TAUBERIAN CONDITIONS FOR GEOMETRIC MAXIMAL OPERATORS

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Abstract. Let $B$ be a collection of measurable sets in $\mathbb{R}^n$. The associated geometric maximal operator $M_B$ is defined on $L^1(\mathbb{R}^n)$ by $M_B f(x) = \sup_{x \in R \in B} \frac{1}{|R|} \int_R |f|$. If $\alpha > 0$, $M_B$ is said to satisfy a Tauberian condition with respect to $\alpha$ if there exists a finite constant $C$ such that for all measurable sets $E \subset \mathbb{R}^n$ the inequality $|\{x : M_B \chi_E(x) > \alpha\}| \leq C|E|$ holds. It is shown that if $B$ is a homothecy invariant collection of convex sets in $\mathbb{R}^n$ and the associated maximal operator $M_B$ satisfies a Tauberian condition with respect to some $0 < \alpha < 1$, then $M_B$ must satisfy a Tauberian condition with respect to $\gamma$ for all $\gamma > 0$ and moreover $M_B$ is bounded on $L^p(\mathbb{R}^n)$ for sufficiently large $p$. As a corollary of these results it is shown that any density basis that is a homothecy invariant collection of convex sets in $\mathbb{R}^n$ must differentiate $L^p(\mathbb{R}^n)$ for sufficiently large $p$.

Let $B$ be a collection of measurable sets in $\mathbb{R}^n$. We define the associated geometric maximal operator $M_B$ on $L^1(\mathbb{R}^n)$ by $M_B f(x) = \sup_{x \in R \in B} \frac{1}{|R|} \int_R |f|$. The operator $M_B$ is said to satisfy a Tauberian condition with respect to $\alpha$ if there exists a finite constant $C$ such that for any measurable set $E \subset \mathbb{R}^n$ the inequality $|\{x : M_B \chi_E(x) > \alpha\}| \leq C|E|$ holds.

This is a very weak condition on a maximal operator - weaker in fact than a restricted weak type $(1,1)$ estimate. This is a useful condition on a maximal operator, however, as was shown by A. Córdoba and R. Fefferman in their work relating the $L^p$ bounds of certain multiplier operators to the weak type $(\frac{p}{2}, \frac{p}{2})'$ bounds of associated geometric maximal operators. (See [2] for complete details.)

Now, suppose we are given a maximal operator $M_B$ satisfying a Tauberian condition such as, for instance,

$$ \left| \left\{ x : M_B \chi_E(x) > \frac{3}{4} \right\} \right| \leq C|E|. $$

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One might wonder whether or not $M_B$ must be bounded on $L^p(\mathbb{R}^n)$ for $p > 1$ or whether or not $M_B$ must satisfy any given stronger Tauberian estimate, say, $\left| \left\{ x : M_B \chi_E(x) > \frac{1}{2} \right\} \right| \leq C |E|$. That neither of the above holds, even in the case that $B$ is a homothecy invariant collection of sets, can be seen by the following example. (Recall that a collection of sets in $\mathbb{R}^n$ is said to be homothecy invariant if and only if any translate or dilate of any member of the collection also lies in the collection.)

**Example.** Let $B$ be the collection of sets in $\mathbb{R}^1$ of the form $I_1 \cup I_2$, where $I_1$ and $I_2$ are intervals and $|I_2| = 2|I_1|$. Note $B$ is homothecy invariant. $M_B$ is not bounded on $L^p(\mathbb{R}^1)$ for $1 < p < \infty$, as $M_B \chi_{[0,1]}(x) \geq \frac{1}{3}$ for all $x$ in $\mathbb{R}^1$. Moreover $\left| \left\{ x : M_B \chi_{[0,1]}(x) > \frac{1}{2} \right\} \right| = \infty$, and so $M_B$ does not satisfy a Tauberian condition with respect to $\frac{3}{4}$.

$M_B$ does satisfy a Tauberian condition with respect to $\frac{3}{4}$, however. To see this, let $E$ be a set of finite measure, and let $\{A_j\} \subset B$ be such that $\frac{1}{|A_j|} \int_{A_j} \chi_E > \frac{3}{4}$ for each $j$. Now, each $A_j$ is of the form $A_j = A_{j,1} \cup A_{j,2}$, where $A_{j,1}$ and $A_{j,2}$ are intervals and $2|A_{j,1}| = |A_{j,2}|$. Since $\frac{1}{|A_j|} \int_{A_j} \chi_E > \frac{3}{4}$, we must have $\frac{1}{|A_{j,1}|} \int_{A_{j,1}} \chi_E > \frac{1}{4}$ and $\frac{1}{|A_{j,2}|} \int_{A_{j,2}} \chi_E > \frac{1}{4}$.

So by the Vitali Covering Theorem we must have $|\cup A_{j,1}| \leq 12|E|$ and $|\cup A_{j,2}| \leq 12|E|$. Therefore $|\cup A_j| \leq 24|E|$ and hence $\left| \left\{ x : M_B \chi_{E}(x) > \frac{3}{4} \right\} \right| \leq 24|E|$. Note that in the above example the elements of $B$ are not all convex. The primary purpose of this paper is to show that if $B$ is a homothecy invariant collection of convex sets in $\mathbb{R}^n$ and the associated maximal operator $M_B$ satisfies a Tauberian condition with respect to some $0 < \alpha < 1$, then $M_B$ must satisfy a Tauberian condition with respect to $\gamma$ for every $\gamma > 0$. As a corollary of the proof we shall see that if $B$ is a homothecy invariant collection of convex sets and $M_B$ satisfies a Tauberian condition with respect to $\alpha$ for some $0 < \alpha < 1$, then $M_B$ must be bounded on $L^p(\mathbb{R}^n)$ for sufficiently large $p$. As a further corollary we shall see that any density basis that is a homothecy invariant collection of convex sets in $\mathbb{R}^n$ must differentiate $L^p(\mathbb{R}^n)$ for sufficiently large $p$.

Our proof will consist of two main parts. First we shall show the desired result in the special case that $B$ is a homothecy invariant collection of rectangular parallelepipeds. Secondly we shall reduce the general case involving homothecy invariant collections of convex sets to this special case.

**Proposition 1.** Let $B$ be a homothecy invariant collection of rectangular parallelepipeds in $\mathbb{R}^n$. Suppose for some $0 < \gamma < 1$ there exists $0 < C_\gamma < \infty$ such that

$$\left| \left\{ x : M_B \chi_E(x) > \gamma \right\} \right| \leq C_\gamma |E|$$

holds for all measurable sets $E$ in $\mathbb{R}^n$. Then if $\alpha > 0$, there exists $0 < C_{\alpha, \gamma} < \infty$ such that

$$\left| \left\{ x : M_B \chi_E(x) > \alpha \right\} \right| \leq C_{\alpha, \gamma} |E|$$

holds for all measurable sets $E$ in $\mathbb{R}^n$, where $C_{\alpha, \gamma}$ depends only on $C_\gamma$, $\alpha$, $\gamma$, and the dimension $n$.

**Proof.** If $\alpha \geq \gamma$, then we may trivially set $C_{\alpha, \gamma} = C_\gamma$. So we assume without loss of generality that $0 < \alpha < \gamma$. Let $E$ be a measurable set in $\mathbb{R}^n$. We inductively
Lemma 1. Suppose for \( c > 1 \), we let \( H \) define \( Q \) that none of the cubes for some constant \( \alpha, \gamma \) depending only on \( n \), \( \alpha \), and \( \gamma \).

Proof. Let \( Q \) denote the unit \( n \)-cube \([0, 1]^n\) in \( \mathbb{R}^n \). Now, since \( R \) is a rectangular parallelepiped, there exists a linear bijection \( \Lambda : \mathbb{R}^n \to \mathbb{R}^n \) such that \( \{ \Lambda(x) : x \in \mathbb{R} \} = Q \).

For each set \( S \in \mathbb{R}^n \) let

\[
S_\Lambda = \{ \Lambda(x) : x \in S \}.
\]

Also, let

\[
B_\Lambda = \{ S_\Lambda : S \in B \}.
\]

Note if \( U \) and \( V \) are measurable sets in \( \mathbb{R}^N \) and \( |V| \neq 0 \), then \( \frac{|U|}{|V|} = \frac{|U \cap V|}{|V|} \). Hence \( M_{\Lambda \chi E} \chi E \geq \alpha \) on a set \( S \in B \) if and only if \( M_{\Lambda \chi E} \chi E \geq \alpha \) on \( S_\Lambda \). Now, if \( \{ x : M_{\Lambda \chi E} \chi E(x) \geq \alpha \} = \cup S_j \), it follows that \( \{ x : M_{\Lambda \chi E} \chi E(x) \geq \alpha \} = \cup S_j \). As \( (\cup S_j)_\Lambda = \cup S_j \Lambda \), one then sees that

\[
(\mathcal{H}_{E_\Lambda}^k(\gamma))(E_\Lambda) = \mathcal{H}_{E_\Lambda}^k(\gamma)(E_\Lambda)
\]

holds for any positive integer \( k \). As \( R_\Lambda = Q \) we realize it suffices to prove

\[
Q \subset \mathcal{H}_{E_\Lambda}^{K_{n, \gamma}}(E_\Lambda)
\]

for some constant \( K_{n, \gamma} \) depending only on \( n \), \( \alpha \), and \( \gamma \). As \( \int_Q \chi E_\Lambda > \alpha \) and \( Q \in B_\Lambda \) we then realize it suffices to prove the lemma in the special case that \( R = Q \). Note that as \( B \) is homothety invariant we may also assume without loss of generality that any \( n \)-cube in \( \mathbb{R}^n \) with sides parallel to the axes lies in \( B \).

So, we now suppose without loss of generality that \( R = Q \), all \( n \)-cubes in \( \mathbb{R}^n \) whose sides are parallel to the axes lie in \( B \), and \( \int_Q \chi E = \alpha \). We take the Calderon-Zygmund decomposition of \( \chi_{E \cap Q} \) with respect to \( \gamma \) yielding a collection of cubes \( \{ Q_j \} \) in \( Q \) with sides parallel to the axes. In particular the collection of cubes \( \{ Q_j \} \) is such that \( \frac{1}{|Q_j|} \int_{Q_j} \chi E > \gamma \) for each \( j \) and \( E \cap Q \subset \cup Q_j \) almost everywhere. Note that none of the cubes \( Q_j \) is \( Q \) itself, as \( \frac{1}{|Q_j|} \int_{Q_j} \chi E = \alpha < \gamma \). Also note that each \( Q_j \) is a dyadic cube and hence has a unique parent dyadic cube. For any constant \( c > 1 \), we let \( cQ_j \) denote the cube containing \( Q_j \) that has sidelength \( c \) times that of \( Q_j \) and also has a common corner with \( Q_j \) and the parent cube of \( Q_j \).

Let now \( E_0 = E \cap Q \), \( E_1 = \cup Q_j \), and, for \( k \geq 2 \),

\[
E_k = \bigcup_j \left( \frac{1}{c} \right)^{(k-1)/n} Q_j.
\]

Note that since

\[
\frac{|\left( \frac{1}{c} \right)^{k-1} Q_j|}{\left| \left( \frac{1}{c} \right)^{k-1} Q_j \right|} = \gamma,
\]

define \( \mathcal{H}_{E_\Lambda}^k(\gamma)(E_\Lambda) \) for \( k = 0, 1, 2, \ldots \) by setting \( \mathcal{H}_{E_\Lambda}^0(\gamma)(E_\Lambda) = E \) and

\[
\mathcal{H}_{E_\Lambda}^k(\gamma)(E_\Lambda) = \left\{ x : M_{\Lambda \chi E_\Lambda} \chi_{E_\Lambda} \chi_{E_\Lambda}(x) \geq \gamma \right\}
\]

for \( k \geq 1 \).
we have $M_{B \chi E_k} \geq \gamma$ on $E_{k+1}$. Also observe that since the average of $\chi_E$ over each $Q_k$ exceeds $\gamma$ we have $E_1 \subset H_{B, \gamma}^1(E)$, and as $M_{B \chi E_k} \geq \gamma$ on $E_{k+1}$ we have $E_k \subset H_{B, \gamma}^k(E)$ for each $k$.

Now let $N$ be a positive integer such that $\left(\frac{1}{\gamma}\right)^N \geq \gamma \cdot 2^n$. Let $Q_j^*$ denote the parent cube of $Q_j$. Now, since

$$\left|\left\{ x \in Q : M_{B \chi E_{N+1}}(x) \geq \gamma \right\}\right| \geq \left|\bigcup Q_j^*\right|$$

we have

$$\frac{|E_{N+1} \cap Q_j^*|}{|Q_j^*|} \geq \gamma,$$

and so $M_{B \chi E_{N+1}} \geq \gamma$ on $Q_j^*$.

Now let $Q_{j_1}, Q_{j_2}, \ldots$ be elements of $\{Q_j\}$ such that the $Q_j^*$ have disjoint interiors and such that $|\bigcup Q_j^*| = |\bigcup Q_i|$. Note that each $Q_j^*$ is contained in $Q$ since $Q \notin \{Q_i\}$. Note also that $|E \cap Q_j| \leq \gamma |Q_j^*|$, as otherwise $Q_j^*$ would have been a selected $Q_j$. Hence we have

$$\left|\left\{ x \in Q : M_{B \chi E_{N+1}}(x) \geq \gamma \right\}\right| \geq \left|\bigcup Q_j^*\right|$$

$$= \sum |Q_j^*| \geq \frac{1}{\gamma} \sum |E \cap Q_j^*| \geq \frac{1}{\gamma} |E_0|.$$

In particular,

$$\left|\left\{ x \in Q : M_{B \chi E_{N+1}}(x) \geq \gamma \right\}\right| \geq \frac{1}{\gamma} |E_0|.$$

Note that if $\left|\left\{ x \in Q : M_{B \chi E_{N+1}}(x) \geq \gamma \right\}\right| \geq \gamma$ we have $Q \subset H_{B, \gamma}^{(N+2)+1}(E)$. Otherwise by the above argument we may obtain

$$\left|\left\{ x \in Q : M_{B \chi E_{N+1}}(x) \geq \gamma \right\}\right| \geq \frac{1}{\gamma} \left|H_{B, \gamma}^{N+2}(E) \cap Q\right| \geq \frac{1}{\gamma} |E_0|.$$

More generally, if $\left|\left\{ x \in Q : M_{B \chi E_{N+1}}(x) \geq \gamma \right\}\right| \geq \gamma$ we have $Q \subset H_{B, \gamma}^{(N+2)+1}(E)$, or otherwise we may obtain

$$\left|\left\{ x \in Q : M_{B \chi E_{N+1}}(x) \geq \gamma \right\}\right| \geq \frac{1}{\gamma} |E_0|.$$

Now, let $\tilde{N}$ be a positive integer such that $\alpha \cdot \left(\frac{1}{\gamma}\right)^\tilde{N} \geq \gamma$. As $|E_0| = \alpha$ we have

$$\left(\frac{1}{\gamma}\right)^\tilde{N} |E_0| \geq \gamma.$$ Hence for some $m \leq (N + 2) \cdot \tilde{N}$ we have $|Q \cap H_{B, \gamma}^m(E)| \geq \gamma$. In particular, $Q \subset H_{B, \gamma}^{(N+2)\tilde{N}+1}(E)$. As any integer greater than or equal to $\frac{\log^{k}(2^n)}{\log(\frac{1}{\gamma})}$
would be acceptable for $N$ and any integer greater than or equal to \( \frac{-\log(\frac{\lambda}{n})}{\log \gamma} \) would be acceptable for $\tilde{N}$, we obtain the lemma, where

(1) \[
K_{\alpha, \gamma} = \left\lfloor \frac{-\log\left(\frac{\lambda}{n}\right)}{\log \gamma} \right\rfloor \cdot \left[ 2 + \frac{\log^+ (\gamma \cdot 2^n)}{\log \left(\frac{1}{\gamma}\right)} \right] + 1.
\]

\[
\Box
\]

We now complete the proof of Proposition 1. As \( \{|x : M_{B \chi E}(x) > \gamma| \leq C |E| \) for every measurable set $E$ if and only if \( \{|x : M_{B \chi E}(x) \geq \gamma| \leq C |E| \) for every measurable set $E$, by the Tauberian condition on $M_B$ we have that

\[
|H_{B, \gamma}^{k+1}(E)| \leq C_1 |H_{B, \gamma}^k(E)|
\]

holds for any positive integer $k$ and any measurable set $E$. An immediate consequence of the above lemma is that \( \{|x : M_{B \chi E}(x) > \alpha| \leq \alpha \in H_{B, \gamma}^{K_{\alpha, \gamma}}(E), \) and hence

\[
\{|x : M_{B \chi E}(x) > \alpha| \leq \left| H_{B, \gamma}^{K_{\alpha, \gamma}}(E) \right| \leq C_\gamma \left| H_{B, \gamma}^{K_{\alpha, \gamma}-1}(E) \right| \leq \ldots \leq C_\gamma^{K_{\alpha, \gamma}} |E|.
\]

So \( \{|x : M_{B \chi E}(x) > \alpha| \leq C_{\alpha, \gamma} |E|, \) where \( C_{\alpha, \gamma} = C_\gamma^{K_{\alpha, \gamma}} \) and \( K_{\alpha, \gamma} \) is as in (1).

\[
\Box
\]

In Proposition 1 $B$ is a homothecy invariant collection of rectangular parallelepipeds. The following theorem is a generalization of Proposition 1 in that we allow $B$ to be a homothecy invariant collection of convex sets.

**Theorem 1.** Let $B$ be a homothecy invariant collection of convex sets in $\mathbb{R}^n$. Suppose for some $0 < \alpha < 1$ there exists $0 < C_\alpha < \infty$ such that

\[
\{|x : M_{B \chi E}(x) > \alpha| \leq C_\alpha |E|
\]

holds for all measurable sets $E$ in $\mathbb{R}^n$. Then if $\delta > 0$, there exists $0 < C_{\alpha, \delta} < \infty$ such that

\[
\{|x : M_{B \chi E}(x) > \delta| \leq C_{\alpha, \delta} |E|
\]

holds for all measurable sets $E$ in $\mathbb{R}^n$, where $C_{\alpha, \delta}$ depends only on $C_\alpha$, $\alpha$, $\delta$, and the dimension $n$.

**Proof.** Given an ellipsoid $E$ in $\mathbb{R}^n$ and $c > 0$, we let $cE$ denote the $c$-fold dilate of $E$ that has the same center and orientation as $E$.

Let $S \in B$. As was proven by F. John in [4] (see also the related article [1] by K. Ball), since $S$ is convex there exists an ellipsoid $E$ contained in $S$ such that $S \subset nE$. Let $R_S$ be a rectangular parallelepiped containing $nE$ of smallest possible volume. Note that $|R_S| < 2^n |nE|$ and hence $|R_S| < 2^n \cdot n^n |S|$. Moreover, letting $cS$ denote the $c$-fold dilate of $S$ about the center of $E_S$ we have $R_S \subset 2nS$, since $R_S \subset 2nE$ and $2nE \subset 2nS$.

Let $B = \{R_S : S \in B\}$. We may assume without loss of generality that the $E_S$ and $R_S$ above are such that $\tilde{B}$ is homothecy invariant.

Note that $M_B f(x) \leq 2^n \cdot n^n M_B f(x)$.

We now fix $\gamma$ such that $0 < \alpha < \gamma < 1$. 


Let \( \rho = \frac{1}{2^n n^2} \). Also let

\[
\epsilon = \frac{\gamma - \alpha}{2 - \gamma - \alpha} \rho \quad \text{and} \quad N = \left\lfloor \frac{\log \left( 1 - \frac{2(1-\gamma)}{2 - \gamma - \alpha} \rho \right)}{\log \left( 1 - \rho - \frac{\gamma - \alpha}{2 - \gamma - \alpha} \rho \right)} \right\rfloor.
\]

One can show that

\[
\rho \frac{1 - (1 - \rho - \epsilon)^{N+1}}{\rho + \epsilon} > \frac{1 - \gamma}{1 - \alpha}.
\]

We will need the following technical lemma.

**Lemma 2.** Let \( \epsilon > 0 \) be as above and \( S \) be a convex set in \( Q = [0,1]^n \).

Let \( m \in \mathbb{N} \) be the unique positive integer such that

\[
\frac{\epsilon}{4n} \leq \sqrt{n} 2^{-m} < \frac{\epsilon}{2n}.
\]

Then there exists a set of cubes \( \{Q_j\} \) of sidelength \( 2^{-m} \) such that

i) all the cubes \( Q_j \) lie in \( Q \) and are members of the mesh \( \mathcal{M}_m \) of dyadic cubes of sidelength \( 2^{-m} \),

ii) each \( Q_j \) is disjoint from \( S \), and

iii) \( |\bigcup Q_j \cup S| \geq 1 - \epsilon \).

**Proof.** Let \( C \) be the set of cubes in the mesh \( \mathcal{M}_m \) that lie in \( Q \) and are disjoint from \( S \). Suppose \( x \in Q \setminus S \) and \( d(x,S) > \frac{\epsilon}{2n} \). Then as the diameter of any cube in \( \mathcal{M}_m \) is less than \( \frac{1}{2^n} \), we have \( x \in Q_j \) for some \( Q_j \) in \( C \). So

\[
\left\{ x \in Q : d(x,S) > \frac{\epsilon}{2n} \right\} \subset \bigcup_{Q_j \in C} Q_j.
\]

Now, since \( S \) is convex,

\[
\left| \left\{ x \in Q : 0 < d(x,S) < \frac{\epsilon}{2n} \right\} \right| < 2n \cdot \frac{\epsilon}{2n} = \epsilon,
\]

so the desired result holds. \( \square \)

If \( S \) is a set in \( \mathbb{R}^n \) and \( \tau \) is a translation operator given by \( \tau f(x) = f(x - \sigma) \) for some \( \sigma \in \mathbb{R}^n \), we let \( \tau S \) denote the set such that \( \chi_{\tau S}(x) = \chi_S(x - \sigma) \). For each \( c > 0 \) and set \( S \) in \( \mathbb{R}^n \) we define the set \( \delta_c S \) to be such that \( \chi_{\delta_c S}(x) = \chi_S \left( \frac{x}{c} \right) \).

**Lemma 3.** Suppose \( R \in \hat{B} \). Let \( S \in B \) such that \( S \subset R \), \( |R| < 2^n \cdot n^2 |S| \), and \( R \subset 2nS \). Then there exists an a.e. disjoint collection \( \{S_j\} \) of translates of dilates of \( S \) and a collection of translation operators \( \{\tau_j\} \) such that \( S_j \subset R \) for each \( j \), \( |\bigcup S_j| > \frac{1}{1-\alpha} |R| \), and \( R \subset \tau_j \delta_{2^{n+m}} S_j \) for each \( j \). (Here \( m \) is as given by Lemma 2 and \( N \) is as in (2).)

**Proof.** As the techniques of this proof are invariant under affine transformation, we may assume without loss of generality that \( R = Q = [0,1]^n \).

Note that \( \frac{\epsilon}{2n} > \rho \).

By Lemma 2, there exists a collection \( \{Q_j\} \) of (a.e.) disjoint \( n \)-cubes contained in \( R \) and disjoint from \( S \) lying in the mesh \( \mathcal{M}_m \) such that \( |\bigcup Q_j \cup S| \geq 1 - \epsilon \).

Now let \( \{\tau_j\} \) be a collection of translation operators such that \( Q_j = \tau_j \delta_{2^{-n}} R \) for each \( j \).
Let \( S_{1,j} = \tau_j \delta_{2^{-m}} S \). Note that
\[
|S \cup (US_{1,j})| \geq \rho + (1 - \rho - \epsilon) \rho
\]
since \(|(Q_j) \cup S| \geq 1 - \epsilon\) and \(|S| > \rho\).

Let \( S_1 = S \cup (US_{1,j}) \) and let \( S_{2,j} = \tau_j \delta_{2^{-m}} S_1 \). Observe that
\[
|S \cup (US_{2,j})| \geq \rho + (1 - \rho - \epsilon) \rho + (1 - \rho - \epsilon)^2 \rho.
\]

Let \( S_2 = S \cup (US_{2,j}) \).

We proceed by induction. \( S_{k+1,j} \) and \( S_{k+1} \) may be obtained from \( S_k \) via
\[
S_{k+1,j} = \tau_j \delta_{2^{-m}} S_k
\]
and
\[
S_{k+1} = S \cup (US_{k+1,j}) .
\]

Note that
\[
|S \cup (U_jS_{k+1,j})| \geq \rho + (1 - \rho - \epsilon) \rho + \ldots + (1 - \rho - \epsilon)^{k+1} \rho.
\]

Now recall \( N \) is such that
\[
\frac{1 - (1 - \rho - \epsilon)^{N+1}}{\rho + \epsilon} > \frac{1 - \gamma}{1 - \alpha} .
\]
So
\[
|S_N| \geq \rho + (1 - \rho - \epsilon) \rho + \ldots + (1 - \rho - \epsilon)^N \rho = \rho \frac{1 - (1 - \rho - \epsilon)^{N+1}}{1 - (1 - \rho - \epsilon)} = \rho \frac{1 - (1 - \rho - \epsilon)^{N+1}}{\rho + \epsilon} > \frac{1 - \gamma}{1 - \alpha} .
\]

Note also that there exists a collection of translation operators \( \tau_{j,k} \) such that
\[
S_N = S \cup (U_{j=1}^N \cup_k \tau_{j,k} \delta_{2^{-m}} S) ,
\]
where the union above is disjoint. So in particular \( S_N \) may be expressed as the disjoint union \( \cup S'_j \), where \(|\cup S'_j| > \frac{1 - \gamma}{1 - \alpha}\) and each \( S'_j \) is a translate of a dilate of \( S \) such that \( S'_j \subset R \). Moreover there exists a set of translation operators \( \{\tau_j'\} \) such that \( S \subset \tau_j' \delta_{2^{N+m}} S'_j \) for each \( j \). Since \( R \subset 2nS \), there also exists a collection of translation operators \( \{\tau_j''\} \) such that \( R \subset \tau_j'' \delta_{2^{N+m+n}} S'_j \) for each \( j \). Relabeling \( \{S'_j\} \) as \( \{S_j\} \) and \( \{\tau_j''\} \) as \( \{\tau_j\} \), we complete the proof of the lemma.

The following lemma shows that, since \( M_B \) satisfies a Tauberian condition with respect to \( \alpha \), the maximal operator \( M_B \) satisfies a Tauberian condition with respect to any \( \gamma \) greater than \( \alpha \).

**Lemma 4.** If \( \alpha < \gamma < 1 \), there exists \( 0 < C'_{\alpha,\gamma} < \infty \) such that
\[
|\{x : M_B \chi_E(x) > \gamma\}| \leq C'_{\alpha,\gamma} |E|
\]
holds for all measurable sets \( E \) in \( \mathbb{R}^n \), where \( C'_{\alpha,\gamma} \) depends only on \( C_\alpha \), \( \alpha \), \( \gamma \), and the dimension \( n \).
Proof. Let $E$ be a measurable set in $\mathbb{R}^n$. Suppose $R \in \tilde{\mathcal{B}}$ and $\frac{1}{|R|} \int_R \chi_E > \gamma$. Let $\{S_j\}$ be as in Lemma 3. Then there exists $\tilde{S} \in \{S_j\}$ such that $\frac{1}{|\tilde{S}|} \int_{\tilde{S}} \chi_E > \alpha$, as otherwise

$$|E \cap R| \leq \left( \frac{1 - \gamma}{1 - \alpha} \cdot \alpha + 1 \cdot \left( 1 - \frac{1 - \gamma}{1 - \alpha} \right) \right) |R| = \gamma |R|,$$

contradicting the fact that $|E \cap R|/|R| > \gamma$. By Lemma 3 we have $R \subset \tau \delta_{2^j} \tilde{S}$ for some translation operator $\tau$. We now define $\Delta_{\alpha, \gamma}$ by

$$\Delta_{\alpha, \gamma} = 1 + n \log 2 \frac{\log \left( 1 - \frac{2(1 - \gamma)}{2 - \gamma - \alpha} \right)}{\log \left( 1 - \frac{1}{2^j n^n} - \frac{1}{2 - \gamma - \alpha} \frac{1}{2^n n^n} \right)} \left[ \frac{\log \left( \frac{2 - \gamma - \alpha}{2^n - n^n} + \frac{1}{n^n} \right)}{\log 2} + n \right].$$

One can show that $\Delta_{\alpha, \gamma}$ satisfies the inequality

$$\left( \frac{1}{\alpha} \right)^{\frac{1}{2}(\Delta_{\alpha, \gamma} - 1)} \geq 2^{N_m + n}.$$

Note then that $R \subset \mathcal{H}_{\mathcal{B}, \alpha}^{\Delta_{\alpha, \gamma}} (\tilde{S})$ and in particular that $R \subset \mathcal{H}_{\mathcal{B}, \alpha}^{\Delta_{\alpha, \gamma}} (E)$. As $R$ is arbitrary in $\tilde{\mathcal{B}}$ subject to the condition that $\frac{1}{|R|} \int_R \chi_E > \gamma$, we then have

$$\{ x : M_{\mathcal{B}} \chi_E (x) > \gamma \} \subset \mathcal{H}_{\mathcal{B}, \alpha}^{\Delta_{\alpha, \gamma}} (E).$$

By the Tauberian condition on $M_{\mathcal{B}}$ we then have that

$$| \{ x : M_{\mathcal{B}} \chi_E (x) > \gamma \} | \leq C_{\alpha, \gamma} |E|.$$

As $C_{\alpha, \gamma}$ depends only on $C_{\alpha}$, $\alpha$, $\gamma$, and $n$, and the desired result holds.

We now come to the end of the proof of the main theorem. We may assume $0 < \delta < \alpha$ without loss of generality. The hypotheses of the theorem and Lemma 4 and its proof imply that $| \{ x : M_{\mathcal{B}} \chi_E (x) > \gamma \} | \leq C_{\alpha}^{\Delta_{\alpha, \gamma}} |E|$ for $\alpha < \gamma < 1$. We now set $\gamma = \tilde{\gamma} = \frac{1}{2^n n^n}$. Since $\tilde{\mathcal{B}}$ is a homothecy invariant collection of rectangular parallelepipeds, by the closing comments of the proof of Proposition 1 we have that for any measurable set $E$ in $\mathbb{R}^n$

$$\left| \left\{ x : M_{\mathcal{B}} \chi_E (x) > \frac{\delta}{2^n n^n} \right\} \right| \leq C_{\alpha}^{\Delta_{\alpha, \tilde{\gamma}} K_{\frac{1}{2^n n^n}, \tilde{\gamma}}} |E|.$$

Since $M_{\mathcal{B}} f(x) \leq 2^n n^n M_{\mathcal{B}} f(x)$ we then have

$$| \{ x : M_{\mathcal{B}} \chi_E (x) > \delta \} | \leq C_{\alpha}^{\Delta_{\alpha, \tilde{\gamma}} K_{\frac{1}{2^n n^n}, \tilde{\gamma}}} |E|.$$

As $\Delta_{\alpha, \tilde{\gamma}}$ and $K_{\frac{1}{2^n n^n}, \tilde{\gamma}}$ depend only on $\alpha$, $\delta$, and $n$, the desired result holds.

We now show that the proof of the above result implies that, if $\mathcal{B}$ is a homothecy invariant collection of convex sets in $\mathbb{R}^n$ and the associated maximal operator $M_{\mathcal{B}}$ satisfies a Tauberian condition with respect to some $0 < \alpha < 1$, then $M_{\mathcal{B}}$ must be bounded on $L^p(\mathbb{R}^n)$ for sufficiently large $p$. 

\[ \square \]
Corollary 1. Let \( \mathcal{B} \) be a homothecy invariant collection of convex sets in \( \mathbb{R}^n \). Suppose for some \( 0 < \alpha < 1 \) there exists a positive finite constant \( C_\alpha \) such that
\[
|\{x : M_\mathcal{B} \chi_E(x) > \alpha\}| \leq C_\alpha |E|
\]
holds for every measurable set \( E \) in \( \mathbb{R}^n \). Then \( M_\mathcal{B} \) is bounded on \( L^p(\mathbb{R}^n) \) for sufficiently large \( p \). In particular, there exists \( p_\alpha < \infty \) depending only on \( \alpha, n, \) and \( C_\alpha \) such that \( M_\mathcal{B} \) is bounded on \( L^p(\mathbb{R}^n) \) for all \( p > p_\alpha \).

Proof. Let \( \delta < \min\left(\frac{1}{100}, \alpha\right) \). By the closing remarks of the proof of Theorem 1 we have that
\[
|\{x : M_\mathcal{B} \chi_E(x) > \delta\}| \leq C_\alpha \Delta_{\alpha, \bar{\alpha}} K_{n, \bar{\alpha}} |E|
\]
\[
\leq C_\alpha \Delta_{\alpha, \bar{\alpha}} \left(2 + \frac{\log (\frac{2\alpha}{\delta})}{\log \bar{\alpha}}\right) |E|
\]
\[
\leq C_\alpha \Delta_{\alpha, \bar{\alpha}} \left(\frac{2 + \log (\frac{2\alpha}{\delta})}{\log \bar{\alpha}}\right) |E|
\]
\[
\leq C_\alpha \Delta_{\alpha, \bar{\alpha}} \left(\frac{2 + \log (\frac{2\alpha}{\delta})}{\log \bar{\alpha}}\right) \Delta_{\alpha, \bar{\alpha}} |E|
\]
Hence \( M_\mathcal{B} \) is of restricted weak type \((p_\alpha, p_\alpha)\), where
\[
p_\alpha = \frac{\log C_\alpha}{\log \bar{\alpha}} \left(2 + \frac{\log (\frac{2\alpha}{\delta})}{\log \bar{\alpha}}\right) \Delta_{\alpha, \bar{\alpha}},
\]
and hence \( M_\mathcal{B} \) is bounded on \( L^p(\mathbb{R}^n) \) for any \( p > p_\alpha \). As \( p_\alpha \) depends only on \( \alpha, n, \) and \( C_\alpha \), the desired result follows.

Recall that a collection of sets in \( \mathbb{R}^n \) is said to be a density basis if it differentiates \( L^\infty(\mathbb{R}^n) \). We conclude this paper by observing the rather striking result that any density basis consisting of a homothecy invariant collection of convex sets in \( \mathbb{R}^n \) must differentiate \( L^p(\mathbb{R}^n) \) for sufficiently large \( p \).

Corollary 2. Let \( \mathcal{B} \) be a density basis that is a homothecy invariant collection of convex sets in \( \mathbb{R}^n \). Then \( \mathcal{B} \) differentiates \( L^p(\mathbb{R}^n) \) for sufficiently large \( p \).

Proof. Suppose \( \mathcal{B} \) is a density basis that is a homothecy invariant collection of convex sets in \( \mathbb{R}^n \). Then since \( \mathcal{B} \) is a Busemann-Feller basis that is invariant by homothecies, we know for some \( 0 < C < \infty \) that
\[
|\{x : M_\mathcal{B} \chi_E(x) > \frac{1}{2}\}| \leq C |E|
\]
holds for all measurable sets \( E \) in \( \mathbb{R}^n \). (See p. 69 of [3] for a proof of this result.) By Corollary 1 we then have that \( M_\mathcal{B} \) is bounded on \( L^p(\mathbb{R}^n) \) for sufficiently large \( p \) and hence \( \mathcal{B} \) differentiates \( L^p(\mathbb{R}^n) \) for sufficiently large \( p \).

References

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