denote by $\Delta_n(f;G)$ the least deviation of f in $L_\infty(\Gamma)$ from the class $\mathcal{M}_n(G)$; that is,

$$\Delta_n(f;G) = \inf_{h\in\mathcal{M}_n(G)} ||f-h||_\infty. \tag{2.2}$$

The AAK theorem (see [1]) asserts that for the unit disk $\mathbb{D}$ and $f \in C(\partial\mathbb{D})$, $s_n(A_f) = \Delta_n(f;\mathbb{D})$ for all $n \ge 0$. One of our tools is the following generalization of the AAK theorem obtained by the second author (see [11]).

If G is a bounded domain whose boundary Γ consists of N closed analytic Jordan curves and if $f \in C(\Gamma)$, then

$$s_n(A_f) \le \Delta_n(f;G), \quad n = 0,1,2,\ldots, \tag{2.3}$$

and

$$\Delta_{n+N-1}(f;G) \le s_n(A_f), \quad \textit{for} \quad n \ge N-1. \tag{2.4}$$

3. Meromorphic approximation and Hankel operators

Before proving the main results of this section, Theorems 5 and 6, we need some auxiliary results from the theory of Hankel operators. For the sake of simplicity and further references we define the following notation which will be used throughout this paper.

Definition. An open subset of the complex plane G is called an *m–domain* ($1 \le m < \infty$) if G is the union of m bounded domains $G_1,\ldots,G_m$ with disjoint closures such that the boundary of each G_i, denoted by Γ_i, consists of finitely many closed analytic Jordan curves. Furthermore, we let $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ denote the boundary of G.

For an m–domain G, denote by $E_2(G)$ the direct sum of the Smirnov classes $E_2(G_i)$, $1 \le i \le m$; that is,

$$E_2(G) = E_2(G_1) \oplus \cdots \oplus E_2(G_m).$$

Let $f \in C(\Gamma)$. We define the operator $A_f = A_{f,G} : E_2(G) \to E_2^\perp(G_1)\oplus\cdots\oplus E_2^\perp(G_m)$ as the direct sum of the Hankel operators $A_{f_i} : E_2(G_i) \to E_2^\perp(G_i)$, $1 \le i \le m$:

$$A_f = A_{f_1} \oplus \cdots \oplus A_{f_m}, \tag{3.1}$$

where $f_i = f|_{\Gamma_i}$ is the restriction of f to Γ_i. Since each A_{f_i} is compact, it follows that A_f is a compact operator. We also mention the following facts regarding A_f :

$$||A_f|| = \max(||A_{f_1}||,\ldots,||A_{f_m}||) \tag{3.2}$$

and

$$A_f^* A_f = A_{f_1}^* A_{f_1} \oplus \cdots \oplus A_{f_m}^* A_{f_m}. \tag{3.3}$$

Equality (3.3) shows that if s is a singular number of the operator A_f, then s must be a singular number for at least one of the operators A_{f_i}. Actually more can be said. The sequence $\{s_n(A_f)\}_{n\geq 0}$ of the singular numbers of A_f can be put into a one–to–one correspondence with the rearrangement (counting multiplicities) of the sequences $\{s_n(A_{f_1})\}_{n\geq 0}, \ldots, \{s_n(A_{f_m})\}_{n\geq 0}$ in a nonincreasing order. The next lemma gives the precise statement of this fact.

Lemma 3. *Suppose $G = \bigcup_{i=1}^m G_i$ is an m–domain. If f is continuous on $\Gamma = \bigcup_{i=1}^m \Gamma_i$, then the following statements hold.*

(a) *For each $n \geq 0$*

$$s_n(A_f) = \min_{k_1+\cdots+k_m\leq n} \max\{s_{k_1}(A_{f_1}), \ldots, s_{k_m}(A_{f_m})\}. \tag{3.4}$$

(b) *There is a one–to–one correspondence between the sequence of singular numbers $\{s_n(A_f)\}_{n\geq 0}$ and the sequence $\{\mu_n\}_{n\geq 0}$ obtained by rearranging (counting multiplicities) the sequences $\{s_n(A_{f_1})\}_{n\geq 0}, \ldots, \{s_n(A_{f_m})\}_{n\geq 0}$ in a nonincreasing order.*

Proof. We first observe that formula (2.1) together with equality (3.2) imply

$$s_n(A_f) = \inf\ \max(||A_{f_1} - L_1||, \ldots, ||A_{f_m} - L_m||), \tag{3.5}$$

where the infimum is taken over all operators $L_i : E_2(G_i) \to E_2^\perp(G_i)$, $i = 1, \ldots, m$, of rank at most k_i such that $k_1 + \cdots + k_m \leq n$.

Since, by (2.1), $||A_{f_i} - L_i|| \geq s_{k_i}(A_{f_i})$ holds for any operator $L_i : E_2(G_i) \to E_2^\perp(G_i)$ of rank at most k_i, where $1 \leq i \leq m$, it is easily seen that

$$s_n(A_f) \geq \min_{k_1+\cdots+k_m\leq n} \max(s_{k_1}(A_{f_1}), \ldots, s_{k_m}(A_{f_m})).$$

To prove the reverse inequality, fix $k_1 \geq 0, \ldots, k_m \geq 0$ such that $k_1+\cdots+k_m \leq n$. Let $M_i : E_2(G_i) \to E_2^\perp(G_i)$, $i = 1, \ldots, m$, denote operators of rank at most k_i for which $s_{k_i}(A_{f_i}) = ||A_{f_i} - M_i||$ (see the citation after the formula (2.1)). According to (3.5), we can write

$$s_n(A_f) \leq \max(||A_{f_1} - M_1||, \ldots, ||A_{f_m} - M_m||) = \max(s_{k_1}(A_{f_1}), \ldots, s_{k_m}(A_{f_m})).$$

Since the above inequality is valid for all $k_1 \geq 0, \ldots, k_m \geq 0$ such that $k_1+\cdots+k_m \leq n$, we have

$$s_n(A_f) \leq \min_{k_1+\cdots+k_m\leq n} \max(s_{k_1}(A_{f_1}), \ldots, s_{k_m}(A_{f_m})).$$

This proves part (a).

To prove part (b), we only need to show that if $s = s_n(A_f)$ is a singular number of A_f with multiplicity α, then s is repeated exactly α times in the sequence $\{\mu_n\}_{n\geq 0}$. Let $I \subseteq \{1, \ldots, m\}$ be the set of all indices i such that s is the singular number of the operators A_{f_i}. As mentioned earlier, (3.3) implies that s is a singular number for at least one of the operators A_{f_i}; therefore, $I \neq \emptyset$. Next, for each $i \in I$, we let α_i denote the multiplicity of the singular number s for the corresponding operator A_{f_i}. Thus we have to show $\alpha = \sum_{i\in I} \alpha_i$.

For each fixed $i \in I$, one can find an orthogonal system in $E_2(G_i)$ of the eigenfunctions $Q_{i1}, \ldots, Q_{i\alpha_i}$ of the operator $(A_{f_i}^* A_{f_i})^{1/2}$ corresponding to the eigenvalue s. Let

$$\mathcal{L} = \{Q = (0, \ldots, Q_{ik}, \ldots, 0) \in E_2(G) : i \in I \text{ and } 1 \leq k \leq \alpha_i\}.$$

The set $\mathcal{L}$ consists of $\sum_{i\in I}\alpha_i$ orthogonal functions in $E_2(G)$, each of which, by (3.3), is an eigenfunction of the operator $(A_f^*A_f)^{1/2}$ corresponding to the eigenvalue s; therefore, $\alpha \geq \sum_{i\in I}\alpha_i$.

Next suppose $\alpha > \sum_{i\in I}\alpha_i$. Consequently, there must exist an eigenfunction $R = (R_1,\dots,R_m) \in E_2(G), R \neq 0$, of $(A_f^*A_f)^{1/2}$ corresponding to s that is orthogonal to each function Q in $\mathcal{L}$. But this would imply that

$$0 = \langle R, Q\rangle_{L_2(\Gamma)} = \langle R_i, Q_{ik}\rangle_{L_2(\Gamma_i)}, \tag{3.6}$$

for all $i \in I$ and $1 \leq k \leq \alpha_i$. Formula (3.3) implies that each nonzero R_i, $1 \leq i \leq m$, is an eigenfunction of the operator $(A_{f_i}^*A_{f_i})^{1/2}$ corresponding to the eigenvalue s. From this and formula (3.6) it follows that $R_i = 0$ for each $i \in I$. Now since $R \neq 0$, we can conclude that s is a singular value for some operator A_{f_i} with $i \notin I$. But, this contradicts the definition of I. Thus $\alpha = \sum_{i\in I}\alpha_i$ and we are done. □

Remark. We remark that since for each $1 \leq i \leq m$ the sequence $\{s_n(A_{f_i})\}_{n\geq 0}$ is nonincreasing, (3.4) directly implies

$$s_n(A_f) = \min_{k_1+\cdots+k_m=n} \max\left\{s_{k_1}(A_{f_1}),\dots,s_{k_m}(A_{f_m})\right\}. \tag{3.7}$$

In order to state our next lemma, we need the following definition that extends the notion of error in meromorphic approximation (2.2) to an m–domain.

Let $G = \bigcup_{i=1}^m G_i$ be an m–domain with the boundary $\Gamma = \bigcup_{i=1}^m \Gamma_i$, and suppose $f \in C(\Gamma)$. For $n \geq 0$ define

$$\Delta_n(f;G) = \min_{k_1+\cdots+k_m\leq n} \max\{\Delta_{k_1}(f_1;G_1),\dots,\Delta_{k_m}(f_m;G_m)\}, \tag{3.8}$$

where $\Delta_{k_i}(f_i;G_i)$, $1 \leq i \leq m$, are defined as in definition (2.2) and $f_i = f|_{\Gamma_i}$ denotes the restriction of f to $\Gamma_i = \partial G_i$. The following result is a direct consequence of Lemma 3, inequalities (2.3) and (2.4), and definition (3.8). However, for the sake of completeness, we also include a proof.

Lemma 4. *Let $G = \bigcup_{i=1}^m G_i$ be an m–domain and suppose that f is continuous on the boundary $\Gamma = \bigcup_{i=1}^m \Gamma_i$.*

(a) *For all $n = 0, 1, 2, \dots$*

$$s_n(A_f) \leq \Delta_n(f;G).$$

(b) *Suppose each G_i, $1 \leq i \leq m$, consists of N_i closed analytic Jordan curves, and put $N = N_1 + \cdots + N_m$. Then there is a positive integer n^* such that*

$$\Delta_{n+N-m}(f;G) \leq s_n(A_f) \quad \textit{for all} \quad n \geq n^*.$$

Proof. Noting that part (a) follows trivially from formula (3.4), inequality (2.3) and definition (3.8), we only give a proof of part (b).

For each fixed $n \geq 0$, by (3.7), there are nonnegative integers $k_1,\dots,k_m$ such that $k_1 + \cdots + k_m = n$ and

$$s_n(A_f) = \max(s_{k_1}(A_{f_1}),\dots,s_{k_m}(A_{f_m})). \tag{3.9}$$

Hence, for some $1 \leq i \leq m$, we have

$$s_n(A_f) = s_{k_i}(A_{f_i}) \geq s_{k_j}(A_{f_j}) \quad \text{for all} \quad j = 1,\dots,m. \tag{3.10}$$

In light of Lemma 3, part (b), we can choose $n^* \geq 1$ sufficiently large such that whenever $n \geq n^*$ and $s_n(A_f) = s_k(A_{f_i})$ for some k and $1 \leq i \leq m$, we have $k \geq N_i - 1$. Now if we let $n \geq n^*$, then from (3.10) and Lemma 3, part (b), it follows that $k_j \geq N_j - 1$ for all $1 \leq j \leq m$. Since, by inequality (2.4), $s_{k_j}(A_{f_j}) \geq \Delta_{k_j+N_j-1}(f_j; G_j)$, where $j = 1, \ldots, m$, we can deduce with the help of (3.9) that

$$\begin{aligned} s_n(A_f) &\geq \max(\Delta_{k_1+N_1-1}(f_1; G_1), \ldots, \Delta_{k_m+N_m-1}(f_m; G_m)) \\ &\geq \min_{l_1+\cdots+l_m=n+N-m} \max(\Delta_{l_1}(f_1; G_1), \ldots, \Delta_{l_m}(f_m; G_m)) \\ &\geq \Delta_{n+N-m}(f; G). \end{aligned}$$

This completes the proof of part (b). $\square$

Let $\varphi, \psi \in E_2(G)$ and $f \in C(\Gamma)$. Here and in what follows, we use the notation

$$\int_\Gamma (\varphi\psi f)(\xi) d\xi = \sum_{k=1}^{m} \int_{\Gamma_k} (\varphi_k \psi_k f_k)(\xi) d\xi,$$

where $\varphi_k = \varphi_{|G_k}$, $\psi_k = \psi_{|G_k}$, and $f_k = f_{|\Gamma_k}$, $k = 1, \ldots, m$.

Now we can state the first result of this section. In fact, Theorem 5 is an extension of the second author's result (see [13]) proved for the case $m = 1$.

Theorem 5. *Suppose G is an m–domain with its boundary denoted by Γ. If f is continuous on Γ and $\varphi_0, \ldots, \varphi_n, \psi_0, \ldots, \psi_n$ belong to $E_2(G)$, then the following estimate for the absolute value of the Hadamard type determinant of order $n+1$ holds:*

$$\left| \left| \int_\Gamma (\varphi_i \psi_j f)(\xi) d\xi \right|_{i,j=0}^{n} \right| \leq \prod_{k=0}^{n} s_k(A_f) \left(|\langle \varphi_i, \varphi_j \rangle_{L_2(\Gamma)}|_{i,j=0}^{n} \right)^{1/2} \left(|\langle \psi_i, \psi_j \rangle_{L_2(\Gamma)}|_{i,j=0}^{n} \right)^{1/2}$$

(*with Gram determinants of order $n+1$ on the right–hand side*).

Proof. It should be mentioned that if one follows Weyl's original proof using antisymmetric tensor products (see e.g. B. Simon [17], pp. 6–7), then one gets the desired inequality (see also [15]). However, the construction developed in our proof (see below) is needed and is referred to in Theorem 6. Therefore, for the sake of completeness and the mentioned fact, we also include a proof.

It may be assumed that Γ is positively oriented with respect to G. For each $1 \leq i \leq m$, it is known (see [12] for exact details) that there are orthonormal systems $\{q_{ik}\}_{k\geq 0}$ and $\{\alpha_{ik}\}_{k\geq 0}$ of the eigenfunctions of the operator $(A_{f_i}^* A_{f_i})^{1/2}$ in $E_2(G_i)$, corresponding to the singular numbers $\{s_k(A_{f_i})\}_{k\geq 0}$, such that

$$(A_{f_i} q_{ik})(\xi) = s_k(A_{f_i}) \overline{\alpha_{ik}(\xi)} |d\xi|/d\xi \quad \text{a.e. on } \Gamma_i. \tag{3.11}$$

Let $n \geq 0$. By Lemma 3, part (b), there is a pair (i, k) such that $s_n(A_f) = s_k(A_{f_i})$. Define $q_n = (0, \ldots, q_{ik}, \ldots, 0)$ and $\alpha_n = (0, \ldots, \alpha_{ik}, \ldots, 0)$; i.e. q_n and α_n have only one nonzero entry in their i–th positions, namely q_{ik} and α_{ik}, and zero elsewhere. By (3.3), $\{q_n\}_{n\geq 0}$ and $\{\alpha_n\}_{n\geq 0}$ are orthonormal systems in $E_2(G)$ of eigenfunctions of the operator $(A_f^* A_f)^{1/2}$ corresponding to the sequence of the singular numbers

$\{s_n(A_f)\}_{n\geq 0}$. In view of (3.11) and the definitions of q_n and α_n, we get

$$(A_f q_n)(\xi) = s_n(A_f)\overline{\alpha_n(\xi)}|d\xi|/d\xi \quad \text{a.e. on } \Gamma. \tag{3.12}$$

We can represent (see, for example, [5]) φ_i and ψ_j $(i, j = 0, 1, \ldots, n)$ as

$$\varphi_i = \sum_{k=0}^{\infty} c_{ik} q_k + \eta_i \quad \text{and} \quad \psi_j = \sum_{k=0}^{\infty} b_{jk}\alpha_k + \omega_j,$$

where $c_{ik} = \langle \varphi_i, q_k\rangle_{L_2(\Gamma)}$, $b_{jk} = \langle \psi_j, \alpha_k\rangle_{L_2(\Gamma)}$, $k = 0, 1, \ldots$. Moreover, we have $\eta_i, \omega_j \in \operatorname{Ker}(A_f)$, $\langle \varphi_i - \eta_i, \eta_i\rangle_{L_2(\Gamma)} = 0$, and $\langle \psi_j - \omega_j, \omega_j\rangle_{L_2(\Gamma)} = 0$ for $i, j = 0, 1, \ldots$.

Now if we let φ_{ik}, ψ_{jk} denote the restrictions of φ_i, ψ_j to G_k, then

$$\int_\Gamma (\varphi_i\psi_j f)(\xi)d\xi = \sum_{k=1}^{m}\int_{\Gamma_k}(A_{f_k}\varphi_{ik})(\xi)\psi_{jk}(\xi)d\xi = \int_\Gamma (A_f\varphi_i)(\xi)\psi_j(\xi)d\xi. \tag{3.13}$$

Since $A_f\varphi_i = \sum_{k=0}^{\infty} c_{ik}A_f q_k$, we can conclude with the help of (3.12) that

$$\int_\Gamma (\varphi_i\psi_j f)(\xi)d\xi = \sum_{k=0}^{\infty} s_k(A_f)c_{ik}b_{jk};$$

that is,

$$J = \left|\left|\int_\Gamma (\varphi_i\psi_j f)(\xi)d\xi\right|_{i,j=0}^{n}\right| = \left|\left|\sum_{k=0}^{\infty} s_k(A_f)c_{ik}b_{jk}\right|_{i,j=0}^{n}\right|.$$

The last expression in the above equality can be expanded using the Binet-Cauchy formula (see [4]),

$$J = \frac{1}{(n+1)!}\left|\sum_{k_0=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} s_{k_0}(A_f)\ldots s_{k_n}(A_f)\left|c_{ik_j}\right|_{i,j=0}^{n}\left|b_{ik_j}\right|_{i,j=0}^{n}\right|. \tag{3.14}$$

By virtue of the Cauchy-Schwarz inequality, we get

$$\begin{aligned} J \leq &\left(\frac{1}{(n+1)!}\sum_{k_0=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} s_{k_0}(A_f)\ldots s_{k_n}(A_f)\left|\left|c_{ik_j}\right|_{i,j=0}^{n}\right|^2\right)^{1/2} \\ &\times\left(\frac{1}{(n+1)!}\sum_{k_0=0}\cdots\sum_{k_n=0} s_{k_0}(A_f)\ldots s_{k_n}(A_f)\left|\left|b_{ik_j}\right|_{i,j=0}^{n}\right|^2\right)^{1/2}. \end{aligned}$$

Since the sequence $\{s_n(A_f)\}_{n\geq 0}$ of the singular numbers of A_f is decreasing, the last inequality implies

$$\begin{aligned} J &\leq s_0(A_f)\ldots s_n(A_f)\left(\left|\sum_{k=0}^{\infty} c_{ik}\overline{c_{jk}}\right|_{i,j=0}^{n}\right)^{1/2}\left(\left|\sum_{k=0}^{\infty} b_{ik}\overline{b_{jk}}\right|_{i,j=0}^{n}\right)^{1/2} \\ &= \prod_{k=0}^{n} s_k(A_f)\left(\left|\langle \varphi_i - \eta_i, \varphi_j - \eta_j\rangle_{L_2(\Gamma)}\right|_{i,j=0}^{n}\right)^{1/2} \\ &\quad\times\left(\left|\langle \psi_i - \omega_i, \psi_j - \omega_j\rangle_{L_2(\Gamma)}\right|_{i,j=0}^{n}\right)^{1/2}. \end{aligned}$$

Finally, since $\langle \varphi_i - \eta_i, \eta_j\rangle_{L_2(\Gamma)} = 0$ and $\langle \psi_i - \omega_i, \omega_j\rangle_{L_2(\Gamma)} = 0$, where $i, j, = 0, 1, 2, \ldots$, the properties of the Gram determinants (see, for example, [4]) imply

$$\left|\langle \varphi_i - \eta_i, \varphi_j - \eta_j\rangle_{L_2(\Gamma)}\right|_{i,j=0}^{n} \leq \left|\langle \varphi_i, \varphi_j\rangle_{L_2(\Gamma)}\right|_{i,j=0}^{n}$$

and

$$|\langle \psi_i - \omega_i, \psi_j - \omega_j\rangle_{L_2(\Gamma)}|_{i,j=0}^n \le |\langle \psi_i, \psi_j\rangle_{L_2(\Gamma)}|_{i,j=0}^n.$$

This completes the proof of the theorem. $\square$

Our next theorem has an important consequence (Corollary 7) with respect to the estimates of errors in meromorphic approximation.

Theorem 6. *Suppose G is an m–domain with its boundary denoted by Γ and let F denote a compact subset of G. If f is a continuous function on Γ which has an analytic extension to $G\setminus F$ and if D is an m_1–domain such that $F \subset D$ and $\overline{D} \subset G$, then*

$$\prod_{k=0}^n s_k(A_{f,G}) \le \prod_{k=0}^n s_k(A_{f,D}) \prod_{k=0}^n s_k^2(J),$$

where $s_k(J)$ denotes the k–th singular number of the restriction operator $J : E_2(G) \to L_2(\partial D)$ defined by $J\varphi = \varphi_{|\partial D}$ for all $\varphi \in E_2(G)$.

Proof. Let $\{q_n\}, \{\alpha_n\}, n = 0, 1, 2, \ldots,$ denote the orthonormal systems of eigenfunctions of the operator $(A^*_{f,G}A_{f,G})^{1/2}$ corresponding to the sequence of singular numbers $\{s_n(A_{f,G})\}$ as in the proof of Theorem 5. From (3.13) (with $\varphi_i = q_i$ and $\psi_j = \alpha_j$) and formula (3.12) (with $n = i$), together with the fact that $\{\alpha_n\}_{n\ge 0}$ is an orthonormal system in $E_2(G)$, it follows that

$$\int_\Gamma (q_i\alpha_j)(\xi)f(\xi)d\xi = s_i(A_{f,G})\delta_{i,j}, \quad i, j = 0, 1, 2, \ldots,$$

where $\delta_{i,j}$ is Kronecker's symbol. Thus the product of singular numbers can be written as a determinant of order $n + 1$:

$$\prod_{k=0}^n s_k(A_{f,G}) = \left|\int_\Gamma (q_i\alpha_j)(\xi)f(\xi)d\xi\right|_{i,j=0}^n.$$

Let γ denote the boundary of D. We may also assume that Γ and γ are positively oriented with regard to G and D, respectively. Since $q_i, \alpha_j, i, j = 0, 1, 2, \ldots,$ belong to $E_2(G)$ and f is analytic on $G \setminus F$, the Cauchy formula yields

$$\prod_{k=0}^n s_k(A_{f,G}) = \left|\int_\gamma (q_i\alpha_j)(t)f(t)dt\right|_{i,j=0}^n.$$

As a consequence of Theorem 5, one can estimate the right–hand side of the above equality to obtain

$$\prod_{k=0}^n s_k(A_{f,G}) \le \prod_{k=0}^n s_n(A_{f,D}) \left(\left|\langle q_i, q_j\rangle_{L_2(\gamma)}\right|_{i,j=0}^n\right)^{1/2} \left(\left|\langle \alpha_i, \alpha_j\rangle_{L_2(\gamma)}\right|_{i,j=0}^n\right)^{1/2}.$$

Noting that J is a compact operator, the Weyl-Horn Theorem (see, for example, [5]) together with the fact $\langle \alpha_i, \alpha_j\rangle_{L_2(\Gamma)} = \langle q_i, q_j\rangle_{L_2(\Gamma)} = \delta_{i,j}$ imply

$$\left|\langle q_i, q_j\rangle_{L_2(\gamma)}\right|_{i,j=0}^n = \left|\langle Jq_i, Jq_j\rangle_{L_2(\gamma)}\right|_{i,j=0}^n \le \prod_{k=0}^n s_k^2(J)\left|\langle q_i, q_j\rangle_{L_2(\Gamma)}\right|_{i,j=0}^n = \prod_{k=0}^n s_k^2(J)$$

and

$$\begin{aligned}\left|\langle\alpha_i,\alpha_j\rangle_{L_2(\gamma)}\right|_{i,j=0}^n &= \left|\langle J\alpha_i,J\alpha_j\rangle_{L_2(\gamma)}\right|_{i,j=0}^n \\ &\le \prod_{k=0}^n s_k^2(J)\left|\langle\alpha_i,\alpha_j\rangle_{L_2(\Gamma)}\right|_{i,j=0}^n = \prod_{k=0}^n s_k^2(J).\end{aligned}$$

Thus the theorem is proved. □

In view of Theorem 6 and Lemma 4, we obtain the following.

Corollary 7. *Under the assumptions of Theorem* 6*, if the boundary of each* G_i $(1 \le i \le m)$ *consists of* N_i *closed analytic Jordan curves and* $N = N_1 + \cdots + N_m$*, then there is a positive integer* n^* *such that for* $n \ge n^*$

$$\prod_{k=0}^{n^*-1} s_k(A_{f,G}) \prod_{k=n^*}^{n} \Delta_{k+N-m}(f;G) \le \prod_{k=0}^{n} \Delta_k(f;D) \prod_{k=0}^{n} s_k^2(J).$$

We end this section with a result regarding the rate with which the product of the singular numbers of the restriction operator decreases.

Lemma 8. *Let* G *be an* m*–domain and suppose* D *is an* m_1*–domain such that* $\overline{D} \subset G$*. Then*

$$\limsup_{n\to\infty}\left(\prod_{k=0}^n s_k^2(J)\right)^{1/n^2} \le \exp\bigl(-1/C(\partial D,\partial G)\bigr), \tag{3.15}$$

where $\{s_n(J)\}, n = 0,1,2,\dots,$ *denotes the sequence of the singular numbers of the restriction operator* $J : E_2(G) \to L_2(\partial D)$*.*

Proof. If G is a domain $(m = 1)$, it follows from the result of Zaharjuta and Skiba regarding the n–widths (see [20] and also [9]) that

$$\lim_{n\to\infty} s_n^{1/n}(J) = \exp\bigl(-1/C(\partial D,\partial G)\bigr).$$

From the above result, (3.15) follows easily. To see how this is done in details (see also [12]), denote by $\{\varphi_n\}, n = 0,1,2,\dots,$ the orthonormal sequence of eigenfunctions of J^*J corresponding to the sequence $\{s_n(J)\}, n = 0,1,2,\dots.$ Since

$$\langle J\varphi_i, J\varphi_j\rangle_{L_2(\partial D)} = s_i^2(J)\langle\varphi_i,\varphi_j\rangle_{L_2(\partial G)} = s_i^2(J)\delta_{ij},$$

we have that

$$\begin{aligned}\prod_{k=0}^n s_k^2(J) &= \left|\int_{\partial D}(\varphi_i\overline{\varphi_j})(t)|dt|\right|_{i,j=0}^n \\ &= \frac{1}{(n+1)!}\int_{\partial D}\cdots\int_{\partial D}\left|\left|\varphi_i(t_j)\right|_{i,j=0}^n\right|^2 |dt_0|\dots|dt_n|.\end{aligned} \tag{3.16}$$

Next, let U be any Jordan domain such that $\overline{D} \subset U \subset \overline{U} \subset G$, and denote by $g(z,\zeta)$ the Green function (see, for example, [6]) of the domain U with singularity at $\zeta \in U$. Using the fact that $\|\varphi_i\|_{L_2(\partial G)} = 1$ $(i = 0,1,\dots)$, we get $\|\varphi_i\|_{\partial U} \le C$ for some positive constant C. Consequently,

$$\max_{t_i\in\partial U}\left|\left|\varphi_i(t_j)\right|_{i,j=0}^n\right| \le (n+1)!C^{n+1}. \tag{3.17}$$

Moreover, it is easily seen that

$$F_n(t_0,\dots,t_n) = \ln\Big| |\varphi_i(t_j)|_{i,j=0}^n \Big|^2 + 2 \sum_{0\le i<j\le n} g(t_i,t_j)$$

defines a subharmonic function in U for each t_i. Now the maximum principle for subharmonic functions together with (3.16) and (3.17) implies

$$\prod_{k=0}^n s_k^2(J) \le (n+1)! C_1^{n+1} \exp(-\tau_n),$$

where C_1 denotes a positive constant and

$$\tau_n = \min_{t_i\in\partial D}\big(2 \sum_{0\le i<j\le n} g(t_i,t_j)\big).$$

Using the fact (see, for example, [6])

$$\lim_{n\to\infty} \tau_n/n^2 = 1/C(\partial D,\partial U),$$

we obtain the desired inequality

$$\limsup_{n\to\infty} \left(\prod_{k=0}^n s_k^2(J)\right)^{1/n^2} \le \exp\big(-1/C(\partial D,\partial U)\big).$$

The result now follows from the general properties of capacity together with the fact that U is an arbitrary Jordan domain satisfying $\overline{D}\subset U\subset\overline{U}\subset G$.

For the general case ($m>1$), let G be a union of domains $G_1,\dots,G_m$ with disjoint closures. Set $D_k = G_k\cap D$. We may further assume that $D_k\ne\emptyset$ for all $k=1,\dots,m$. Denote by $J_k : E_2(G_k)\to L_2(\partial D_k)$, $1\le k\le m$, the corresponding restriction operator. Since $J = J_1\oplus\cdots\oplus J_m$, it follows from a similar argument as in Lemma 3 that

$$s_n(J) = \min_{k_1+\cdots+k_m\le n} \max\{s_{k_1}(J_1),\dots,s_{k_m}(J_m)\}. \tag{3.18}$$

Now for each $i=1,\dots,m$, the simple case $m=1$ implies

$$\limsup_{n\to\infty} \left(\prod_{k=0}^n s_k^2(J_i)\right)^{1/n^2} \le \exp\big(-1/C(\partial D_i,\partial G_i)\big). \tag{3.19}$$

Furthermore (see [12], Lemma 3)

$$C(\partial D,\partial G) = \sum_{i=1}^m C(\partial D_i,\partial G_i), \tag{3.20}$$

which together with (3.18) and (3.19) implies (see [12], Lemma 2)

$$\limsup_{n\to\infty} \left(\prod_{k=0}^n s_k^2(J)\right)^{1/n^2} \le \exp\left(-\sum_{i=1}^m \frac{w_i^2}{C(\partial D_i,\partial G_i)}\right), \tag{3.21}$$

where $w_i\ge 0$ for all $1\le i\le m$ and $\sum_{i=1}^m w_i = 1$. For any $\theta_i\ge 0$ we have (see [12], Lemma 4)

$$\frac{1}{\sum_{i=1}^m \theta_i} \le \sum_{i=1}^m \frac{w_i^2}{\theta_i}.$$

Finally, letting $\theta_i = C(\partial D_i,\partial G_i)$, the result follows from (3.20) and (3.21). $\square$

4. Proof of Theorem A

The proof of Theorem A is given in two parts.

Part I (special case). Here we will use the obtained results from the theory of Hankel operators (Section 3) to prove Theorem A under the assumption that K and E are bounded by finitely many disjoint closed analytic Jordan curves. First of all we remark that in view of the mapping $w = 1/(z-a)$, where a is some fixed point of the interior of K, we can confine ourselves to the case where the complement of K, denoted by G, is bounded.

Denote the interior of E by Ω, and let $w(z)$ be the solution of the Dirichlet problem with respect to the boundary values 1 on ∂K and 0 on $\partial\Omega$. Extend $w(z)$ by continuity to $\overline{\mathbb{C}}$ such that $w(z) = 1$ for $z \in K$ and $w(z) = 0$ for $z \in \overline{\mathbb{C}} \setminus E$. Furthermore, for any $0 < \varepsilon < 1$, let $G(\varepsilon) = \{z : w(z) < \varepsilon\}$ and $\gamma(\varepsilon) = \{z : w(z) = \varepsilon\}$, where it is assumed that $\gamma(\varepsilon)$ is positively oriented with respect to the open set $G(\varepsilon)$.

Next choose $0 < \varepsilon < \varepsilon_1 < 1$, sufficiently close to 0 and 1, respectively, so that $\gamma(\varepsilon)$ and $\gamma(\varepsilon_1)$ consist of finitely many closed analytic Jordan curves. It is not hard to see that $G(\varepsilon)$ and $G(\varepsilon_1)$ are m-domains satisfying $\overline{G(\varepsilon)} \subset G(\varepsilon_1)$. We also assume that $G(\varepsilon_1)$ and $G(\varepsilon)$ consist of m and m' connected components, respectively. Denote the components of $G(\varepsilon_1)$ by $G_1, \ldots, G_m$, where the boundary of each G_i consists of N_i closed analytic Jordan curves. Put $N = N_1 + \cdots + N_m$. Since f is analytic in Ω, we can assert with the aid of Corollary 7 that there exists a positive integer n^* such that for all $n \geq n^*$

$$\prod_{k=n^*}^{n} \Delta_{k+N-m}(f; G(\varepsilon_1)) \leq C_1 \prod_{k=0}^{n} \Delta_k(f; G(\varepsilon)) \prod_{k=0}^{n} s_k^2(J), \tag{4.1}$$

where C_1 is a positive constant independent of n and $s_k(J)$ is the k–th singular number of the restriction operator $J : E_2(G(\varepsilon_1)) \to L_2(\gamma(\varepsilon))$ (see Corollary 7).

Next we claim there is a constant $C_2 > 0$ such that

$$\rho_k(f; K) \leq C_2\, \Delta_k(f; G(\varepsilon_1)), \quad k = 0, 1, \ldots. \tag{4.2}$$

To see this, fix a nonnegative integer k. It follows from the definition (3.8) that there are nonnegative integers $k_1, \ldots, k_m$ such that $k_1 + \cdots + k_m \leq k$ and

$$\Delta_k(f; G(\varepsilon_1)) = \max(\Delta_{k_1}(f_1; G_1), \ldots, \Delta_{k_m}(f_m; G_m)).$$

Let $h_i, 1 \leq i \leq m$, denote the function of best approximation of $f_i = f|\partial G_i$ in $L_\infty(\partial G_i)$ by meromorphic functions from the class $\mathcal{M}_{k_i}(G_i)$; that is,

$$||f - h_i||_\infty = \Delta_{k_i}(f_i; G_i).$$

Now (4.2) follows from the above observation together with the estimation of the Cauchy integral formula

$$(r - f)(z) = \sum_{i=1}^{m} \frac{1}{2\pi i} \int_{\partial G_i} \frac{(f - h_i)(\xi) d\xi}{\xi - z},$$

which holds for some $r \in \mathcal{R}_k$ and all $z \in K$.

In view of (4.1) and (4.2), we obtain

$$\prod_{k=0}^{n} \rho_k(f;K) \leq C^n \prod_{k=0}^{n} \Delta_k(f;G(\varepsilon)) \prod_{k=0}^{n} s_k^2(J), \tag{4.3}$$

where $C > 0$ is a constant independent of n.

Moreover, it is easy to verify that

$$\Delta_k(f;G(\varepsilon)) \leq \rho_k(f;\gamma(\varepsilon)). \tag{4.4}$$

First we note that for any rational function $r \in \mathcal{R}_k$ with poles off $\gamma(\varepsilon)$ and for any connected component $D_i, i = 1, \ldots, m'$, of $G(\varepsilon)$ we have

$$||f - r||_{\gamma(\varepsilon)} \geq ||f - r||_{\partial D_i} \geq \Delta_{l_i}(f_i;\partial D_i),$$

where l_i is the number of poles (counted with multiplicities) of r inside D_i and $f_i = f_{|\partial D_i}$. Therefore

$$||f - r||_{\gamma(\varepsilon)} \geq \max(\Delta_{l_1}(f_1;\partial D_1), \ldots, \Delta_{l_{m'}}(f_{m'};\partial D_{m'})).$$

Since $l_1 + \cdots + l_{m'} \leq k$, we get $||f - r||_{\gamma(\varepsilon)} \geq \Delta_k(f;G(\varepsilon))$. Thus (4.4) follows from the fact that r is an arbitrary rational function in the class $\mathcal{R}_k$ with poles off $\gamma(\varepsilon)$.

Using (4.4), together with the fact $\gamma(\varepsilon) \subseteq E$, we have

$$\rho_k(f;\gamma(\varepsilon)) \leq \rho_k(f;E) \quad \text{and} \quad \Delta_k(f;G(\varepsilon)) \leq \rho_k(f;E).$$

Therefore it follows from (4.3) that

$$\prod_{k=0}^{n} \rho_k(f;K) \leq C^n \prod_{k=0}^{n} \rho_k(f;E) \prod_{k=0}^{n} s_k^2(J). \tag{4.5}$$

Combining (4.5) and (3.15), we get

$$\limsup_{n\to\infty} \left(\prod_{k=0}^{n} \rho_k(f;K) / \prod_{k=0}^{n} \rho_k(f;E) \right)^{1/n^2} \leq \exp(-1/C(\gamma(\varepsilon), \gamma(\varepsilon_1))). \tag{4.6}$$

Since the left–hand side of (4.6) does not depend on ε and ε_1, the proof of Theorem A, in the special case, follows from the properties of capacities that (see [8], [14])

$$\lim_{\varepsilon\to 0,\ \varepsilon_1\to 1} C(\gamma(\varepsilon), \gamma(\varepsilon_1)) = C(\partial E, K).$$

Part II (general case). Here we consider the general case, where E is an arbitrary compact set with nonempty interior Ω and K is an arbitrary compact subset of Ω.

We start our proof by observing that Ω is an open cover of K. Therefore, there are finitely many open connected components Ω_i of Ω such that $\Omega_i \cap K \neq \emptyset$. Let

$$\Omega' = \bigcup \{\Omega_i : \Omega_i \cap K \neq \emptyset\}.$$

From the properties of the capacity (see [8] or [16]), we have $C(\partial\Omega', K) = C(\partial\Omega, K)$. Since $\partial\Omega \subset \partial E$, it follows that

$$C(\partial\Omega', K) \leq C(\partial E, K). \tag{4.7}$$

It is well known (see [6]) that one can construct two sequences of compact sets $\{K_m\}_{m\geq 1}$ and open sets $\{\Omega_m\}_{m\geq 1}$ which tend monotonically to K and Ω', respectively. Furthermore, we may also arrange the sequences such that both K_m and

Ω_m are bounded by finitely many closed analytic Jordan curves, and $K_m \subset \Omega_m$ for $m = 1, 2, \ldots$. More precisely,

$$K_1 \supset K_2 \supset \cdots \supset K, \quad \bigcap_{m=1}^{\infty} K_m = K, \quad \Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega', \text{ and } \bigcup_{m=1}^{\infty} \Omega_m = \Omega'.$$

Set $E_m = \overline{\Omega}_m$. Since $K \subset K_m$ and $E_m \subset E$,

$$\rho_n(f; K) \le \rho_n(f; K_m) \quad \text{and} \quad \rho_n(f; E_m) \le \rho_n(f; E), \quad \text{for all } n \ge 0. \tag{4.8}$$

For each fixed $m \ge 1$, using the fact that K_m and E_m are both bounded by finitely many closed analytic Jordan curves, it follows from Part I that

$$\limsup_{n\to\infty} \left(\prod_{k=0}^{n} \rho_k(f; K_m) / \prod_{k=0}^{n} \rho_k(f; E_m) \right)^{1/n^2} \le \exp(-1/C(\partial E_m, K_m)).$$

As a consequence of (4.8), we have

$$\limsup_{n\to\infty} \left(\prod_{k=0}^{n} \rho_k(f; K) / \prod_{k=0}^{n} \rho_k(f; E) \right)^{1/n^2} \le \exp(-1/C(\partial E_m, K_m)). \tag{4.9}$$

Now the definition of E_m and K_m together with the properties of the capacity (see Part I) imply

$$\lim_{m\to\infty} C(\partial E_m, K_m) = C(\partial \Omega', K).$$

Hence, taking the limit on the right–hand side of the inequality (4.9) as $m \to \infty$, we obtain

$$\limsup_{n\to\infty} \left(\prod_{k=0}^{n} \rho_k(f; K) / \prod_{k=0}^{n} \rho_k(f; E) \right)^{1/n^2} \le \exp(-1/C(\partial \Omega', K)).$$

Finally use (4.7) to conclude the proof of Theorem A.

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