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REPRESENTATIONS OF LIE GROUPS AND RANDOM MATRICES

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ABSTRACT. We study the asymptotics of representations of a fixed compact Lie group. We prove that the limit behavior of a sequence of such representations can be described in terms of certain random matrices; in particular operations on representations (for example: tensor product, restriction to a subgroup) correspond to some natural operations on random matrices (respectively: sum of independent random matrices, taking the corners of a random matrix). Our method of proof is to treat the canonical block matrix associated to a representation as a random matrix with non-commutative entries.

1. INTRODUCTION

1.1. **Need for an asymptotic theory of representations.**

One of the main questions in representation theory of Lie groups and Lie algebras is to understand multiplicities, commutant spaces and the structure of representations arising in various natural situations, such as restriction to a subgroup or tensor product of representations.

There are many reasons to study asymptotic versions of such questions in the limit when the representation (and, possibly, also the Lie group) tends in some sense to infinity.

- From the viewpoint of probability theory it is natural to consider the limit theorems (such as laws of large numbers, central limit theorem, etc.) in order to study the limits of probability measures on a given set. Reducible representations of a given group, the subject of this article, can be alternatively described as probability measures on the set of irreducible representations.
- Even though for nearly all problems in representation theory there are explicit answers [FH91, GW98], they are based on some combinatorial algorithms which are too cumbersome to be tractable asymptotically. For this reason in the asymptotic theory of representations one has to look for non-combinatorial tools such as random matrix theory [Meh91] or Voiculescu's free probability theory [VDN92].

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- In theoretical physics it is a natural question as well. Indeed, finite-dimensional representation theory is described as a nice discrete object which can be scaled in some thermodynamic limit to model continuous phenomena [KSW96a, KSW96b].
- Many important questions in the theory of operator algebras concern free group factors. One of the foundations of Voiculescu's free probability theory was the fact that free products may be approximated in some sense by random matrices. It was observed by Biane [Bia95] that also representations may provide such finite-dimensional approximants with an adventage of being fully constructive and non-random.

1.2. **Asymptotics of representations of a fixed Lie group.** Let G be a fixed compact Lie group (in the following we consider the example when $G = U(d)$ is the unitary group). One of our motivations is to study the problem of decomposing a given concrete reducible representation of G into irreducible components. For example we wish to study the following interesting examples of reducible representations:

(1) restriction of a given irreducible representation of G' to its subgroup G, where G' is a given compact Lie group (for example: $G = U(d)$ and $G' = U(d')$ with $d < d'$);
(2) Kronecker tensor product of two given irreducible representations of G.

The irreducible representations of compact Lie groups are uniquely determined by their highest weights (cf. Section 3 for definitions); in the example of $G = U(d)$ the irreducible representations are indexed by sequences of integers $\lambda_1 \geq \cdots \geq \lambda_d$. For explicit answers to the above questions there are well-known algorithms involving combinatorial manipulations on the the highest weights. For example, the decomposition of an irreducible representation of $G' = U(d+1)$ to a subgroup $G = U(d)$ is given by Weyl's branching rule; in the general case the answer to this question is given by counting certain Littelmann paths [Lit95]. However, when we are interested in the situation when the dimension of the representations tends to infinity, these combinatorial algorithms become very cumbersome. In particular, the direct study of the multiplicities in the above problems seems to be rather difficult.

In order to avoid such difficulties we concentrate on approximate asymptotic answers. More explicitly, for a given sequence (ρ_n) of representations of G for which the highest weights of the irreducible components tend to infinity, we study the limit distribution of a (rescaled) highest weight of a randomly chosen irreducible component of ρ_n.

This situation in the asymptotic theory of representations in many ways resembles statistical mechanics: when the number of particles in a physical system grows, the complexity of its description also grows so that its exact solution becomes quickly intractable. However, a more modest approach in which we are interested only in some macroscopic quantities may result in a surprisingly simple description. Similarly, in the asymptotic theory of representations when we abandon the attempts to find exact combinatorial solutions and restrict ourselves to a statistical description we may expect dramatic simplifications. Since representations have a highly non-commutative structure and the combinatorial algorithms behind are so cumbersome, therefore we can expect quite surprising results.

1.3. **The main result: Representations of Lie groups and random matrices.** For a fixed compact Lie group G and a sequence (ρ_n) of representations we construct a random matrix whose joint eigenvalues distribution depends on the asymptotic behavior of the highest weights of ρ_n. The symmetry of this random matrix depends on the group G; for example when $G = U(d)$ is the unitary group the corresponding random matrix will be a Hermitian $d \times d$ random matrix which additionally is invariant under conjugation by unitary matrices.

We prove that some operations on the representations (such as Kronecker tensor product, restriction to a subgroup) correspond to some natural operations on the appropriate random matrices (sum of independent random matrices, taking the corner of a random matrix, respectively). In this way problems concerning asymptotic properties of representations are reduced to much simpler analytic problems concerning random matrices.

Some results proved in this paper were already considered by Heckman [Hec82]. However, as far as we could understand, his proofs are very different from ours. Our methods are probabilistic and lead to many new applications and examples; see Section 5.

1.4. **The main tool: Random matrices with non-commutative entries.** Let ρ be a representation of a compact Lie group G; from here on we shall restrict our attention to the corresponding representation of the Lie algebra $\mathfrak{g}$ (if G is connected the representation of the Lie group is uniquely determined by the representation of the Lie algebra).

The family of matrices $\{\rho(x) : x \in \mathfrak{g}\}$ can be viewed as a family of *non-commutative random variables*; in other words, ρ can be viewed as a *non-commutative random vector* in $\mathfrak{g}^\star$. We prove that asymptotically, when the representation ρ tends to infinity, such a non-commutative random vector converges in distribution (after some rescaling) to a classical (*commutative*) random vector in $\mathfrak{g}^\star$ and hence—in many cases—can be identified with a random matrix.

This idea is closely related to the work of Kuperberg [Kup02, Kup05] who—among other results—gives a new, conceptual proof of the result of Johansson [Joh01] (see Theorem 5.4 below). Kuperberg's idea is to treat elements of the Lie algebra as non-commutative random variables and to show that for the tensor product of representations a non-commutative central limit theorem can be applied. The results of this article can therefore be viewed as an extension of some of Kuperberg's results [Kup02] from the central limit theorem related to the tensor product of representations to other operations on representations such as restrictions or tensor products of a fixed number of representations.

1.5. **Asymptotics of representations for a series of Lie groups and free probability.** A variation of the above problem appears when we replace the fixed group G by a classical series of compact Lie groups (G_n) and we consider a series (ρ_n), where ρ_n is a representation of G_n; we are interested in solving the analogues of the problems (1)–(2) from Section 1.2. For example, we may take $G_n = U(n)$ to be the series of the unitary groups. This case was studied in detail by Biane [Bia95] who found a connection between asymptotics of such representations and Voiculescu's free probability theory [VDN92].

In a subsequent paper [CŚ08] we show that the method of random matrices with non-commuting entries can also be applied to this situation and the results obtained in this way are significantly stronger than the ones of Biane [Bia95].

1.6. **Asymptotics of representations of symmetric groups.** It turns out that the ideas presented in this article can also be applied to the case of the symmetric groups. A canonical matrix associated to a representation of the symmetric group was given by Biane [Bia98] and it turns out that the recent results of the second-named author [Śni06] were proved by treating (in a very concealed way) this matrix as a permutationally-invariant random matrix with non-commuting entries. A subsequent paper [ŚS08] will present the details.

1.7. **Overview of this article.** In Section 2 we introduce some notation concerning non-commutative random variables. In Section 3 we state some facts about representation theory and fix some notation. In Section 4 we prove the main result and in Section 5 we investigate its new consequences.

2. Non-commutative random variables

2.1. **Non-commutative probability spaces.** Let $(\Omega, \mathfrak{M}, P)$ be a Kolmogorov probability space. We consider an algebra

$$\mathcal{L}^{\infty-}(\Omega) = \bigcap_{n \geq 1} \mathcal{L}^n(\Omega)$$

of random variables with all moments finite. This algebra is equipped with a functional $\mathbb{E} : \mathcal{L}^{\infty-}(\Omega) \to \mathbb{R}$ which to a random variable associates its mean value.

We consider a generalization of the above setup in which the commutative algebra $\mathcal{L}^{\infty-}(\Omega)$ is replaced by any (possibly non-commutative) $\star$-algebra $\mathfrak{A}$ with a unit and $\mathbb{E} : \mathfrak{A} \to \mathbb{R}$ is any linear functional which is normalized (i.e. $\mathbb{E}(1) = 1$), positive (i.e. $\mathbb{E}(x^\star x) > 0$ for all $x \in \mathfrak{A}$ such that $x \neq 0$) and tracial (i.e. $\mathbb{E}(xy) = \mathbb{E}(yx)$ for all $x, y \in \mathfrak{A}$). The elements of $\mathfrak{A}$ are called *non-commutative random variables* and the functional $\mathbb{E}$ is called the mean value or expectation. We also say that $(\mathfrak{A}, \mathbb{E})$ is a non-commutative probability space [VDN92].

Here are two motivating examples, which will be used in the following.

Example 2.1. For any Kolmogorov probability space, the corresponding pair $\big(\mathcal{L}^{\infty-}(\Omega), \mathbb{E}\big)$ is a non-commutative probability space.

Example 2.2. For any Kolmogorov probability space and integer $d \geq 1$ we consider the algebra $\mathcal{L}^{\infty-}(\Omega) \otimes \mathcal{M}_d = \mathcal{L}^{\infty-}(\Omega; \mathcal{M}_d)$ of $d \times d$ random matrices and we equip it with a functional

$$\mathbb{E}(x) = E(\operatorname{tr} x),$$

for any random matrix x, where E on the right-hand side denotes the mean value and

$$\operatorname{tr} x = \frac{1}{d} \operatorname{Tr} x$$

is the normalized trace. In this way $\big(\mathcal{L}^{\infty-}(\Omega; \mathcal{M}_d), \mathbb{E}\big)$ is a non-commutative probability space.

Any non-commutative probability space $(\mathfrak{A}, \mathbb{E})$ can be equipped with the corresponding L^2 norm:

$$\|x\|_{L^2} = \sqrt{\mathbb{E}(xx^\star)}.$$

Notice that the above definitions of non-commutative probability spaces and random variables do not require any analytic notions other than positivity. In particular, as we shall see in the remaining part of this section, by the distribution of a non-commutative random variable we understand the collection of its mixed moments. While this approach turns out to be very useful to state and prove our results and it has the advantage of encompassing many non-bounded random variables, it has a couple of drawbacks. For instance, the convergence of non-commutative distributions as defined in Section 2.2 does not coincide in the commutative case with the weak convergence of probability measures; therefore some of our corollaries concerning convergence in distribution of classical random variables will be formulated and proved not in the most desirable weak topology of probability measures but with respect to the moments convergence. This issue and the way to fix it in the cases which are of our interest (so that the convergence in the weak topology of probability measures in fact holds true) are discussed in Section 4.3.

2.2. **Random vectors.** Let V be a finite-dimensional (real) vector space. If $v : \Omega \to V$ is a random variable valued in the space V we say that v *is a (classical) random vector in* V. We say that v has all moments finite if $\mathbb{E}\|v\|^k < \infty$ holds true for any exponent $k \geq 1$. Notice that this definition does not depend on the choice of the norm $\|\cdot\|$ on V. For a random vector v with all moments finite we define its *moments*

$$m_k = m_k^{\mathbb{E}}(v) = \mathbb{E}\,\underbrace{v \otimes \cdots \otimes v}_{k \text{ times}} \in V^{\otimes k}. \tag{2.1}$$

In the case when $V = \mathbb{R}$, v becomes a usual number-valued random variable; furthermore $V^{\otimes k} = \mathbb{R}^{\otimes k} \cong \mathbb{R}$ and the moments $m_k = \mathbb{E}v^k \in \mathbb{R}$ are just real numbers and this definition coincides with the usual notion of the moments of a random variable. In the following we are interested in the space

$$\left\{v : \Omega \to V \text{ such that } \mathbb{E}\|v\|^k < \infty \text{ for each } k \geq 1\right\}$$

of random vectors with all moments finite, which we will view as a tensor product

$$V \otimes \mathcal{L}^{\infty-}(\Omega). \tag{2.2}$$

2.3. **Non-commutative random vectors.** Let $(\mathfrak{A}, \mathbb{E})$ be a non-commutative probability space; in analogy to (2.2) we call the elements of $V \otimes \mathfrak{A}$ *non-commutative random vectors in* V *(over a non-commutative probability space* $(\mathfrak{A}, \mathbb{E})$*)*.

Given $v_1 = x_1 \otimes a_1 \in V_1 \otimes \mathfrak{A}$ and $v_2 = x_2 \otimes a_2 \in V_2 \otimes \mathfrak{A}$ we define

$$v_1 \hat{\otimes} v_2 = (x_1 \otimes a_1) \hat{\otimes} (x_2 \otimes a_2) = (x_1 \otimes x_2 \otimes a_1 a_2) \in V_1 \otimes V_2 \otimes \mathfrak{A}$$

and its linear extension on non-elementary tensors. Whenever $v_1 = v_2$ with $V_1 = V_2$ one shortens the notation as $v^{\hat{\otimes} 2} \in V^{\otimes 2} \otimes \mathfrak{A}$ and one extends it by recursion to the definition of

$$v^{\hat{\otimes} k} \in V^{\otimes k} \otimes \mathfrak{A}.$$

Observe that this definition matches the definition of the tensor product of compact quantum groups of Woronowicz [Wor87] provided that $\mathfrak{A}$ is a quantum group and V a representation of $\mathfrak{A}$.

The *k-th order vector moment* $m_k^{\mathbb{E}}(v)$ is defined as

$$m_k^{\mathbb{E}}(v) = (\mathrm{Id} \otimes \mathbb{E}) v^{\hat{\otimes} k} \in V^{\otimes k}.$$

If there is no ambiguity we might remove the superscript $\mathbb{E}$. Note that in the case of Example 2.1 when $\mathfrak{A}$ is commutative the moments, as defined above, coincide with the old definition (2.1) of the moments of a random vector.

We define the distribution of a non-commutative random vector as its sequence $(m_k^{\mathbb{E}}(v))_{k=1,2,\dots}$ of moments. Accordingly, convergence in distribution of non-commutative random vectors is to be understood as convergence of the moments.

The above definitions can be made more explicit as follows: let $e_1,\dots,e_d$ be a base of the vector space V. Then a (classical) random vector v in V can be viewed as

$$v = \sum_i a_i e_i, \tag{2.3}$$

where the a_i are the (random) coordinates. Then a non-commutative random vector can be viewed as the sum (2.3) in which the a_i are replaced by non-commutative random variables. One can easily see that the sequence of moments

$$m_k(v) = \sum_{i_1,\dots,i_k} \mathbb{E}(a_{i_1}\cdots a_{i_k})\; e_{i_1}\otimes\cdots\otimes e_{i_k}$$

contains nothing else but the information about the mixed moments of the non-commutative coordinates $a_1,\dots,a_d$, and the convergence of moments is equivalent to the convergence of the mixed moments of $a_1,\dots,a_d$.

The following result provides a necessary and sufficient condition for a sequence of moments to be those of a commutative vector.

Proposition 2.3. *A non-commutative random vector v actually arises from a commutative probability space iff for each value of $k\in\{1,2,\dots\}$ the tensor $m_k^{\mathbb{E}}(v)\in V^{\otimes k}$ is invariant under the action of the symmetric group.*

Proof. The necessity is trivial. For sufficiency, if $m_k^{\mathbb{E}}(v)$ are invariant under the action by conjugation of the symmetric group, this implies that the GNS representation of $\mathfrak{A}$ with respect to $\mathbb{E}$ is its abelianized quotient. Since $\mathbb{E}$ is supposed to be faithful the proof is complete. □

3. Preliminaries of representation theory

3.1. **Structure of compact Lie groups.** In this section we recall some facts about Lie groups and their algebras [BtD95, FH91]. Let G be a compact Lie group and let $H\subseteq G$ be a maximal abelian subgroup and let $\mathfrak{h}\subseteq\mathfrak{g}$ be the corresponding Lie algebras. We consider the adjoint action of $\mathfrak{h}$ on the vector space $\mathfrak{g}$; then there is a decomposition

$$\mathfrak{g} = \mathfrak{h}\oplus\bigoplus_{\alpha\in\mathfrak{h}^\star}\mathfrak{g}_\alpha \tag{3.1}$$

into eigenspaces. The non-zero elements α for which the corresponding eigenspace $\mathfrak{g}_\alpha$ is non-trivial are called *roots*. We identify $\mathfrak{h}^\star$ as a set of functionals on $\mathfrak{g}$ which vanish on all root spaces $\mathfrak{g}_\alpha$; thus

$$\mathfrak{h}^\star\subseteq\mathfrak{g}^\star. \tag{3.2}$$

Suppose that $\mathfrak{g}$ is equipped with a G-invariant scalar product; in this way $\mathfrak{g}\cong\mathfrak{g}^\star$ and $\mathfrak{h}\cong\mathfrak{h}^\star$. The above isomorphisms and the inclusion $\mathfrak{h}\subseteq\mathfrak{g}$ allow us to consider the inclusion $\mathfrak{h}^\star\subseteq\mathfrak{g}^\star$. This inclusion does not depend on the choice of the G-invariant scalar product on $\mathfrak{g}$ and it coincides with the inclusion (3.2) considered above. More generally, throughout all this paper, whenever we state a result related to $\mathfrak{g}$ but do not mention the invariant scalar product, this means that the result does not depend on the choice of the scalar product.

By definition, the Weyl group $W = \{g \in G : gHg^{-1} = H\}$ is the set of the elements which normalize H. The Weyl group W is always finite.

Every element $x \in \mathfrak{g}$ is conjugate to some element $y \in \mathfrak{h}$ (i.e. there exists an element $g \in G$ such that $\mathrm{Ad}_g(x) = y$, where $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g})$ denotes the adjoint representation) and y is determined only up to the action of the Weyl group. Similarly, every element $x \in \mathfrak{g}^\star$ is conjugate to some element $y \in \mathfrak{h}^\star$ (which is determined only up to the action of the Weyl group).

Example 3.1. For $G = U(n)$ we may take H to be the group of diagonal unitary matrices. Then $\mathfrak{g}$ is the set of $n \times n$ anti-Hermitian matrices and $\mathfrak{h}$ is the set of diagonal matrices with imaginary entries. In this case W is the group of permutation matrices and hence is isomorphic to the symmetric group S_n. We equip $\mathfrak{g}$ with a G-invariant scalar product $\langle x, y\rangle = \mathrm{Tr}\, xy^\star$ and thus we identify $\mathfrak{g} \cong \mathfrak{g}^\star$ and $\mathfrak{h} \cong \mathfrak{h}^\star$. Clearly, every element $x \in \mathfrak{g} \cong \mathfrak{g}^\star$ is conjugate by a unitary matrix to some diagonal matrix $y \in \mathfrak{h} \cong \mathfrak{h}^\star$ which is determined by the eigenvalues of x. The freedom of choosing the order of the diagonal elements of y corresponds to the action of the Weyl group S_n.

Remark 3.2. In random matrix theory it is more customary to work with the space of Hermitian matrices instead of the space of anti-Hermitian matrices; for this reason we may consider a simple isomorphism between these spaces given by multiplication by i.

The above example heuristically motivates our thinking that for a given element $x \in \mathfrak{g}^\star$ the corresponding element $y \in \mathfrak{h}^\star$ contains the information about the *eigenvalues* of x.

3.2. **Irreducible representations and highest weights.** Let $\rho : \mathfrak{g} \to \mathrm{End}(V)$ be an irreducible representation. The representation space can be decomposed into the eigenspaces of the maximal abelian subalgebra $\mathfrak{h}$:

$$V = \bigoplus_{\alpha \in \mathfrak{h}^\star} V_\alpha; \tag{3.3}$$

in other words, $h \in \mathfrak{h}$ acts on V_α as multiplication by $\alpha(h)$. The elements $\alpha \neq 0$ for which the corresponding eigenspace V_α is non-trivial are called *weights*.

Let us fix some element $t \in \mathfrak{h}$ which is generic in a sense that for any non-zero weight α we have $\alpha(t) \neq 0$. Now a weight α is called *positive* if $\alpha(t) > 0$; otherwise it is called *negative*. A similar convention concerns roots as well. The *highest weight* of a representation is the weight α for which $\alpha(t)$ takes the maximal value. In fact, any irreducible representation is uniquely determined (up to equivalence) by its highest weight. The weight space $\mathfrak{g}_\alpha$ corresponding to the highest weight is always one-dimensional. Clearly, the above definitions depend on the choice of t; nevertheless the notion of highest weight is well defined (up to the action of the Weyl group).

3.3. **Enveloping algebra.** The enveloping algebra of $\mathfrak{g}$, denoted by $\bar{\mathfrak{g}}$, is the free algebra generated by the elements of $\mathfrak{g}$ quotiented by relations $gh - hg = [g, h]$ for any $g, h \in \mathfrak{g}$. Usually we work over the complex (complexified) algebra. This algebra is naturally endowed with a $\star$-algebra structure obtained by declaring that the elements of $\mathfrak{g}$ (before complexification) are anti-Hermitian elements. For our purposes, it has the following two important features:

- an irreducible representation (ρ, V) of $\mathfrak{g}$ gives rise to an onto algebra homomorphism $\bar{\mathfrak{g}} \to \mathrm{End}(V)$ and conversely;
- therefore, irreducible representations of $\mathfrak{g}$ are in one-to-one correspondence with minimal tracial states on $\bar{\mathfrak{g}}$.

We will need the following theorem, known as the Poincaré-Birkhoff-Witt theorem.

Theorem 3.3. *If $h_1, \dots, h_n$ is a basis of $\mathfrak{g}$ as a vector space, then*

$$(h_1^{\alpha_1} \cdots h_n^{\alpha_n})_{\alpha_1,\dots,\alpha_n \in \mathbb{N}}$$

is a basis of $\bar{\mathfrak{g}}$ as a vector space. In particular, there is a filtration on $\bar{\mathfrak{g}}$ defined as follows: the degree of $p \in \bar{g}$ is the smallest k such that p is a sum of monomials of elements in g with at most k factors.

3.4. **Reducible representations and random highest weights.** Let $\rho : G \to \mathrm{End}(V)$ be a (possibly reducible) representation of G on a finite-dimensional vector space V. We may decompose ρ as a sum of irreducible representations:

$$\rho = \bigoplus_{\lambda \in \mathfrak{h}^\star} n_\lambda \rho_\lambda,$$

where ρ_λ denotes the irreducible representation of G with the highest weight λ and $n_\lambda \in \{0, 1, 2, \dots\}$ denotes its multiplicity in ρ.

We define a probability measure μ_ρ on $\mathfrak{h}^\star$ such that the probability of $\lambda \in \mathfrak{h}^\star$ is equal to

$$\frac{n_\lambda \cdot (\text{dimension of } \rho_\lambda)}{(\text{dimension of } V)};$$

in other words, it is proportional to the total dimension of all the summands of type $[\rho_\lambda]$ in ρ. In this way the probability measure μ_ρ encodes in a compact way the information about the decomposition of ρ into irreducible components. We define the *random highest weight* associated to the representation ρ as a random variable distributed accordingly.

Remark 3.4. If we literally follow the above definition, then μ_ρ is a certain probability measure on the Weyl chamber. This definition has the disadvantage that it depends on the choice of the Weyl chamber; therefore sometimes it will be convenient to understand by μ_ρ the W-invariant probability measure on $\mathfrak{h}^\star$ obtained by symmetrizing the above measure by the action of the Weyl group W.

3.5. **Random matrix with specified eigenvalues.** For an element $\lambda \in \mathfrak{h}^\star$ we consider a random vector in $\mathfrak{g}^\star$ given by $\tilde{\lambda} = \mathrm{Ad}_g \lambda \in \mathfrak{g}^\star$, where g is a random element of G, distributed according to the Haar measure on G. We will say that $\tilde{\lambda}$ is a *G-invariant random matrix with the eigenvalues* λ. This terminology was motivated by the following example.

Example 3.5. If $G = U(n)$ is the group of the unitary matrices and $\lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \in \mathfrak{h}^\star$ is a diagonal matrix, then the distribution of $\tilde{\lambda}$ is indeed equal to the uniform measure on the manifold of anti-Hermitian matrices with the eigenvalues $\lambda_1, \dots, \lambda_n$.

We may extend the above definition to the case when $\lambda \in \mathfrak{h}^\star$ is random; in such a case we will additionally assume that g and λ are independent. If μ is a probability measure on $\mathfrak{h}^\star$ we may treat it as a distribution of the random variable λ; in this case we say that $\tilde{\lambda}$ is a *G-invariant random matrix with the distribution of eigenvalues given by μ*. We denote by $\tilde{\mu}$ the corresponding distribution of $\tilde{\lambda}$.

Remark 3.6. It would be more appropriate to call λ a G-invariant random *vector* in $\mathfrak{g}^\star$; nevertheless in the most interesting examples the Lie groups under consideration carry some canonical matrix structure. Hence the elements of $\mathfrak{g}^\star$ can be indeed viewed as matrices.

The moments of G-invariant matrices are characterized by the following lemma.

Lemma 3.7. *Let μ be a probability measure on $\mathfrak{h}^\star$ with all moments finite and let $\tilde{\mu}$ be the corresponding distribution of a G-invariant random matrix.*

For each k the moment $m_k = m_k(\tilde{\mu}) \in (\mathfrak{g}^\star)^{\otimes k}$ is the unique element such that:

1. *m_k is invariant under the adjoint action of G,*
2. *actions of $m_k : \mathfrak{g}^{\otimes k} \to \mathbb{R}$ and $m_k(\mu) : \mathfrak{g}^{\otimes k} \to \mathbb{R}$ coincide on G-invariant tensors in $\mathfrak{g}^{\otimes k}$ (where above $m_k(\mu)$ denotes the canonical extension of $m_k(\mu) \in (\mathfrak{h}^\star)^{\otimes k}$ which is possible thanks to the inclusion $\mathfrak{h}^\star \subseteq \mathfrak{g}^\star$).*

Proof. Point (1) follows easily from the invariance of the Haar measure.

For any elements $x_1, x_2 \in \mathfrak{g}^\star$ which are conjugate to each other the restrictions of the maps $x_1^{\otimes k}, x_2^{\otimes k} : \mathfrak{g}^{\otimes k} \to \mathbb{R}$ to G-invariant tensors coincide. In this way point (2) follows easily.

For $x \in \mathfrak{g}^{\otimes k}$ let

$$x' = \int_G \mathrm{Ad}_g^{\otimes k}(x) \, dg$$

be the average over the Haar measure on G. Clearly, x' is a G-invariant tensor. Since

$$\big(m_k(\tilde{\mu})\big)(x) = \big(m_k(\tilde{\mu})\big)(x'),$$

therefore the values of the functional $m_k(\tilde{\mu})$ are uniquely determined by its values on G-invariant tensors, which shows the uniqueness of m_k. □

Remark 3.8. Similarly as in Remark 3.6, it is sometimes more convenient to understand the distribution of the eigenvalues of a random element of $\mathfrak{g}^\star$ as a W-invariant measure on $\mathfrak{h}^\star$.

4. Representations and random matrices with non-commutative entries

4.1. **The main result.** Let $\rho : \mathfrak{g} \to \mathrm{End}(V)$ be a representation of $\mathfrak{g}$. We shall view ρ as an element of $\mathfrak{g}^\star \otimes \mathrm{End}(V)$; in other words, ρ is a non-commutative random vector in $\mathfrak{g}^\star$ over a non-commutative probability space $\big(\mathrm{End}(V), \mathrm{tr}\big)$. The sequence of its moments $m_k = m_k(\rho) \in (\mathfrak{g}^\star)^{\otimes k}$, or equivalently, $m_k : \mathfrak{g}^{\otimes k} \to \mathbb{C}$ is given explicitly by

$$m_k(g_1 \otimes \cdots \otimes g_k) = \mathrm{tr}\left[\rho(g_1) \cdots \rho(g_k)\right].$$

The following theorem is the main result of this article.

Theorem 4.1. *Let (ϵ_n) be a sequence of real numbers which converges to zero. For each n let $\rho_n : \mathfrak{g} \to \operatorname{End}(V_n)$ be a representation of $\mathfrak{g}$ and let λ_n be the corresponding random highest weight. Let A be a G-invariant random matrix with all moments finite.*

Then the following conditions are equivalent:

1. *the distributions of the random variables $\epsilon_n \lambda_n$ converge in moments to the distribution of eigenvalues of A;*
2. *the sequence $\epsilon_n \rho_n$ of non-commutative random matrices converges in distribution to A.*

Proof. Suppose that condition (1) holds true. It is enough to prove that from any subsequence $(\epsilon_{k(n)}\rho_{k(n)})$ one can choose a subsequence $(\epsilon_{k(l(n))}\rho_{k(l(n))})$ which converges in distribution to the random matrix A.

Let a subsequence $(\epsilon_{k(n)}\rho_{k(n)})$ be given; by Lemma 4.2 and the compactness argument it follows that there exists a subsequence $(\epsilon_{k(l(n))}\rho_{k(l(n))})$ which converges in distribution to some random matrix $M \in \mathfrak{g}^\star \otimes \mathfrak{A}$ with non-commutative entries (for some non-commutative probability space $(\mathfrak{A}, \mathbb{E})$). It is enough to prove that M is the G-invariant random matrix in $\mathfrak{g}^\star$ with the same distribution of the eigenvalues as for A. In order to keep the notation simple, instead of $(\epsilon_{k(l(n))}\rho_{k(l(n))})$ we will write $(\epsilon_n \rho_n)$.

Firstly, observe that Lemma 4.2 shows that

$$\begin{aligned} \big\| [\epsilon_n \rho_n(x_1), \epsilon_n \rho_n(x_2)] \big\|_{L^2} &= \big\| \epsilon_n^2 \rho_n\big([x_1, x_2]\big) \big\|_{L_2} \\ &\leq \left(\mathbb{E} \| \epsilon_n \lambda_n \|^2 \right)^{\frac{1}{2}} \epsilon_n \big\| [x_1, x_2] \big\| \to 0, \end{aligned}$$

where we used the fact that the first factor on the right-hand side converges to some constant depending on the distribution of eigenvalues of A. This shows that the elements $\{M(x) \in \mathfrak{A} : x \in \mathfrak{g}\}$ commute; hence M can be identified with a classical random variable (valued in $\mathfrak{g}$).

Secondly, since each of the random matrices $\epsilon_n \rho_n$ is G-invariant, the random matrix M is also G-invariant.

Thirdly, let $x \in \mathfrak{g}^{\otimes k}$ be invariant under the action of $\operatorname{Ad}^{\otimes k} : G \to \operatorname{End}(\mathfrak{g}^{\otimes k})$. For every irreducible representation ρ we have that $\rho^{\otimes k}(x)$ is a multiple of the identity, hence can be identified with a complex number. The exact value of this number is equal to $\rho^{\otimes k}(x)\big|_{V_\lambda}$, which can be estimated with the help of Lemma 4.3. Therefore for a (possibly reducible) representation ρ_n we have

$$\big(m_k(\epsilon_n \rho_n) \big)(x) = \mathbb{E} \Lambda_n^{\otimes k}(x) + (\text{terms of degree at least 1 in } \epsilon),$$

where $\Lambda_n = \epsilon_n \lambda_n$, where λ_n is the random highest weight associated to the representation ρ_n. Hence

$$\big(m_k(M) \big)(x) = \lim_{n \to \infty} \big(m_k(\epsilon_n \rho_n) \big)(x) = \lim_{n \to \infty} \big(m_k(\epsilon_n \lambda_n) \big)(x) = \big(m_k(A) \big)(x).$$

The above equality and Lemma 3.7 show that the distribution of the eigenvalues of M coincides with the distribution of eigenvalues of A, which finishes the proof that (1) $\Longrightarrow$ (2).

Suppose that condition (2) holds true. In order to prove (1)—similarly as in the proof of the implication (1) $\Longrightarrow$ (2)—it is enough to show that from any subsequence $(\epsilon_{k(n)}\lambda_{k(n)})$ one can choose a subsequence $(\epsilon_{k(l(n))}\lambda_{k(l(n))})$ which converges in moments to the distribution of the eigenvalues of A.

Lemma 4.4 can be used to show that if k is even, then the sequence of moments $m_k(\epsilon_n \lambda_n)$ is bounded; hence for every k the sequence of moments $m_k(\epsilon_n \lambda_n)$ is bounded. Again, by a compactness argument we can find a subsequence which converges in moments to the distribution of eigenvalues of some random matrix A'; from the implication (1) $\Longrightarrow$ (2) it follows that the random matrices A and A' must have equal moments of their entries. From Lemma 4.5 it follows that the eigenvalues of A' have the same moments as the eigenvalues of A, which finishes the proof. □

4.2. **Key lemmas.** We start with the following estimate:

Lemma 4.2. *We equip $\mathfrak{g}$ with a G-invariant scalar product; in this way we equip $\mathfrak{g} \cong \mathfrak{g}^\star$ with the corresponding norm.*

Let a unitary representation of a group G on a finite-dimensional Hilbert space be given and let ρ be the corresponding representation of a Lie algebra $\mathfrak{g}$. If ρ is irreducible with highest weight λ, then for any $x \in \mathfrak{g}$,

$$\|\rho(x)\| \leq \|x\| \; \|\lambda\|, \tag{4.1}$$

where the norm on the left-hand side denotes the operator norm.

Proof. Since $x \in \mathfrak{g}$ is conjugate to some element of $\mathfrak{h}$, it is enough to prove (4.1) for $x \in \mathfrak{h}$. For such elements the action of $\rho(x)$ is diagonal with respect to the decomposition (3.3) and

$$\|\rho(x)\| = \max_{\alpha} |\alpha(x)| \leq |\lambda(x)|,$$

where the maximum runs over the set of roots $\alpha \in \mathfrak{g}^\star$ contributing to (3.3). □

For a representation $\rho : \mathfrak{g} \to \operatorname{End}(V)$ we define $\rho^k : \mathfrak{g}^{\otimes k} \to \operatorname{End}(V)$ on simple tensors by

$$\rho^k(g_1 \otimes \cdots \otimes g_k) = \rho(g_1) \cdots \rho(g_k)$$

and extend it to the general case by linearity.

Lemma 4.3. *Let ϵ be a number, let $\rho : \mathfrak{g} \to \operatorname{End}(V)$ be an irreducible representation of $\mathfrak{g}$ and let λ be the corresponding highest weight.*

Let $z \in \mathfrak{g}^{\otimes k}$ be given; by a small abuse of notation we will denote by $\rho^k(z)\big|_{V_\lambda} : V_\lambda \to V_\lambda$ the restriction of $\rho^k(z)$ to V_λ projected again onto V_λ. Since the highest-weight space V_λ in the decomposition (3.3) is one-dimensional, we will identify $\rho^k(z)\big|_{V_\lambda}$ with a complex number. Furthermore, $\epsilon^k \rho^k(z)\big|_{V_\lambda}$ is a polynomial in $\{\Lambda(x) : x \in \mathfrak{g}\}$ and ϵ, where $\Lambda = \epsilon\lambda$. This polynomial can be written as

$$\epsilon^k \rho^k(z)\big|_{V_\lambda} = \Lambda^{\otimes k}(z) + (\textit{terms of degree at least } 1 \textit{ in } \epsilon).$$

Proof. It is enough to prove that

$$\epsilon^k \rho(g_1) \cdots \rho(g_k)\big|_{V_\lambda} = \begin{cases} \Lambda(g_1) \cdots \Lambda(g_k) + (\text{terms of degree at least 1 in } \epsilon) & \text{if } g_1, \dots, g_k \in \mathfrak{h}, \\ (\text{terms of degree at least 1 in } \epsilon) & \text{otherwise} \end{cases}$$

holds true for all k-tuples $g_1, \dots, g_k \in \mathfrak{g}$ such that each g_i belongs either to $\mathfrak{h}$ or to one of the root spaces $\mathfrak{g}_\alpha$ in the decomposition (3.1). We use induction over k: assume that the lemma holds true for all $k' < k$.

Let $\pi \in S_k$ be a permutation. Thanks to the commutation relations $xy = yx + [x,y]$ in the universal enveloping algebra of $\mathfrak{g}$ we may write in the universal enveloping algebra:

$$g_1 \cdots g_k = g_{\pi(1)} \cdots g_{\pi(k)} + (\text{summands with at most } k-1 \text{ factors});$$

hence

$$\epsilon^k \rho(g_1) \cdots \rho(g_k)\big|_{V_\lambda} = \epsilon^k \rho(g_{\pi(1)}) \cdots \rho(g_{\pi(k)})\big|_{V_\lambda} + \epsilon^k \rho(\text{summands with at most } k-1 \text{ factors})\big|_{V_\lambda}.$$

The induction hypothesis can be applied to the second summand on the right-hand side and it shows that it is of degree at least 1 in ϵ; therefore

$$\epsilon^k \rho(g_1) \cdots \rho(g_k)\big|_{V_\lambda} = \epsilon^k \rho(g_{\pi(1)}) \cdots \rho(g_{\pi(k)})\big|_{V_\lambda} + (\text{terms of degree at least 1 in } \epsilon).$$

It follows that it is enough to consider the case when the elements $g_1, \dots, g_k$ are sorted in such a way that for some $r, s, t \geq 0$ such that $r+s+t = k$ the initial r elements $g_1, \dots, g_r$ belong to root spaces corresponding to the negative roots, the next s elements $g_{r+1}, \dots, g_{r+s}$ belong to $\mathfrak{h}$ and the final t elements $g_{r+s+1}, \dots, g_{r+s+t}$ belong to root spaces corresponding to the positive roots. A direct calculation shows that

$$\epsilon^k \rho(g_1) \cdots \rho(g_k)\big|_{V_\lambda} = \begin{cases} \epsilon^k \lambda(g_1) \cdots \lambda(g_k) & \text{if } g_1, \dots, g_k \in \mathfrak{h}, \\ 0 & \text{otherwise} \end{cases}$$

and thus the inductive step follows. □

Lemma 4.4. *For a given irreducible representation ρ^λ of $\mathfrak{g}$ we denote*

$$M_\lambda = -\sum_i \rho^\lambda(x_i)^2,$$

where (x_i) denotes an orthogonal basis of $\mathfrak{g}$ (regarded as a real vector space). M_λ is a multiple of the identity and hence can be identified with a complex number.

There exists a constant C with the property that

$$|\lambda|^2 \leq 2M_\lambda + C$$

for any value of λ.

Proof. Let (e_i) be some linear basis of $\mathfrak{g}$ (this time regarded as a complex vector space) and (f_i) be its dual base. Then

$$M_\lambda = -\sum_i \rho^\lambda(e_i) \rho^\lambda(f_i).$$

It will be convenient for us to take as (e_i) a union of two families: firstly from each root space $\mathfrak{g}_\alpha$ we select some non-zero vector (the corresponding dual vector f_i belongs to $\mathfrak{g}_{-\alpha}$) and secondly we select some base of $\mathfrak{h}$.

Since M_λ is a multiple of the identity, Lemma 4.3 can be used to evaluate it. It is easy to check that there is some element $x \in \mathfrak{h}$ (which does not depend on the choice of λ) with the property that

$$M_\lambda = |\lambda|^2 + \lambda(x).$$

The estimate

$$2\lambda(x) \geq -|\lambda|^2 - |x|^2$$

finishes the proof. □

Lemma 4.5. *Let λ be a random vector in $\mathfrak{h}$ which is invariant under the action of the Weyl group W and let A be the corresponding G-invariant random vector in $\mathfrak{g}$. Then each moment $m_k(\lambda)$ is a polynomial function in the moment $m_k(A)$.*

Proof. Let us assume for simplicity that $\mathfrak{g}$ is semisimple. The Harish-Chandra isomorphism (see [Kna02] for a reference) is an isomorphism between $Z(\bar{\mathfrak{g}})$ (the center of the enveloping algebra $\bar{g}$) and $S(\mathfrak{h})^W$ (the W-invariant part of the symmetric algebra $S(\mathfrak{h})$).

The isomorphism of vector spaces $\bar{\mathfrak{g}} \cong \bigoplus_{k\geq 0} \mathrm{Sym}^k(\mathfrak{g})$ implies that

$$Z(\bar{\mathfrak{g}}) \cong \bigoplus_{k\geq 0} \left[\mathrm{Sym}^k(\mathfrak{g})\right]^G; \tag{4.2}$$

similarly

$$S(\mathfrak{h})^W \cong \bigoplus_{k\geq 0} \left[\mathrm{Sym}^k(\mathfrak{h})\right]^W. \tag{4.3}$$

The Harish-Chandra isomorphism provides an isomorphism between the right-hand sides of (4.2) and (4.3). Since it preserves the gradation it is the required map.

The general case follows from the fact that the Lie algebra $\mathfrak{g}$ can be written as a direct sum of its center and a semisimple Lie algebra. □

4.3. **Weak topology and uniqueness.** In view of Theorem 4.1 it is natural to address the question whether the notion of convergence of moments considered there could be replaced by some more probabilistic notion of convergence.

The notion of convergence in moments is very effective in the framework of bounded operators, as can be seen for example in Voiculescu's free probability theory [VDN92]. Indeed, the convergence in moments fully determines the von Neumann algebra generated by the limiting operator, and in the commutative case the moments of a bounded random variable determine its distribution by the Stone–Weierstrass theorem.

Unfortunately, if the random variables under consideration are not bounded, then their moments might not be finite; even if the latter case holds, then in general the moments do not determine the distribution of a random variable. Therefore it would be desirable to use some more refined description of the joint distribution of random variables. In the context of classical probability theory such a description is given by an appropriate probability measure; unfortunately, it is not clear what would be a good notion of distribution of unbounded operators in the non-commutative case. Some attempts to define weak convergence of the joint distribution in the context of $W^\star$–probability spaces have been made in very specific examples (see [Mey93] and the references therein). Unfortunately, it is not clear to us how one can adapt these definitions in our setting. A promising approach to this problem via Gromov-Hausdorff distance was presented by Rieffel [Rie04]; however for the moment it is not clear if this method can be succesfully used for our purposes.

To summarize the above discussion: we have no candidate for some kind of convergence which would replace the convergence in moments in point (2) of Theorem 4.1 since we deal here with a joint distribution of a family of non-commuting random variables.

Nevertheless, in point (1) of Theorem 4.1 we deal with classical random variables. Therefore it makes sense to consider convergence in some other sense, such as weak convergence of probability measures. Unfortunately, in general there is no connection between convergence in moments and weak convergence of probability measures. In particular, in order for convergence of measures in moments to imply their weak convergence we need to assume, for example, that the limit measure is uniquely determined by its moments.

It seems to be hard to incorporate the weak convergence of probability measures to condition (1) and preserve the equivalence of conditions (1) and (2). Therefore our strategy will be to keep Theorem 4.1 unchanged and in the study of its applications to pay attention to the weak convergence of probability measures (e.g. Theorem 5.1, item (2) and Theorem 5.3, item (2)).

5. Applications of the main theorem

5.1. **Restriction of representations and tensor product of representations.**

In this section we investigate a few remarkable consequences of Theorem 4.1.

Theorem 5.1. *Let $G \subset G'$ be Lie groups and $\mathfrak{g} \subset \mathfrak{g}'$ be the corresponding Lie algebras, let (ϵ_n) be a sequence of real numbers which converges to zero. Let (ρ'_n) be a sequence of representations of $\mathfrak{g}'$; by λ'_n and λ_n we denote the random highest weight corresponding to representations ρ'_n and $\rho'_n|_{\mathfrak{g}}$, respectively.*

(1) *Assume that $\epsilon_n \lambda'_n$ converges in moments to the distribution of a G'-invariant random vector A with values in $\mathfrak{g}'$. Then the sequence $\epsilon_n \lambda_n$ converges in moments towards the G-invariant random vector $\Pi_{\mathfrak{g}}(A)$, where $\Pi_{\mathfrak{g}} : \mathfrak{g}' \to \mathfrak{g}$ is the orthogonal projection.*
(2) *Assume that $\epsilon_n \lambda'_n$ converges weakly to the distribution of a G'-invariant random vector A with values in $\mathfrak{g}'$. Then the sequence $\epsilon_n \lambda_n$ converges weakly towards the G-invariant random vector $\Pi_{\mathfrak{g}}(A)$.*

Proof. Notice that the non-commutative random vector $\rho'_n|_{\mathfrak{g}}$ is a projection of the non-commutative random vector ρ'_n onto $\mathfrak{g}$. It follows that the random matrix which is the entrywise limit of $\epsilon_n \rho'_n|_{\mathfrak{g}}$ is a projection of A to $\mathfrak{g}$. Theorem 4.1 can be applied twice: for the sequence (ρ'_n) and for the sequence $(\rho'_n|_{\mathfrak{g}})$, which finishes the proof of part (1).

In order to prove (2) it is enough to show that for every $\varepsilon > 0$ and every subsequence $\epsilon_{k(n)} \lambda_{k(n)}$ we can chose a subsequence $\epsilon_{k(l(n))} \lambda_{k(l(n))}$ which converges weakly to some limit distribution with the property that its variation distance from the distribution of $\Pi_{\mathfrak{g}}(A)$ is smaller than ε.

Let $\varepsilon > 0$ and a subsequence $\epsilon_{k(n)} \lambda_{k(n)}$ be fixed. For simplicity, in the following instead of $\epsilon_{k(n)} \lambda_{k(n)}$ we shall consider just the sequence $\epsilon_n \lambda_n$. We can find a sequence of representations $(\tilde{\rho}'_n)$ for which the corresponding rescaled random highest weights $\epsilon_n \tilde{\lambda}'_n$ have a common compact support and the total variation distance between the distribution of $\epsilon_n \lambda'_n$ and $\epsilon_n \tilde{\lambda}'_n$ is smaller than ε (such a sequence $\tilde{\rho}'_n$ can be constructed by truncating the distribution of $\epsilon_n \tilde{\lambda}'_n$ to some sufficiently large compact set).

By a compactness argument we can select a subsequence $\epsilon_{l(n)} \tilde{\lambda}'_{l(n)}$ which converges weakly (hence in moments) to some limit; let $\tilde{A}$ be a G-invariant random vector in $\mathfrak{g}^\star$ with this distribution of eigenvalues. The total variation distance between the distribution of $\tilde{A}$ and A is bounded by ε.

The first part of the theorem can be applied to the subsequence of representations $\epsilon_{l(n)} \tilde{\rho}_{l(n)}$; it follows that the highest weights $\epsilon_{l(n)} \tilde{\lambda}_{l(n)}$ (corresponding to the restrictions of $\tilde{\rho}_{l(n)}$ to $\mathfrak{g}$) converge in moments to the distribution of the eigenvalues of $\Pi_{\mathfrak{g}}(\tilde{A})$. Since the random weights have a common compact support the convergence holds also in the weak sense. We finish the proof by observing that the total variation distance between the distributions of $\Pi_{\mathfrak{g}}(\tilde{A})$ and $\Pi_{\mathfrak{g}}(A)$ is smaller than ε. □

In the case of the inclusion of the groups $U(d) \subseteq U(d')$ for $d < d'$ the above theorem takes the following concrete form:

Corollary 5.2. *Let $d < d'$ be positive integers and let (ϵ_n) be a sequence of real numbers which converges to zero. Let $A = (A_{ij})_{1\leq i,j\leq d'}$ be a Hermitian $U(d)$-invariant random matrix and let ρ_n be a sequence of representations of $U(d')$ with the property that the distribution of $\epsilon_n \lambda'_n$ converges to the joint distribution of eigenvalues of A, where $\lambda'_n \in \mathbb{Z}_{d'}$ is a random weight associated to ρ_n.*

Then the distribution of $\epsilon_n \lambda_n$ converges to the joint distribution of eigenvalues of the corner $(A_{ij})_{1\leq i,j\leq d}$, where $\lambda_n \in \mathbb{Z}_d$ is a random weight associated to the restriction $\epsilon_n \rho_n|_{\mathfrak{u}_d}$.

Observe that in the above result we used Remark 3.2 in order to work with Hermitian random matrices. A similar concrete interpretation in the case of unitary groups is also possible for the other results presented in this section.

We leave to the reader to investigate the other such simple inclusions: of orthogonal groups $O(d) \subset O(d')$ and of symplectic groups $Sp(d) \subset Sp(d')$.

Theorem 5.3. *Let (ϵ_n) be a sequence of real numbers which converges to zero, let $(\rho_n^{(1)})$, $(\rho_n^{(2)})$ be two sequences of representations of $\mathfrak{g}$ and let $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$ be the corresponding sequences of random highest weights. Furthermore, let λ_n be the sequence of random highest weights corresponding to the tensor products $\rho_n^{(1)} \otimes \rho_n^{(2)}$. Let $A^{(1)}$, $A^{(2)}$ be independent G-invariant random matrices in $\mathfrak{g}^\star$.*

1. *If for each $i \in \{1,2\}$ the sequence $\epsilon_n \lambda_n^{(i)}$ converges in moments to the distribution of eigenvalues of $A^{(i)}$, then $\epsilon_n \lambda_n$ converges in moments to the distribution of eigenvalues of $A^{(1)} + A^{(2)}$.*
2. *If for each $i \in \{1,2\}$ the sequence $\epsilon_n \lambda_n^{(i)}$ converges weakly to the distribution of eigenvalues of $A^{(i)}$, then $\epsilon_n \lambda_n$ converges weakly to the distribution of eigenvalues of $A^{(1)} + A^{(2)}$.*

Proof. Let $\rho_n^{(3)} := \rho_n^{(1)} \otimes \rho_n^{(2)}$. Then

$$\rho_n^{(3)}(x) = \rho_n^{(1)}(x) \otimes 1 + 1 \otimes \rho_n^{(2)}(x). \tag{5.1}$$

It follows that $\epsilon_n \rho_n^{(3)}$ viewed as a non-commutative random vector is a sum of two non-commutative random vectors: $\big(\epsilon_n \rho_n^{(1)}(x)\big) \otimes 1$ and $1 \otimes \big(\epsilon_n \rho_n^{(2)}(x)\big)$. Theorem 4.1 implies that the first summand converges in moments to $A^{(1)}$ and the second summand converges in moments to $A^{(2)}$. The coordinates of the first vector (viewed as non-commutative random variables) commute with the coordinates of the second vector; since we consider them with respect to a state (i.e. the normalized trace) which is a tensor product of the original states (i.e. normalized traces), it follows that their sum converges to the sum of the independent random matrices, which finishes the proof of part (1).

Part (2) follows from part (1) in a similar way as in Theorem 5.1. □

5.2. **Central limit theorem.** The following theorem was already proved by Kuperberg [Kup02] in a slightly different setting. We include this theorem here because we believe that the proof in this framework (using Theorem 4.1) is new.

Theorem 5.4. *Let G be a Lie group and let ρ be its representation. We define $c \in \mathfrak{g}^\star$ given by $c(x) = \operatorname{tr} \rho(x)$.*

Let λ_n be the random highest weight corresponding to the representation $\rho^{\otimes n}$. Then the sequence

$$\frac{1}{\sqrt{n}} \big[\lambda_n - cn \big] \tag{5.2}$$

converges (both in moments and weakly) to the distribution of eigenvalues of a certain centered Gaussian random matrix in $\mathfrak{g}^\star$.

The covariance of the above Gaussian random matrix is given by

$$\int_{\mathfrak{g}^\star} \lambda(x)\ \lambda(y) d\mu(\lambda) = \operatorname{tr} \rho(x)\rho(y)$$

for any $x, y \in \mathfrak{g}$.

Proof. Similarly as in Eq. (5.1) we have that

$$\rho^{\otimes n}(x) = \rho(x) \otimes 1 \otimes \cdots + 1 \otimes \rho(x) \otimes 1 \otimes \cdots + \cdots ,$$

regarded as a non-commutative random variable, is a sum of n commuting summands. Therefore the non-commutative central limit theorem of Giri and von Waldenfels [GvW78] can be applied. It follows that the distribution of the non-commutative random vector $\frac{1}{\sqrt{n}} \big[\rho^{\otimes n} - cn \big]$ converges in moments to the distribution of a certain centered Gaussian random variable X, which takes values in $\mathfrak{g}^\star$. We apply Theorem 4.1; it follows that the distribution of the random weight (5.2) converges in moments to the distribution of the eigenvalues of X.

Let $\| \cdot \|$ be the norm on $\mathfrak{g}^\star$ associated to any G-invariant scalar product. The distribution of X is multidimensional Gaussian. Therefore there exists a constant $d > 0$ such that the tail estimate

$$P\left(\|X\| > t \right) < e^{-dt^2} \tag{5.3}$$

holds true for sufficiently large values of t. For any element of $\mathfrak{g}^\star$ its norm is equal to the norm of the element of $\mathfrak{h}^\star$ corresponding to its eigenvalues; it follows that the estimate (5.3) remains true if the random variable X is replaced by its eigenvalues. The estimate (5.3) shows therefore that the even moments of the eigenvalue distribution of X are dominated by the even moments of a Gaussian distribution.

This implies that $\mathbb{E}(e^{\|X\|}) < \infty$ and by Corollary 2.2 and Theorem 2.1 of [DLY02] it follows that the distribution of eigenvalues of X is uniquely determined by its moments. It follows by a standard compactness argument that the distribution of the random weights (5.2) converges weakly to the distribution of the eigenvalues of X. □

Often we have some additional information about the considered Lie group G which restricts the number of G–invariant Gaussian measures on $\mathfrak{g}^\star$. In the case of $G = U(d)$ it is convenient to consider a centered Hermitian Gaussian random matrix $g = (g_{ij})_{1 \leq i,j \leq d}$ defined by the convariance

$$\mathbb{E} g_{ij} g_{kl} = 0, \qquad \mathbb{E} g_{ij} \overline{g_{kl}} = \delta_{il} \delta_{jk};$$

this kind of random matrix (and the corresponding measure on the space of $d \times d$ Hermitian matrices) is called a *Gaussian Unitary Ensemble* (GUE) and plays an important role in random matrix theory. For any $v \geq 0$ we define GUE_v as the distribution of a random matrix

$$g - \operatorname{tr} g + x,$$

where x is an independent centered Gaussian variable with the variance v.

Under the isomorphism from Remark 3.2, GUE_v becomes a measure on the Lie algebra $u(d)$ and it is not very difficult to check that (except for degenerate cases) every $U(d)$–invariant Gaussian measure (up to dilation by some number) is of this form. In particular, we get the following result.

Corollary 5.5. *Let ρ be a representation of $U(d)$. There exist constants c_1, c_2 with the property that if $\lambda_n = (\lambda_{n,1}, \dots, \lambda_{n,d})$ is a random weight associated to $\rho^{\otimes n}$, then the joint distribution of the components of the vector*

$$\frac{c_1}{\sqrt{n}} (\lambda_n - nc_2)$$

converges to the joint distribution of the eigenvalues of the GUE_v random matrix.

5.3. **Toy example: Representations of $SO(3)$ and $SU(2)$.** The above results in the simplest non-trivial case of $G = SO(3)$ should not be very surprising from the viewpoint of quantum mechanics. Each quantum-mechanical system in three-dimensional space can be viewed as a (possibly reducible) representation of $SO(3)$ (or its universal cover $\mathrm{Spin}(3) = SU(2)$) on some Hilbert space V. The irreducible components of this representation have a nice physical interpretation as physical states with a well-defined length $|J|$ of the angular momentum. For simplicity, we assume that V itself is irreducible; hence V is finite-dimensional. The information concerning the state of the physical system is encoded by a state ϕ on the algebra generated by observables. We are interested in the situation when the physical state of the system is $SO(3)$-invariant; it follows that ϕ is the normalized trace on $\mathrm{End}(V)$.

The physicist's question pertaining to the distribution of a component J_z of the angular momentum can be reformulated in the language of mathematics as a question of the decomposition into irreducible components of the restriction $V \downarrow^{SO(3)}_{SO(2)}$ of the representation V to a subgroup $SO(2)$ (or, more generally, restriction of the representation V of $\mathrm{Spin}(3) = SU(2)$ to its subgroup $\mathrm{Spin}(2) = U(1)$); namely it is the uniform measure on the set of integers (or half-integers)

$$\big\{ -|J|, -|J|+1, \dots, |J|-1, |J| \big\} \tag{5.4}$$

(for simplicity we use the system of units in which Planck's constant $\hbar = 1$).

On the other side it is well known that when the size of our system becomes macroscopic, then quantum mechanics may be approximated by classical mechanics, where the angular momentum $\vec{J} = (J_x, J_y, J_z)$ is just a usual vector consisting of numbers. Our assumptions on irreducibility and $SO(3)$-invariance imply that in the classical limit $\vec{J}$ is a random vector with a uniform distribution on a sphere of a fixed length $|J|$. The physicist's question concerning the distribution of a component J_z of the angular momentum in this context is answered by the theorem of Archimedes; namely it is the uniform distribution on the interval $[-|J|, |J|]$. In the limit $|J| \to \infty$, after appropriate rescaling, the uniform measure on the set (5.4) indeed converges to the uniform measure on the set $[-|J|,|J|]$; hence the answer given by classical mechanics is indeed the limit of the answer given by quantum mechanics.

It is also a consequence of this article: indeed we view the representation ρ of the Lie algebra $so(3)$ as an element of $\big(so(3)\big)^\star \otimes \operatorname{End}(V)$; in our current case this takes the concrete form of a matrix

$$J = \begin{bmatrix} 0 & J_z & -J_y \\ -J_z & 0 & J_x \\ J_y & -J_x & 0 \end{bmatrix} \tag{5.5}$$

(the form of this matrix depends on the particular choice of the identification of $\big(so(3)\big)^\star$ with certain 3×3 matrices). Theorem 4.1 states that asymptotically the matrix (5.5) behaves like a random 3×3 antisymmetric matrix with eigenvalues $0, |J|, -|J|$.

We leave the analysis of the central limit theorem in this case to the reader.

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