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ENTIRE FUNCTIONS MAPPING UNCOUNTABLE DENSE SETS OF REALS ONTO EACH OTHER MONOTONICALLY

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Abstract. When A and B are countable dense subsets of $\mathbb{R}$, it is a well-known result of Cantor that A and B are order-isomorphic. A theorem of K.F. Barth and W.J. Schneider states that the order-isomorphism can be taken to be very smooth, in fact the restriction to $\mathbb{R}$ of an entire function. J.E. Baumgartner showed that consistently $2^{\aleph_0} > \aleph_1$ and any two subsets of $\mathbb{R}$ having $\aleph_1$ points in every interval are order-isomorphic. However, U. Abraham, M. Rubin and S. Shelah produced a ZFC example of two such sets for which the order-isomorphism cannot be taken to be smooth. A useful variant of Baumgartner's result for second category sets was established by S. Shelah. He showed that it is consistent that $2^{\aleph_0} > \aleph_1$ and second category sets of cardinality $\aleph_1$ exist while any two sets of cardinality $\aleph_1$ which have second category intersection with every interval are order-isomorphic. In this paper, we show that the order-isomorphism in Shelah's theorem can be taken to be the restriction to $\mathbb{R}$ of an entire function. Moreover, using an approximation theorem of L. Hoischen, we show that given a nonnegative integer n, a nondecreasing surjection $g\colon \mathbb{R} \to \mathbb{R}$ of class C^n and a positive continuous function $\epsilon\colon \mathbb{R} \to \mathbb{R}$, we may choose the order-isomorphism f so that for all $i = 0, 1, \ldots, n$ and for all $x \in \mathbb{R}$, $|D^i f(x) - D^i g(x)| < \epsilon(x)$.

1. Introduction

When A and B are countable dense subsets of $\mathbb{R}$, it is a well-known result of Cantor [Ca, §9] that A and B are order-isomorphic. Notice that an order-isomorphism between dense subsets of $\mathbb{R}$ extends to an order-isomorphism of $\mathbb{R}$. We record this simple fact as a proposition for ease of reference.

Proposition 1.1. *If $K, L \subseteq \mathbb{R}$ are dense and $h\colon K \to L$ is an order isomorphism, then h extends to an order isomorphism of $\mathbb{R}$.*

The extension to an order-isomorphism of $\mathbb{R}$ of an isomorphism between countable dense sets given by Cantor's theorem is in particular a monotone function and hence differentiable almost everywhere. The question of improving the isomorphism was examined by Franklin [Fr], who showed that it can be taken to be real-analytic. Motivated by the problem of finding order-isomorphisms of $[0, 1]$ which map each of

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the sets of rational, algebraic and transcendental numbers onto themselves, Melzak [Me] observes that Franklin's methods show that if $\{A_n\}_{n<\omega}$ and $\{B_n\}_{n<\omega}$ are each a sequence of pairwise disjoint countable dense subsets of $(0,1)$, then there is an analytic order-isomorphism f of $[0,1]$ such that for each $n<\omega$ the function f maps A_n onto B_n. Moreover, given any function g of class C^n whose derivative is bounded away from zero, f can be chosen so that its first n derivatives are uniformly approximated by those of g. The map in Franklin's result was improved to being the restriction to $\mathbb{R}$ of an entire function by Barth and Schneider [BS], thereby solving [Er, Problem 24]. They also state without proof that their method gives the generalization to sequences of pairwise disjoint countable dense sets as obtained by Melzak for analytic functions, but that "the massive amount of bookkeeping involved in this proof is such as to make it impractical to include it in this paper". A variation on the problem of Erdős referred to above, interpreted so that it refers to countable dense subsets of $\mathbb{C}$ rather than of $\mathbb{R}$, was solved by Maurer [Ma]. An elegant proof of the Barth-Schneider result based on Maurer's work was given by Sato and Rankin [SR]. (See also [NT], which contains a variation on the same argument.) They make no comment about the result for sequences of pairwise disjoint countable dense sets, but their proof easily yields that version as well.

In [Ba], a nonempty set S of real numbers is said to be $\aleph_1$-*dense* if S is without endpoints and there are exactly $\aleph_1$ members of S between any two distinct points of S. In particular, if $S\cap I$ has cardinality $\aleph_1$ for every nonempty open interval I, then S is $\aleph_1$-dense. We shall use the term only in this more restricted sense. Baumgartner proved [Ba] that if ZFC is consistent, then so is the theory ZFC + the statement "all $\aleph_1$-dense sets of reals are order-isomorphic", which is a natural stepping-up by one cardinal of Cantor's theorem. Following Abraham, Rubin and Shelah [ARS], we denote the statement in quotes by BA. (The two meanings of $\aleph_1$-dense are equivalent for the purposes of BA.) It is shown in [ARS] that the function inducing the order-isomorphisms in BA cannot in general be taken to be smooth. We reproduce the argument from [ARS], giving a few more details. Our statement of the result is slightly different; see Remark 1.4.

Proposition 1.2 ([ARS, Proposition 9.4]). *There are $\aleph_1$-dense sets $A,B\subseteq\mathbb{R}$ such that for no nonconstant C^1 function $f\colon\mathbb{R}\to\mathbb{R}$ do we have $f[A]\subseteq B$.*

Proof. Fix a one-to-one enumeration $\{r_n : n<\omega\}$ of $\mathbb{Q}$. We define strictly increasing functions $g,h\colon\mathbb{Q}\to\mathbb{R}$ and, letting $a_n=g(r_n)$ and $b_n=h(r_n)$, satisfy

$$\min\{|a_i-a_j|,\,|b_k-b_\ell|\}<\max\{|a_i-a_j|,\,|b_k-b_\ell|\}^2 \tag{1.1}$$

whenever $i\neq j$ and $k\neq\ell$. The definition proceeds by inductively choosing $a_0,b_0,a_1,b_1,\ldots$. Choose a_0 and b_0 arbitrarily. If $n\geq 1$ and a_i, b_i have been chosen for $i<n$ and (1) holds whenever the indices are smaller than n, pick a_n and b_n as follows. Choose an index $i_0<n$ such that r_{i_0} is adjacent to r_n in $\{r_i : i\leq n\}$. The choice of a_n must be made so that the instances of (1.1) of the form

$$\min\{|a_i-a_n|,\,|b_k-b_\ell|\}<\max\{|a_i-a_n|,\,|b_k-b_\ell|\}^2, \tag{1.2}$$

where $i<n$ and $k,\ell<n$ are distinct, hold. When $i\neq i_0$, the fact that

$$\min\{|a_i-a_{i_0}|,\,|b_k-b_\ell|\}<\max\{|a_i-a_{i_0}|,\,|b_k-b_\ell|\}^2$$

ensures that as long as a_n is close enough to a_{i_0}, (1.2) will hold. When $i=i_0$, as long as a_n is close enough to a_{i_0}, we have $|a_{i_0}-a_n|<|b_k-b_\ell|$, and moreover

$|a_{i_0} - a_n| < |b_k - b_\ell|^2$ whenever $k, \ell < n$ are distinct. Thus choosing a_n close enough to a_{i_0} gives (1.1) for all distinct indices $i, j \le n$ and all distinct indices $k, \ell < n$. Hence we choose a_n so that it is close enough to a_{i_0}, as just described, and $g \upharpoonright \{r_i : i \le n\}$ is order-preserving. (When $n = 1$, the first part is vacuous since there do not exist distinct $k, \ell < n$.) The choice of b_n is completely analogous. Now let $L = \mathrm{cl}(g[\mathbb{Q}])$, $M = \mathrm{cl}(h[\mathbb{Q}])$. (cl denotes closure in $\mathbb{R}$.) The points of L can be approximated arbitrarily well by points a_n and those of M by points b_n. It follows that for any points $a, a' \in L$ and $b, b' \in M$ we have

$$\min\{|a - a'|, |b - b'|\} \le \max\{|a - a'|, |b - b'|\}^2. \tag{1.3}$$

(Notice that (1.3) is trivially true when $a = a'$ or $b = b'$. Use (1.1) for the case $a \ne a'$ and $b \ne b'$.) Since the sets $g[\mathbb{Q}]$ and $h[\mathbb{Q}]$ have the order type of the rationals, their closures are uncountable. (In each set, distinct initial segments have distinct suprema.) Choose sets of cardinality $\aleph_1$, $A_L \subseteq L$ and $B_M \subseteq M$. Let $A = A_L + \mathbb{Q}$, $B = B_M + \mathbb{Q}$.

Consider a continuous function $f \colon \mathbb{R} \to \mathbb{R}$ such that $f[A] \subseteq B$.

Claim 1.3. In each nonempty open interval I, either f has divided differences $|f(a) - f(b)|/|a - b|$, $a \ne b$, which are arbitrarily large, or f has divided differences which are arbitrarily close to zero.

Proof of Claim. Fix a nonempty open interval I. For each $x \in A \cap I$, there are $q_x, r_x \in \mathbb{Q}$ such that $x \in A_L + q_x$, $f(x) \in B_M + r_x$. Since $A \cap I$ is uncountable, there are $q, r \in \mathbb{Q}$ and an uncountable set $S \subseteq A \cap I$ such that for all $x \in S$, $q_x = q$ and $r_x = r$. Choose any accumulation point $a \in I$ for the set S, and pick a sequence of distinct points $\{s_n\} \subseteq S$ converging to a. Then $\{f(s_n)\}$ converges to $f(a)$ since f is continuous. Given any $\varepsilon > 0$, fix a large enough n so that $|s_n - s_{n+1}| < \varepsilon$ and $|f(s_n) - f(s_{n+1})| < \varepsilon$. By (1.3),

$$\begin{aligned}\min\{|s_n - s_{n+1}|, |f(s_n) - f(s_{n+1})|\} &\le \max\{|s_n - s_{n+1}|, |f(s_n) - f(s_{n+1})|\}^2 \\ &\le \varepsilon \max\{|s_n - s_{n+1}|, |f(s_n) - f(s_{n+1})|\},\end{aligned}$$

which implies that the divided difference $d_\varepsilon = |f(s_n) - f(s_{n+1})|/|s_n - s_{n+1}|$ is either $\le \varepsilon$ or $\ge 1/\varepsilon$. At least one of $d_\varepsilon \le \varepsilon$ or $d_\varepsilon \ge 1/\varepsilon$ must occur for values of ε arbitrarily close to zero. This establishes Claim 1.3. □

The conclusion now follows from the claim since a nonconstant C^1 function will have intervals over which the values of its derivative, and hence also its divided differences, are in a bounded closed interval not containing zero. □

Remark 1.4. Consider the case of the proposition where f is an order-isomorphism. The point a in the above argument can be taken to be in S and to be a two-sided accumulation point of S. The sequence $\{s_n\}$ can then be chosen so that s_n and s_{n+1} are on different sides of a. Then $|f(s_n) - f(s_{n+1})|/|s_n - s_{n+1}| > 1/\varepsilon$ gives either $|f(s_n) - f(a)|/|s_n - a| > 1/\varepsilon$ or $|f(a) - f(s_{n+1})|/|a - s_{n+1}| > 1/\varepsilon$. Similarly in the $< \varepsilon$ situation. Hence either $Df(a)$ does not exist or $Df(a) = 0$. In [ARS], the stronger conclusion that for some $a \in A$, $Df(a)$ does not exist is stated. We do not see why this stronger conclusion holds.

Remark 1.5. Since order-isomorphisms of $\mathbb{R}$ are homeomorphisms, one consequence of Proposition 1.1 is that dense subsets of $\mathbb{R}$ which are order-isomorphic must be indistinguishable topologically as subspaces of $\mathbb{R}$. In particular, a dense first category set cannot be order isomorphic to a dense second category set. There is always an $\aleph_1$-dense first category set. (Cf. the proof of Proposition 1.2.) Hence, BA implies that all sets of cardinality $\aleph_1$ are first category.

Notice that the sets A and B given by the proof of Proposition 1.2 are first category. Shelah proved the following theorem as part of the proof of [Sh1980, Theorem 4.7], which states that if ZFC is consistent, then so is ZFC + $2^{\aleph_0} = \aleph_2$ + "there is a universal (linear) order of power $\aleph_1$".

Theorem 1.6 ([Sh1980])**.** *If ZFC is consistent, then so is ZFC + both of the following statements.*

(a) *There is a second category set in* $\mathbb{R}$ *of cardinality* $\aleph_1$.
(b) *Let* A *and* B *be everywhere second category subsets of* $\mathbb{R}$ *of cardinality* $\aleph_1$. *Then* A *and* B *are order-isomorphic.*

(A set $A \subseteq \mathbb{R}$ is *everywhere second category* if $A \cap I$ is second category for every nonempty open interval I.)

An examination of Shelah's model shows that, by a simple genericity argument, the functions witnessing (b) fail to be differentiable at any constructible real. In this paper we show that the order-isomorphism in Theorem 1.6 can be taken to be the restriction to $\mathbb{R}$ of an entire function. The following is the main result of this paper.

Theorem 1.7. *If ZFC is consistent, then so is ZFC +* $2^{\aleph_0} = \aleph_2$ *+ the following statements.*

(a) *Every second category set in* $\mathbb{R}$ *has a second category subset of cardinality* $\aleph_1$.
(b) *For any two sequences,* $\langle A_\alpha : \alpha < \omega_1\rangle$ *and* $\langle B_\alpha : \alpha < \omega_1\rangle$, *each consisting of pairwise disjoint dense subsets of* $\mathbb{R}$, *if* A_α *and* B_α *are countable for* $\alpha < \omega$ *and are everywhere second category sets of cardinality* $\aleph_1$ *for* $\omega \leq \alpha < \omega_1$, *then there is an entire function* $f\colon \mathbb{C} \to \mathbb{C}$ *which restricts to an order-isomorphism of* $\mathbb{R}$ *such that* $f[A_\alpha] = B_\alpha$ *for every* $\alpha < \omega_1$.
(c) *Suppose that in* (b) *we are additionally given a positive continuous function* $\varepsilon\colon \mathbb{R} \to \mathbb{R}$ *and a nondecreasing surjection* $g\colon \mathbb{R} \to \mathbb{R}$.
    (i) *If* n *is a nonnegative integer and* g *is of class* C^n, *then we may ask that for all* $i = 0, 1, \ldots, n$ *and all* $x \in \mathbb{R}$, $|D^i f(x) - D^i g(x)| < \varepsilon(x)$.
    (ii) *If* $0 \leq c_0 \leq c_1 \leq \ldots$ *satisfies* $\lim_{i\to\infty} c_i = \infty$ *and* g *is of class* C^∞, *then we may ask that for every* $i < \omega$, *and each* $x \in \mathbb{R}$ *such that* $|x| \geq c_i$, $|D^i f(x) - D^i g(x)| < \varepsilon(x)$.

Remark 1.8. The theorem applies to sequences of length less than ω_1 as well. For example, to apply it to a single pair of everywhere second category sets A and B of cardinality $\aleph_1$, inductively define A_α, $\alpha < \omega_1$, as follows. The sets A_α for $\alpha < \omega$ are any countable family of pairwise disjoint countable dense subsets of $\mathbb{R} \setminus A$. Set $A_\omega = A$ and for $\omega < \alpha < \omega_1$, take A_α to be any translate $A + r$ which is disjoint from all the sets A_β, $\beta < \alpha$. (Because $2^{\aleph_0} > \aleph_1$, there is an $r \notin \{a_1 - a_2 : a_1 \in \bigcup_{\beta<\alpha} A_\beta, a_2 \in A\}$.) Similarly define B_α, $\alpha < \omega_1$ (with $B_\omega = B$) and now (b) (or (b) and (c)) applied to these sequences produces the desired order-isomorphism of A and B.

From Theorem 1.7, we can deduce a version of the Barth-Schneider result with the ability to approximate derivatives. (Alternatively, a direct proof of the corollary can be extracted from the proof of the theorem.)

Corollary 1.9. *For any two sequences, $\langle A_n : n < \omega\rangle$ and $\langle B_n : n < \omega\rangle$, each consisting of pairwise disjoint countable dense subsets of $\mathbb{R}$, there is an entire function $f \colon \mathbb{C} \to \mathbb{C}$ which restricts to an order-isomorphism of $\mathbb{R}$ such that $f[A_n] = B_n$ for every $n < \omega$ and f can be chosen to approximate a given nondecreasing surjection as in Theorem 1.7(c).*

Proof. For a given choice of ground model parameters A_n, B_n, g, c_i (thinking of g as a (finite or infinite) sequence of Borel codes $\langle g, Dg, D^2g, \dots\rangle$ for its derivatives), the hypothesized properties of these parameters are $\mathbf{\Pi^1_1}$ and hence continue to hold in the forcing extension which produces the model of Theorem 1.7. (The model is built starting with a model of $V = L$, but it is easily seen (and well known) that the required consequences of $V = L$, namely the existence of diamond sequences on ω_1 and on the limits of cofinality ω_1 in ω_2, can be forced.) The existence of an entire function f satisfying the conclusion is a $\mathbf{\Sigma^1_2}$ property of the parameters and hence holds in the ground model by the Shoenfield absoluteness theorem. □

The following problem seems to be open.

Problem 1.10. [1] Are there (in ZFC) two sequences, $\langle A_\alpha : \alpha < \omega_1\rangle$ and $\langle B_\alpha : \alpha < \omega_1\rangle$, each consisting of pairwise disjoint countable dense subsets of $\mathbb{R}$ and such that for no order-isomorphism $f \colon \mathbb{R} \to \mathbb{R}$ do we have $f[A_\alpha] = B_\alpha$ for all $\alpha < \omega_1$?

Note that since every $\aleph_1$-dense subset of $\mathbb{R}$ can be partitioned into $\aleph_1$ countable dense subsets, in any model for the negative answer to Problem 1.10, any two $\aleph_1$-dense subsets of $\mathbb{R}$ are order-isomorphic, i.e., BA holds. In particular, sets of reals of cardinality $\aleph_1$ are first category (Remark 1.5). Thus, in part (b) of Theorem 1.7, the restriction that only countably many of the pairs (A_α, B_α) consist of countable sets cannot be relaxed to allow for uncountably many such pairs as this would contradict part (a).

Our main tool for approximating differentiable real functions by entire ones is the following strengthening of Carleman's theorem due to Hoischen. It generalizes to entire functions the approximation given in [Wh, Lemma 6] (which deals with approximation by analytic functions).

Theorem 1.11 ([Ho, Satz 1 + Satz 2])**.** *Let $n < \omega$. If $g \colon \mathbb{R} \to \mathbb{R}$ is a function of class C^n and $\varepsilon \colon \mathbb{R} \to \mathbb{R}$ is a positive continuous function, then there exists an entire function f such that $f[\mathbb{R}] \subseteq \mathbb{R}$ and for all $i = 0, \dots, n$ and all $x \in \mathbb{R}$, $|D^i f(x) - D^i g(x)| < \varepsilon(x)$. Furthermore, if g is of class C^∞ and $\{c_i\}_{i<\omega}$ is any nondecreasing sequence of nonnegative real numbers with $\lim c_i = \infty$, then, for every positive continuous ε on $\mathbb{R}$ there exists an entire function f such that $f[\mathbb{R}] \subseteq \mathbb{R}$ and for $i = 0, 1, 2 \dots$ and all $x \in \mathbb{R}$ such that $|x| \geq c_i$, $|D^i f(x) - D^i g(x)| < \varepsilon(x)$.*

(Hoischen doesn't state that $f[\mathbb{R}] \subseteq \mathbb{R}$ but his proof gives this immediately.)

The derivative of a function f is denoted exclusively by Df in this paper. In particular, f' does not denote the derivative of f. $\mathbb{N}$ denotes the set of positive integers. We refer the reader to [Je] or [Ku] for set-theoretic notation and results

[1] S. Shelah has found an elegant example. (Email message 2004/07/06.)

not explained here. We assume that the reader is familiar with the oracle-cc forcing technique as explained in [Sh1998]. We recall the basic definitions and properties.

Definition 1.12. A sequence $\overline{M} = \langle M_\delta : \delta < \omega_1 \rangle$ is called an *oracle* if each M_δ is a countable transitive model of a sufficiently large fragment of ZFC, $\delta \in M_\delta$ is countable in M_δ and for each $A \subseteq \omega_1$, $\{\delta : A \cap \delta \in M_\delta\}$ is stationary in ω_1.

The existence of an oracle is equivalent to $\diamondsuit$ (see [Ku, Theorem II 7.14]) and hence implies CH. Associated with an oracle $\overline{M}$, there is a filter $\mathrm{Trap}(\overline{M})$ generated by the sets

$$\{\delta < \omega_1 : \delta \text{ is a limit ordinal and } A \cap \delta \in M_\delta\}, \quad A \subseteq \omega_1.$$

This is a proper normal filter containing all closed unbounded sets.

The definition of the $\overline{M}$-chain condition which follows is valid only for partial orders of cardinality $\aleph_1$. This case suffices for our purposes.

Definition 1.13. A partial order P *satisfies the $\overline{M}$-chain condition*, or simply *is $\overline{M}$-cc*, if there is a one-to-one function $f\colon P \to \omega_1$ such that the set of limit ordinals $\delta < \omega_1$ such that every predense subset of $f^{-1}(\delta)$ of the form $f^{-1}[A]$, where $A \subseteq \delta$ and $A \in M_\delta$, is predense in P belongs to $\mathrm{Trap}(\overline{M})$.

It is not hard to verify that if P is $\overline{M}$-cc, then P is ccc. Also, any one-to-one function $g\colon P \to \omega_1$ can replace f in the definition.

Recall the following properties of oracle-cc forcing. See [Sh1998, Chapter IV] for more details.

Proposition 1.14. *Assume $\diamondsuit$. Let A be a second category subset of $\mathbb{R}$. Then there is an oracle $\overline{M} = \langle M_\delta : \delta < \omega_1 \rangle$ such that if P is any partial order satisfying the $\overline{M}$-cc, then $\Vdash_P$ "A is second category".*

Proposition 1.15. *The $\overline{M}$-cc satisfies the following properties.*

(1) *If $\alpha < \omega_2$ is a limit ordinal, $\langle \langle P_\beta \rangle_{\beta \le \alpha}, \langle \dot{Q}_\beta \rangle_{\beta < \alpha} \rangle$ is a finite-support α-stage iteration of partial orders, and for each $\beta < \alpha$, P_β is $\overline{M}$-cc, then P_α is $\overline{M}$-cc.*

(2) *If P is $\overline{M}$-cc, then there is a P-name $\overline{M}^*$ for an oracle such that for each P-name $\dot{Q}$ for a partial order, if $\Vdash_P$ "$\dot{Q}$ is $\overline{M}^*$-cc" then $P * \dot{Q}$ is $\overline{M}$-cc.*

(3) *If $\overline{M}_\alpha$, $\alpha < \omega_1$, are oracles, then there is an oracle $\overline{M}$ such that for any partial order P, if P is $\overline{M}$-cc, then P is $\overline{M}_\alpha$-cc for all $\alpha < \omega_1$.*

The proof of the main theorem involves many technical arguments for which the intuition may not be clear upon a first reading. The reader may find it helpful to read the proof of Theorem 1.6 upon which the proofs given here build. The proof in [Sh1980] is only a brief sketch, but the argument is presented in detail in [BM]. In Section 2, we provide a direct proof of a consequence of the main theorem for which many of the technical difficulties do not arise. Briefly, the difference is that the consequence deals with C^∞ functions rather than entire ones and the former are very much more flexible as a class than the latter. In particular, a C^∞ function can be zero outside a compact interval without being identically zero. The proof of the main theorem makes no reference to results in Section 2, so the reader who wants to do so can skip that section. We ask the indulgence of the readers of Section 2 for the repetition of some of the arguments and remarks in the proof of the main theorem.

In Section 3, we show how to reduce Theorem 1.7 to the case where the function g in part (c) is the restriction to $\mathbb{R}$ of an entire function and has a strictly positive derivative. In Section 4, we define the class of entire functions which will be used in the proof of Theorem 1.7 and establish some of its properties. The main lemma needed for the proof of Theorem 1.7 is established in Section 5. The deduction of the theorem from the main lemma is standard oracle-cc technique which we sketch in the final section of the paper.

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2. A special case

We begin with a proof of the following special case of the main theorem. Results from this section are not used elsewhere in the paper.

Theorem 2.1. *If ZFC is consistent, then so is ZFC + $2^{\aleph_0} = \aleph_2$ + the following statements.*

(a) *Every second category set in $\mathbb{R}$ has a second category subset of cardinality $\aleph_1$.*
(b) *For any two everywhere second category sets $A, B \subseteq \mathbb{R}$ of cardinality $\aleph_1$, there is a C^∞ order-isomorphism $f\colon \mathbb{R} \to \mathbb{R}$ such that $f[A] = B$.*

We shall use the following standard fact.

Proposition 2.2. *Let $f_n\colon \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$ be C^∞ functions such that for each $i < \omega$, the sequence $\{D^i f_n\}_{n=1}^\infty$ converges uniformly to a function g_i. Then g_0 is a C^∞ function and $D^i g_0 = g_i$ for each $i < \omega$.*

Proof. See [Zi, Example 1.1.10 and Proposition 1.1.13]. □

We begin by defining a family of C^∞ bump functions. We will call an interval I of the form $I = (a, b)$, where $a, b \in \mathbb{Q}$ and $a < b$, a *rational interval*. Let $\mathcal{I}$ denote the set of all rational intervals.

For $I = (a, b)$ with $a, b \in \mathbb{R}$ and $a < b$, let $g_I\colon \mathbb{R} \to \mathbb{R}$ be the following C^∞ bump function

$$g_I(x) = \begin{cases} e^{-[(x-a)(b-x)]^{-1}} & a < x < b, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{G}^* = \{g_I : I \in \mathcal{I}\}$.

The following proposition provides a means for modifying functions in $\mathcal{G}^*$ without significantly altering their derivatives, and also for approximating members of $\mathcal{G}^*$ by elements of $\mathcal{G}^*$ from a given model.

Proposition 2.3. *Let N be an elementary submodel of H_θ for some regular $\theta > \mathfrak{c}$. Let ε be a positive rational number. Let $i_0 \in \mathbb{N}$. Let $f_0\colon \mathbb{R} \to \mathbb{R}$ be the identity function.*

Suppose that we are given countable dense subsets B and C of $\mathbb{R}$ with $B, C \in N$. Let $u \in \mathbb{R}$. Let $f \in f_0 + \operatorname{span} \mathcal{G}^*$. Let $K_0, K_1 \subseteq \mathbb{R}$ be finite such that $K_0, f[K_1] \in N$. Let $h \subseteq f$ be finite such that $h \in N$. Assume that the sets $\operatorname{dom} h$, K_0, K_1, $\{u\}$ are pairwise disjoint. Then there is a function $f' \in N \cap (f_0 + \operatorname{span} \mathcal{G}^*)$ such that *for some rational interval I and positive rational number r, the following properties hold:*

(a) $h \subseteq f'$;
(b) *for all* $x \in K_0$, $f'(x) \in C$;
(c) *for all* $x \in K_1$, *there is a* $b_x \in B$ *such that* $|b_x - x| < \varepsilon$ *and* $f'(b_x) = f(x)$;
(d) *for all* $i \leq i_0$, $r\|D^i g_I\|_\infty < \varepsilon$ *and for all* $\sigma \in \mathbb{R}$ *such that* $|\sigma| \leq r$,

$$\|D^i f - D^i(f' + \sigma g_I)\|_\infty < \varepsilon;$$

(e) $g_I(u) \neq 0$ *and for some real number* σ *such that* $|\sigma| < r$, *we have*

$$f(u) = f'(u) + \sigma g_I(u).$$

Proof. Let $K = K_0 \cup K_1 \cup \operatorname{dom} h$. Choose pairwise disjoint rational intervals I_x such that $x \in I_x$ for $x \in K \cup \{u\}$. For each $x \in K_1$, choose a sequence of points $b_{x,m} \in B \cap I_x$, $m \in \mathbb{N}$, such that $|b_{x,m} - x| < 1/m$. For each $x \in K_0$, choose a sequence of points $c_{x,m} \in C$, $m \in \mathbb{N}$, such that $|c_{x,m} - f(x)| < 1/m$. Let

$$f = f_0 + \sum_{I \in \mathcal{I}'} \lambda_I g_I,$$

where $\mathcal{I}'$ is a finite subset of $\mathcal{I}$ and $\lambda_I \in \mathbb{R}$ for each $I \in \mathcal{I}'$. Consider functions f', $\bar{f}$ of the form

$$f' = f_0 + \sum_{I \in \mathcal{I}'} \mu_I g_I + \sum_{x \in K} \sigma_x g_{I_x}$$

and

$$\bar{f} = f_0 + \sum_{I \in \mathcal{I}'} \mu_I g_I + \sum_{x \in K} \sigma_x g_{I_x} + \sigma g_{I_u} = f' + \sigma g_{I_u},$$

where $\mu_I, \sigma_x, \sigma \in \mathbb{R}$ ($I \in \mathcal{I}'$, $x \in K$). For $a \in K$, among the functions $g_{I_{a'}}$, $a' \in K$, only the one with $a' = a$ is not zero at a and for $x \in K_1$, only g_{I_x} is not zero at $b_{x,m}$. This leads to the following observations.

(1) For $a \in \operatorname{dom} h$ and each choice of $m \in \mathbb{N}$ and $\vec{\mu} = (\mu_I)_{I \in \mathcal{I}'}$, there is a unique $\sigma_a = \sigma_a(m, \vec{\mu})$ for which $f'(a) = f(a) = h(a)$, namely

$$\sigma_a(m, \vec{\mu}) = \frac{\sum_{I \in \mathcal{I}'} (\lambda_I - \mu_I) g_I(a)}{g_{I_a}(a)}.$$

($\sigma_a(m, \vec{\mu})$ does not depend on m, but it is convenient to denote it this way for uniformity of the notation.)

(2) For each $x \in K_0$ and each choice of $m \in \mathbb{N}$ and $\vec{\mu} = (\mu_I)_{I \in \mathcal{I}'}$, there is a unique $\sigma_x = \sigma_x(m, \vec{\mu})$ for which $f'(x) = c_{x,m}$, namely

$$\begin{aligned}\sigma_x(m, \vec{\mu}) &= \frac{c_{x,m} - f_0(x) - \sum_{I \in \mathcal{I}'} \mu_I g_I(x)}{g_{I_x}(x)} \\ &= \frac{c_{x,m} - f(x) + \sum_{I \in \mathcal{I}'} (\lambda_I - \mu_I) g_I(x)}{g_{I_x}(x)}.\end{aligned}$$

(3) For each $x \in K_1$ and each choice of $m \in \mathbb{N}$ and $\vec{\mu} = (\mu_I)_{I\in\mathcal{I}'}$, there is a unique $\sigma_x = \sigma_x(m, \vec{\mu})$ for which $f'(b_{x,m}) = f(x)$, namely

$$\sigma_x(m,\vec{\mu}) = \frac{f(x) - f_0(b_{x,m}) - \sum_{I\in\mathcal{I}'} \mu_I g_I(b_{x,m})}{g_{I_x}(b_{x,m})}$$
$$= \frac{f_0(x) - f_0(b_{x,m}) + \sum_{I\in\mathcal{I}'} \lambda_I g_I(x) - \sum_{I\in\mathcal{I}'} \mu_I g_I(b_{x,m})}{g_{I_x}(b_{x,m})}.$$

(4) There is a unique value of $\sigma = \sigma(m, \vec{\mu})$ for which $\bar{f}(u) = f(u)$, namely

$$\sigma(m,\vec{\mu}) = \frac{\sum_{I\in\mathcal{I}'}(\lambda_I - \mu_I) g_I(u)}{g_{I_u}(u)}.$$

Note that for functions f', if $\sigma_a = \sigma_a(m, \vec{\mu})$ for each $a \in K$, then (a), (b) hold and (c) holds (with $b_x = b_{x,m}$) if m is large enough. Also, as $\vec{\mu} = (\mu_I)_{I\in\mathcal{I}'} \to (\lambda_I)_{I\in\mathcal{I}'}$ and $m \to \infty$, we have $\sigma_a(m, \vec{\mu}) \to 0$ for each $a \in K$ and $\sigma(m, \vec{\mu}) \to 0$.

Let $L = [-\ell, \ell]$ be a compact interval such that each of the intervals $I \in \mathcal{I}'$, I_a for $a \in K$ and I_u are contained in L.

Henceforth, we limit ourselves to functions f', $\bar{f}$ for which we have $\sigma_a = \sigma_a(m, \vec{\mu})$, $a \in K$. Consider the following facts:

(5) for $i \le i_0$, $D^i f(x) = D^i \bar{f}(x)$ $(= D^i f_0(x))$ when $x \in \mathbb{R} \setminus L$;
(6) for $i \le i_0$ and $x \in L$, we have

$$\begin{aligned} |D^i f(x) - D^i \bar{f}(x)| &\le \sum_{I\in\mathcal{I}'} |\lambda_I - \mu_I|\,|D^i g_I(x)| + \sum_{a\in K} |\sigma_a(m,\vec{\mu})|\,|D^i g_{I_a}(x)| \\ &\quad + |\sigma|\,|D^i g_{I_u}(x)| \\ &\le C\left[\sum_{I\in\mathcal{I}'} |\lambda_I - \mu_I| + \sum_{a\in K} |\sigma_a(m,\vec{\mu})| + |\sigma|\right], \end{aligned}$$

where $C = \sup_{x\in L,\, i\le i_0} \max\{|D^i g_I(x)| : I \in \mathcal{I}' \cup \{I_a : a \in K\} \cup \{I_u\}\}$.

We may choose $m \in \mathbb{N}$, $\mu_I \in \mathbb{Q}$ for each $I \in \mathcal{I}'$ and $r > 0$ so that (d) and (e) are satisfied with $b_{x,m}$ in the place of b_x and I_u in the place of I. (First get a neighborhood of $\vec{\lambda}$, an m_0 and an r so that (d) is satisfied for $\vec{\mu}$ in the given neighborhood of $\vec{\lambda}$ and $m \ge m_0$. Then choose such a $\vec{\mu}$ so that $|\sigma(m, \vec{\mu})| < r$, giving (e).) Note that we have $f' \in N$ because $h, H, K_0, f[K_1], \{b_{x,m} : x \in K_1\}, \{c_{x,m} : x \in K_0\} \in N$, each μ_I is rational (and hence belongs to N) and each $\sigma_a(m, \vec{\mu})$, for $a \in K$, is uniquely determined by the condition that $f'(a) = h(a)$ for $a \in \operatorname{dom} h$, $f'(x) = c_{x,m}$ for $a = x \in K_0$ and $f'(b_{x,m}) = f(x)$ for $a = b_{x,m}$ $(x \in K_1)$, and hence belongs to N by elementarity. □

We now proceed to prove the main technical lemma for the oracle-cc iteration. The reader familiar with oracle-cc iteration will see that this lemma completes the proof. In the final section of the paper we have indicated how this standard oracle-cc argument proceeds in the case of our main theorem, so we omit it in the present section.

Lemma 2.4. *Let $\overline{M} = \langle M_\delta : \delta < \omega_1\rangle$ be an oracle. Let $A, B \subseteq \mathbb{R}$ be everywhere second category sets of cardinality $\aleph_1$. There is a forcing notion P satisfying the*

$\overline{M}$-cc such that for every $G \subseteq P$ generic over V, $V[G] \models$ there is a C^∞ function $f\colon \mathbb{R} \to \mathbb{R}$ for which the following properties hold:

(i) *f is an order-isomorphism of $\mathbb{R}$;*
(ii) *$f[A] = B$.*

Proof. Let $f_0\colon \mathbb{R} \to \mathbb{R}$ be the identity map. For the rest of the proof, fix a suitably large regular cardinal θ. Let $\langle I_n : n < \omega\rangle$ list all the nonempty open intervals with rational endpoints. Fix a well-ordering of $\mathbb{R}$ in type ω_1. (CH holds because there is an oracle.) Let Q denote the set $\omega_1 \times \omega \times 2$ equipped with the lexicographical order which we denote by $\lhd$. We will inductively define partial orders $P(u)$, $u \in Q$, from the following class of partial orders.

Definition 2.5. Let $N \prec H_\theta$. Let $\bar{a} = \langle a_\xi : \xi < \alpha\rangle$, $\bar{b} = \langle b_\xi : \xi < \beta\rangle$ be one-to-one sequences of real numbers, $\alpha, \beta \leq \omega_1$. We write, for $\delta < \omega_1$,

$$\bar{a}^\delta = \{a_{\omega\delta+n} : n < \omega,\ \omega\delta + n < \alpha\} \quad \text{and} \quad \bar{b}^\delta = \{b_{\omega\delta+n} : n < \omega,\ \omega\delta + n < \beta\}.$$

$P(\bar{a}, \bar{b}, N)$ denotes the partial order consisting of conditions $p = (h_p, f_p, \varepsilon_p, n_p)$ such that

(i) h_p is a finite partial order-preserving map from $\{a_\xi : \xi < \alpha\}$ to $\{b_\xi : \xi < \beta\}$;
(ii) $h_p[\bar{a}^\delta] \subseteq \bar{b}^\delta$ for all $\delta < \omega_1$;
(iii) $f_p \in (f_0 + \operatorname{span} \mathcal{G}^*) \cap N$;
(iv) $h_p \subseteq f_p$;
(v) $\|D(f_p - f_0)\|_\infty < 1 - \varepsilon_p$;
(vi) ε_p is a rational number, $0 < \varepsilon_p < 1$, and $1 \leq n_p < \omega$.

The order is given by $p \leq q$ if and only if

(vii) $h_p \supseteq h_q$, $n_p \geq n_q$;
(viii) for all $i \leq n_q$, $\|D^i f_p - D^i f_q\|_\infty + \varepsilon_p \leq \varepsilon_q$. (This gives in particular $\varepsilon_p \leq \varepsilon_q$.)

The order relation in Definition 2.5 is transitive because if $p \leq q \leq r$, then $h_p \supseteq h_q \supseteq h_r$, $n_p \geq n_q \geq n_r$ and for $i \leq n_r$, we have $i \leq n_q$ as well and hence

$$\begin{aligned}\|D^i f_p - D^i f_r\|_\infty + \varepsilon_p &\leq \|D^i f_q - D^i f_r\|_\infty + (\|D^i f_p - D^i f_q\|_\infty + \varepsilon_p)\\ &\leq \|D^i f_q(z) - D^i f_r\|_\infty + \varepsilon_q \leq \varepsilon_r.\end{aligned}$$

Fix a function $\gamma\colon \omega_1 \to 2$ so that for each $i < 2$, $|\gamma^{-1}(i)| = \aleph_1$.

To each $u = (\delta, n, j) \in Q$ we associate a pair of ordinals $(\alpha, \beta) = (\alpha(u), \beta(u))$ as follows. Let $\gamma(\delta) = i$. If $j = 0$, let (α, β) be the pair of ordinals $(\omega\delta + n, \omega\delta + n)$. If $j = 1$, let (α, β) be the pair $(\omega\delta + n + 1, \omega\delta + n)$ if $i = 0$, and $(\omega\delta + n, \omega\delta + n + 1)$ if $i = 1$.

(Notice that the pair $(\alpha(u), \beta(u))$ uniquely determines u: from (α, β) we can clearly recover δ and n, and we have $j = 0$ if $\alpha = \beta$ and $j = 1$ otherwise.)

We inductively define one-to-one enumerations $\bar{a} = \langle a_\xi : \xi < \omega_1\rangle$ of A and $\bar{b} = \langle b_\xi : \xi < \omega_1\rangle$ of B and a continuous $\in$-increasing sequence $\langle N_u : (1,0,0) \unlhd u \in Q\rangle$ of countable elementary submodels of H_θ, and then for each $u \in Q$ such that $(1,0,0) \unlhd u$ we define

$$P(u) = P(\bar{a} \restriction \alpha(u), \bar{b} \restriction \beta(u), N_u).$$

We simplify the notation by writing $P(u) = P(\bar{a} \restriction \alpha(u), \bar{b} \restriction \beta(u))$, omitting the explicit mention of N_u. No confusion should arise since $\alpha(u)$ and $\beta(u)$ uniquely determine u, and hence N_u. The indexing of the induction is such that for fixed $\delta < \omega_1$ such that $\gamma(\delta) = i$, at stage $u = (\delta, n, j)$ of the induction, if $i = 0$, then we pick $a_{\omega\delta+n} \in A$ if $j = 0$ and $b_{\omega\delta+n} \in B$ if $j = 1$, whereas if $i = 1$, then we pick $b_{\omega\delta+n} \in B$ if $j = 0$ and $a_{\omega\delta+n} \in A$ if $j = 1$. In other words, the elements of $\bar{a}^\delta$ and $\bar{b}^\delta$ are chosen alternately from A and B, respectively, starting with an element of A when $i = 0$ and with an element of B when $i = 1$, as indicated in Tables 1 and 2.

TABLE 1. Indexing when $i = \gamma(\delta) = 0$:

Stage $u = (\delta, n, j)$	Element chosen at stage u	$\alpha(u)$	$\beta(u)$
$(\delta, 0, 0)$	$a_{\omega\delta} \in A$	$\omega\delta$	$\omega\delta$
$(\delta, 0, 1)$	$b_{\omega\delta} \in B$	$\omega\delta + 1$	$\omega\delta$
$(\delta, 1, 0)$	$a_{\omega\delta+1} \in A$	$\omega\delta + 1$	$\omega\delta + 1$
$(\delta, 1, 1)$	$b_{\omega\delta+1} \in B$	$\omega\delta + 2$	$\omega\delta + 1$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$(\delta, n, 0)$	$a_{\omega\delta+n} \in A$	$\omega\delta + n$	$\omega\delta + n$
$(\delta, n, 1)$	$b_{\omega\delta+n} \in B$	$\omega\delta + n + 1$	$\omega\delta + n$
$\cdots$	$\cdots$	$\cdots$	$\cdots$

TABLE 2. Indexing when $i = \gamma(\delta) = 1$:

Stage $u = (\delta, n, j)$	Element chosen at stage u	$\alpha(u)$	$\beta(u)$
$(\delta, 0, 0)$	$b_{\omega\delta} \in B$	$\omega\delta$	$\omega\delta$
$(\delta, 0, 1)$	$a_{\omega\delta} \in A$	$\omega\delta$	$\omega\delta + 1$
$(\delta, 1, 0)$	$b_{\omega\delta+1} \in B$	$\omega\delta + 1$	$\omega\delta + 1$
$(\delta, 1, 1)$	$a_{\omega\delta+1} \in A$	$\omega\delta + 1$	$\omega\delta + 2$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$(\delta, n, 0)$	$b_{\omega\delta+n} \in B$	$\omega\delta + n$	$\omega\delta + n$
$(\delta, n, 1)$	$a_{\omega\delta+n} \in A$	$\omega\delta + n$	$\omega\delta + n + 1$
$\cdots$	$\cdots$	$\cdots$	$\cdots$

For technical reasons, we also define a second sequence $\langle N'_u : u \in Q \rangle$ of countable elementary submodels of H_θ and functions e_δ, $1 \leq \delta < \omega_1$.

Fix countable dense sets $A_0 \subseteq A$ and $B_0 \subseteq B$.

We will arrange that the following conditions hold for all $u = (\delta, n, j) \in Q$, with $\gamma(\delta) = i$.

(1) For $\delta = 0$, the only requirements are that $\bar{a}^0 = A_0$ and $\bar{b}^0 = B_0$, where A_0 and B_0 are the countable dense sets fixed above (i.e., we define $\{a_n : n < \omega\}$ and $\{b_n : n < \omega\}$ to be any one-to-one enumerations of A_0 and B_0, respectively).
(2) For $\delta \geq 1$, $N_u \in N'_u$ are countable elementary submodels of H_θ.
(3) f_0, $\langle c_i : i < \omega \rangle$, η, A_0, B_0, A, B are all elements of $N_{(0,0,0)}$.

(4) If $\delta \geq 1$ and $(n,j) = (0,0)$, then
   (i) $N_u = \bigcup\{N_v : v \lhd u\}$;
   (ii) if $i = 0$, then $a_{\omega\delta}$ is the least element of $A \setminus \{a_\xi : \xi < \omega\delta\}$;
   (iii) if $i = 1$, then $b_{\omega\delta}$ is the least element of $B \setminus \{b_\xi : \xi < \omega\delta\}$.

   In (ii) and (iii), "least" refers to the well-ordering of $\mathbb{R}$ fixed earlier.
(5) If $\delta \geq 1$ and $(n,j) \neq (0,0)$, then
   (i) e_δ, $\langle a_\xi : \xi < \alpha(u)\rangle$, $\langle b_\xi : \xi < \beta(u)\rangle$, $\langle M_{\delta'} : \delta' \leq \delta\rangle$, and $\langle N_v : v \lhd u\rangle$ all belong to N_u.
   (ii) If $i = 0$, $j = 0$ or $i = 1$, $j = 1$, then $a_{\omega\delta+n} \in A \cap I_n$ is a Cohen real over N'_u.
   (iii) If $i = 0$, $j = 1$ or $i = 1$, $j = 0$, then $b_{\omega\delta+n} \in B \cap I_n$ is a Cohen real over N'_u.

   The point of using N'_u rather than N_u in (ii) and (iii) is that it will be useful later to have $P(u)$ belonging to the model over which the Cohen reals are chosen. (See the paragraphs immediately following the proofs of Claim 2.8 and Claim 2.10.)
(6) If $\delta \geq 1$, e_δ is a bijective map of $P(\bar{a} \restriction \omega\delta, \bar{b} \restriction \omega\delta)$ onto $\omega\delta$.
(7) For each δ' such that $\delta' < \delta$, $e_{\delta'} \subseteq e_\delta$.
(8) If $\delta \geq 1$, the predense subsets of $P(\bar{a} \restriction \omega\delta, \bar{b} \restriction \omega\delta)$ which have the form $e_\delta^{-1}[S]$ for some $S \in \bigcup_{\eta\leq\delta} M_\eta$ are predense in $P(\bar{a} \restriction \omega(\delta+1), \bar{b} \restriction \omega(\delta+1))$.

Remark 2.6. (a) From (1) and (5)((ii)+(iii)), it follows that the sets $\bar{a}^\delta$ and $\bar{b}^\delta$ are dense in $\mathbb{R}$. From (4)((ii)+(iii)) and (5)((ii)+(iii)), we get $\bar{a}^\delta \subseteq A$ and $\bar{b}^\delta \subseteq B$. From the same clauses together with (5)(i), we get that the enumerations $\langle a_\xi : \xi < \omega_1\rangle$ and $\langle b_\xi : \xi < \omega_1\rangle$ are one-to-one. (Note that $\alpha(u) = \omega\delta + n$ in (5)(ii) and $\beta(u) = \omega\delta + n$ in (5)(iii).)

(b) From 4(ii) and the fact that $\gamma^{-1}(0)$ is uncountable, it follows that $A = \{a_\xi : \xi < \omega_1\}$. Similarly, we get $B = \{b_\xi : \xi < \omega_1\}$.

(c) It follows inductively, using (5)(i) at successor stages, that $\{a_\xi : \xi < \alpha(u)\} \subseteq N_u$ and $\{b_\xi : \xi < \beta(u)\} \subseteq N_u$. (At a limit stage $u = (\delta,0,0)$, we have $\alpha(u) = \beta(u) = \omega\delta$. If $\xi < \omega\delta$, then $\xi = \omega\delta' + n'$ for some $\delta' < \delta$ and $n' < \omega$. Then a_ξ and b_ξ are defined (in an order depending on $\gamma(\delta')$) at stages $(\delta', n', 0)$ and $(\delta', n', 1)$. By the induction hypothesis, $a_\xi, b_\xi \in N_{(\delta',n'+1,0)} \subseteq N_u$.) Hence, $h_p \in N_u$ for each $p \in P(u)$.

(d) From (4)(i) and (5)(i), it follows that the sequence $\langle N_u : (1,0,0) \unlhd u \in Q\rangle$ is $\in$-increasing and continuous at limits. This gives in particular that for each limit ordinal $\delta > \omega$, $P(\bar{a} \restriction \omega\delta, \bar{b} \restriction \omega\delta) = \bigcup_{\delta'<\delta} P(\bar{a} \restriction \omega\delta', \bar{b} \restriction \omega\delta')$.

(e) Set $P = P(\bar{a} \restriction \omega_1, \bar{b} \restriction \omega_1, \bigcup_{u\in Q} N_u)$. In the third coordinate we could put the universe (more precisely, H_θ) since $\bigcup_{u\in Q} N_u$ includes all C^∞ functions by (5)(i) and the assumption on the M_δ's. The conditions (6)–(8) ensure that P is $\overline{M}$-cc. To see this, let $e = \bigcup_{\omega\leq\delta<\omega_1} e_\delta \colon P \to \omega_1$. For any infinite $\delta < \omega_1$ we have $e^{-1}[\omega\delta] = P(\bar{a} \restriction \omega\delta, \bar{b} \restriction \omega\delta)$ and for each $S \subseteq \omega\delta$ belonging to M_δ, whenever a set E of the form $e^{-1}[S] = e_\delta^{-1}[S]$ is predense in $P(\bar{a} \restriction \omega\delta, \bar{b} \restriction \omega\delta)$, a simple induction on δ' using (8) shows that if $\delta \leq \delta' < \omega_1$, then E is predense in $P(\bar{a} \restriction \omega\delta', \bar{b} \restriction \omega\delta')$. Thus, E is predense in P. For a club of $\delta < \omega_1$ we have $\omega\delta = \delta$, so this shows that P satisfies the $\overline{M}$-cc.

We begin by arranging (1)–(7) by induction on $u = (\delta, n, j)$. This is straightforward and we leave most of it to the reader. Notice that because the reals a_ξ and b_ξ being constructed are not indexed directly by u, we need to check that clauses (4)((ii)+(iii)), (5) and (6) make sense. For 5(ii) for example, it is important that at stage u, $a_{\omega\delta+n}$ has not yet been defined. But at an earlier stage $v = (\delta', n', j')$, we defined $a_{\omega\delta'+n'}$ or $b_{\omega\delta'+n'}$. If (δ', n') lexicographically precedes (δ, n), then $\omega\delta' + n' < \omega\delta + n$. If $(\delta', n') = (\delta, n)$, then necessarily $(j', j) = (0, 1)$. Since $j = 1$, the assumption of 5(ii) gives $i = 1$. So at stage v we defined $b_{\omega\delta+n}$, not $a_{\omega\delta+n}$. A similar argument holds for 5(iii) and (4)((ii)+(iii)). Similarly, we can check that in (5)(i), a_ξ for $\xi < \alpha(u)$ and b_ξ for $\xi < \beta(u)$ were defined before stage u. For (6), we observe that the partial order $P(\bar{a} \restriction \omega\delta, \bar{b} \restriction \omega\delta)$ is defined because we have reached or passed the stage $(\delta, 0, 0)$ and $N_{(\delta,0,0)}$ has been defined. The function e_δ is chosen at stage $(\delta, 0, 0)$. By Remark 2.6(d), the choice of e_δ is dictated by (7) when $\delta > \omega$ is a limit ordinal. The function $e_{\delta+1}$ can be taken to be an arbitrary extension of e_δ satisfying (6).

We must check that the construction gives (8). Let E be a predense subset of $P(\bar{a} \restriction \omega\delta, \bar{b} \restriction \omega\delta)$ of the appropriate form, i.e., $E = e_\delta^{-1}[S]$ for some $S \in \bigcup_{\eta \leq \delta} M_\eta$. We will show by induction on $u = (\delta, n, j) \in Q$ such that $(\delta, 0, 0) \trianglelefteq u \triangleleft (\delta + 1, 0, 0)$ that E remains predense in $P(u + 1)$, where $u + 1$ denotes the successor of u in Q, i.e., $(\delta, n, 1)$ if $j = 0$ and $(\delta, n + 1, 0)$ if $j = 1$. (This establishes (8) since each member of $P(\bar{a} \restriction \omega(\delta + 1), \bar{b} \restriction \omega(\delta + 1))$ belongs to $P(\bar{a} \restriction \omega\delta + n, \bar{b} \restriction \omega\delta + n)$ for some $n < \omega$.)

Remark 2.7. At the stage where $n = 0$ and $j = 0$, we consider the passage from $P(\bar{a} \restriction \omega\delta, \bar{b} \restriction \omega\delta)$ to either $P(\bar{a} \restriction \omega\delta + 1, \bar{b} \restriction \omega\delta)$ or $P(\bar{a} \restriction \omega\delta, \bar{b} \restriction \omega\delta + 1)$ (depending on whether $i = 0$ or $i = 1$). These two partial orders have the same allowable finite parts h_p for their conditions because, by Definition 2.5(ii), there is no legal value for either of $a_{\omega\delta}$ or $b_{\omega\delta}$ to correspond to until the other is chosen.

Let $p \in P(u+1) \setminus P(u)$. We must show that p is compatible with some member of E.

Case 1. $h_p \in N_u$.

Proposition 2.3 gives a function $f' \in (f_0 + \operatorname{span} \mathcal{G}^*) \cap N_u$ such that

1(a) $h_p \subseteq f'$;

1(d) for all $i \leq n_p$, $\|D^i f_p - D^i f'\|_\infty < \frac{1}{2}\varepsilon_p$ (and hence $\|D(f' - f_0)\|_\infty < 1 - \frac{1}{2}\varepsilon_p$).

The letters in the labels here correspond to those in the statement of Proposition 2.3. The number 1 is a reference to Case 1. We will use similar notation in the rest of the proof when applying this proposition. Then

$$q = (h_p, f', \varepsilon_p/2, n_p)$$

belongs to $P(u)$. Also, q and some $r \in E$ have a common extension $q' \in P(u)$. Then $q' \leq p$ since for each $i \leq n_p$, we have also $i \leq n_q$ (since $n_q = n_p$) and hence

$$\begin{aligned} \|D^i f_{q'} - D^i f_p\|_\infty + \varepsilon_{q'} &\leq \|D^i f_q - D^i f_p\|_\infty + \|D^i f_{q'} - D^i f_q\|_\infty + \varepsilon_{q'} \\ &= \|D^i f' - D^i f_p\|_\infty + \|D^i f_{q'} - D^i f_q\|_\infty + \varepsilon_{q'} \\ &\leq \varepsilon_p/2 + \|D^i f_{q'} - D^i f_q\|_\infty + \varepsilon_{q'} \\ &\leq \varepsilon_p/2 + \varepsilon_q = \varepsilon_p. \end{aligned}$$

Case 2. $h_p \notin N_u$.

By Remark 2.7, we have $(n, j) \neq (0, 0)$. Hence (by (5)(i)), $e_\delta, \langle M_{\delta'} : \delta' \leq \delta \rangle \in N_u$ which gives $S \in N_u$ and hence $E = e_\delta^{-1}[S] \in N_u$.

Subcase 2a. $i = 0$, $j = 1$ or $i = 1$, $j = 0$.

In this subcase, h_p has the form $h \cup \{(a, b_{\omega\delta+n})\}$ for some $h \subseteq N_u$ and $a \in \{a_{\omega\delta+m} : m < n + 1 - i\}$.

Proposition 2.3 gives a rational interval I_1, a function $f' \in (f_0 + \operatorname{span} \mathcal{G}^*) \cap N_u$ and a rational number $\lambda_0 > 0$ such that

- 2a(a) $h \subseteq f'$;
- 2a(d) for all $i \leq n_p$ we have $\lambda_0 \|D^i g_{I_1}\|_\infty < \varepsilon_p/4$ and for all $\lambda \in \mathbb{R}$ such that $|\lambda| \leq \lambda_0$,
$$\|D^i f_p - D^i(f' + \lambda g_{I_1})\|_\infty < \tfrac{1}{2}\varepsilon_p$$
(and hence in particular $\|D((f' + \lambda g_{I_1}) - f_0)(x)\| < 1 - \frac{1}{2}\varepsilon_p$);
- 2a(e) $g_{I_1}(a) \neq 0$ and for some number λ such that $|\lambda| < \lambda_0$, we have $h_p \subseteq f' + \lambda g_{I_1}$.

For functions $g_0 \colon \mathbb{R} \to \mathbb{R}$, numbers $\mu > 0$, and rational intervals I, define

$$V(g_0, \mu, I) = \{(g_0 + \lambda g_I)(a) : |\lambda| < \mu\}.$$

As long as $g_I(a) \neq 0$, $V(g_0, \mu, I)$ is a nonempty open interval in $\mathbb{R}$. Consider the open interval $U = V(f', \lambda_0, I_1)$. By (e), $f_p(a) = b_{\omega\delta+n} \in U$. Define

$$q_0 = (h,\, f',\, \varepsilon_p/2,\, n_p)$$

and notice that $q_0 \in P(u)$.

The idea now is to find a common extension of p and an element of E by taking instead a common extension of q_0 and an element of E. The latter exists by the induction hypothesis since $q_0 \in P(u)$. Unfortunately, such an extension might not even be compatible with p. Claim 2.8 establishes that the common extensions of q_0 and an element of E exist in sufficient profusion that one of them must be compatible with p. (The reader who has read Shelah's original argument will recognize that this is a version of the argument in [Sh1980, Case 3, p. 565] (or see [BM, Claim 4.7]) adapted to the present context.)

Claim 2.8. The union of the open sets $V(f_q, \mu, I)$ such that

1. (1) $q \in P(u)$ is a common extension of q_0 and an element of E,
2. (2) $g_I(a) \neq 0$ (i.e., $a \in I$),
3. (3) $\mu > 0$ is rational, $m \in \mathbb{N}$, I is a rational interval,
4. (4) $V(f_q, \mu, I) \subseteq U$,
5. (5) for all $i \leq n_q$, $\mu \|D^i g_I\|_\infty \leq \frac{1}{2}\varepsilon_q$
(and hence for all λ such that $|\lambda| \leq \mu$ we have
$$\|D^i((f_q + \lambda g_I) - f_0)\|_\infty < 1 - \tfrac{1}{2}\varepsilon_q)$$

is dense in U.

Proof of Claim 2.8. Fix $\lambda_1 \in \mathbb{Q}$ such that $|\lambda_1| < \lambda_0$. Define

$$w = \bar{f}'(a) \in U, \text{ where } \bar{f}' = f' + \lambda_1 g_{I_1}.$$

Note that the numbers w of this form, as λ_1 runs over all rational numbers such that $|\lambda_1| < \lambda_0$, are dense in U. Fix $\delta > 0$ such that $(w - \delta, w + \delta) \subseteq U$. Let $x_1, x_2 \in A_0 \setminus \operatorname{dom} h$ satisfy $x_1 < a < x_2$ and $\bar{f}'(x_1), \bar{f}'(x_2) \in (w - \delta, w + \delta)$. Apply

Proposition 2.3 to get a function $f'' \in (f_0 + \operatorname{span} \mathcal{G}^*) \cap N_u$ and a rational number $\lambda_2 > 0$ such that

2.8(a) $h \subseteq f''$,

2.8(b) $y_1 = f''(x_1)$ and $y_2 = f''(x_2)$ are both members of B_0,

2.8(d)$_1$ $y_1, y_2 \in (w - \delta, w + \delta)$.
(This holds as long as the quantities $|(\bar{f}' - f'')(x_j)|$, $j = 1, 2$, are small enough.)

2.8(d)$_2$ For all $i \leq n_p$,

$$\|D^i \bar{f}' - D^i f''\|_\infty < \tfrac{1}{8}\varepsilon_p$$

(and hence in particular $\|D(f'' - f_0)\|_\infty < 1 - \frac{1}{4}\varepsilon_p$).

Note that f'' is increasing (by 2.8(d)$_2$, $\|Df'' - Df_0\|_\infty < 1$ and hence $Df'' > 0$) and hence we have $y_1 < y_2$. Then $q_1 \in P(u)$, where

$$q_1 = (h \cup \{(x_1, y_1), (x_2, y_2)\},\ f'',\ \varepsilon_p/8,\ n_p).$$

Also, q_1 extends q_0 because if $i \leq n_p$, then

$$\begin{aligned}\|D^i f_{q_1} - D^i f_{q_0}\|_\infty + \varepsilon_{q_1} &= \|D^i f'' - D^i f'\|_\infty + \varepsilon_p/8\\ &\leq \|D^i f'' - D^i \bar{f}'\|_\infty + \|D^i \bar{f}' - D^i f'\|_\infty + \varepsilon_p/8\\ &\leq \|D^i \bar{f}' - D^i f'\|_\infty + \varepsilon_p/4 \leq \varepsilon_p/2 = \varepsilon_{q_0}.\end{aligned}$$

By the induction hypothesis, there is a common extension $q \in P(u)$ of q_1 and some $r \in E$. Since a does not belong to the domains of h_{q_1} or h_r ($r \in E \subseteq P(\delta, 0, 0)$ and hence $\operatorname{dom} h_r \subseteq \{a_\xi : \xi < \omega\delta\}$), we may discard it from the domain of h_q if necessary to get $a \notin \operatorname{dom} h_q$. Choose a rational interval I such that $a \in I$ and $I \cap \operatorname{dom} h_q = \emptyset$. For $\mu > 0$ small enough we have that part 5 of the claim holds.

Then the functions in the definition of $V(f_q, \mu, I)$ are increasing on $\mathbb{R}$. Since $\{(x_1, y_1), (x_2, y_2)\} \subseteq h_q$ and $I \cap \operatorname{dom} h_q = \emptyset$, we have $V(f_q, \mu, I) \subseteq (y_1, y_2) \subseteq (w - \delta, w + \delta) \subseteq U$ and, in particular, part 4 of the claim holds.

This proves Claim 2.8. □

The dense open subset of U given by Claim 2.8 is coded in N'_u. (As noted earlier, $P(u) \in N'_u$ when u is not a limit stage.) By (5)(iii), there are q, μ, I satisfying Claim 2.8(1–5) for which $b_{\omega\delta+n} \in V(f_q, \mu, I)$. Choosing λ with $|\lambda| < \mu$ so that $(f_q + \lambda g_I)(a) = b_{\omega\delta+n}$, we get that

$$q' = (h_q \cup \{(a, b_{\omega\delta+n})\},\ f_q + \lambda g_I,\ \varepsilon_q/2,\ n_q)$$

belongs to $P(u+1)$ (using clause 5 of Claim 2.8) and extends both q and p. It extends q by clause 5 of Claim 2.8. To see that $q' \leq p$, note that for each $i \leq n_p$,

$$\begin{aligned}\|D^i f_{q'} - D^i f_p\|_\infty + \varepsilon_{q'} &\leq \|D^i f_q - D^i f_p\|_\infty + \|D^i f_{q'} - D^i f_q\|_\infty + \varepsilon_{q'}\\ &\leq \|D^i f_q - D^i f_p\|_\infty + \varepsilon_q\\ &\leq \|D^i f_{q_0} - D^i f_p\|_\infty + \|D^i f_q - D^i f_{q_0}\|_\infty + \varepsilon_q\\ &\leq \|D^i f_{q_0} - D^i f_p\|_\infty + \varepsilon_{q_0}\\ &\leq \varepsilon_p/2 + \varepsilon_p/2 = \varepsilon_p.\end{aligned}$$

Thus, p is compatible with q and hence with some element of E.

Subcase 2b. $i = 0$, $j = 0$ or $i = 1$, $j = 1$.

In this subcase, h_p has the form $h \cup \{(a_{\omega\delta+n}, b)\}$ for some $h \subseteq N_u$ and $b \in \{b_{\omega\delta+m} : m < n + i\}$.

Proposition 2.3 gives a rational interval I_1, a function $f' \in (f_0 + \operatorname{span} \mathcal{G}^*) \cap N_u$ and a rational number $\lambda_0 > 0$ such that

2b(a) $h \subseteq f'$;
2b(d) for all $i \leq n_p$, $\lambda_0 \|D^i g_{I_1}\|_\infty < \frac{1}{4}\varepsilon_p$ and for all $\lambda \in \mathbb{R}$ such that $|\lambda| \leq \lambda_0$,
$$\|D^i f_p - D^i(f' + \lambda g_{I_1})\|_\infty < \tfrac{1}{2}\varepsilon_p$$
(and hence in particular $\|D((f' + \lambda g_{I_1}) - f_0)\|_\infty < 1 - \frac{1}{2}\varepsilon_p$);
2b(e) for some number λ such that $|\lambda| < \lambda_0$, we have $h_p \subseteq f' + \lambda g_{I_1}$.

(This list agrees with the one for Subcase 2a except for (e).)

Remark 2.9. Note that 2b(d) ensures that for $|\lambda| \leq \lambda_0$, $\frac{1}{2}\varepsilon_p < D(f' + \lambda g_{I_1})$ and hence $f' + \lambda g_{I_1}$ is an order-isomorphism.

For functions $g_0 \colon \mathbb{R} \to \mathbb{R}$, numbers $\mu > 0$ and rational intervals, define
$$W(g_0, \mu, I) = \{(g_0 + \lambda g_I)^{-1}(b) : |\lambda| < \mu\}.$$
The definition makes sense if $g_0 + \lambda g_I$ is an order-isomorphism whenever $|\lambda| < \mu$. By Proposition 4.1, $W(g_0, \mu, I)$ is an open interval in $\mathbb{R}$ as long as there is no $a \in \mathbb{R}$ such that $g_0(a) = b$ and $g_I(a) = 0$. When g_0 is invertible, then only the value $a = g_0^{-1}(b)$ is relevant. Consider the open interval $U = W(f', \lambda_0, I_1)$. By (e), $a_{\omega\delta+n} = f_p^{-1}(b) \in U$. Define
$$q_0 = (h,\ f',\ \varepsilon_p/2,\ n_p)$$
and notice that $q_0 \in P(u)$.

Claim 2.10. The union of the open sets $W(f_q, \mu, I)$ such that

(1) $q \in P(u)$ is a common extension of q_0 and an element of E;
(2) $g_I(a) \neq 0$ where $a = f_q^{-1}(b)$;
(3) $\mu > 0$ is rational, I is a rational interval;
(4) $W(f_q, \mu, I) \subseteq U$;
(5) for all $i \leq n_q$, $\mu \|D^i g_I\|_\infty \leq \varepsilon_q/2$
(and hence for all λ such that $|\lambda| \leq \mu$,
$$\|D((f_q + \lambda g_I) - f_0)\|_\infty < 1 - \tfrac{1}{2}\varepsilon_q)$$

is dense in U.

Proof of Claim 2.10. Fix $\lambda_1 \in \mathbb{Q}$ such that $|\lambda_1| < \lambda_0$. Define
$$w = (\bar{f}')^{-1}(b) \in U, \text{ where } \bar{f}' = f' + \lambda_1 g_{I_1}.$$
By Remark 2.9, $\bar{f}'$ is an order-isomorphism. Note that $w \notin \operatorname{dom} h$ since $b \notin \operatorname{range} h$ and $h \subseteq \bar{f}'$. Note that the numbers w of the given form, as λ_1 runs over all rational numbers such that $|\lambda_1| < \lambda_0$, are dense in U. Fix $\delta > 0$ such that $(w - \delta, w + \delta) \subseteq U$ and $(w - \delta, w + \delta) \cap \operatorname{dom} h = \emptyset$. Let $y_1, y_2 \in B_0$ satisfy $y_1 < b < y_2$ and $(\bar{f}')^{-1}(y_1), (\bar{f}')^{-1}(y_2) \in (w - \delta, w + \delta)$. Apply Proposition 2.3 to get a rational interval I_2, a function $f'' \in (f_0 + \operatorname{span} \mathcal{G}^*) \cap N_u$ and a rational number $\lambda_2 > 0$ such that

2.10(a) $h \subseteq f''$;
2.10(c) $y_1 = f''(x_1)$ and $y_2 = f''(x_2)$, where x_1, x_2 are both members of A_0 and $x_1, x_2 \in (w - \delta, w + \delta)$;

2.10(d) for all $i \le n_p$,

$$\|D^i \bar{f}' - D^i f''\|_\infty < \tfrac{1}{8}\varepsilon_p$$

(and hence in particular $\|D(f'' - f_0)\|_\infty < 1 - \frac{1}{4}\varepsilon_p$).

By 2.10(d), f'' is increasing and hence $x_1 < x_2$. Then $q_1 \in P(u)$, where

$$q_1 = (h \cup \{(x_1, y_1), (x_2, y_2)\},\ f'',\ \varepsilon_p/8,\ n_p).$$

Exactly as in the proof of Claim 2.8, q_1 extends q_0. By the induction hypothesis, there is a common extension $q \in P(u)$ of q_1 and some $r \in E$. The number $a = f_q^{-1}(b)$ does not belong to the domains of h_{q_1} or h_r.
[From the fact that f_q is increasing and hence injective, we see that because $f_q(x_i) = y_i$, $i = 1, 2$, and $f_q(a) = b$, we have $a \neq x_i$, $i = 1, 2$. Because $h \subseteq f_q$ and $b \notin \operatorname{range} h$, we have $a \notin \operatorname{dom} h$. Thus, $a \notin \operatorname{dom} h_{q_1}$. Since $h_r \subseteq h_q \subseteq f_q$, if a were in $\operatorname{dom} h_r$, then $h_r(a) = b$, which is not possible because $r \in E \subseteq P(\delta, 0, 0)$ and hence $\operatorname{range} h_r \subseteq \{b_\xi : \xi < \omega\delta\}$.]

We may thus discard it from the domain of h_q if necessary to get $a \notin \operatorname{dom} h_q$. Choose a rational interval I such that $a \in I$ and $I \cap \operatorname{dom} h_q = \emptyset$. As in the proof of Claim 2.8, for $\mu > 0$ small enough we have that part 5 of the claim holds. Then the functions in the definition of $W(f_q, \mu, I)$ are order-isomorphisms. Since $\{(x_1, y_1), (x_2, y_2)\} \subseteq h_q$ and $I \cap \operatorname{dom} h_q = \emptyset$, we have $W(f_q, \mu, I) \subseteq (x_1, x_2) \subseteq (w - \delta, w + \delta) \subseteq U$ and in particular part 4 of the claim holds.

This proves Claim 2.10. □

The dense open subset of U given by Claim 2.10 is coded in N'_u. By (5)(ii), there are q, μ, I satisfying Claim 2.10(1–5) for which $a_{\omega\delta+n} \in W(f_q, \mu, I)$. Choosing λ with $|\lambda| < \mu$ so that $(f_q + \lambda g_I)(a_{\omega\delta+n}) = b$, we get, as in Subcase 2a, that

$$q' = (h_q \cup \{(a_{\omega\delta+n}, b)\},\ f_q + \lambda g_I,\ \varepsilon_q/2,\ n_q)$$

belongs to $P(u+1)$ and extends both q and p. Thus, p is compatible with q and hence with some element of E. This completes the proof of (8).

We now have an $\overline{M}$-cc partial order $P = P(\bar{a} \restriction \omega_1, \bar{b} \restriction \omega_1)$ as in Remark 2.6(e). It remains to check that forcing with P adds the desired entire function f. Let $h = \bigcup\{h_p : p \in G\}$. For each $x \in A$, it follows easily using Proposition 2.3(b) that conditions with $f_p(x) \in B$ are dense and then (by extending such conditions further) so are conditions with $x \in \operatorname{dom} h_p$. Similarly, for each $y \in B$, the conditions with $y \in \operatorname{range} h_p$ are dense (using Proposition 2.3(c) this time). Hence $\operatorname{dom} h = A$, $\operatorname{range} h = B$ and h is clearly an order-isomorphism. For $k \in \mathbb{N}$, choose $p_k \in G$ such that $p_{k+1} \le p_k$, $n_{p_k} \ge k$ and $\varepsilon_{p_k} < 1/k$. The sequences $\{D^i f_{p_k}\}_{k=1}^\infty$ are uniformly Cauchy because for all k, ℓ with $k < \ell$ and k large enough so that $i \le n_k$, the fact that $p_\ell \le p_k$ gives $\|D^i f_{p_\ell} - D^i f_{p_k}\|_\infty \le \varepsilon_{p_k} < 1/k$. Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = \lim_{k\to\infty} f_{p_k}(x)$. For all $i < \omega$, we have $D^i f(x) = \lim_{k\to\infty} D^i f_{p_k}(x)$ by Proposition 2.2. Also, for each $a \in A$, we can choose k and $p \in G$ such that $p \le p_k$ and $a \in \operatorname{dom} h_p$. Then

$$|f_{p_k}(a) - h(a)| = |f_{p_k}(a) - h_p(a)| = |f_{p_k}(a) - f_p(a)| \le \varepsilon_{p_k} < 1/k,$$

and hence $f(a) = \lim_{k\to\infty} f_{p_k}(a) = h(a)$.

This completes the proof of the lemma. □

3. Reduction to the approximation of entire functions

The results of this section are elementary and are surely known, at least in some form. We sketch the proofs for completeness.

Proposition 3.1. *For any interval $[a, b]$, $a < b$, and for any $\varepsilon > 0$, there is a C^∞ bump function which is zero outside $[a, b]$, positive on (a, b) and whose derivative has a unique zero in (a, b) inside the interval $(b - \varepsilon, b)$.*

Proof. For $n \in \mathbb{N}$, the unique zero in (a, b) of the derivative of $x \mapsto \exp(-[(x-a)^n(b-x)]^{-1})$ occurs at $\frac{1}{n+1}a + \frac{n}{n+1}b$. □

The statements of Proposition 3.2 and Proposition 3.5 hold with C^1 replaced by C^∞ but the C^1 case suffices for us.

Proposition 3.2. *Let $a \in \mathbb{R}$. Let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $Dg(x) > 0$ and $D^2g(x) = 0$ for all $x \neq a$. Let U be an open neighborhood of the point $(a, g(a))$ in $\mathbb{R}^2$. There is a C^1 function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \subseteq g \cup U$ and $Df(x) > 0$ for all $x \in \mathbb{R}$.*

Proof. This is a calculus exercise. We may assume $a = 0$ and $g(0) = 0$. Write $g = g_1 + g_2$ where $g_1(x) = \min(g(x), 0)$ and $g_2(x) = \max(g(x), 0)$. For some $c_i > 0$, $i = 1, 2$, we have $g_1(x) = c_1x$ for all $x \leq 0$ and $g_2(x) = c_2x$ for all $x \geq 0$. We may assume that U is an open square centered at $(0, 0)$. It is enough to find an f which works for $U = (-R, R) \times (-R, R)$ for some $R > 0$, because then given a smaller neighborhood $U_0 = (-r, r) \times (-r, r)$, the function $f_0(x) = (r/R)f(Rx/r)$ works for U_0. Let f_1 be obtained from g_1 by replacing the portion of the graph over the interval $[-\frac{1}{4}c_1, \frac{1}{4}c_1]$ by $x \mapsto -(x - \frac{1}{4}c_1)^2$. Let f_2 be obtained from g_2 by replacing the portion of the graph over the interval $[-\frac{1}{4}c_2, \frac{1}{4}c_2]$ by $x \mapsto (x + \frac{1}{4}c_2)^2$. Then $f = f_1 + f_2$ is as desired. □

Proposition 3.3. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a nondecreasing C^1 surjection. Let $\varepsilon\colon \mathbb{R} \to \mathbb{R}$ be a positive continuous function. There is a C^1 order-isomorphism $g\colon \mathbb{R} \to \mathbb{R}$ having a strictly positive derivative and such that for all $x \in \mathbb{R}$, $|f(x) - g(x)| < \varepsilon(x)$ and $|Df(x) - Dg(x)| < \varepsilon(x)$. If f is moreover a C^n function, then we may also ask that g be a C^n function and that for all $i \leq n$ and all $x \in \mathbb{R}$, $|D^if(x) - D^ig(x)| < \varepsilon(x)$. If f is moreover C^∞ and $0 \leq c_0 \leq c_1 \leq \dots$ satisfies $\lim_{i\to\infty} c_i = \infty$, then we may ask that g be C^∞ and that for all $i < \omega$ and all $x \in \mathbb{R}$ such that $|x| \geq c_i$, $|D^if(x) - D^ig(x)| < \varepsilon(x)$.*

Proof. Because $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$, we can find a strictly increasing sequence of real numbers $\langle a_i : i \in \mathbb{Z}\rangle$ such that $\lim_{i\to\infty} a_i = \infty$, $\lim_{i\to-\infty} a_i = -\infty$ and $Df(a_i) > 0$ for each $i \in \mathbb{Z}$. Since Df is continuous, we can pick positive numbers δ_i such that the intervals $[a_i - \delta_i, a_i + \delta_i]$, $i \in \mathbb{Z}$, are pairwise disjoint and Df is positive everywhere on $[a_i - \delta_i, a_i + \delta_i]$. For each $i \in \mathbb{Z}$, fix a C^∞ bump function g_i which is zero outside $[a_i, a_{i+1}]$, positive on (a_i, a_{i+1}) and whose derivative has a unique zero in (a_i, a_{i+1}) inside the interval $(a_{i+1} - \delta_{i+1}, a_{i+1})$. (Use Proposition 3.1.) Hence, g_i has a strictly positive derivative on $(a_i, a_{i+1} - \delta_{i+1})$. We shall take $g = f + \sum_{i\in\mathbb{Z}} \lambda_i g_i$ for suitably small positive numbers λ_i. Since Df is positive everywhere on $[a_{i+1} - \delta_{i+1}, a_{i+1} + \delta_{i+1}]$, it is bounded below by a positive constant on this interval. Thus, if λ_i is small enough, $f + \lambda_i g_i$ will have a positive derivative on this interval. Then, on this same interval, we have that

$g = f + \lambda_i g_i + \lambda_{i+1} g_{i+1}$ has a positive derivative for any choice of $\lambda_{i+1} > 0$. It is clear that the constants λ_i may be chosen small enough so that, in addition, for all $x \in \mathbb{R}$ we have $|f(x) - g(x)| < \varepsilon(x)$ and $|Df(x) - Dg(x)| < \varepsilon(x)$. The modifications to the foregoing argument needed to establish the last two clauses are straightforward and left to the reader. □

Proposition 3.4. *Let $a, b \in \mathbb{R}$ be such that $a < b$ and let $\varepsilon > 0$. Let $f\colon [a,b] \to \mathbb{R}$ be a nondecreasing continuous map such that $f(a) < f(b)$. Let y_a, y_b be such that $f(a) < y_a < f(a)+\varepsilon$, $f(b) < y_b < f(b)+\varepsilon$ and $y_a < y_b$. There is a strictly increasing piecewise linear continuous map $g\colon [a,b] \to \mathbb{R}$ such that $g(a) = y_a$, $g(b) = y_b$ and for all $x \in [a,b]$, $f(x) \le g(x) < f(x) + \varepsilon$.*

Proof. We may assume that for the value of $\lambda \in (0,1)$ for which $y_b = f(b) + \lambda\varepsilon$, we also have $y_a = f(a) + \lambda\varepsilon$. (Note that because $f(a) < f(b)$, this substitution preserves $y_a < y_b$.) For, given g a strictly increasing piecewise linear continuous map such that $g(a) = f(a) + \lambda\varepsilon$ and $g(b) = y_b = f(b) + \lambda\varepsilon$, we get the desired map by modifying the definition of g near a as follows. If $y_a < f(a)+\lambda\varepsilon$, then pick $\delta > 0$ such that for $x \in [a, a+\delta]$ we have $f(x) < y_a$. Then replace $g \restriction [a, a+\delta]$ by the straight line segment joining (a, y_a) and $(a+\delta, g(a+\delta))$. If $f(a)+\lambda\varepsilon < y_a$, then find the unique $a' > a$ for which $g(a') = y_a$. (a' exists because $g(a) < y_a < y_b = g(b)$.) For a suitably small $\delta > 0$ we have $g(a'+\delta) < f(a) + \varepsilon \le f(x) + \varepsilon$ for all $x \in [a,b]$. Replace $g \restriction [a, a'+\delta]$ by the straight line segment joining (a, y_a) and $(a'+\delta, g(a'+\delta))$.

Now let $a' = \sup\{x \in [a,b] : f(a) = f(x)\}$, $b' = \inf\{x \in [a,b] : f(x) = f(b)\}$. Let $c = \sup A$ where $A = \{u \in [a', b'] :$ for all $v \in (a', u]$, there is a piecewise linear strictly increasing continuous function h_v defined on $[a,v]$ satisfying $h_v(a) = y_a = f(a) + \lambda\varepsilon$, $h_v(v) = f(v) + \lambda\varepsilon$ and $f(x) < h_v(x) < f(x) + \varepsilon$ for $x \in [a,v]\}$. The straight line segments joining (a, y_a) to $(v, f(v) + \lambda\varepsilon)$ for $a' < v < a' + \delta$ for small enough δ show that $c > a'$. If $c \ge b'$, then for $u \in (a', b') \cap A$ close enough to b' and $v \in [b', b]$, a function h_u as in the definition of $u \in A$ and the straight line segment $k_{u,v}$ joining $(u, f(u) + \lambda\varepsilon)$ to $(v, f(v) + \lambda\varepsilon)$ $(= (v, y_b))$ define a function $k = h_u \cup k_{u,v}$ satisfying $f(x) < k(x) < f(x) + \lambda\varepsilon$ for $x \in [a,v]$. This shows that $b \in A$ which completes the proof in this case.

Now suppose $c < b'$. Let $c' = \inf\{x \in [a,c] : f(x) = f(c)\}$, $c'' = \sup\{x \in [c,b] : f(c) = f(x)\}$. Note that $a' < c' \le c \le c'' < b'$. For $u \in (a', c')$ close enough to c' and $v \in [c', c'']$, the functions $h_u \cup k_{u,v}$ defined as above show that $c'' \in A$ and hence $c = c''$. Then for $u \in (a', c)$ and $v \in (c, b')$ both close enough to c, the functions $h_u \cup k_{u,v}$ show that A contains elements larger than c, contradicting the definition of c. □

Proposition 3.5. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a nondecreasing surjection. Let $\varepsilon\colon \mathbb{R} \to \mathbb{R}$ be a positive continuous function. There is a C^1 order-isomorphism $g\colon \mathbb{R} \to \mathbb{R}$ such that $Dg(x) > 0$ and $f(x) < g(x) < f(x) + \varepsilon(x)$ for all $x \in \mathbb{R}$.*

Proof. The assumption on f ensures that f is continuous and that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$. By replacing $\varepsilon(x)$ by $\min(\varepsilon(x), 1)$, we may assume that $\varepsilon(x) \le 1$ for all $x \in \mathbb{R}$. Inductively choose integers $0 = n_0 < n_1 < n_2 < \dots$ such that for each nonnegative integer i,

$$f(n_i) + 1 < f(n_{i+1}).$$

Then inductively choose integers $0 = n_0 > n_{-1} > n_{-2} > \dots$ such that the same inequality holds for negative integers i as well. For each integer i, choose y_i so that

$$f(n_i) < y_i < f(n_i) + \inf_{n_{i-1} \le x \le n_{i+1}} \varepsilon(x).$$

The choice of the n_i's ensures that we have $y_i < y_{i+1}$ for each i. For each integer i, apply Proposition 3.4 to the interval $[n_i, n_{i+1}]$ with $\varepsilon = \inf_{n_i \le x \le n_{i+1}} \varepsilon(x)$ and piece together the resulting functions to get a strictly increasing piecewise linear continuous function $g\colon \mathbb{R} \to \mathbb{R}$ satisfying $f(x) < g(x) < f(x) + \varepsilon(x)$ for all $x \in \mathbb{R}$. Then modify g to round off the corners of its graph using Proposition 3.2. □

Proposition 3.6. *Let $\varepsilon\colon \mathbb{R} \to \mathbb{R}$ be a positive continuous function and let $g\colon \mathbb{R} \to \mathbb{R}$ be a nondecreasing continuous surjection.*

(i) *If n is a nonnegative integer and g is C^n, there is an entire function $f\colon \mathbb{C} \to \mathbb{C}$ such that $f[\mathbb{R}] \subseteq \mathbb{R}$, $f \restriction \mathbb{R}$ is an order-isomorphism of $\mathbb{R}$ with a strictly positive derivative and for all $i = 0, 1, \dots, n$ and all $x \in \mathbb{R}$, $|D^i f(x) - D^i g(x)| < \varepsilon(x)$.*

(ii) *If $0 \le c_0 \le c_1 \le \dots$ is such that $\lim_{i\to\infty} c_i = \infty$ and g is C^∞, there is an entire function $f\colon \mathbb{C} \to \mathbb{C}$ such that $f[\mathbb{R}] \subseteq \mathbb{R}$, $f \restriction \mathbb{R}$ is an order-isomorphism of $\mathbb{R}$ with a strictly positive derivative and for every $i < \omega$, and each $x \in \mathbb{R}$ such that $|x| \ge c_i$, $|D^i f(x) - D^i g(x)| < \varepsilon(x)$.*

Proof. Consider first the case of (i) with $n = 0$. By Proposition 3.5, there is a C^1 order-isomorphism $\bar{g}$ with a strictly positive derivative such that $|\bar{g}(x) - g(x)| < \frac{1}{2}\varepsilon(x)$ for all $x \in \mathbb{R}$. Then by Theorem 1.11, there is an entire function f such that $f[\mathbb{R}] \subseteq \mathbb{R}$ and for all $x \in \mathbb{R}$, we have $|f(x) - \bar{g}(x)| < \frac{1}{2}\varepsilon(x)$ and $|Df(x) - D\bar{g}(x)| < D\bar{g}(x)$ (and hence $Df(x) > 0$). We may assume that ε is bounded, so that the fact that $|f(x) - g(x)| < \varepsilon(x)$ ensures that f is surjective.

The case of (i) where $n \ge 1$ and (ii) are handled similarly, using Proposition 3.3 instead of Proposition 3.5. □

4. Preliminary results

The main goal of this section is to define a family of entire functions and prove Proposition 4.5, which shows how members of this family can be approximated in smaller models of set theory and perturbed slightly to alter their values at certain points. We begin with a technical fact which will be useful in the next section.

Proposition 4.1. *Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be continuous functions such that g is bounded. Assume that for some open interval I and every $\lambda \in I$, $f + \lambda g$ is an order-isomorphism of $\mathbb{R}$. Fix $b \in \mathbb{R}$. Then the function $h\colon I \to \mathbb{R}$ given by $h(\lambda) = (f + \lambda g)^{-1}(b)$ is continuous. Moreover, h is either constant or strictly monotonic, with the first alternative happening precisely when there is an x such that $f(x) = b$ and $g(x) = 0$.*

Proof. Let λ and λ_n, $n \in \mathbb{N}$, be elements of I such that $\lambda_n \to \lambda$ as $n \to \infty$. We want to show that $h(\lambda_n) \to h(\lambda)$. Let $x_n = h(\lambda_n)$, so that $(f + \lambda_n g)(x_n) = b$. Notice that

$$(f + \lambda g)(x_n) = (f + \lambda_n g)(x_n) + (\lambda - \lambda_n) g(x_n) = b + (\lambda - \lambda_n) g(x_n) \to b \text{ as } n \to \infty$$

since g is bounded. Since $f + \lambda g$ is an order-isomorphism, it is a homeomorphism and hence $h(\lambda_n) = x_n \to (f + \lambda g)^{-1}(b) = h(\lambda)$ as $n \to \infty$. This proves the first part of the proposition.

For the second, note first that if there is an x_0 such that $f(x_0) = b$ and $g(x_0) = 0$, then for any $\lambda \in I$ we have $(f + \lambda g)(x_0) = b$ and hence $h(\lambda) = x_0$. Now suppose that there is no such x_0. Since h is continuous and defined on an interval, if we show that h is one-to-one, then it will follow that h is stricly monotonic. Suppose that for some λ_1, λ_2 we have $h(\lambda_1) = h(\lambda_2)$, i.e., letting $x = h(\lambda_1)$,

$$f(x) + \lambda_1 g(x) = b = f(x) + \lambda_2 g(x).$$

Then $\lambda_1 = \lambda_2$ follows as long as $g(x) \neq 0$. If we had $g(x) = 0$, then the displayed equations would give $f(x) = b$, contradicting our assumption that no such x exists. $\square$

The function H given by the following proposition will serve as an envelope which controls the behavior at infinity of the members of the family of entire functions defined below.

Proposition 4.2. *Let $\zeta\colon \mathbb{R} \to \mathbb{R}$ be given by $\zeta(x) = (1 + |x|)^{-1}$. For any non-decreasing sequence $\{c_i\}_{i<\omega}$ of nonnegative real numbers with $\lim c_i = \infty$ and any positive continuous function $\varepsilon\colon \mathbb{R} \to \mathbb{R}$, there is an entire function H such that*

(a) *$H[\mathbb{R}] \subseteq \mathbb{R}$,*
(b) *for all $x \in \mathbb{R}$, $H(x) > 0$,*
(c) *for all $i < \omega$ and all $x \in \mathbb{R}$ such that $|x| \geq c_i$, $|D^i H(x)| < 2^{-i}\zeta(x)\varepsilon(x)$.*

Remark 4.3. (1) Concerning the choice of ζ, all that matters is that ζ is a continuous function such that $0 < \zeta(x) \leq 1$ for all $x \in \mathbb{R}$ and $\lim_{x\to\pm\infty} \zeta(x) = 0$. (2) It would be equivalent to state the proposition with $\varepsilon(x)$ instead of $2^{-i}\zeta(x)\varepsilon(x)$ in part (c), but the present formulation is more convenient for our purposes.

Proof. Choose a positive continuous function ε' such that for all $x \in \mathbb{R}$, if $|x| \geq c_i$, then $\varepsilon'(x) < 2^{-i}\zeta(x)\varepsilon(x)$. Consider functions of the form $H' = \sum_{n=-\infty}^{\infty} \varepsilon_n g_{(\frac{n}{2}, \frac{n}{2}+1)}$, where $\varepsilon_n > 0$ for each $n \in \mathbb{Z}$ and for each pair of real numbers $a < b$, $g_{(a,b)}$ is any C^∞ function which is positive on (a, b) and zero elsewhere. It is clear that for a suitable choice of the coefficients ε_n, we have that H' is a positive C^∞ function satisfying $|D^i H'(x)| < \frac{1}{2}\varepsilon'(x)$ for all $i < \omega$ and all $x \in \mathbb{R}$ such that $|x| \geq c_i$. By Theorem 1.11, there is an entire function H such that $H[\mathbb{R}] \subseteq \mathbb{R}$, $H(x) > 0$ for all $x \in \mathbb{R}$, and for all $i < \omega$ and all $x \in \mathbb{R}$ such that $|x| \geq c_i$, $|D^i H(x) - D^i H'(x)| < \frac{1}{2}\varepsilon'(x)$. Then, for all $i < \omega$ and all $x \in \mathbb{R}$ such that $|x| \geq c_i$, $|D^i H(x)| < \varepsilon'(x)$ and hence (c) holds. $\square$

Let $\mathcal{G}$ be the family of entire functions $g(n, A)$ where $n \in \mathbb{N}$, $A \subseteq \mathbb{R}$ is a nonvoid finite set, and for all $z \in \mathbb{C}$,

$$g(n, A)(z) = \prod_{a\in A} \sin\left(\frac{z - a}{n}\right).$$

Let $\mathcal{G}_0$ be the subfamily consisting of those functions $g(n, A)$ for which $n \geq 4|A|$. The next proposition gathers some simple properties of the collection $\mathcal{G}$.

Proposition 4.4. *The family $\mathcal{G}$ satisfies the following properties.*

(a) *For all $i < \omega$ and all $x \in \mathbb{R}$, $|D^i g(n, A)(x)| \leq (|A|/n)^i$.*
(b) *Let I_a, $a \in A$, be pairwise disjoint open intervals such that $a \in I_a$, for each $a \in A$. For each $a \in A$, let $(r_{a,m} : m \in \mathbb{N})$ be a sequence in I_a such that for*

each $m \in \mathbb{N}$ *we have* $|a - r_{a,m}| < 1/m$. *Let* $A(m) = \{r_{a,m} : a \in A\}$. *Then for all* $i < \omega$, $m \in \mathbb{N}$ *and* $x \in \mathbb{R}$,

$$|D^i g(n, A)(x) - D^i g(n, A(m))(x)| \leq m^{-1}(|A|/n)^{i+1}$$

and for all $z \in \mathbb{C}$, $m \in \mathbb{N}$ *and* $\lambda, \mu \in \mathbb{R}$, *if* $M > 0$ *and* $|z| \leq M$, *then* $|g(n, A(m))(z)| \leq T_1$ *and*

$$|\lambda g(n, A)(z) - \mu g(n, A(m))(z)| \leq T_2(|\lambda - \mu| + |\mu|/m),$$

where T_1 *and* T_2 *are constants which depend only on* n, A *and* M.

(c) *Let* $\{c_i\}_{i<\omega}$ *be a nondecreasing sequence of nonnegative real numbers with* $\lim c_i = \infty$ *and let* $\varepsilon \colon \mathbb{R} \to \mathbb{R}$ *be a positive continuous function. Let* $H \colon \mathbb{C} \to \mathbb{C}$ *be as given by Proposition* 4.2. *Write* $H \operatorname{span} \mathcal{G}_0$ *for the set of functions of the form* HG *where* $G \in \operatorname{span} \mathcal{G}_0$ *and* $\operatorname{span} \mathcal{G}_0$ *is the set of all real linear combinations of elements of* $\mathcal{G}_0$. *Then for all* $i < \omega$, *for all* $f \in H \operatorname{span} \mathcal{G}_0$ *and for all* $x \in \mathbb{R}$ *such that* $|x| \geq c_i$, *we have* $|D^i f(x)| \leq (\sum_{g \in \mathcal{G}'} |\lambda_g|)(3/4)^i \zeta(x)\varepsilon(x)$, *where* $f = H \sum\{\lambda_g g : g \in \mathcal{G}'\}$ *for some finite* $\mathcal{G}' \subseteq \mathcal{G}_0$ *and* $\lambda_g \in \mathbb{R}$, $g \in \mathcal{G}'$. *(*ζ *is as in Proposition* 4.2.*)*

Proof. (a) By induction on i, it follows that the i-th derivative of $g(n, A)$ can be expressed as a sum of $|A|^i$ terms, each of which is equal to $\pm 1/n^i$ times a product indexed by A in which the factor corresponding to $a \in A$ is equal to $\sin((z-a)/n)$ or $\cos((z-a)/n)$. The inequality in (a) follows since for $z \in \mathbb{R}$, the factors $\sin((z-a)/n)$ and $\cos((z-a)/n)$ are bounded in absolute value by 1.

(b) Note that for any natural number k and for any numbers $u_i, v_i \in \mathbb{C}$ of modulus at most μ_0, $i = 1, \ldots, k$, we have

$$\begin{aligned} (4.1) \qquad & |(u_1 u_2 \ldots u_k) - (v_1 v_2 \ldots v_k)| \\ &= \left| \sum_{i=1}^{k} (v_1 \ldots v_{i-1} u_i u_{i+1} \ldots u_k) - (v_1 \ldots v_{i-1} v_i u_{i+1} \ldots u_k) \right| \\ &\leq \sum_{i=1}^{k} |v_1 \ldots v_{i-1}(u_i - v_i) u_{i+1} \ldots u_k| \\ &\leq \sum_{i=1}^{k} \mu_0^{k-1} |u_i - v_i|. \end{aligned}$$

For $x \in \mathbb{R}$, the difference between $D^i g(n, A)(x)$ and $D^i g(n, A(m))(x)$ can be expressed, as in the argument for (a), as a sum of $|A|^i$ terms, each of which is equal to $\pm 1/n^i$ times a difference of the form

$$\prod_{a \in A} f_a\left(\frac{x - a}{n}\right) - \prod_{a \in A} f_a\left(\frac{x - r_{a,m}}{n}\right),$$

where each f_a is either a sine or a cosine. Applying (4.1) with $\mu_0 = 1$ to these differences and using that, by the Mean Value Theorem, for $x \in \mathbb{R}$ and $a \in A$ we have

$$\left| f_a\left(\frac{x - a}{n}\right) - f_a\left(\frac{x - r_{a,m}}{n}\right) \right| \leq (1/n)|r_{a,m} - a|,$$

we get that for each $x \in \mathbb{R}$,

$$|D^i g(n,A)(x) - D^i g(n,A(m))(x)| \le \frac{|A|^i}{n^i} \sum_{a\in A} (1/n)|r_{a,m} - a| \le \frac{|A|^{i+1}}{m\, n^{i+1}}.$$

This takes care of the first part of (b). For the second, we have that if $|z| \le M$, then $|z - r_{a,m}| \le |z-a| + |a - r_{a,m}| \le M + |a| + 1$. Hence, letting T_1 denote the supremum of $|\prod_{a\in A} \sin(z_a/n)|$ over all choices of $z_a \in \mathbb{C}$ such that $|z_a| \le M + |a| + 1$, $a \in A$, we get $|g(n,A(m))(z)| \le T_1$. For the remaining part of (b), we use

$$\sin\left(\frac{z-a}{n}\right) - \sin\left(\frac{z - r_{a,m}}{n}\right) = 2\cos\left(\frac{2z - r_{a,m} - a}{2n}\right)\sin\left(\frac{r_{a,m} - a}{2n}\right)$$

and apply (4.1) to the difference $g(n,A)(z) - g(n,A(m))(z)$ with μ_0 taken to be the supremum of 1 and all the quantities $|\sin((z-r)/n)|$, $|\cos((2z-r-a)/(2n))|$ and $|g(n,A)(z)|$, where z, r, a range over the values given by $|z| \le M$, $r \in \bigcup_{a\in A}[a-1, a+1]$, $a \in A$. This gives that for $|z| \le M$ and $m \in \mathbb{N}$ we have

$$\begin{aligned} |g(n,A)(z) - g(n,A(m))(z)| &\le \mu_0^{|A|-1} \sum_{a\in A} \left|\sin\left(\frac{z-a}{n}\right) - \sin\left(\frac{z-r_{a,m}}{n}\right)\right| \\ &\le 2\mu_0^{|A|} \sum_{a\in A} \left|\sin\left(\frac{r_{a,m}-a}{2n}\right)\right|. \end{aligned}$$

Since $|\sin((r_{a,m} - a)/(2n))| \le |(r_{a,m} - a)/(2n)| \le 1/m$, we have

$$\begin{aligned} |\lambda g(n,A)(z) - \mu g(n,A(m))(z)| &\le |\lambda - \mu|\,|g(n,A)(z)| \\ &\quad + |\mu|\,|g(n,A)(z) - g(n,A(m))(z)| \\ &\le \mu_0|\lambda - \mu| + 2|A|\mu_0^{|A|}|\mu|/m \\ &\le 2|A|\mu_0^{|A|}(|\lambda - \mu| + |\mu|/m). \end{aligned}$$

(c) Let $f \in H \operatorname{span} \mathcal{G}_0$ and let $i < \omega$. Write

$$f = H \sum \{\lambda_s s : s \in \mathcal{G}'\}$$

for some finite $\mathcal{G}' \subseteq \mathcal{G}_0$ and $\lambda_s \in \mathbb{R}$, $s = g(n_s, A_s) \in \mathcal{G}'$. We have

$$D^i f = \sum \{\lambda_s D^i(Hs) : s \in \mathcal{G}'\}.$$

For each $s \in \mathcal{G}'$ we have, using (a), the bound (for $x \in \mathbb{R}$)

$$|D^i(Hs)(x)| \le \sum_{k=0}^{i} \binom{i}{k} |D^k H(x)|\,|D^{i-k}s(x)| \le \sum_{k=0}^{i} \binom{i}{k} |D^k H(x)|\,(|A_s|/n_s)^{i-k}.$$

Using $n_s \ge 4|A_s|$, this gives

$$|D^i f(x)| \le \sum_{s\in\mathcal{G}'} |\lambda_s| \sum_{k=0}^{i} \binom{i}{k} |D^k H(x)| \frac{1}{4^{i-k}}.$$

When $|x| \geq c_i$, we have $|x| \geq c_k$, $0 \leq k \leq i$ and so

$$|D^i f(x)| \leq \sum_{s \in \mathcal{G}'} |\lambda_s| \sum_{k=0}^{i} \binom{i}{k} \frac{1}{2^k} \frac{1}{4^{i-k}} \zeta(x)\varepsilon(x) = \left(\sum_{s \in \mathcal{G}'} |\lambda_s| \right) (3/4)^i \zeta(x)\varepsilon(x).$$

□

Proposition 4.5. *Let N be an elementary submodel of H_θ for some regular $\theta > \mathfrak{c}$. Let $\{c_i\}_{i<\omega} \in N$ be a nondecreasing sequence of nonnegative real numbers with $\lim c_i = \infty$. Let $\varepsilon \colon \mathbb{R} \to \mathbb{R}$ be positive and continuous, $\varepsilon \in N$. Let $H \colon \mathbb{C} \to \mathbb{C}$ be as given by Proposition* 4.2*, $H \in N$. Let $M > 0$. Let B, C be countable dense subsets of $\mathbb{R}$, $B, C \in N$. Let $u \in \mathbb{R}$. Let f_0 be an entire function such that $f_0[\mathbb{R}] \subseteq \mathbb{R}$, $f_0 \in N$. Let $f \in f_0 + H \operatorname{span} \mathcal{G}_0$. Let $K_0, K_1 \subseteq \mathbb{R}$ be finite such that $K_0, f[K_1] \in N$. Let $h \subseteq f$ be finite such that $h \in N$. Assume that the sets $\operatorname{dom} h$, K_0, K_1, $\{u\}$ are pairwise disjoint. Then there is a function $f' \in N \cap (f_0 + H \operatorname{span} \mathcal{G}_0)$ such that for some $n \in \mathbb{N}$ and some positive rational number r, the following properties hold. In this list, K denotes the set $K_0 \cup \{b_x : x \in K_1\} \cup \operatorname{dom} h$ (b_x is defined in* (c) *below).*

- (a) *$h \subseteq f'$.*
- (b) *For all $x \in K_0$, $f'(x) \in C$.*
- (c) *For all $x \in K_1$, there is a $b_x \in B$ such that $|b_x - x| < \varepsilon(0)$ and $f'(b_x) = f(x)$.*
- (d) *$n \geq 4|K|$ and for all $i < \omega$ and for all $\sigma \in \mathbb{R}$ such that $|\sigma| \leq r$, we have that for $x \in \mathbb{R}$ such that $|x| \geq c_i$,*
$$|D^i f(x) - D^i(f' + \sigma H g(n, K))(x)| < \varepsilon(x).$$
- (e) *$g(n, K)(u) \neq 0$ and for some real number σ such that $|\sigma| < r$, we have*
$$f(u) = f'(u) + \sigma H(u) g(n, K)(u).$$
- (f) *For all $z \in \mathbb{C}$ such that $|z| \leq M$ and for all $\sigma \in \mathbb{R}$ such that $|\sigma| \leq r$,*
$$r|H(z)g(n, K)(z)| < \varepsilon(0) \quad \text{and} \quad |f(z) - (f' + \sigma H g(n, K))(z)| < \varepsilon(0).$$

Remark 4.6. Under the hypotheses of the proposition, we shall have occasion to want the conclusion for $\lambda\varepsilon$ instead of ε, where λ is a rational number such that $0 < \lambda < 1$. To see that this modified conclusion holds, we argue as follows. First notice that it follows from properties (a), (b) and (c) of Proposition 4.2 that the same properties hold for $(\lambda\varepsilon, \lambda H)$ instead of (ε, H). Since λ is rational, $\lambda\varepsilon, \lambda H \in N$. Since $(\lambda H) \operatorname{span} \mathcal{G}_0 = H \operatorname{span} \mathcal{G}_0$, we have $f \in f_0 + (\lambda H) \operatorname{span} \mathcal{G}_0$. The proposition therefore gives $f' \in f_0 + (\lambda H) \operatorname{span} \mathcal{G}_0 = f_0 + H \operatorname{span} \mathcal{G}_0$, $f' \in N$, $n \in \mathbb{N}$ and a positive rational number r such that (a)–(f) hold with $\lambda\varepsilon$ and λH in the place of ε and H, respectively. But now notice that everywhere H is mentioned, i.e., in (d), (e) and (f), it is multiplied by r or by some σ such that $|\sigma| \leq r$. Equivalent statements of these clauses are obtained by restoring λH to H and replacing r by λr, giving the desired modification of the conclusion.

Proof. Choose pairwise disjoint open intervals I_x such that $x \in I_x$ and $I_x \cap (\{u\} \cup K_0 \cup \operatorname{dom} h) = \emptyset$, for $x \in K_1$. For each $x \in K_1$, choose a sequence of points $b_{x,m} \in B \cap I_x$, $m \in \mathbb{N}$, such that $|b_{x,m} - x| < 1/m$. Let

$$\bar{K} = K_0 \cup K_1 \cup \operatorname{dom} h, \tag{4.2}$$

$$\bar{K}_m = K_0 \cup \{b_{x,m} : x \in K_1\} \cup \operatorname{dom} h. \tag{4.3}$$

For each $x \in K_0$, choose a sequence of points $c_{x,m} \in C$, $m \in \mathbb{N}$, such that $|c_{x,m} - f(x)| < 1/m$. Let

$$f = f_0 + H \sum \{\lambda_s s : s \in \mathcal{G}'\},$$

where $\mathcal{G}'$ is a finite subset of $\mathcal{G}_0$ and $\lambda_s \in \mathbb{R}$ for each $s \in \mathcal{G}'$. For each $s \in \mathcal{G}'$, let $n(s) \in \mathbb{N}$ and $A(s) \subseteq \mathbb{R}$ be such that $s = g(n(s), A(s))$. For each $s \in \mathcal{G}'$, pick pairwise disjoint open intervals I_a^s such that $a \in I_a^s$, for $a \in A(s)$. For each $a \in A(s)$, choose a sequence of rational numbers $r_{a,m}^s \in I_a^s$, $m \in \mathbb{N}$, such that for each $m \in \mathbb{N}$ we have $|a - r_{a,m}^s| < 1/m$. Let $A(s,m) = \{r_{a,m}^s : a \in A(s)\}$. Choose $n \in \mathbb{N}$ large enough so that

(i) u is not a zero of $g(n, \bar{K})$ or any $g(n, \bar{K}_m)$, $m \in \mathbb{N}$,
(ii) for each $a \in \bar{K}_m$, a is not a zero of $g(n, \bar{K}_m \setminus \{a\})$,
(iii) for each $x \in \bar{K}$, x is not a zero of $g(n, \bar{K} \setminus \{x\})$,
(iv) $n \geq 4|\bar{K}_m| = 4(|K_0| + |K_1| + |\operatorname{dom} h|)$.

(This is possible because the sets $\{b_{x,m} : m \in \mathbb{N}\}$ are bounded so that for large enough n we have $|a - b_{x,m}|, |u - b_{x,m}| < n\pi$ ($a \in \bar{K}_m$, $x, \in K_1$) for all m simultaneously.) Consider functions f', $\bar{f}$ of the form

$$f' = f_0 + H \sum \{\mu_s g(n(s), A(s,m)) : s \in \mathcal{G}'\} + H \sum \{\sigma_a g(n, \bar{K}_m \setminus \{a\}) : a \in \bar{K}_m\}$$

and

$$\begin{aligned}\bar{f} &= f_0 + H \sum \{\mu_s g(n(s), A(s,m)) : s \in \mathcal{G}'\} + H \sum \{\sigma_a g(n, \bar{K}_m \setminus \{a\}) : a \in \bar{K}_m\} \\ &\quad + \ \sigma H g(n, \bar{K}_m) \\ &= f' + \sigma H g(n, \bar{K}_m),\end{aligned}$$

where $m \in \mathbb{N}$, $\mu_s, \sigma_a, \sigma \in \mathbb{R}$ ($s \in \mathcal{G}'$, $a \in \bar{K}_m$). For $a \in \bar{K}_m$, among the functions $g(n, \bar{K}_m \setminus \{a'\})$, $a' \in \bar{K}_m$, only the one with $a' = a$ is not zero at a. This leads to the following observations.

(1) For $a \in \operatorname{dom} h$ and for each choice of $m \in \mathbb{N}$ and $\vec{\mu} = (\mu_s)_{s \in \mathcal{G}'}$, there is a unique $\sigma_a = \sigma_a(m, \vec{\mu})$ for which $f'(a) = f(a) = h(a)$, namely

$$\sigma_a(m, \vec{\mu}) = \frac{\sum \{\lambda_s g\big(n(s), A(s)\big)(a) - \mu_s g\big(n(s), A(s,m)\big)(a) : s \in \mathcal{G}'\}}{g(n, \bar{K}_m \setminus \{a\})(a)}.$$

(2) For each $x \in K_0$ and each choice of $m \in \mathbb{N}$ and $\vec{\mu} = (\mu_s)_{s \in \mathcal{G}'}$, there is a unique $\sigma_x = \sigma_x(m, \vec{\mu})$ for which $f'(x) = c_{x,m}$, namely

$$\begin{aligned}\sigma_x(m, \vec{\mu}) &= \frac{c_{x,m} - f_0(x) - H(x) \sum \{\mu_s g\big(n(s), A(s,m)\big)(x) : s \in \mathcal{G}'\}}{H(x) g(n, \bar{K}_m \setminus \{x\})(x)} \\ &= \frac{c_{x,m} - f(x) + H(x) \sum \{\lambda_s g\big(n(s), A(s)\big)(x) - \mu_s g\big(n(s), A(s,m)\big)(x) : s \in \mathcal{G}'\}}{H(x) g(n, \bar{K}_m \setminus \{x\})(x)}.\end{aligned}$$

(3) For each $x \in K_1$ and each choice of $m \in \mathbb{N}$ and $\vec{\mu} = (\mu_s)_{s\in\mathcal{G}'}$, there is a unique $\sigma_{b_{x,m}} = \sigma_{b_{x,m}}(m,\vec{\mu})$ for which $f'(b_{x,m}) = f(x)$, namely

$$\sigma_{b_{x,m}}(m,\vec{\mu}) = \frac{f(x) - f_0(b_{x,m}) - H(b_{x,m}) \sum_{s\in\mathcal{G}'} \mu_s g(n(s), A(s,m))(b_{x,m})}{H(b_{x,m}) g(n, \bar{K}_m \setminus \{b_{x,m}\})(b_{x,m})}$$

$$= \frac{\begin{array}{l} f_0(x) - f_0(b_{x,m}) + H(x) \sum_{s\in\mathcal{G}'} \lambda_s g(n(s), A(s))(x) \\ \qquad - H(b_{x,m}) \sum_{s\in\mathcal{G}'} \mu_s g(n(s), A(s,m))(b_{x,m}) \end{array}}{H(b_{x,m}) g(n, \bar{K}_m \setminus \{b_{x,m}\})(b_{x,m})}$$

$$= \frac{\begin{array}{l} f_0(x) - f_0(b_{x,m}) + [H(x) - H(b_{x,m})] \sum_{s\in\mathcal{G}'} \lambda_s g(n(s), A(s))(x) \\ \quad + H(b_{x,m}) \sum_{s\in\mathcal{G}'} \lambda_s \Big(g(n(s), A(s))(x) - g(n(s), A(s))(b_{x,m}) \Big) \\ \quad + H(b_{x,m}) \sum_{s\in\mathcal{G}'} \Big(\lambda_s g(n(s), A(s))(b_{x,m}) - \mu_s g(n(s), A(s,m))(b_{x,m}) \Big) \end{array}}{H(b_{x,m}) g(n, \bar{K}_m \setminus \{b_{x,m}\})(b_{x,m})}.$$

(4) Given the assignment of values $\sigma_a = \sigma_a(m,\vec{\mu})$, $a \in \bar{K}_m$, there is a unique value of $\sigma = \sigma(m,\vec{\mu})$ for which $\bar{f}(u) = f(u)$, namely

$$\sigma(m,\vec{\mu}) = \frac{\begin{array}{c} \sum_{s\in\mathcal{G}'} \big[\lambda_s g\big(n(s), A(s)\big)(u) - \mu_s g\big(n(s), A(s,m)\big)(u) \big] \\ - \sum_{a\in\bar{K}_m} \sigma_a(m,\vec{\mu}) g(n, \bar{K}_m \setminus \{a\})(u) \end{array}}{g(n, \bar{K}_m)(u)}.$$

Note that for functions f', if $\sigma_a = \sigma_a(m,\vec{\mu})$ for each $a \in \bar{K}_m$, then (a), (b) hold and (c) holds (with $b_x = b_{x,m}$) if m is large enough. Also, as $\vec{\mu} = (\mu_s)_{s\in\mathcal{G}'} \to (\lambda_s)_{s\in\mathcal{G}'}$ and $m \to \infty$, we have $\sigma_a(m,\vec{\mu}) \to 0$ for each $a \in K_0 \cup \operatorname{dom} h$, $\sigma_{b_{x,m}}(m,\vec{\mu}) \to 0$ for each $x \in K_1$, and then it follows that $\sigma(m,\vec{\mu}) \to 0$ as well. To check this, use the consequences of Proposition 4.4 (part (a) and the first part of (b), both with $i = 0$) that whenever $n' \geq |A'|$, we have for all $x \in \mathbb{R}$, $|g(n', A')(x)| \leq 1$ and $|g(n', A'(m))(x) - g(n', A')(x)| \leq 1/m$. In the present circumstances, we have $\bar{K}$ and $\bar{K}_m$ (as well as the pairs $(\bar{K} \setminus \{a\}, \bar{K}_m \setminus \{a\})$, $a \in K_0 \cup \operatorname{dom} h$, and $(\bar{K} \setminus \{x\}, \bar{K}_m \setminus \{b_{x,m}\})$, $x \in K_1$) playing the role of A' and $A'(m)$. Note in particular that the denominators in (1)–(4) are bounded away from zero. For example, for the denominator in (3) we have that $H(b_{x,m}) \to H(x) > 0$ and, by the choice of n, $g(n, \bar{K} \setminus \{x\})(x) \neq 0$ and

$$\begin{aligned}|g(n, \bar{K}_m \setminus \{b_{x,m}\})(b_{x,m}) - g(n, \bar{K} \setminus \{x\})(x)|\\ &\le |g(n, \bar{K}_m \setminus \{b_{x,m}\})(b_{x,m}) - g(n, \bar{K} \setminus \{x\})(b_{x,m})|\\ &\quad + |g(n, \bar{K} \setminus \{x\})(b_{x,m}) - g(n, \bar{K} \setminus \{x\})(x)|\\ &\le 1/m + |g(n, \bar{K} \setminus \{x\})(b_{x,m}) - g(n, \bar{K} \setminus \{x\})(x)|\\ &\quad \to 0 \text{ as } m \to \infty.\end{aligned}$$

For the rest of the proof, we restrict our attention to choices of the coefficients in the definitions of f' and $\bar{f}$ such that $|\mu_s - \lambda_s| \le 1$, $s \in \mathcal{G}'$, $|\sigma_a| \le 1$, $a \in \bar{K}_m$, and $|\sigma| \le 1$. Given these restrictions, let $i_0 < \omega$ be such that for all $i > i_0$, for all $m \in \mathbb{N}$, $\vec{\mu} = (\mu_s)_{s \in \mathcal{G}'}$ and $x \in \mathbb{R}$, we have that if $|x| \ge c_i$, then $|D^i(\bar{f} - f)(x)| < \varepsilon(x)$. (Such an i_0 exists by Proposition 4.4(c) which, because $\zeta(x) \le 1$, yields $|D^i(\bar{f} - f)(x)| < C(3/4)^i \varepsilon(x)$ whenever $|x| \ge c_i$, where C is a constant independent of the choice of the coefficients in the definition $\bar{f}$ as long as the restrictions we just placed on these coefficients are respected.)

Now let L be a compact interval such that for all $i \le i_0$, for all $m \in \mathbb{N}$, for all $\vec{\mu} = (\mu_s)_{s \in \mathcal{G}'}$ and for all $x \in \mathbb{R} \setminus L$ (and given the above restrictions on the choice of coefficients for $\bar{f}$), we have that if $|x| \ge c_i$, then $|D^i(\bar{f} - f)(x)| < \varepsilon(x)$. (Such an L exists by Proposition 4.4(c) again. This time we use the fact that for each $i < \omega$ and x such that $|x| \ge c_i$, $|D^i(\bar{f} - f)(x)| < C(3/4)^i \zeta(x)\varepsilon(x) \le C\zeta(x)\varepsilon(x) < \varepsilon(x)$ for $|x|$ large enough since $\lim_{x \to \pm\infty} \zeta(x) = 0$.)

Henceforth, we limit ourselves to functions f', $\bar{f}$ for which, in addition to the restrictions imposed above, we have $\sigma_a = \sigma_a(m, \vec{\mu})$, $a \in \bar{K}_m$. (So we now consider only coefficients so that $|\sigma| \le 1$ and for $a \in \bar{K}_m$, $\sigma_a = \sigma_a(m, \vec{\mu})$ with $\vec{\mu}$ close enough to $\vec{\lambda}$ and m large enough so that the conditions $|\mu_s - \lambda_s| \le 1$, $s \in \mathcal{G}'$ and $|\sigma_a| \le 1$, $a \in \bar{K}_m$ are satisfied.) Consider the following facts.

(5) For $|x| \ge c_i$, $|D^i f(x) - D^i \bar{f}(x)| < \varepsilon(x)$ can only fail if $i \le i_0$ and $x \in L$.
(6) For some $\delta > 0$, we have $\varepsilon(x) > \delta$ for all $x \in L$.
(7) For $|z| \le M$, we have

$$\begin{aligned}&|f(z) - \bar{f}(z)|\\ &\le |H(z)| \Big[\sum \{ \big| \lambda_s g\big(n(s), A(s)\big)(z) - \mu_s g\big(n(s), A(s,m)\big)(z) \big| : s \in \mathcal{G}' \}\\ &\quad + \sum \{ |\sigma_a(m, \vec{\mu})| \, |g(n, \bar{K}_m \setminus \{a\})(z)| : a \in \bar{K}_m \} + |\sigma| \, |g(n, \bar{K}_m)(z)| \Big]\\ &\le \left(\sup_{|z| \le M} |H(z)| \right) \Big[\sum \{ T_{s,M}(|\lambda_s - \mu_s| + |\mu_s|/m) : s \in \mathcal{G}' \}\\ &\quad + \sum \{ |\sigma_a(m, \vec{\mu})| T_{n,M} : a \in \bar{K}_m \} + |\sigma| T_{n,M} \Big],\end{aligned}$$

where $T_{n,M}$, $T_{s,M}$ are constants independent of m as in Proposition 4.4(b).

(8) For $i \leq i_0$ and $x \in L$, we have

$$\begin{aligned}
&|D^i f(x) - D^i \bar{f}(x)| \\
&\leq \sum_{k=0}^{i} \binom{i}{k} |D^{i-k} H(x)| \Bigg[\sum_{s\in\mathcal{G}'} \big|\lambda_s D^k g\big(n(s), A(s)\big)(x) \\
&\qquad\qquad - \mu_s D^k g\big(n(s), A(s,m)\big)(x)\big| \\
&\quad + \sum_{a\in\bar{K}_m} |\sigma_a(m,\vec{\mu})|\, |D^k g(n, \bar{K}_m \setminus \{a\})(x)| + |\sigma|\, |D^k g(n, \bar{K}_m)(x)| \Bigg] \\
&\leq \sum_{k=0}^{i} \binom{i}{k} |D^{i-k} H(x)| \Bigg[\sum_{s\in\mathcal{G}'} \big(|\lambda_s - \mu_s|\, |D^k g\big(n(s), A(s)\big)(x)| + |\mu_s|/m\big) \\
&\quad + \sum_{a\in\bar{K}_m} |\sigma_a(m,\vec{\mu})|\, |D^k g(n, \bar{K}_m \setminus \{a\})(x)| + |\sigma|\, |D^k g(n, \bar{K}_m)(x)| \Bigg] \\
&\leq \sum_{k=0}^{i} \binom{i}{k} C_H \Bigg[\sum_{s\in\mathcal{G}'} (|\lambda_s - \mu_s| + |\mu_s|/m) + \sum_{a\in\bar{K}_m} |\sigma_a(m,\vec{\mu})| + |\sigma| \Bigg],
\end{aligned}$$

where $C_H = \sup_{x\in L,\, i\leq i_0} |D^i H(x)|$ and the second and third inequalities used the first part of Proposition 4.4(b), the fact that $n(s) \geq |A(s)|$, Proposition 4.4(a) and the fact that $n \geq |\bar{K}_m|$.

We may choose $m \in \mathbb{N}$, $\mu_s \in \mathbb{Q}$ for each $s \in \mathcal{G}'$ and $r > 0$ so that (d), (e) and (f) are satisfied with $b_{x,m}$ in the place of b_x. (First get a neighborhood of $\vec{\lambda}$, an m_0 and an r so that (d), (f) are satisfied for $\vec{\mu}$ in the given neighborhood of $\vec{\lambda}$ and $m \geq m_0$. Then choose such $\vec{\mu}$ and m so that $|\sigma(m,\vec{\mu})| < r$, giving (e).) Note that we have $f' \in N$ because $f_0, h, H, K_0, f[K_1], \{b_{x,m} : x \in K_1\}, \{c_{x,m} : x \in K_0\} \in N$, each μ_s is rational (and hence belongs to N) and each $\sigma_a(m,\vec{\mu})$, for $a \in \bar{K}_m$, is uniquely determined by the condition that $f'(a) = h(a)$ for $a \in \operatorname{dom} h$, $f'(x) = c_{x,m}$ for $a = x \in K_0$ and $f'(b_{x,m}) = f(x)$ for $a = b_{x,m}$ $(x \in K_1)$, and hence belongs to N by elementarity. □

5. Main lemma

The next result is the main technical lemma for the oracle-cc iteration.

Lemma 5.1. *Let $\overline{M} = \langle M_\delta : \delta < \omega_1\rangle$ be an oracle. Let $\langle A_\alpha : \alpha < \omega_1\rangle$ and $\langle B_\alpha : \alpha < \omega_1\rangle$ each be a sequence of pairwise disjoint subsets of $\mathbb{R}$ such that for $\alpha < \omega$, A_α and B_α are countable dense sets and for $\omega \leq \alpha < \omega_1$, A_α and B_α are everywhere second category sets of cardinality $\aleph_1$. Let $\{c_i\}_{i<\omega}$ be a nondecreasing sequence of nonnegative real numbers with $\lim c_i = \infty$. Let $\eta\colon \mathbb{R} \to \mathbb{R}$ be a positive continuous function. Let $f_0\colon \mathbb{C} \to \mathbb{C}$ be an entire function such that $f_0[\mathbb{R}] \subseteq \mathbb{R}$, $f_0 \restriction \mathbb{R}$ is an order-isomorphism and $Df_0(x) > 0$ for all $x \in \mathbb{R}$. Then there is a forcing notion P satisfying the $\overline{M}$-cc such that for every $G \subseteq P$ generic over V,*

$V[G] \models$ *there is an entire function* $f\colon \mathbb{C} \to \mathbb{C}$ *for which the following properties hold:*

(i) $f \restriction \mathbb{R}$ *is an order-isomorphism of* $\mathbb{R}$*,*
(ii) $f[A_\alpha] = B_\alpha$ *for all* $\alpha < \omega_1$*,*
(iii) *for all* $i < \omega$ *and all* $x \in \mathbb{R}$ *such that* $|x| \geq c_i$*,* $|D^i f(x) - D^i f_0(x)| \leq \eta(x)$*.*

Proof. By replacing $\eta(x)$ by $\min\{1, \eta(x), Df_0(x)\}$, we may assume that for all $x \in \mathbb{R}$, $\eta(x) \leq 1$ and $\eta(x) \leq Df_0(x)$. We may also assume that $c_0 = c_1 = 0$.

Let H be as in Proposition 4.2 for $\{c_i\}_{i<\omega}$ and η. For the rest of the proof, fix a suitably large regular cardinal θ. Let $\langle I_n : n < \omega\rangle$ list all the nonempty open intervals with rational endpoints. Fix a well-ordering of $\mathbb{R}$ in type ω_1. (CH holds because there is an oracle.) Let Q denote the set $\omega_1 \times \omega \times 2$ equipped with the lexicographical order, which we denote by $\lhd$. We will inductively define partial orders $P(u)$, $u \in Q$, from the following class of partial orders.

Definition 5.2. Let $N \prec H_\theta$. Let $\bar{a} = \langle a_\xi : \xi < \alpha\rangle$, $\bar{b} = \langle b_\xi : \xi < \beta\rangle$ be one-to-one sequences of real numbers, $\alpha, \beta \leq \omega_1$. We write, for $\delta < \omega_1$,

$$\bar{a}^\delta = \{a_{\omega\delta+n} : n < \omega,\, \omega\delta + n < \alpha\} \quad \text{and} \quad \bar{b}^\delta = \{b_{\omega\delta+n} : n < \omega,\, \omega\delta + n < \beta\}.$$

$P(\bar{a}, \bar{b}, N)$ denotes the partial order consisting of conditions $p = (h_p, f_p, \varepsilon_p, n_p)$ such that

(i) h_p is a finite partial order-preserving map from $\{a_\xi : \xi < \alpha\}$ to $\{b_\xi : \xi < \beta\}$,
(ii) $h_p[\bar{a}^\delta] \subseteq \bar{b}^\delta$ for all $\delta < \omega_1$,
(iii) $f_p \in (f_0 + H \operatorname{span} \mathcal{G}_0) \cap N$,
(iv) $h_p \subseteq f_p$,
(v) for all $i < \omega$ and all $x \in \mathbb{R}$ such that $|x| \geq c_i$, $|D^i(f_p - f_0)(x)| < (1-\varepsilon_p)\eta(x)$,
(vi) ε_p is a rational number, $0 < \varepsilon_p < 1$, and $n_p < \omega$.

The order is given by $p \leq q$ if and only if

(vii) $h_p \supseteq h_q$, $n_p \geq n_q$,
(viii) for all $z \in \mathbb{C}$ such that $|z| \leq n_q$, $|f_p(z) - f_q(z)| + \varepsilon_p \leq \varepsilon_q$.
(This gives in particular $\varepsilon_p \leq \varepsilon_q$.)

This order relation is transitive because if $p \leq q \leq r$, then $h_p \supseteq h_q \supseteq h_r$, $n_p \geq n_q \geq n_r$ and for $z \in \mathbb{C}$ such that $|z| \leq n_r$, we have $|z| \leq n_q$ as well and hence

$$\begin{aligned} |f_p(z) - f_r(z)| + \varepsilon_p &\leq |f_q(z) - f_r(z)| + (|f_p(z) - f_q(z)| + \varepsilon_p) \\ &\leq |f_q(z) - f_r(z)| + \varepsilon_q \\ &\leq \varepsilon_r. \end{aligned}$$

Remark 5.3. The clauses of the definition together with the assumptions at the beginning of the proof ensure that $f_p \restriction \mathbb{R}$ is an order-isomorphism. We are assuming $c_1 = 0$ so that by (v), for all $x \in \mathbb{R}$, $|Df_p(x) - Df_0(x)| < (1 - \varepsilon_p)\eta(x)$. Thus,

$$Df_p(x) > Df_0(x) - (1 - \varepsilon_p)\eta(x) \geq \eta(x) - (1 - \varepsilon_p)\eta(x) = \varepsilon_p\eta(x) > 0.$$

Hence, f_p is increasing. Because $c_0 = 0$ we also have, for all $x \in \mathbb{R}$, $|f_p(x) - f_0(x)| < (1 - \varepsilon_p)\eta(x) < \eta(x) \leq 1$. Because f_0 is onto, it follows that f_p is as well.

Fix a function $\gamma\colon \omega_1 \to (\omega_1 \setminus \omega) \times 2$ so that for each $(\alpha, i) \in (\omega_1 \setminus \omega) \times 2$, $|\gamma^{-1}(\alpha, i)| = \aleph_1$.

To each $u = (\delta, n, j) \in Q$ we associate a pair of ordinals $(\alpha, \beta) = (\alpha(u), \beta(u))$ as follows. Let $\gamma(\delta) = (\alpha', i)$. If $j = 0$, let (α, β) be the pair of ordinals $(\omega\delta+n, \omega\delta+n)$.

If $j=1$, let (α,β) be the pair $(\omega\delta+n+1,\omega\delta+n)$ if $i=0$, and $(\omega\delta+n,\omega\delta+n+1)$ if $i=1$.

(Notice that the pair $(\alpha(u),\beta(u))$ uniquely determines u: from (α,β) we can clearly recover δ and n, and we have $j=0$ if $\alpha=\beta$ and $j=1$ otherwise.)

We inductively define one-to-one enumerations $\bar{a}=\langle a_\xi:\xi<\omega_1\rangle$ of $\bigcup_{\alpha<\omega_1}A_\alpha$ and $\bar{b}=\langle b_\xi:\xi<\omega_1\rangle$ of $\bigcup_{\alpha<\omega_1}B_\alpha$ and a continuous $\in$-increasing sequence $\langle N_u: (\omega,0,0)\trianglelefteq u\in Q\rangle$ of countable elementary submodels of H_θ, and then for each $u\in Q$ such that $(\omega,0,0)\trianglelefteq u$, we define

$$P(u)=P(\bar{a}\restriction\alpha(u),\bar{b}\restriction\beta(u),N_u).$$

We simplify the notation by writing $P(u)=P(\bar{a}\restriction\alpha(u),\bar{b}\restriction\beta(u))$, omitting the explicit mention of N_u. No confusion should arise since $\alpha(u)$ and $\beta(u)$ uniquely determine u, and hence N_u. The indexing of the induction is such that for fixed $\delta<\omega_1$ such that $\gamma(\delta)=(\alpha,i)$, at stage $u=(\delta,n,j)$ of the induction, if $i=0$, then we pick $a_{\omega\delta+n}\in A_\alpha$ if $j=0$ and $b_{\omega\delta+n}\in B_\alpha$ if $j=1$, whereas if $i=1$, then we pick $b_{\omega\delta+n}\in B_\alpha$ if $j=0$ and $a_{\omega\delta+n}\in A_\alpha$ if $j=1$. In other words, the elements of $\bar{a}^\delta$ and $\bar{b}^\delta$ are chosen alternately from A_α and B_α, respectively, starting with an element of A_α when $i=0$ and with an element of B_α when $i=1$. For technical reasons, we also define a second sequence $\langle N'_u:(\omega,0,0)\trianglelefteq u\in Q\rangle$ of countable elementary submodels of H_θ and functions e_δ, $\omega\le\delta<\omega_1$.

We will arrange that the following conditions hold for all $u=(\delta,n,j)\in Q$, with $\gamma(\delta)=(\alpha,i)$.

(1) For $\delta<\omega$, the only requirements are that $\bar{a}^\delta=A_\delta$ and $\bar{b}^\delta=B_\delta$, where A_δ and B_δ are the countable dense sets from the hypothesis.
(2) For $\delta\ge\omega$, $N_u\in N'_u$ are countable elementary submodels of H_θ.
(3) f_0, $\langle c_i:i<\omega\rangle$, η, H, $\langle A_\alpha:\alpha<\omega_1\rangle$, $\langle B_\alpha:\alpha<\omega_1\rangle$ are all elements of $N_{(\omega,0,0)}$.
(4) If $\delta\ge\omega$ and $(n,j)=(0,0)$, then
  (i) $N_u=\bigcup\{N_v:(\omega,0,0)\trianglelefteq v\vartriangleleft u\}$;
  (ii) if $i=0$, then $a_{\omega\delta}$ is the least element of $A_\alpha\setminus\{a_\xi:\xi<\omega\delta\}$;
  (iii) if $i=1$, then $b_{\omega\delta}$ is the least element of $B_\alpha\setminus\{b_\xi:\xi<\omega\delta\}$.
  In (ii) and (iii), "least" refers to the well-ordering of $\mathbb{R}$ fixed earlier.
(5) If $\delta\ge\omega$ and $(n,j)\ne(0,0)$, then
  (i) e_δ, $\langle a_\xi:\xi<\alpha(u)\rangle$, $\langle b_\xi:\xi<\beta(u)\rangle$, $\langle M_{\delta'}:\delta'\le\delta\rangle$, and $\langle N_v:v\vartriangleleft u\rangle$ all belong to N_u.
  (ii) If $i=0$, $j=0$ or $i=1$, $j=1$, then $a_{\omega\delta+n}\in A_\alpha\cap I_n$ is a Cohen real over N'_u.
  (iii) If $i=0$, $j=1$ or $i=1$, $j=0$, then $b_{\omega\delta+n}\in B_\alpha\cap I_n$ is a Cohen real over N'_u.
  The point of using N'_u rather than N_u in (ii) and (iii) is that it will be useful later to have $P(u)$ belonging to the model over which the Cohen reals are chosen.
(6) If $\delta\ge\omega$, e_δ is a bijective map of $P(\bar{a}\restriction\omega\delta,\bar{b}\restriction\omega\delta)$ onto $\omega\delta$.
(7) For each δ' such that $\delta'<\delta$, $e_{\delta'}\subseteq e_\delta$.
(8) If $\delta\ge\omega$, the predense subsets of $P(\bar{a}\restriction\omega\delta,\bar{b}\restriction\omega\delta)$ which have the form $e_\delta^{-1}[S]$ for some $S\subseteq\omega\delta$ such that $S\in\bigcup_{\eta\le\delta}M_\eta$ are predense in $P(\bar{a}\restriction\omega(\delta+1),\bar{b}\restriction\omega(\delta+1))$.

Remark 5.4. (a) From (1) and (5)((ii)+(iii)), it follows that the sets $\bar{a}^\delta$ and $\bar{b}^\delta$ are dense in $\mathbb{R}$. From (4)((ii)+(iii)) and (5)((ii)+(iii)), we get $\bar{a}^\delta \subseteq A_\alpha$ and $\bar{b}^\delta \subseteq B_\alpha$. From the same clauses together with (5)(i), we get that the enumerations $\langle a_\xi : \xi < \omega_1 \rangle$ and $\langle b_\xi : \xi < \omega_1 \rangle$ are one-to-one. (Note that $\alpha(u) = \omega\delta + n$ in (5)(ii) and $\beta(u) = \omega\delta + n$ in (5)(iii).)

(b) From 4(ii) and the fact that $\gamma^{-1}(\alpha, 0)$ is uncountable, it follows that $A_\alpha \subseteq \{a_\xi : \xi < \omega_1\}$. Similarly, we get $B_\alpha \subseteq \{b_\xi : \xi < \omega_1\}$.

(c) It follows inductively, using (5)(i) at successor stages, that $\{a_\xi : \xi < \alpha(u)\} \subseteq N_u$ and $\{b_\xi : \xi < \beta(u)\} \subseteq N_u$. (At a limit stage $u = (\delta, 0, 0)$, we have $\alpha(u) = \beta(u) = \omega\delta$. If $\xi < \omega\delta$, then $\xi = \omega\delta' + n'$ for some $\delta' < \delta$ and $n' < \omega$. Then a_ξ and b_ξ are defined (in an order depending on $\gamma(\delta')$) at stages $(\delta', n', 0)$ and $(\delta', n', 1)$. By the induction hypothesis, $a_\xi, b_\xi \in N_{(\delta', n'+1, 0)} \subseteq N_u$.)

Hence, $h_p \in N_u$ for each $p \in P(u)$.

(d) From (4)(i) and (5)(i), it follows that the sequence $\langle N_u : (\omega, 0, 0) \trianglelefteq u \in Q \rangle$ is $\in$-increasing and continuous at limits. This gives in particular that for each limit ordinal $\delta > \omega$, $P(\bar{a} \restriction \omega\delta, \bar{b} \restriction \omega\delta) = \bigcup_{\delta' < \delta} P(\bar{a} \restriction \omega\delta', \bar{b} \restriction \omega\delta')$.

(e) Set $P = P(\bar{a} \restriction \omega_1, \bar{b} \restriction \omega_1, \bigcup_{u \in Q} N_u)$. In the third coordinate we could put the universe (more precisely, H_θ) since $\bigcup_{u \in Q} N_u$ includes all entire functions by (5)(i) and the assumption on the M_δ's. The conditions (6)–(8) ensure that P is $\overline{M}$-cc. To see this, let $e = \bigcup_{\omega \le \delta < \omega_1} e_\delta \colon P \to \omega_1$. For any infinite $\delta < \omega_1$ we have $e^{-1}[\omega\delta] = P(\bar{a} \restriction \omega\delta, \bar{b} \restriction \omega\delta)$ and for each $S \subseteq \omega\delta$ belonging to M_δ, whenever a set E of the form $e^{-1}[S] = e_\delta^{-1}[S]$ is predense in $P(\bar{a} \restriction \omega\delta, \bar{b} \restriction \omega\delta)$, a simple induction on δ' using (8) shows that if $\delta \le \delta' < \omega_1$, then E is predense in $P(\bar{a} \restriction \omega\delta', \bar{b} \restriction \omega\delta')$. Thus, E is predense in P. For a club of $\delta < \omega_1$ we have $\omega\delta = \delta$, so this shows that P satisfies the $\overline{M}$-cc.

We begin by arranging (1)–(7) by induction on $u = (\delta, n, j)$. This is straightforward and we leave most of it to the reader. Notice that because the reals a_ξ and b_ξ being constructed are not indexed directly by u, we need to check that the clauses (4)((ii)+(iii)), (5) and (6) make sense. For 5(ii) for example, it is important that at stage u, $a_{\omega\delta+n}$ has not yet been defined. But at an earlier stage $v = (\delta', n', j')$, we defined $a_{\omega\delta'+n'}$ or $b_{\omega\delta'+n'}$. If (δ', n') lexicographically precedes (δ, n), then $\omega\delta' + n' < \omega\delta + n$. If $(\delta', n') = (\delta, n)$, then necessarily $(j', j) = (0, 1)$. Since $j = 1$, the assumption of 5(ii) gives $i = 1$. So at stage v we defined $b_{\omega\delta+n}$, not $a_{\omega\delta+n}$. Similarly for 5(iii) and (4)((ii)+(iii)). Similarly, we can check that in (5)(i), a_ξ for $\xi < \alpha(u)$ and b_ξ for $\xi < \beta(u)$ were defined before stage u. For (6), we observe that the partial order $P(\bar{a} \restriction \omega\delta, \bar{b} \restriction \omega\delta)$ is defined because we have reached or passed the stage $(\delta, 0, 0)$ and $N_{(\delta,0,0)}$ has been defined. The function e_δ is chosen at stage $(\delta, 0, 0)$. By Remark 5.4(d), the choice of e_δ is dictated by (7) when $\delta > \omega$ is a limit ordinal. The function $e_{\delta+1}$ can be taken to be an arbitrary extension of e_δ satisfying (6).

We must check that the construction gives (8). Let E be a predense subset of $P(\bar{a} \restriction \omega\delta, \bar{b} \restriction \omega\delta)$ of the appropriate form, i.e., $E = e_\delta^{-1}[S]$ for some $S \subseteq \omega\delta$ such that $S \in \bigcup_{\eta \le \delta} M_\eta$. We will show by induction on $u = (\delta, n, j) \in Q$ such that $(\delta, 0, 0) \trianglelefteq u \triangleleft (\delta + 1, 0, 0)$ that E remains predense in $P(u + 1)$, where $u + 1$ denotes the successor of u in Q, i.e., $(\delta, n, 1)$ if $j = 0$ and $(\delta, n + 1, 0)$ if $j = 1$.

(This establishes (8) since each member of $P(\bar{a}\restriction\omega(\delta+1),\bar{b}\restriction\omega(\delta+1))$ belongs to $P(\bar{a}\restriction\omega\delta+n,\bar{b}\restriction\omega\delta+n)$ for some $n<\omega$.)

Remark 5.5. At the stage where $n=0$ and $j=0$, we consider the passage from $P(\bar{a}\restriction\omega\delta,\bar{b}\restriction\omega\delta)$ to either $P(\bar{a}\restriction\omega\delta+1,\bar{b}\restriction\omega\delta)$ or $P(\bar{a}\restriction\omega\delta,\bar{b}\restriction\omega\delta+1)$ (depending on whether $i=0$ or $i=1$). These two partial orders have the same allowable finite parts h_p for their conditions, because, by Definition 5.2(ii), there is no legal value for either of $a_{\omega\delta}$ or $b_{\omega\delta}$ to correspond to until the other is chosen.

Let $p\in P(u+1)\setminus P(u)$. We must show that p is compatible with some member of E.

Case 1. $h_p\in N_u$.

Proposition 4.5 gives a function $f'\in(f_0+H\operatorname{span}\mathcal{G}_0)\cap N_u$ such that

1(a) $h_p\subseteq f'$;

1(d) for all $i<\omega$ and all $x\in\mathbb{R}$ such that $|x|\geq c_i$, $|D^i f_p(x)-D^i f'(x)|<\frac{1}{2}\varepsilon_p\eta(x)$ (and hence, using Definition 5.2(v), $|D^i(f'-f_0)(x)|<(1-\frac{1}{2}\varepsilon_p)\eta(x)$);

1(f) for all $z\in\mathbb{C}$ such that $|z|\leq n_p$, $|f_p(z)-f'(z)|<\varepsilon_p/2$.

The letters in the labels here correspond to those in the statement of Proposition 4.5. The number 1 is a reference to Case 1. We will use similar notation in the rest of the proof when applying this proposition. (To get f', in Proposition 4.5, take $\varepsilon(x)=\eta(x)$, but get the conclusion for $\frac{1}{2}\varepsilon_p\eta$ instead of η. See Remark 4.6. Note that $\frac{1}{2}\varepsilon_p\eta\leq\frac{1}{2}\varepsilon_p$ because $\eta\leq 1$.) Then

$$q=(h_p,f',\varepsilon_p/2,n_p)$$

belongs to $P(u)$. Also, q and some $r\in E$ have a common extension $q'\in P(u)$. Then $q'\leq p$ since for each $z\in\mathbb{C}$ such that $|z|\leq n_p$, we have also $|z|\leq n_q$ (since $n_q=n_p$) and hence

$$\begin{aligned}|f_{q'}(z)-f_p(z)|+\varepsilon_{q'} &\leq |f_q(z)-f_p(z)|+|f_{q'}(z)-f_q(z)|+\varepsilon_{q'}\\ &= |f'(z)-f_p(z)|+|f_{q'}(z)-f_q(z)|+\varepsilon_{q'}\\ &\leq \varepsilon_p/2+|f_{q'}(z)-f_q(z)|+\varepsilon_{q'}\\ &\leq \varepsilon_p/2+\varepsilon_q=\varepsilon_p.\end{aligned}$$

Case 2. $h_p\notin N_u$.

By Remark 5.5, we have $(n,j)\neq(0,0)$. Hence (by (5)(i)), $e_\delta,\langle M_{\delta'}:\delta'\leq\delta\rangle\in N_u$ which gives $S\in N_u$ and hence $E=e_\delta^{-1}[S]\in N_u$.

Subcase 2a. $i=0$, $j=1$ or $i=1$, $j=0$.

In this subcase, h_p has the form $h\cup\{(a,b_{\omega\delta+n})\}$ for some $h\subseteq N_u$ and $a\in\{a_{\omega\delta+m}:m<n+1-i\}$.

Proposition 4.5 gives $n_1\in\mathbb{N}$, a function $f'\in(f_0+H\operatorname{span}\mathcal{G}_0)\cap N_u$ and a rational number $\lambda_0>0$ such that

2a(a) $h\subseteq f'$;

2a(d) $n_1\geq 4|\operatorname{dom}h|$ and for all $i<\omega$, for all $x\in\mathbb{R}$ such that $|x|\geq c_i$ and for all $\lambda\in\mathbb{R}$ such that $|\lambda|\leq\lambda_0$,

$$|D^i f_p(x)-D^i(f'+\lambda Hg(n_1,\operatorname{dom}h))(x)|<\tfrac{1}{2}\varepsilon_p\eta(x)$$

(and hence in particular $|D^i(f'+\lambda Hg(n_1,\operatorname{dom}h)-f_0)(x)|<(1-\frac{1}{2}\varepsilon_p)\eta(x)$);

2a(e) $g(n_1,\operatorname{dom}h)(a)\neq 0$ and for some number λ such that $|\lambda|<\lambda_0$, we have $h_p\subseteq f'+\lambda Hg(n_1,\operatorname{dom}h)$;

2a(f) for all $z \in \mathbb{C}$ such that $|z| \leq n_p$ we have $\lambda_0 |H(z) g(n_1, \operatorname{dom} h)(z)| < \varepsilon_p/4$ and for all λ such that $|\lambda| < \lambda_0$, $|f_p(z) - (f' + \lambda H g(n_1, \operatorname{dom} h))(z)| < \varepsilon_p/4$.

For functions $g_0 \colon \mathbb{C} \to \mathbb{C}$ such that $g_0[\mathbb{R}] \subseteq \mathbb{R}$, numbers $\mu > 0$, $m \in \mathbb{N}$ and finite sets $A \subseteq \mathbb{R}$, define

$$V(g_0, \mu, m, A) = \{(g_0 + \lambda H g(m, A))(a) : |\lambda| < \mu\}.$$

As long as $g(m, A)(a) \neq 0$, $V(g_0, \mu, m, A)$ is a nonempty open interval in $\mathbb{R}$. Consider the open interval $U = V(f', \lambda_0, n_1, \operatorname{dom} h)$. By 2a(e), $f_p(a) = b_{\omega\delta+n} \in U$. Define

$$q_0 = (h,\ f',\ \varepsilon_p/2,\ n_p)$$

and notice that $q_0 \in P(u)$.

Claim 5.6. The union of the open sets $V(f_q, \mu, m, \operatorname{dom} h_q)$ such that

(1) $q \in P(u)$ is a common extension of q_0 and an element of E,
(2) $g(m, \operatorname{dom} h_q)(a) \neq 0$,
(3) $\mu > 0$ is rational, $m \in \mathbb{N}$, $m \geq 4|\operatorname{dom} h_q|$,
(4) $V(f_q, \mu, m, \operatorname{dom} h_q) \subseteq U$,
(5) for all $z \in \mathbb{C}$ such that $|z| \leq n_q$, $\mu |H(z) g(m, \operatorname{dom} h_q)(z)| \leq \varepsilon_q/2$ and for all $i < \omega$, for all $x \in \mathbb{R}$ such that $|x| \geq c_i$, $\mu |D^i(H g(m, \operatorname{dom} h_q))(x)| < \frac{1}{2}\varepsilon_q \eta(x)$ (and hence for all λ such that $|\lambda| \leq \mu$ we have

$$|D^i(f_q + \lambda H g(m, \operatorname{dom} h_q) - f_0)(x)| < (1 - \tfrac{1}{2}\varepsilon_q)\eta(x))$$

is dense in U.

Proof of Claim 5.6. Fix $\lambda_1 \in \mathbb{Q}$ such that $|\lambda_1| < \lambda_0$. Define

$$w = \bar{f}'(a) \in U, \text{ where } \bar{f}' = f' + \lambda_1 H g(n_1, \operatorname{dom} h).$$

Note that the numbers w of this form, as λ_1 runs over all rational numbers such that $|\lambda_1| < \lambda_0$, are dense in U. Fix $\delta > 0$ such that $(w - \delta, w + \delta) \subseteq U$. Let $x_1, x_2 \in A_0 \setminus \operatorname{dom} h$ satisfy $x_1 < a < x_2$ and $\bar{f}'(x_1), \bar{f}'(x_2) \in (w - \delta, w + \delta)$. Apply Proposition 4.5 to get a function $f'' \in (f_0 + H \operatorname{span} \mathcal{G}_0) \cap N_u$ and a rational number $\lambda_2 > 0$ such that

5.6(a) $h \subseteq f''$,
5.6(b) $y_1 = f''(x_1)$ and $y_2 = f''(x_2)$ are both members of B_0,
5.6(d)$_1$ $y_1, y_2 \in (w - \delta, w + \delta)$.
(This holds as long as the quantities $|(\bar{f}' - f'')(x_j)|$, $j = 1, 2$, are small enough.)
5.6(d)$_2$ For all $i < \omega$ and for all $x \in \mathbb{R}$ such that $|x| \geq c_i$,

$$|D^i \bar{f}'(x) - D^i f''(x)| < \tfrac{1}{4}\varepsilon_p \eta(x)$$

(and hence in particular $|D^i(f'' - f_0)(x)| < (1 - \frac{1}{4}\varepsilon_p)\eta(x)$).
5.6(f) For all $z \in \mathbb{C}$ such that $|z| \leq n_p$, $|\bar{f}'(z) - f''(z)| < \varepsilon_p/8$.

Note that $f'' \restriction \mathbb{R}$ is increasing (by 5.6(d)$_2$, for all $x \in \mathbb{R}$ $|Df''(x) - Df_0(x)| < \eta(x) \leq Df_0(x)$ and hence $Df''(x) > 0$) and hence we have $y_1 < y_2$. Then $q_1 \in P(u)$, where

$$q_1 = (h \cup \{(x_1, y_1), (x_2, y_2)\},\ f'',\ \varepsilon_p/8,\ n_p).$$

Also, q_1 extends q_0 because if $|z| \le n_p$, then

$$\begin{aligned}|f_{q_1}(z) - f_{q_0}(z)| + \varepsilon_{q_1} &= |f''(z) - f'(z)| + \varepsilon_p/8 \\ &\le |f''(z) - \bar{f}'(z)| + |\bar{f}'(z) - f'(z)| + \varepsilon_p/8 \\ &\le |\bar{f}'(z) - f'(z)| + \varepsilon_p/4 \le \varepsilon_p/2 = \varepsilon_{q_0}.\end{aligned}$$

By the induction hypothesis, there is a common extension $q \in P(u)$ of q_1 and some $r \in E$. Since a does not belong to the domains of h_{q_1} or h_r ($r \in E \subseteq P(\delta, 0, 0)$ and hence $\operatorname{dom} h_r \subseteq \{a_\xi : \xi < \omega\delta\}$), we may discard it from the domain of h_q if necessary to get $a \notin \operatorname{dom} h_q$. Then choose $m \ge 4|\operatorname{dom} h_q|$ large enough so that part 2 of the claim holds. For $\mu > 0$ small enough we have that part 5 of the claim holds.
[For the first assertion of part 5 this is clear. For the second, proceed as follows. By Proposition 4.4(c), whenever $|x| \ge c_i$ we have $|D^i(Hg(m, \operatorname{dom} h_q))(x)| \le (3/4)^i \zeta(x)\eta(x)$. For all large enough i, say $i > i_0$, we have $(3/4)^i \le \frac{1}{2}\varepsilon_q$ and hence, for $|x| \ge c_i$, $|D^i(Hg(m, \operatorname{dom} h_q))(x)| \le (3/4)^i \zeta(x)\eta(x) \le \frac{1}{2}\varepsilon_q \eta(x)$. Moreover, for all x outside some compact interval L we have $\zeta(x) \le \frac{1}{2}\varepsilon_q$ and hence, for $|x| \ge c_i$, $|D^i(Hg(m, \operatorname{dom} h_q))(x)| \le (3/4)^i \zeta(x)\eta(x) \le \frac{1}{2}\varepsilon_q \eta(x)$. For μ small enough we will also have $\mu|D^i(Hg(m, \operatorname{dom} h_q))(x)| < \frac{1}{2}\varepsilon_q \eta(x)$ for each $i \le i_0$ and each $x \in L$, giving part 5 of the claim.]

Then the functions in the definition of $V(f_q, \mu, m, \operatorname{dom} h_q)$ are increasing on $\mathbb{R}$. Since $\{(x_1, y_1), (x_2, y_2)\} \subseteq h_q$, we have $V(f_q, \mu, m, \operatorname{dom} h_q) \subseteq (y_1, y_2) \subseteq (w - \delta, w + \delta) \subseteq U$ and, in particular, part 4 of the claim holds.

This proves Claim 5.6. □

The dense open subset of U given by Claim 5.6 is coded in N'_u. (As noted earlier, $P(u) \in N'_u$ when u is not a limit stage.) By (5)(iii), there are q, μ, m satisfying Claim 5.6(1–5) for which $b_{\omega\delta+n} \in V(f_q, \mu, m, \operatorname{dom} h_q)$. Choosing λ with $|\lambda| < \mu$ so that

$$(f_q + \lambda Hg(m, \operatorname{dom} h_q))(a) = b_{\omega\delta+n}$$

(note that this equation uniquely determines λ and hence $\lambda \in N_{u+1}$), we get that

$$q' = (h_q \cup \{(a, b_{\omega\delta+n})\},\ f_q + \lambda Hg(m, \operatorname{dom} h_q),\ \varepsilon_q/2,\ n_q)$$

belongs to $P(u+1)$ (using the second part of clause 5 of Claim 5.6) and extends both q and p. It extends q by the first part of clause 5 of Claim 5.6. To see that $q' \le p$, note that for each $z \in \mathbb{C}$ such that $|z| \le n_p$,

$$\begin{aligned}|f_{q'}(z) - f_p(z)| + \varepsilon_{q'} &\le |f_q(z) - f_p(z)| + |f_{q'}(z) - f_q(z)| + \varepsilon_{q'} \\ &\le |f_q(z) - f_p(z)| + \varepsilon_q \\ &\le |f_{q_0}(z) - f_p(z)| + |f_q(z) - f_{q_0}(z)| + \varepsilon_q \\ &\le |f_{q_0}(z) - f_p(z)| + \varepsilon_{q_0} \\ &\le \varepsilon_p/4 + \varepsilon_p/2 < \varepsilon_p.\end{aligned}$$

Thus, p is compatible with q and hence with some element of E.
Subcase 2b. $i = 0$, $j = 0$ or $i = 1$, $j = 1$. This subcase is similar to the previous one. In the setting of [Sh1980], the corresponding two subcases are symmetric. In our context they are not, however, because the entire functions f_p are not invertible.

In this subcase, h_p has the form $h \cup \{(a_{\omega\delta+n}, b)\}$ for some $h \subseteq N_u$ and $b \in \{b_{\omega\delta+m} : m < n + i\}$.

Proposition 4.5 gives $n_1 \in \mathbb{N}$, a function $f' \in (f_0 + H \operatorname{span} \mathcal{G}_0) \cap N_u$ and a rational number $\lambda_0 > 0$ such that

2b(a) $h \subseteq f'$;

2b(d) $n_1 \geq 4|\operatorname{dom} h|$ and for all $i < \omega$, for all $x \in \mathbb{R}$ such that $|x| \geq c_i$ and for all $\lambda \in \mathbb{R}$ such that $|\lambda| \leq \lambda_0$,
$$|D^i f_p(x) - D^i(f' + \lambda Hg(n_1, \operatorname{dom} h))(x)| < \tfrac{1}{2}\varepsilon_p \eta(x)$$
(and hence in particular $|D^i(f' + \lambda Hg(n_1, \operatorname{dom} h) - f_0)(x)| < (1 - \frac{1}{2}\varepsilon_p)\eta(x)$);

2b(e) for some number λ such that $|\lambda| < \lambda_0$, we have $h_p \subseteq f' + \lambda Hg(n_1, \operatorname{dom} h)$;

2b(f) for all $z \in \mathbb{C}$ such that $|z| \leq n_p$ we have $\lambda_0 |H(z)g(n_1, \operatorname{dom} h)(z)| < \varepsilon_p/4$ and for all λ such that $|\lambda| < \lambda_0$, $|f_p(z) - (f' + \lambda Hg(n_1, \operatorname{dom} h))(z)| < \varepsilon_p/4$.

(This list agrees with the one for Subcase 2a except for (e).)

Remark 5.7. Note that 2b(d) ensures that for $|\lambda| \leq \lambda_0$, the restriction to $\mathbb{R}$ of $f' + \lambda Hg(n_1, \operatorname{dom} h)$ is an order-isomorphism. Indeed, as in the argument after 5.6(f) above, $f' + \lambda Hg(n_1, \operatorname{dom} h)$ is increasing. The fact that for $|x| \geq c_0$ we have $|(f' + \lambda Hg(n_1, \operatorname{dom} h))(x) - f_0(x)| < \eta(x) \leq 1$ ensures that $f' + \lambda Hg(n_1, \operatorname{dom} h)$ is onto.

For functions $g_0 \colon \mathbb{C} \to \mathbb{C}$ such that $g_0[\mathbb{R}] \subseteq \mathbb{R}$, numbers $\mu > 0$, $m \in \mathbb{N}$ and finite sets $A \subseteq \mathbb{R}$, define
$$W(g_0, \mu, m, A) = \{(g_0 + \lambda Hg(m, A))^{-1}(b) : |\lambda| < \mu\}.$$
The definition makes sense if the restriction to $\mathbb{R}$ of $g_0 + \lambda Hg(m, A)$ is an order-isomorphism whenever $|\lambda| < \mu$. (In the definition, $(g_0 + \lambda Hg(m, A))^{-1}$ means the inverse of this restriction.) By Proposition 4.1, $W(g_0, \mu, m, A)$ is an open interval in $\mathbb{R}$ as long as there is no $a \in \mathbb{R}$ such that $g_0(a) = b$ and $g(m, A)(a) = 0$. When $g_0 \restriction \mathbb{R}$ is invertible, then only the value $a = g_0^{-1}(b)$ is relevant. Consider the open interval $U = W(f', \lambda_0, n_1, \operatorname{dom} h)$. By (e), $a_{\omega\delta+n} = f_p^{-1}(b) \in U$. Define
$$q_0 = (h,\ f',\ \varepsilon_p/2,\ n_p)$$
and notice that $q_0 \in P(u)$.

Claim 5.8. The union of the open sets $W(f_q, \mu, m, \operatorname{dom} h_q)$ such that

(1) $q \in P(u)$ is a common extension of q_0 and an element of E;
(2) $g(m, \operatorname{dom} h_q)(a) \neq 0$ where $a = f_q^{-1}(b)$;
(3) $\mu > 0$ is rational, $m \in \mathbb{N}$, $m \geq 4|\operatorname{dom} h_q|$;
(4) $W(f_q, \mu, m, \operatorname{dom} h_q) \subseteq U$;
(5) for all $z \in \mathbb{C}$ such that $|z| \leq n_q$, $\mu |H(z)g(m, \operatorname{dom} h_q)(z)| \leq \varepsilon_q/2$ and for all $i < \omega$, for all $x \in \mathbb{R}$ such that $|x| \geq c_i$, $\mu |D^i(Hg(m, \operatorname{dom} h_q))(x)| < \frac{1}{2}\varepsilon_q \eta(x)$ (and hence for all λ such that $|\lambda| \leq \mu$,
$$|D^i(f_q + \lambda Hg(m, \operatorname{dom} h_q) - f_0)(x)| < (1 - \tfrac{1}{2}\varepsilon_q)\eta(x))$$

is dense in U.

Proof of Claim 5.8. Fix $\lambda_1 \in \mathbb{Q}$ such that $|\lambda_1| < \lambda_0$. Define
$$w = (\bar{f}')^{-1}(b) \in U, \text{ where } \bar{f}' = f' + \lambda_1 Hg(n_1, \operatorname{dom} h).$$
By Remark 5.7, $\bar{f}'$ is an order-isomorphism. Note that $w \notin \operatorname{dom} h$ since $b \notin \operatorname{range} h$ and $h \subseteq \bar{f}'$. Note that the numbers w of the given form, as λ_1 runs over all rational numbers such that $|\lambda_1| < \lambda_0$, are dense in U. Fix $\delta > 0$ such that $(w - \delta, w + \delta) \subseteq$

U and $(w-\delta, w+\delta) \cap \operatorname{dom} h = \emptyset$. Let $y_1, y_2 \in B_0$ satisfy $y_1 < b < y_2$ and $(\bar{f}')^{-1}(y_1), (\bar{f}')^{-1}(y_2) \in (w-\delta, w+\delta)$. Apply Proposition 4.5 to get $n_2 \in \mathbb{N}$, a function $f'' \in (f_0 + H \operatorname{span} \mathcal{G}_0) \cap N_u$ and a rational number $\lambda_2 > 0$ such that

5.8(a) $h \subseteq f''$;

5.8(c) $y_1 = f''(x_1)$ and $y_2 = f''(x_2)$, where x_1, x_2 are both members of A_0 and $x_1, x_2 \in (w-\delta, w+\delta)$;

5.8(d) for all $i < \omega$ and for all $x \in \mathbb{R}$ such that $|x| \geq c_i$,

$$|D^i \bar{f}'(x) - D^i f''(x)| < \tfrac{1}{4}\varepsilon_p \eta(x)$$

(and hence in particular $|D^i(f'' - f_0)(x)| < (1 - \frac{1}{4}\varepsilon_p)\eta(x)$);

5.8(f) for all $z \in \mathbb{C}$ such that $|z| \leq n_p$, $|\bar{f}'(z) - f''(z)| < \varepsilon_p/8$.

By 5.8(d), $f'' \restriction \mathbb{R}$ is increasing and hence $x_1 < x_2$. Then $q_1 \in P(u)$, where

$$q_1 = (h \cup \{(x_1, y_1), (x_2, y_2)\},\ f'',\ \varepsilon_p/8,\ n_p).$$

Exactly as in the proof of Claim 5.6, q_1 extends q_0. By the induction hypothesis, there is a common extension $q \in P(u)$ of q_1 and some $r \in E$. The number $a = f_q^{-1}(b)$ does not belong to the domains of h_{q_1} or h_r.
[From the fact that $f_q \restriction \mathbb{R}$ is increasing and hence injective, we see that because $f_q(x_i) = y_i$, $i = 1, 2$, and $f_q(a) = b$, we have $a \neq x_i$, $i = 1, 2$. Because $h \subseteq f_q$ and $b \notin \operatorname{range} h$, we have $a \notin \operatorname{dom} h$. Thus, $a \notin \operatorname{dom} h_{q_1}$. Since $h_r \subseteq h_q \subseteq f_q$, if a were in $\operatorname{dom} h_r$, then $h_r(a) = b$, which is not possible because $r \in E \subseteq P(\delta, 0, 0)$ and hence $\operatorname{range} h_r \subseteq \{b_\xi : \xi < \omega\delta\}$.]

We may thus discard it from the domain of h_q if necessary to get $a \notin \operatorname{dom} h_q$. Choose $m \geq 4|\operatorname{dom} h_q|$ large enough so that part 2 of the claim holds. As in the proof of Claim 5.6, for $\mu > 0$ small enough we have that part 5 of the claim holds. Then the functions in the definition of $W(f_q, \mu, m, \operatorname{dom} h_q)$ are order-isomorphisms. Since $\{(x_1, y_1), (x_2, y_2)\} \subseteq h_q$, we have $W(f_q, \mu, m, \operatorname{dom} h_q) \subseteq (x_1, x_2) \subseteq (w-\delta, w+\delta) \subseteq U$ and in particular part 4 of the claim holds.

This proves Claim 5.8. □

The dense open subset of U given by Claim 5.8 is coded in N'_u. By (5)(ii), there are q, μ, m satisfying Claim 5.8(1–5) for which $a_{\omega\delta+n} \in W(f_q, \mu, m, \operatorname{dom} h_q)$. Choosing λ with $|\lambda| < \mu$ so that $(f_q + \lambda H g(m, \operatorname{dom} h_q))(a_{\omega\delta+n}) = b$, we get, as in Subcase 2a, that

$$q' = (h_q \cup \{(a_{\omega\delta+n}, b)\},\ f_q + \lambda H g(m, \operatorname{dom} h_q),\ \varepsilon_q/2,\ n_q)$$

belongs to $P(u+1)$ and extends both q and p. Thus, p is compatible with q and hence with some element of E. This completes the proof of (8).

We now have an $\overline{M}$-cc partial order $P = P(\bar{a} \restriction \omega_1, \bar{b} \restriction \omega_1)$ as in Remark 5.4(e). It remains to check that forcing with P adds the desired entire function f. Let $h = \bigcup\{h_p : p \in G\}$. For each $\alpha < \omega_1$ and each $x \in A_\alpha$, it follows using Proposition 4.5(b) that conditions with $f_p(x) \in B_\alpha$ are dense and then (by extending such conditions further) so are conditions with $x \in \operatorname{dom} h_p$. Similarly, for each $\alpha < \omega_1$ and each $y \in B_\alpha$, the conditions with $y \in \operatorname{range} h_p$ are dense (using Proposition 4.5(c) this time). Hence $\operatorname{dom} h = \bigcup_{\alpha<\omega_1} A_\alpha$, $\operatorname{range} h = \bigcup_{\alpha<\omega_1} B_\alpha$ and h is clearly an order-isomorphism. For $k \in \mathbb{N}$, choose $p_k \in G$ such that $p_{k+1} \leq p_k$, $n_{p_k} \geq k$ and $\varepsilon_{p_k} < 1/k$. The sequence $\{f_{p_k}\}$ is uniformly Cauchy on compact sets because for $\ell > k$ and $|z| \leq k(\leq n_{p_k})$, $|f_{p_\ell}(z) - f_{p_k}(z)| \leq \varepsilon_{p_k} < 1/k$. Define $f\colon \mathbb{C} \to \mathbb{C}$ by $f(z) = \lim_{k\to\infty} f_{p_k}(z)$. For all $i < \omega$, we have $D^i f(z) = \lim_{k\to\infty} D^i f_{p_k}(z)$

uniformly on compact sets [Ru, Theorem 10.28]. Also, for each $a \in \bigcup_{\alpha<\omega_1} A_\alpha$, we can choose k such that $|a| \leq n_{p_k}$, and $p \in G$ such that $p \leq p_k$ and $a \in \operatorname{dom} h_p$. Then

$$|f_{p_k}(a) - h(a)| = |f_{p_k}(a) - h_p(a)| = |f_{p_k}(a) - f_p(a)| \leq \varepsilon_{p_k} < 1/k,$$

and hence $f(a) = \lim_{k\to\infty} f_{p_k}(a) = h(a)$. Also, for each $i < \omega$ and each $x \in \mathbb{R}$ such that $|x| \geq c_i$,

$$|D^i f(x) - D^i f_0(x)| = \lim_{k\to\infty} |D^i f_{p_k}(x) - D^i f_0(x)| \leq \eta(x).$$

This completes the proof of the lemma. □

6. Proof of Theorem 1.7

By Proposition 3.6, it suffices to prove the theorem in the case where the function g in part (c) is the restriction to $\mathbb{R}$ of an entire function and has a strictly positive derivative. Note that in this case, (c)(i) can be omitted as it is covered by (c)(ii). The rest of the proof is standard oracle-cc technique. We sketch the argument. Our sketch is an adaptation to the present context of the corresponding argument in [BM]. Start with a ground model of $V = L$. Fix a diamond sequence

$$\langle (x_\alpha, a_\alpha, b_\alpha, f_\alpha, c_\alpha, e_\alpha) : \alpha < \omega_2,\ \operatorname{cof}(\alpha) = \omega_1 \rangle$$

for trapping sextuples (x, a, b, f, c, e) consisting of:

(1) A function $x\colon \omega_2 \to ([\omega_2]^{\leq\omega})^\omega$. The idea of x is that, with ω_2 identified with the ccc partial order we are about to build, $[\omega_2]^{\leq\omega}$ contains the antichains. Thus, $([\omega_2]^{\leq\omega})^\omega$ contains a name for each real number (construed as a subset of ω). Then for any second category set X in the extension, we can find a ground model function $x\colon \omega_2 \to ([\omega_2]^{\leq\omega})^\omega$ enumerating the names of the elements of X.

(2) Functions $a, b\colon \omega_1 \times \omega_1 \to ([\omega_2]^{\leq\omega})^\omega$ representing (enumerations of the names for the elements of) an ω_1-sequence of sets in which the first ω, represented by $\{a(i,n) : n < \omega\}$ and $\{b(i,n) : n < \omega\}$, $i < \omega$, are countable dense sets and the remainder, represented by $\{a(\alpha,\xi) : \xi < \omega_1\}$ and $\{b(\alpha,\xi) : \xi < \omega_1\}$, $\omega \leq \alpha < \omega_1$, are everywhere second category sets of cardinality ω_1.

(3) A function $f \in ([\omega_2]^{\leq\omega})^\omega$ representing a name for the Borel code of an entire function restricting to an order-isomorphism of $\mathbb{R}$ having a strictly positive derivative.

(4) A function $c\colon \omega \to ([\omega_2]^{\leq\omega})^\omega$ so that $\langle c(i) : i < \omega\rangle$ represents a sequence of names for the terms of a nondecreasing sequence of nonnegative real numbers converging to ∞.

(5) A function $e \in ([\omega_2]^{\leq\omega})^\omega$ intended to represent a name for the Borel code of a positive continuous function.

So for each $\alpha < \omega_2$ of cofinality ω_1, we have $x_\alpha\colon \alpha \to ([\alpha]^{\leq\omega})^\omega$, $a_\alpha, b_\alpha\colon \omega_1 \times \omega_1 \to ([\alpha]^{\leq\omega})^\omega$, $f_\alpha, e_\alpha \in ([\alpha]^{\leq\omega})^\omega$ and $c_\alpha\colon \omega \to ([\alpha]^{\leq\omega})^\omega$. Also, for each (x, a, b, f, c, e) as in (1)–(5), $\{\alpha < \omega_2 : \operatorname{cof}(\alpha) = \omega_1,\ x \restriction \alpha = x_\alpha,\ a = a_\alpha,\ b = b_\alpha,\ f = f_\alpha,\ c = c_\alpha$ and $e = e_\alpha\}$ is stationary in ω_2.

We will inductively define an ω_2-stage finite support iteration

$$\langle \langle P_\alpha \rangle_{\alpha\leq\omega_2}, \langle \dot{Q}_\alpha \rangle_{\alpha<\omega_1} \rangle$$

as well as P_α-names $\overline{M}_\alpha$ for oracles and one-to-one functions $F_\alpha\colon P_\alpha \to \omega_2$ for $\alpha < \omega_2$ such that the range of each F_α is an initial segment of ω_2 which includes α and for $\beta < \alpha < \omega_2$, we have $F_\beta \subseteq F_\alpha$. (At each stage, F_α is any function satisfying these conditions.)

For $\alpha < \omega_2$, we make the following definitions after P_α and F_α are defined.

(6) $\dot{X}_\alpha$ denotes the P_α-name for the set of real numbers whose elements have the names
$$\bigcup_{n<\omega}\{n\} \times F_\alpha^{-1}[x_\alpha(\xi)(n)], \quad \xi < \alpha.$$

(7) For each $i < \omega$, $\dot{A}_{\alpha i}$ and $\dot{B}_{\alpha i}$ denote the ω-sequences of P_α-names for real numbers
$$\left\langle \bigcup_{n<\omega}\{n\} \times F_\alpha^{-1}[a_\alpha(i,j)(n)] : j < \omega \right\rangle$$
and
$$\left\langle \bigcup_{n<\omega}\{n\} \times F_\alpha^{-1}[b_\alpha(i,j)(n)] : j < \omega \right\rangle,$$
respectively, and for each η such that $\omega \le \eta < \omega_1$, $\dot{A}_{\alpha\eta}$ and $\dot{B}_{\alpha\eta}$ denote the ω_1-sequences of P_α-names for real numbers
$$\left\langle \bigcup_{n<\omega}\{n\} \times F_\alpha^{-1}[a_\alpha(\eta,\xi)(n)] : \xi < \omega_1 \right\rangle$$
and
$$\left\langle \bigcup_{n<\omega}\{n\} \times F_\alpha^{-1}[b_\alpha(\eta,\xi)(n)] : \xi < \omega_1 \right\rangle,$$
respectively.

(8) $\dot{c}_\alpha$ denotes the ω-sequence of P_α-names for real numbers
$$\langle \bigcup_{n<\omega}\{n\} \times F_\alpha^{-1}[c_\alpha(i)(n)] : i < \omega \rangle.$$

(9) $\dot{f}_\alpha$ and $\dot{e}_\alpha$ denote respectively the P_α-names for real numbers given by
$$\bigcup_{n<\omega}\{n\} \times F_\alpha^{-1}[f_\alpha(n)] \quad \text{and} \quad \bigcup_{n<\omega}\{n\} \times F_\alpha^{-1}[e_\alpha(n)].$$

At stage $\alpha < \omega_2$ of the construction, if $\operatorname{cof}(\alpha) = \omega_1$ and if
$$\Vdash_{P_\alpha} \dot{X}_\alpha \text{ is second category,}$$
then we use Lemma 1.14 to get a P_α-name $\overline{M}'_\alpha$ for an oracle so that if P is any forcing notion which satisfies the $\overline{M}'_\alpha$-cc, then X_α remains second category after forcing with P. Otherwise, in particular if $\operatorname{cof}(\alpha) \neq \omega_1$, we let $\overline{M}'_\alpha$ be any P_α-name for an oracle.

For $\beta < \alpha$, let $P_{\beta\alpha}$ be the usual P_β-name for a partial order such that P_α is isomorphic to a dense subset of $P_\beta * P_{\beta\alpha}$. Let $\overline{M}_{\beta\alpha}$ be a P_α-name for an oracle such that
$$\begin{gathered}\text{If } \Vdash_{P_\beta} \text{“}P_{\beta,\alpha} \text{ is } \overline{M}_\beta\text{-cc and } \Vdash_{P_{\beta,\alpha}} \dot{Q}_\alpha \text{ is } \overline{M}_{\beta\alpha}\text{-cc”},\\ \text{then } \Vdash_{P_\beta} \text{“}P_{\beta,\alpha+1} = P_{\beta,\alpha} * \dot{Q}_\alpha \text{ is } \overline{M}_\beta\text{-cc”}.\end{gathered} \tag{10}$$

Let $\overline{M}_\alpha$ be a P_α-name for an oracle such that
$$\Vdash_{P_\alpha} \text{“If } \dot{Q}_\alpha \text{ is } \overline{M}_\alpha\text{-cc, then } \dot{Q}_\alpha \text{ is } \overline{M}'_\alpha\text{-cc and } \overline{M}_{\beta\alpha}\text{-cc for all } \beta < \alpha\text{”}. \tag{11}$$

Now, if $\operatorname{cof}(\alpha) = \omega_1$ and if

(12) $\Vdash_{P_\alpha}$ for $i < \omega$, the ranges of $\dot{A}_{\alpha i}, \dot{B}_{\alpha i}$ are dense in $\mathbb{R}$,

(13) $\Vdash_{P_\alpha}$ for η such that $\omega \le \eta < \omega_1$, the ranges of $\dot{A}_{\alpha\eta}, \dot{B}_{\alpha\eta}$ are everywhere second category,

(14) $\Vdash_{P_\alpha} \dot{f}_\alpha$ is an entire function restricting to an order-isomorphism of $\mathbb{R}$ with a strictly positive derivative,
(15) $\Vdash_{P_\alpha} \dot{c}_\alpha(0) \leq \dot{c}_\alpha(1) \leq \dots$ and $\lim_{i\to\infty} \dot{c}_\alpha(i) = \infty$, and
(16) $\Vdash_{P_\alpha} \dot{e}_\alpha$ is a positive continuous function,

then use Lemma 5.1 to get a P_α-name $\dot{Q}_\alpha$ for a partial order satisfying the $\overline{M}_\alpha$-cc and forcing an entire function inducing an order-isomorphism between the $A_{\alpha\eta}$ and $B_{\alpha\eta}$, $\eta < \omega_1$, as described in the statement of the lemma. In all other cases, take $\dot{Q}_\alpha$ to name the partial order Q for adding one Cohen real. We have thus

$$\Vdash_{P_\alpha} \text{ “}\dot{Q}_\alpha \text{ satisfies the } \overline{M}_\alpha\text{-cc”.} \tag{17}$$

Now suppose that for some P_{ω_2}-name $\dot{X}$ we have $\Vdash_{P_{\omega_2}} \dot{X}$ is second category. Fix a name $\dot{x}$ such that $\Vdash_{P_{\omega_2}} \dot{x}\colon \omega_2 \to \dot{X}$ is onto. Then define $x\colon \omega_2 \to ([\omega_2]^{\leq\omega})^\omega$ so that if

$$\tau_\xi = \textstyle\bigcup_{n<\omega}\{n\} \times F^{-1}[x(\xi)(n)],\ \xi < \omega_2,$$

then for each $\xi < \omega_2$, $\Vdash_{P_{\omega_2}} \dot{x}(\xi) = \tau_\xi$. There is a closed unbounded set $C \subseteq \omega_2$ such that for each $\alpha \in C$ of cofinality ω_1 we have:

(18) $x \restriction \alpha\colon \alpha \to ([\alpha]^{\leq\omega})^\omega$,
(19) $\forall\xi < \alpha$, τ_ξ is a P_α-name,
(20) $\Vdash_{P_\alpha} \{\tau_\xi : \xi < \alpha\}$ is second category.

Choose such an α of cofinality ω_1 for which $x \restriction \alpha = x_\alpha$. By (19), the definition of τ_ξ for $\xi < \alpha$ would not change if we used x_α instead of x and F_α instead of F. Then from the definition of $\dot{X}_\alpha$ we get

$$\Vdash_{P_\alpha} \dot{X}_\alpha = \{\tau_\xi : \xi < \alpha\}.$$

So at stage α we chose a P_α-name $\overline{M}_\alpha$ and we arranged that $\Vdash_{P_\alpha}$ “$P_{\alpha,\gamma}$ is $\overline{M}_\alpha$-cc”. (This follow by induction on $\gamma \geq \alpha$ using Proposition 1.15(1) at limits, and using (17), (11) and (10) above at successors.) Hence, by the choice of $\overline{M}_\alpha$, $\Vdash_{P_\alpha}\Vdash_{P_{\alpha,\gamma}}$ “$\dot{X}_\alpha$ is second category” from which it follows that $\Vdash_{P_\alpha}\Vdash_{P_{\alpha,\omega_2}}$ “$\dot{X}_\alpha$ is second category”.

By what we have established, there are guaranteed to be sets of cardinality ω_1 which are second category in any extension by P_{ω_2}. Hence there are guaranteed to be everywhere second category sets of cardinality ω_1. Suppose that for some P_{ω_2}-names $\dot{A}_\eta, \dot{B}_\eta$ for $\eta < \omega_1$, $\dot{c}_i$ for $i < \omega$, and $\dot{f}, \dot{e}$ we have

(21) $\Vdash_{P_{\omega_2}}$ for $i < \omega$, the ranges of $\dot{A}_i, \dot{B}_i\colon \omega \to \mathbb{R}$ are dense in $\mathbb{R}$,
(22) $\Vdash_{P_{\omega_2}}$ for η such that $\omega \leq \eta < \omega_1$, the ranges of $\dot{A}_\eta, \dot{B}_\eta\colon \omega_1 \to \mathbb{R}$ are everywhere second category in $\mathbb{R}$,
(23) $\Vdash_{P_{\omega_2}} \dot{f}$ is an entire function restricting to an order-isomorphism of $\mathbb{R}$ with a strictly positive derivative,
(24) $\Vdash_{P_{\omega_2}} \dot{c}_0 \leq \dot{c}_1 \leq \dots$ and $\lim_{i\to\infty} \dot{c}_i = \infty$, and
(25) $\Vdash_{P_{\omega_2}} \dot{e}$ is a positive continuous function.

Define a, b, f, c, e to be functions as in (2)–(5) above so that letting

(26) $\sigma_{\eta\xi} = \bigcup_{n<\omega}\{n\} \times F^{-1}[a(\eta,\xi)(n)]$ and $\tau_{\eta\xi} = \bigcup_{n<\omega}\{n\} \times F^{-1}[b(\eta,\xi)(n)]$ for $\eta, \xi < \omega$ and for $\omega \leq \eta < \omega_1$, $\xi < \omega_1$, we have that for each such pair η, ξ, $\Vdash_{P_{\omega_2}} \dot{A}_\eta(\xi) = \sigma_{\eta\xi}$ and $\Vdash_{P_{\omega_2}} \dot{B}_\eta(\xi) = \tau_{\eta\xi}$;
(27) $\gamma_i = \bigcup_{n<\omega}\{n\} \times F^{-1}[c(i)(n)]$, $i < \omega$, we have for each $i < \omega$, $\Vdash_{P_{\omega_2}} \dot{c}(i) = \gamma_i$;

(28) $\varphi = \bigcup_{n<\omega}\{n\} \times F^{-1}[f(n)]$ and $\varepsilon = \bigcup_{n<\omega}\{n\} \times F^{-1}[e(n)]$, we have $\Vdash_{P_{\omega_2}} \dot{f} = \varphi$ and $\Vdash_{P_{\omega_2}} \dot{e} = \varepsilon$.

For all large enough $\alpha < \omega_2$, we have

(29) $a, b\colon \omega_1 \times \omega_1 \to ([\alpha]^{\leq\omega})^\omega$, $f\colon \omega \to [\alpha]^{\leq\omega}$, $c\colon \omega \to ([\alpha]^{\leq\omega})^\omega$, and $e\colon \omega \to [\alpha]^{\leq\omega}$;

(30) for all values of the indices for which they are defined, $\sigma_{\eta\xi}$, $\tau_{\eta\xi}$, γ_i, φ and ε are P_α-names.

Choose any such α of cofinality ω_1 for which $(a, b, f, c, e) = (a_\alpha, b_\alpha, f_\alpha, c_\alpha, e_\alpha)$. By (30), the definitions of $\sigma_{\eta\xi}$, $\tau_{\eta\xi}$, γ_i, φ and ε would not change if we used $a_\alpha, b_\alpha, f_\alpha, c_\alpha, e_\alpha$ instead of a, b, f, c, e, respectively, and F_α instead of F. Then from the definitions of $\dot{A}_{\alpha i}, \dot{B}_{\alpha i}$ for $i < \omega$, $\dot{A}_{\alpha\eta}, \dot{B}_{\alpha\eta}$ for $\omega \leq \eta < \omega_1$, $\dot{f}_\alpha$, $\dot{c}_\alpha$, and $\dot{e}_\alpha$, we get that (12)–(16) hold. (Being everywhere second category is trivially downward absolute.) Then $\dot{Q}_\alpha$ was chosen to add an order isomorphism of the desired type and its properties are clearly upward absolute.

This completes the proof of the theorem.

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