

TOPOLOGICAL ENTROPIES OF EQUIVALENT SMOOTH FLOWS

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ABSTRACT. We construct two equivalent smooth flows, one of which has positive topological entropy and the other has zero topological entropy. This provides a negative answer to a problem posed by Ohno.

1. INTRODUCTION

Two flows defined on a smooth manifold are *equivalent* if there exists a homeomorphism of the manifold that sends each orbit of one flow onto an orbit of the other flow while preserving the time orientation. The *topological entropy* of a flow is defined as the entropy of its time-1 map. While topological entropy is an invariant for equivalent homeomorphisms (Theorem 7.2 in [13]), finite non-zero topological entropy for a flow cannot be an invariant because its value is affected by time reparameterization. However, 0 and ∞ topological entropy are invariants for equivalent flows *without* fixed points (see [7], [8], [10], [11]).

In equivalent flows *with* fixed points there exists a counterexample, constructed by Ohno [7], showing that neither 0 nor ∞ topological entropy is preserved by equivalence. The two flows constructed in [7] are suspensions of a transitive subshift and thus are not differentiable. Note that a differentiable flow on a compact manifold cannot have ∞ entropy (see Theorem 7.15 in [13]). These facts led Ohno [7] in 1980 to ask the following:

Is 0 topological entropy an invariant for equivalent differentiable flows?

In this paper, we construct two equivalent C^∞ smooth flows with a singularity, one of which has positive topological entropy while the other has zero topological entropy. This gives a negative answer to Ohno's question.

We begin by supposing that $f : M \rightarrow M$ is a C^∞ diffeomorphism of a smooth compact Riemannian manifold M with $\dim M = m \geq 2$ with the following properties: (1) f has positive topological entropy and (2) f is minimal in the sense that all forward orbits are dense in M (or equivalently closed invariant sets are either empty or the entire space). An example of such an f was constructed by Herman [3]. Using the constant function $I : M \rightarrow \mathbb{R}$, $I(x) \equiv 1$, one gets a suspension manifold Ω

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and a smooth suspension flow $\psi : \Omega \times \mathbb{R} \rightarrow \Omega$ (Definition 2.1). Let X denote the smooth vector field associated with this flow. For any function $\alpha \in C^\infty(\Omega, [0, 1])$, note that αX is a C^∞ vector field on Ω and it thus induces a unique, differentiable flow.

Theorem 1.1 (Main Theorem). *There exist two functions $\alpha, \hat{\alpha} \in C^\infty(\Omega, [0, 1])$ satisfying the following:*

- (1) αX and $\hat{\alpha} X$ induce equivalent flows $\varphi(t)$ and $\hat{\varphi}(t)$;
- (2) $\varphi(t)$ has zero topological entropy and $\hat{\varphi}(t)$ has positive topological entropy.

The key idea used by Ohno and in this work is that the time reparameterization between an orbit in one flow and its image orbit in an equivalent flow can grow super-exponentially near a fixed point. In the present case the main challenge is to ensure that the time parameterizations lead to differentiable flows. Throughout the rest of the paper we fix one point $p = (x_0, 0) \in \Omega$ and consider functions $\alpha, \hat{\alpha} \in C^\infty(\Omega, [0, 1])$ satisfying the following criteria:

(H). We assume that $\alpha(p) = \hat{\alpha}(p) = 0$, $\alpha(q)$ and $\hat{\alpha}(q) > 0$, for $q \neq p \in \Omega$, and there exists a small neighborhood V of p in Ω such that $\alpha(q) = \hat{\alpha}(q) \equiv 1$ if $q \in \Omega \setminus V$.

In our construction $\hat{\varphi}$ will be shown to have an ergodic probability measure that is equivalent to an ergodic probability preserved by the suspension flow of f . Positive entropy of f will then imply positive entropy for $\hat{\varphi}$. On the other hand, in the construction of φ , α will be very flat so that orbits near p will be strongly slowed down. This together with minimality, which ensures fast returns to any neighborhood of p , will imply that ϕ has Dirac measure at p as its unique ergodic probability measure.

2. PRELIMINARIES

2.1. Notation. The symbol $\langle \cdot, \cdot \rangle$ will be used to denote the Riemannian metric on either M or Ω depending on the context. We will denote a ball in M by $B_M(x, \epsilon) = \{y \in M : d(x, y) < \epsilon\}$ and by $B_\Omega(x, \epsilon)$ a ball in Ω . A ball in \mathbb{R}^n centered at the origin we will denote by $B^n(r) = \{x \in \mathbb{R}^n : |x| < r\}$.

Definition 2.1. Suppose that M is a smooth compact Riemannian manifold and that f is a C^∞ diffeomorphism. Consider the space

$$\Omega = M \times [0, 1] / \sim,$$

where \sim is the identification of $(y, 1)$ with $(f(y), 0)$. The *standard suspension* of f is the flow ψ_t on Ω defined by $\psi_t(y, s) = (y, t + s)$, for $0 \leq t + s < 1$.

A standard argument as in [5] shows that Ω is a smooth compact Riemannian manifold and ψ_t is C^∞ . If $f : M \rightarrow M$ is minimal as a homeomorphism, then ψ_t is a minimal flow.

Proposition 2.2. *If α and $\hat{\alpha}$ in $C^\infty(\Omega, [0, 1])$ satisfy **(H)** and X is the suspension vector field on Ω described above, then αX and $\hat{\alpha} X$ induce equivalent differential flows on Ω with one singularity p .*

Proof. That a C^∞ vector field induces a C^∞ flow is a standard result. The identity map on Ω takes orbits of one flow to orbits of the other since the unique singular point p is mapped to itself and elsewhere α and $\hat{\alpha}$ are positive. The assumption that α and $\hat{\alpha}$ are non-negative also implies preservation of time orientation. \square

2.2. Minimal homeomorphisms and uniform recurrence. The following result concerning uniform recurrence of orbits in minimal homeomorphisms will be used in the proof that φ has zero entropy. It guarantees fast return of orbits to a neighborhood of the stopped point p .

Lemma 2.3. *Let M denote a smooth compact Riemannian manifold and let $f : M \rightarrow M$ be a C^∞ diffeomorphism. Suppose (M, f) is a minimal homeomorphism. Then for any $\varepsilon > 0$, there exists $L(\varepsilon) \in \mathbb{N}$ such that, for any $x, y \in M$, $f^l(y) \in B(x, \varepsilon)$, for some l , $0 \leq l \leq L(\varepsilon)$.*

Proof. This is a classical result (see [2]). We present a proof for the convenience of the readers.

For any $x, y \in M$ and $\varepsilon > 0$, there exists $l(x, y, \varepsilon) \in \mathbb{N}$ such that $f^{l(x, y, \varepsilon)}(y) \in B(x, \frac{\varepsilon}{2})$, since f is minimal. It follows that there exists $\delta(y) > 0$ such that $f^{l(x, y, \varepsilon)}(B(y, \delta(y))) \subset B(x, \frac{3\varepsilon}{4}) \subset B(x, \varepsilon)$. By the compactness of M , $\exists y_1, \dots, y_n$ and $L(x, \varepsilon) = \max_i \{l(x, y_i, \varepsilon)\}$ such that

- (i) $\bigcup_{i=1}^n B(y_i, \delta(y_i)) = M$;
- (ii) $\forall y \in M, \exists l(y) \leq L(x, \varepsilon)$ such that $f^{l(y)}(y) \in B(x, \frac{3\varepsilon}{4}) \subset B(x, \varepsilon)$.

Then, for any $\tilde{x} \in B(x, \frac{\varepsilon}{4})$, $f^{l(y)}(y) \in B(\tilde{x}, \varepsilon)$. The compactness of M implies that there exist finite points x_1, \dots, x_{n_0} such that $\bigcup_{i=1}^{n_0} B(x_i, \frac{\varepsilon}{4}) = M$. Then, $L(\varepsilon) = \max_i \{L(x_i, \varepsilon)\}$ satisfies the lemma. □

Uniform recurrence as in Lemma 2.3 allows us to put lower bounds on the measures of sets with respect to ergodic measures.

Lemma 2.4. *Suppose the assumptions of Lemma 2.3 and suppose that μ is an ergodic measure of f . Then*

$$\mu(B_M(x, \varepsilon)) \geq \frac{1}{L(\varepsilon)} > 0, \quad \forall x \in M, \varepsilon > 0,$$

where $L(\varepsilon)$ is as in Lemma 2.3.

Proof. Set

$$Q_\mu(f) = \left\{ x \in M : \lim_{n \rightarrow \pm\infty} \frac{1}{n} \sum_{k=0}^{n-1} \xi(f^k x) = \int_M \xi(x) d\mu(x), \quad \forall \xi \in C^0(M) \right\}.$$

By the Birkhoff ergodic theorem, $Q_\mu(f)$ is an f -invariant and μ -full measure set. For any $x \in M$ and $\varepsilon > 0$, take $L(\varepsilon)$ as in Lemma 2.3. If $y \in Q_\mu(f)$, then by the Birkhoff ergodic theorem and Lemma 2.3,

$$\begin{aligned} \mu(B_M(x, \varepsilon)) &= \lim_{n \rightarrow \infty} \frac{\text{Card}\{i \in \{0, 1, \dots, n-1\} : f^i(y) \cap B_M(x, \varepsilon) \neq \emptyset\}}{n} \\ &\geq \frac{1}{L(\varepsilon)} > 0. \end{aligned} \quad \square$$

2.3. Time-changed systems and entropy. First, we recall the notion of an additive function.

Definition 2.5. Suppose ψ_t is a measurable flow on a Borel probability space $(\Omega, \mathcal{B}, \mu)$ and Ω is divided into disjoint invariant measurable sets A and N such that $\mu(A) = 1$ and $\mu(N) = 0$. Further suppose that $\theta(t, x)$ is a real measurable

function defined on $(-\infty, +\infty) \times (\Omega \setminus N) = \mathbb{R} \times A$ with the following properties for every fixed $x \in A$:

- (1) $\theta(t, x)$ is continuous and non-decreasing in t ;
- (2) $\theta(t + s, x) = \theta(s, x) + \theta(t, \psi_s x)$ for all t and s ;
- (3) $\theta(0, x) = 0$, $\lim_{t \rightarrow \infty} \theta(t, x) = \infty$, $\lim_{t \rightarrow -\infty} \theta(t, x) = -\infty$.

Then θ is called an *additive function of ψ_t with carrier A* . An additive function is said to be *integrable* if it is integrable in Ω for every fixed t .

Lemma 2.6. *If ψ_t is a measurable flow on a Borel probability space $(\Omega, \mathcal{B}, \mu)$ and $a(x)$ is a non-negative, integrable function satisfying*

$$E_\mu(a) = \int_\Omega a(x) d\mu(x) > 0,$$

then the function

$$\theta(t, x) = \int_0^t a(\psi_s x) ds$$

is an integrable additive function.

For a proof see Theorem 3.1 in [12].

Definition 2.7. The function $\theta(t, x)$ in Lemma 2.6 is called the *additive function defined by $a(x)$* .

Let ψ_t be a measurable flow on a Borel probability space $(\Omega, \mathcal{B}, \mu)$ and let θ be an additive function of ψ_t . Define

$$\varphi_t x = \psi_{\tau(t, x)} x, \quad \text{with } \tau(t, x) = \sup\{s : \theta(s, x) \leq t\},$$

for all $-\infty < t < \infty$ and all $x \in A$, where A is the carrier of θ . We call $\widehat{\Omega} = \{x \in A : \theta(t, x) > 0, \forall t > 0\}$ the *regular set* of θ , in other words, the set of non-singular points of φ_t . Let $\widehat{\mathcal{B}} = \mathcal{B} \cap \widehat{\Omega}$. Then $\varphi_t : (\widehat{\Omega}, \widehat{\mathcal{B}}) \rightarrow (\widehat{\Omega}, \widehat{\mathcal{B}})$ is a measurable flow (see Lemma 4.1 in [12]), which is called the *time changed system* of ψ_t by θ .

Let μ be a ψ_t invariant probability measure and $a(x)$ be a non-negative, integrable function. Define

$$\widehat{\mu}(B) = \int_B d\widehat{\mu}(x) = \int_B a(x) d\mu(x)$$

for all $B \in \widehat{\mathcal{B}}$. We get an invariant measure $\widehat{\mu}$ of the time changed system φ_t (see ([6]).

Lemma 2.8 ([12]). *If ψ_t is ergodic, then any time changed flow φ_t of ψ_t is also ergodic.*

The following is Theorem 10.1 of [12].

Theorem 2.9. *Let ψ_t be an arbitrary measurable flow on a Borel probability space $(\Omega, \mathcal{B}, \mu)$, and let θ be any integrable additive function of ψ_t . If the flow φ_t on $(\widehat{\Omega}, \widehat{\mathcal{B}}, \widehat{\mu})$ is the time changed flow of ψ_t by θ , then we have the inequality*

$$h_{\widehat{\mu}}(\varphi_t) \widehat{\mu}(\widehat{\Omega}) \leq h_\mu(\psi_t) \mu(\Omega)$$

for all fixed t , where $h_{\widehat{\mu}}(\varphi_t)$ and $h_\mu(\psi_t)$ denote the measure-theoretic entropies of the homeomorphisms φ_t and ψ_t , respectively. The equality holds when ψ_t is ergodic.

This theorem will be used to establish that the entropy of $\widehat{\varphi}$ is non-zero, by exhibiting an ergodic μ with positive entropy and establishing that $\bar{\mu}$ is finite.

Definition 2.10. An ergodic measure is *atomic* if it is supported on a periodic orbit.

Lemma 2.11. Assume that ψ_t is the standard suspension of f from above, X is the vector field that induces ψ_t and $\alpha \in C^1(\Omega, [0, 1])$ satisfies **(H)** (see §1). Denote by φ_t the flow induced by the vector field αX on Ω . For any $x \in M$, define $\gamma(x)$ by:

$$\begin{cases} \varphi_{\gamma(x)}(x, 0) = \psi_1(x, 0) = (f(x), 0), & (x, 0) \neq f^{-1}(p) \text{ and } (x, 0) \neq p; \\ \gamma(x) = +\infty, & (x, 0) = f^{-1}(p) \text{ or } (x, 0) = p. \end{cases}$$

Then for any non-atomic ergodic measure $\bar{\mu}$ of φ_t , there exists a non-atomic ergodic measure μ of f such that

$$(2.1) \quad E_{\bar{\mu}}(\xi) = \frac{1}{E_{\mu}(\gamma)} E_{\mu} \left(\int_0^{\gamma(x)} \xi(\varphi_t(x, 0)) dt \right), \quad \forall \xi \in C^0(\Omega),$$

where $E_{\bar{\mu}}(\xi) = \int_{\Omega} \xi d\bar{\mu}$ and $E_{\mu}(\gamma) = \int_M \gamma(y) d\mu(y)$.

The proof is similar to Lemma 4 in [7] and is thus omitted.

Corollary 2.12. If $E_{\mu}(\gamma) = +\infty$ for all non-atomic ergodic measures μ of f , then φ_t has only atomic invariant Borel probability measures.

Proof. Otherwise, φ_t should have a non-atomic invariant Borel probability measure $\bar{\mu}$ and by Lemma 2.11, there exists a non-atomic ergodic measure μ of f satisfying equation (2.1). Since $\bar{\mu}$ is non-atomic, one can choose an open set $V \subset \Omega \setminus \{p\}$ such that $\bar{V} \subset \Omega \setminus \{p\}$ and $\bar{\mu}(V) > 0$. Let U be an open set satisfying $\bar{V} \subset U \subset \bar{U} \subset \Omega \setminus \{p\}$.

Define a continuous function $\xi : \Omega \rightarrow [0, 1]$ such that $\xi|_V \equiv 1$ and $\xi|_{\Omega \setminus U} \equiv 0$. Clearly, $E_{\bar{\mu}}(\xi) > 0$. A straightforward calculation also shows that

$$(2.2) \quad E_{\mu} \left(\int_0^{\gamma(x)} \xi(\varphi_t(x, 0)) dt \right)$$

is finite. Thus $\frac{1}{E_{\mu}(\gamma)} E_{\mu} \left(\int_0^{\gamma(x)} \xi(\varphi_t(x, 0)) dt \right) = 0$. This implies that μ and $\bar{\mu}$ do not satisfy equation (2.1) and so we reach a contradiction. \square

Lemma 2.4 will be used to show that $E_{\mu}(\gamma) = +\infty$ for φ and then Corollary 2.12 will imply that φ has only atomic invariant probability measures and thus has zero entropy.

2.4. Measure-theoretic entropies of f and its standard suspension ψ_t .

First we present a link between the invariant measures of f and ψ_t .

Lemma 2.13. For any invariant probability measure μ of f on M , we define

$$\int_{\Omega} \xi d\bar{\mu} := \int_M \int_0^1 \xi(x, t) dt d\mu, \quad \forall \xi \in C^0(\Omega).$$

Then $\bar{\mu}$ is an invariant measure of ψ_t on Ω . Further, $\bar{\mu}$ is ergodic if μ is ergodic.

The proof is elementary and omitted.

That topological entropies of a homeomorphism and its standard suspension coincide is a well-known fact; see e.g. [1]. The same is true for measure-theoretic entropy, although, to our best knowledge, we could not find the original proof in the literature. For convenience of the reader we present a proof (see Proposition 2.15 below) using Katok's definition [4] of metric entropy.

For given $x \in M$, $n \in \mathbb{N}$ and $\varepsilon > 0$, we set

$$D(x, n, \varepsilon, f) = \{y \in M : d(f^i x, f^i y) < \varepsilon, i = 0, 1, \dots, n-1\}$$

and call it an (n, ε, f) -ball.

Definition 2.14 ([4]). Given an ergodic measure μ on $f : M \rightarrow M$ and a real $\delta \in (0, 1)$, let $R(\delta, n, \varepsilon, f)$ denote the smallest number of (n, ε, f) -balls needed to cover a set whose μ -measure is greater than $1 - \delta$. Then the *measure-theoretic entropy* of f , denoted by $h_\mu(f)$, is defined by

$$h_\mu(f) := \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \ln R(\delta, n, \varepsilon, f).$$

The quantity $h_\mu(f)$ in Definition 2.14 is independent of the choice of δ (see [4]). In a similar way we define measure-theoretic entropy for ψ as follows. For $q \in \Omega$, $t \in \mathbb{R}$ and $\varepsilon > 0$, we define a (t, ε, ψ) -ball

$$D(q, t, \varepsilon, \psi) = \{w \in \Omega : d(\psi_s w, \psi_s q) < \varepsilon, 0 \leq s \leq t\}.$$

Given an ergodic measure μ of ψ and $\delta \in (0, 1)$, let $R(\delta, t, \varepsilon, \psi)$ denote the smallest number of (t, ε, ψ) -balls needed to cover a set whose μ -measure is greater than $1 - \delta$. Then the measure-theoretic entropy of ψ , denoted by $h_\mu(\psi)$, is defined by

$$h_\mu(\psi) := \lim_{\varepsilon \rightarrow 0} \limsup_{t \rightarrow \infty} \frac{1}{t} \ln R(\delta, t, \varepsilon, \psi).$$

One shows easily that $h_\mu(\psi)$ defined in this way coincides with $h_\mu(\psi_1)$, the measure-theoretic entropy for the time-one map.

Proposition 2.15. *Let μ be an ergodic measure of f on M . Then we have*

$$h_{\bar{\mu}}(\psi) = h_\mu(f),$$

where $\bar{\mu}$ is defined as in Lemma 2.13.

Proof. For given $q = (x, t) \in \Omega$, $n \in \mathbb{N}$ and $\varepsilon > 0$, we set

$$D(q, n, \varepsilon, \psi) = \{w \in \Omega : d(\psi_i w, \psi_i q) < \varepsilon, i = 0, 1, \dots, n-1\}$$

and

$$\tilde{D}(q, n, \varepsilon, \psi) = \{(y, s) : y \in D(x, n, \varepsilon, f), t - \varepsilon < s < t + \varepsilon\},$$

where $D(x, n, \varepsilon, f)$ is defined before Definition 2.14. Then by Definition 2.1 it follows that

$$\tilde{D}(q, n, \varepsilon, \psi) = \{w \in \Omega : d(\psi_\tau w, \psi_\tau q) < \varepsilon, 0 \leq \tau \leq n\}.$$

We call $\tilde{D}(q, n, \varepsilon, \psi)$ an (n, ε, ψ) -box with center q . Let $\tilde{R}(\delta, n, \varepsilon, \psi)$ denote the smallest number of (n, ε, ψ) -boxes needed to cover a set whose $\bar{\mu}$ -measure is greater than $1 - \delta$. Then,

$$(2.3) \quad h_{\bar{\mu}}(\psi) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \ln \tilde{R}(\delta, n, \varepsilon, \psi).$$

We will prove that $h_{\bar{\mu}}(\psi) = h_\mu(f)$ in two steps.

Step 1. For a subset $A \subset \Omega$, we denote

$$\pi A = \{x \in M : (x, t) \in A, \text{ for some } 0 \leq t < 1\}.$$

We have

$$\bar{\mu}(A) = \int_M \int_0^1 \chi_A(x, t) dt d\mu \leq \int_M \chi_{\pi A}(x) d\mu = \mu(\pi(A)).$$

Let $\varepsilon > 0$ and $0 < \delta < 1$. If the boxes $\tilde{D}(q_1, n, \varepsilon, \psi), \tilde{D}(q_2, n, \varepsilon, \psi), \dots, \tilde{D}(q_{\tilde{R}(\delta, n, \varepsilon, \psi)}, n, \varepsilon, \psi)$ cover a set $A \subset \Omega$ and $\bar{\mu}(A) \geq 1 - \delta$, then $D(x_1, n, \varepsilon, f), D(x_2, n, \varepsilon, f), \dots, D(x_{\tilde{R}(\delta, n, \varepsilon, \psi)}, n, \varepsilon, f)$ cover πA and $\mu(\pi A) \geq 1 - \delta$; here $q_i = (x_i, t_i)$ for $i = 1, 2, \dots, \tilde{R}(\delta, n, \varepsilon, \psi)$. So, $\tilde{R}(\delta, n, \varepsilon, \psi) \geq R(\delta, n, \varepsilon, f)$ and thus $h_{\bar{\mu}}(\psi) \geq h_{\mu}(f)$ by Definition 2.14 and (2.3).

Step 2. Let $\varepsilon > 0$ and $0 < \delta < 1$. If the (n, ε, f) -balls

$$D(x_1, n, \varepsilon, f), D(x_2, n, \varepsilon, f), \dots, D(x_{R(\delta, n, \varepsilon, f)}, n, \varepsilon, f)$$

cover $B \subset M$ with $\mu(B) \geq 1 - \delta$, then the boxes

$$\begin{aligned} &\tilde{D}((x_1, 0), n, \varepsilon, \psi), \quad \dots, \quad \tilde{D}((x_1, k(\varepsilon)), n, \varepsilon, \psi), \\ &\tilde{D}((x_2, 0), n, \varepsilon, \psi), \quad \dots, \quad \tilde{D}((x_2, k(\varepsilon)), n, \varepsilon, \psi), \\ &\dots, \\ &\tilde{D}((x_{R(\delta, n, \varepsilon, f)}, 0), n, \varepsilon, \psi), \quad \dots, \quad \tilde{D}((x_{R(\delta, n, \varepsilon, f)}, k(\varepsilon)), n, \varepsilon, \psi) \end{aligned}$$

cover a set $A \subset \Omega$ with $\bar{\mu}(A) \geq 1 - \delta$, where $k(\varepsilon)$ is the smallest integer larger than $\frac{1}{\varepsilon}$. So,

$$\tilde{R}(\delta, n, \varepsilon, \psi) \leq k(\varepsilon)R(\delta, n, \varepsilon, f).$$

Then by Definition 2.14 and (2.3) we conclude that $h_{\bar{\mu}}(\psi) \leq h_{\mu}(f)$. □

3. PROOF OF MAIN THEOREM

Recall that M denotes a smooth compact Riemannian manifold of dimension at least 2. In the rest of the paper we suppose that $f : M \rightarrow M$ is a C^∞ minimal diffeomorphism, it preserves a measure μ and has positive measure-theoretic entropy $h_\mu(f) > 0$ (and thus has a positive topological entropy). See [3] for the existence of such a diffeomorphism that preserves Riemann volume. By the variational principle we can assume that μ is an ergodic measure.

As in Definition 2.1 let Ω be the suspension manifold with the standard differentiable suspension flow $\psi : \Omega \times \mathbb{R} \rightarrow \Omega$. Denote by X the C^∞ vector field on Ω which induces ψ . We will construct a function $\alpha \in C^\infty(\Omega, [0, 1])$ and a C^∞ vector field $Y = \alpha X$ on Ω with zero topological entropy in Theorem 3.2. In Theorem 3.3 we construct a function $\hat{\alpha} \in C^\infty(\Omega, [0, 1])$ and corresponding C^∞ vector field $\hat{Y} = \hat{\alpha} X$ on Ω with positive topological entropy. Both α and $\hat{\alpha}$ will be constructed satisfying condition **(H)** and by Proposition 2.2 the two differentiable flows induced by Y and \hat{Y} are equivalent, thus completing the Main Theorem.

Before we prove Theorem 3.2 below, we will need the following.

Lemma 3.1. *For a given sequence of positive numbers $1 = \beta_{-1} > \beta_0 > \beta_1 > \dots > \beta_i > \dots$, there exists a C^∞ function $\omega : B^n(2) \rightarrow [0, 1]$ such that*

- (1) $\omega(x) = 0$ if and only if $x = 0$;
- (2) $\|\omega|_{B^n(\frac{1}{i+1})}\|_{C^0} \leq \beta_{i-1}$, $i = 0, 1, 2, \dots$;
- (3) $\omega|_{B^n(2) \setminus B^n(1)} = 1$.

Proof. Without loss of generality, we assume that $\lim_{i \rightarrow \infty} \beta_i = 0$.

Let $h(t)$ be the function:

$$h(t) = \begin{cases} e^{-1/t}, & 0 < t \leq 1, \\ 0, & -1 < t \leq 0. \end{cases}$$

Clearly $h(t)$ is C^∞ smooth and has zero derivatives of all orders at 0. Let $\{\beta_i\}$ be as above and suppose $\{\alpha_i\}$ is any decreasing sequence of positive numbers $1 = \alpha_{-1} > \alpha_0 > \alpha_1 > \dots > \alpha_i > \dots$, with $\lim_{i \rightarrow \infty} \alpha_i = 0$. For $t < \alpha_0$, let $\eta(t)$ be the function on $[-1, \alpha_0]$ defined by the series:

$$\eta(t) = \sum_{i=1}^{\infty} 2^{-i-1} \beta_{i-1} h(t - \alpha_i).$$

This series is monotone increasing in i and converges uniformly. It is zero on $[-1, 0]$ and positive on $(0, \alpha_0]$. For any k and $0 < t < \alpha_k$ we have

$$(3.1) \quad \eta(t) = \sum_{i=k+1}^{\infty} 2^{-i-1} \beta_{i-1} h(t - \alpha_i) < \frac{\beta_k h(1)}{2^k}.$$

Further, since the derivatives of the partial sums of this series converge uniformly, we may take the derivative of the sum and we have that:

$$\eta'(t) = \sum_{i=1}^{\infty} 2^{-i-1} \beta_{i-1} h'(t - \alpha_i).$$

This series is also monotone increasing and converges uniformly. For $0 < t < \alpha_k$ we obtain

$$(3.2) \quad \eta'(t) = \sum_{i=j+1}^{\infty} 2^{-i-1} \beta_{i-1} h'(t - \alpha_i) < \frac{\beta_k h'(1)}{2^k}.$$

Thus, $\lim_{t \rightarrow 0^+} \eta'(t) = 0$. By induction, we may also conclude that

$$\eta^{(k)}(t) = \sum_{i=1}^{\infty} 2^{-i-1} \beta_{i-1} h^{(k)}(t - \alpha_i)$$

converges uniformly and

$$(3.3) \quad \lim_{t \rightarrow 0^+} \eta^{(k)}(t) = 0.$$

We can now clearly extend $\eta(t)$ to the interval $[0, 2]$ in such a way that η is C^∞ and $\eta(t) = 1$ for $t \in [1, 2]$.

To finish the proof of the lemma, we set $\alpha_i = (i + 1)^{-1}$ in the above construction of $\eta(t)$ and let $\omega(x) = \eta(|x|)$ on $B^n(2)$. Because of the construction and equation (3.3) this function is C^∞ smooth at 0 and on $B^n(2)$. \square

Theorem 3.2. *There exists a function $\alpha \in C^\infty(\Omega, [0, 1])$, satisfying **(H)**, such that the flow defined by the vector field $Y = \alpha X$ has zero topological entropy.*

Proof. Recall that $p = [(x_0, 0)] = \pi(x_0, 0)$ in condition **(H)** where π is the quotient map $\pi : M \times \mathbb{R} \rightarrow \Omega$ for the suspension.

Without loss of generality, we can assume the existence of a coordinate chart (\tilde{V}, ξ) of Ω satisfying the following.

- (i) There exists an open set V of Ω , such that $p \in V$ and $\bar{V} \subset \tilde{V}$.
- (ii) $\xi(p) = 0$, $\xi(V) = B^{m+1}(1)$, $\xi(\tilde{V}) = B^{m+1}(2)$, where $2 \leq m = \dim M$.

(iii) $\exists i_1 \in \mathbb{N}$ such that

$$\text{Closure}(\pi(B_M(x_0, i_1^{-1}) \times \{0\})) \subset V$$

and

$$\xi(\pi(B_M(x_0, i_1^{-1}) \times \{0\})) \subset \mathcal{R} = \{x = (x_1, \dots, x_m, x_{m+1}) : x_{m+1} = 0\}.$$

(iv) $\exists i_2 \in \mathbb{N}$ such that

$$B_\Omega(p, i_2^{-1}) \subset V$$

and

$$\xi(B_\Omega(p, i^{-1})) = B^{m+1}(i^{-1})$$

for any $i_2 < i \in \mathbb{N}$.

Under these assumptions, there exists $i_3 \in \mathbb{N}$ and $i_2 < i_3$, with the property that for any $i \geq 0$ there exists $1 \gg l_{i_3+i} > 0$ such that

$$\text{Closure}\left(\pi\left(B_M\left(x_0, \frac{1}{i_3+i}\right) \times [-l_{i_3+i}, 0]\right)\right) \subset B_\Omega\left(p, \frac{1}{i_2+i}\right).$$

We set $i_0 := \max\{i_1, i_2, i_3\}$. For any $i > i_0$, by Lemma 2.4, there exists $L(\frac{1}{i})$ such that for any ergodic measure τ of f , we have

$$\tau(B_M(f^{-1}x_0, \frac{1}{i})) \geq \frac{1}{L(\frac{1}{i})} := \delta(i) > 0.$$

We define $\beta_{-1} := 1$ and $\beta_{i-1} := \frac{l_{i_0+i}}{i_0+i} \delta(i_0 + i)$ for $i \geq 1$.

Using Lemma 3.1, one can find a C^∞ function $\omega : \xi(\tilde{V}) \rightarrow [0, 1]$ with the properties:

- (i) $\omega|_{\xi(\tilde{V} \setminus V)} \equiv 1$;
- (ii) $\|\omega|_{B^{m+1}(\frac{1}{i_2+i})}\|_{C^0} \leq \beta_{i-1}$;
- (iii) $\omega(0) = 0$ and $0 < \omega(a) \leq 1$ for $0 \neq a \in \xi(\tilde{V})$.

We then define a function α in $C^\infty(\Omega, [0, 1])$ as follows:

$$\alpha(q) := \begin{cases} \omega \circ \xi(q), & q \in \tilde{V}; \\ 1, & q \in \Omega \setminus \tilde{V}. \end{cases}$$

Then

$$\left\| \alpha|_{B_\Omega(p, \frac{1}{i_2+i})} \right\|_{C^0} = \sup_{x \in B_\Omega(p, \frac{1}{i_2+i})} \{\alpha(x)\} \leq \beta_{i-1},$$

where we assume, without loss of generality, that $\|\xi\|_{C^0} \leq 1$. We then define $Y := \alpha X$ and let φ_t denote the flow induced by Y .

Recall the function $\gamma : M \rightarrow \mathbb{R} \cup \{\infty\}$ in Lemma 2.11 and observe that for any $x \in B_M(f^{-1}(x_0), \frac{1}{i_2+i})$,

$$l_{i_0+i} = \int_{t(x)}^{\gamma(x)} \sqrt{\langle \alpha(\varphi_s(x))X(\varphi_s(x)), \alpha(\varphi_s(x))X(\varphi_s(x)) \rangle} ds,$$

where $t(x) > 0$ satisfies $\varphi_{t(x)}(x) = \psi_{1-l_{i_0+i}}(x)$. Then

$$\gamma(x) \geq \gamma(x) - t(x) \geq \frac{l_{i_0+i}}{\|\alpha|_{B_\Omega(p, \frac{1}{i_2+t})}\| \|X\|} \geq \frac{l_{i_0+i}}{\beta_{i-1} \|X\|} = \frac{i_0+i}{\delta(i_0+i) \|X\|}$$

for any $x \in B_M(f^{-1}(x_0), \frac{1}{i_0+i})$. Therefore,

$$\gamma|_{B_M(f^{-1}(x_0), \frac{1}{i_0+i})} \geq \frac{i_0+i}{\delta(i_0+i) \|X\|}.$$

Now for any ergodic measure τ of f ,

$$E_\tau(\gamma) = \int_M \gamma(x) d\mu(x) \geq \frac{i_0+i}{\delta(i_0+i) \|X\|} \tau(B_M(f^{-1}(x_0), \frac{1}{i_0+i})) \geq \frac{i_0+i}{\|X\|} \rightarrow +\infty$$

as $i \rightarrow +\infty$. By Corollary 2.12, all ergodic measures of the flow φ induced by $Y = \alpha X$ are atomic. The topological entropy $h(\varphi)$ equals $h(\varphi_1)$ by definition, while $h(\varphi_1)$ coincides with the supremum of the measure-theoretic entropies $h_m(\varphi_1)$ over all φ -invariant and ergodic measures m by Theorem A of [9] (note that a φ -invariant and ergodic measure is φ_1 -invariant but not necessarily φ_1 -ergodic). So the topological entropy of φ is zero. \square

Theorem 3.3. *There exists a function $\hat{\alpha} \in C^\infty(\Omega, [0, 1])$, satisfying **(H)**, such that $\hat{Y} = \hat{\alpha}X$ has positive topological entropy.*

Proof. Take $p \in \Omega$ as in Theorem 3.2. Recall that $f : M \rightarrow M$ is a minimal diffeomorphism which preserves an ergodic measure μ with $h_\mu(f) > 0$. Let $\bar{\mu}$ denote the ergodic measure on (Ω, ψ) defined in Lemma 2.13. By Theorem 2.15, $h_{\bar{\mu}}(\psi) = h_\mu(f) > 0$. Noting that $\dim M \geq 2$ and thus that $\dim \Omega \geq 3$, we can find a function $\hat{\alpha} \in C^\infty(\Omega, [0, 1])$ that satisfies not only the assumption **(H)** stated in Section 2 but also the following:

$$(3.4) \quad \int_\Omega \frac{1}{\hat{\alpha}(x)} d\bar{\mu}(x) = K < \infty.$$

In fact, we can assume that the coordinate (V, ξ) satisfies $\xi(V) = B^{m+1}(2)$ and $\xi(V') = B^{m+1}(1)$; here V' is an open set of Ω satisfying $p \in V'$ and $\bar{V}' \subset V$. We select a smooth function $\omega : \mathbb{R}^{m+1} \rightarrow \mathbb{R}$ satisfying

- (1) $\omega(x) = \|x\|^2 := (x_1)^2 + (x_2)^2 + \dots + (x_{m+1})^2$, if $\|x\| \leq \frac{1}{2}$;
- (2) $\frac{1}{4} < |\omega(x)| < 2$, if $\frac{1}{2} < \|x\| < 1$;
- (3) $\omega(x) = 1$, if $\|x\| \geq 1$.

It is easy to see that $\frac{1}{\omega(x)}$ is integrable w.r.t. Lebesgue measure on $B^{m+1}(2)$ since $m+1 \geq 3$. Then the function

$$\hat{\alpha}(q) = \begin{cases} \omega \circ \xi^{-1} & p \in V; \\ 1 & p \in M \setminus V \end{cases}$$

satisfies assumption **(H)**. If M is equivalent in Riemann volume, as in the example of Herman [3], then $\hat{\alpha}$ satisfies (3.4). If M has singularities, (3.4) still holds provided we choose x_0 where M has finite density. By Lemma 2.6, the function

$$\theta(t, x) = \int_0^t \frac{1}{\hat{\alpha}(\psi_s x)} ds$$

is an integrable additive function. Consider the regular set $\widehat{\Omega}$ of θ and construct the corresponding Borel σ -algebra $\widehat{\mathcal{B}}$ (see the description after Remark 2.7). Denote by $\widehat{\varphi}_t$ the time changed system of ψ_t .

One can easily verify that $\widehat{\varphi}_t$ is the flow induced by the vector field $\widehat{Y} := \widehat{\alpha}X$. As discussed in Section 2, we can define a measure on $(\widehat{\Omega}, \widehat{\mathcal{B}})$:

$$\widehat{\mu}(B) = \int_B d\widehat{\mu}(x) = \int_B \frac{1}{\widehat{\alpha}(x)} d\bar{\mu}(x)$$

for all $B \in \widehat{\mathcal{B}}$. By Lemma 2.8, $\widehat{\mu}$ is an invariant ergodic measure of $\widehat{\varphi}_t$ and

$$\widehat{\mu}(\widehat{\Omega}) \leq \int_{\Omega} \frac{1}{\widehat{\alpha}(x)} d\bar{\mu}(x) = K < \infty.$$

Since $\bar{\mu}$ is ergodic and by Theorem 2.9, we have

$$h_{\widehat{\mu}}(\widehat{\varphi}_1)\widehat{\mu}(\widehat{\Omega}) = h_{\bar{\mu}}(\psi_1)\bar{\mu}(\Omega).$$

So, $h_{\widehat{\mu}}(\widehat{\varphi}) = h_{\widehat{\mu}}(\widehat{\varphi}_1) > 0$. □

Proof of Theorem 1.1. The flow induced by the C^∞ vector field $Y = \alpha X$ has zero entropy and the flow induced by the C^∞ vector field $\widehat{Y} = \widehat{\alpha}X$ has positive entropy. They are equivalent by Proposition 2.2. □

Final note. The map $f : M \rightarrow M$ provided by [3] is real analytic. However, our method of proof clearly cannot be made analytic since α is flat at p . It seems likely that zero is an invariant of equivalent analytic flows.

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