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THE COX RING OF $\overline{M}_{0,6}$

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Abstract. We prove that the Cox ring of the moduli space $\overline{M}_{0,6}$, of stable rational curves with 6 marked points, is finitely generated by sections corresponding to the boundary divisors and divisors which are pull-backs of the hyperelliptic locus in $\overline{M}_3$ via morphisms $\rho : \overline{M}_{0,6} \to \overline{M}_3$ that send a 6-pointed rational curve to a curve with 3 nodes by identifying 3 pairs of points. In particular this gives a self-contained proof of Hassett and Tschinkel's result about the effective cone of $\overline{M}_{0,6}$ being generated by the above mentioned divisors.

1. Introduction

A question of Fulton about the moduli space $\overline{M}_{0,n}$, of stable, n-pointed, rational curves, is whether the cone $\overline{\mathrm{NE}}^k(\overline{M}_{0,n})$ of effective cycles of codimension k in $\overline{M}_{0,n}$ is generated by *k-strata*, i.e., loci in $\overline{M}_{0,n}$ corresponding to reducible curves with at least k nodes. While the case when $k = n - 4$ (i.e., the cone of effective curves) is completely open (and an affirmative result would imply, by results of Gibney, Keel and Morrison [GKM], the similar statement for the moduli space $\overline{M}_{g,n}$, of stable, n-pointed, genus g curves, thus determining the ample cone of $\overline{M}_{g,n}$), the case when $k = 1$ (i.e., the cone of effective divisors) was settled independently by Keel (unpublished; a refference to this may be found in [GKM], p.277) and Vermeire [V]: Fulton's question has a negative answer when $n = 6$ (and therefore for any $n \geq 6$). Hassett and Tschinkel prove in [HT] that the *Keel-Vermeire divisors* (pull-backs of the locus of hyperelliptic curves in the moduli space $\overline{M}_3$, via morphisms $\overline{M}_{0,6} \to \overline{M}_3$ sending a 6-pointed rational curve to a curve with 3 nodes by identifying 3 pairs of points) together with the 2-strata (the boundary) generate the cone of effective divisors in $\overline{M}_{0,6}$. The proof in [HT] is based on a computer check. In this paper we give a proof of Hassett and Tschinkel's result, by proving a stronger statement: we show that the sections corresponding to the above divisors generate the Cox ring of $\overline{M}_{0,6}$.

Recall that if X is a smooth projective variety with Picard group freely generated by divisors $D_1, \ldots, D_r$, then the Cox ring (or total coordinate ring) of X is the multi-graded ring:

$$\mathrm{Cox}(X) = \bigoplus_{(m_1,\ldots,m_r)\in\mathbb{Z}^r} \mathrm{H}^0(X, m_1D_1 + \ldots + m_rD_r).$$

The Cox ring being finitely generated has strong implications for the birational geometry of X (X is a so-called *Mori Dream Space*): the effective cone and the nef

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cone are both polyhedral and there are finitely many *small modifications* of X (i.e., varieties X' isomorphic in codimension one to X) such that any moving divisor on X (i.e., a divisor whose base locus has codimension at least 2) is nef on one of the varieties X' (see [HK] for the precise statements). It has been conjectured by Hu and Keel [HK] that any log-Fano variety has a finitely generated Cox ring. This has been recently proved in the groundbreaking paper [BCHM]. In [HK] Hu and Keel ask the following question:

Question 1.1. Is the Cox ring of $\overline{M}_{0,n}$ finitely generated?

As pointed out in [KM], the moduli space $\overline{M}_{0,n}$ is log-Fano only for $n \leq 6$.

We answer Question 1.1 for $n = 6$ by finding explicit generators. Our hope is that our method for finding generators, which proved to be useful in other circumstances (see [CT]), will eventually help answer Question 1.1 for larger n as well.

Consider the Kapranov description of the moduli space $\overline{M} = \overline{M}_{0,6}$. If $p_1, \ldots, p_5$ are points in linearly general position in $\mathbb{P}^3$, then $\overline{M}$ is the iterated blow-up of $\mathbb{P}^3$ along $p_1, \ldots, p_5$ and along the proper transforms of the lines $l_{ij} = \overline{p_i p_j}$ for all $i \neq j$. If p is a general point in $\mathbb{P}^3$, there is a unique twisted cubic C in $\mathbb{P}^3$ that contains the points $p_1, \ldots, p_5, p$. Then $(C, p_1, \ldots, p_5, p)$ is a 6-pointed rational curve, hence an element of $\overline{M}_{0,6}$. The point p corresponds to the 6'th marking (the so-called *moving point*).

Denote by H the hyperplane class on $\overline{M}$ and by E_i and E_{ij} the exceptional divisors in $\overline{M}$ corresponding to the points p_i and the lines l_{ij}.

Notation 1.2. Let Λ_{ijk} be the class of the proper transform of the plane $\overline{p_i p_j p_k}$:

$$\Lambda_{ijk} = H - E_i - E_j - E_k - E_{ij} - E_{ik} - E_{jk}.$$

If $S \subset \{1, \ldots, 6\}$ and $|S| = 2,$ or 3, let Δ_S be the boundary divisor in $\overline{M}$ with general element a curve with two irreducible components with the partition of the markings given by $S \cup S^c$. In the Kapranov description, the boundary divisors Δ_S have the following classes:

$$\Delta_{i6} = E_i, \quad \Delta_{ij6} = E_{ij}, \quad i, j = 1, \ldots, 5,$$
$$\Delta_{ij} = \Lambda_{abc}, \text{ if } \{i, j, a, b, c\} = \{1, \ldots, 5\}.$$

Notation 1.3. Let $Q_{(ij)(kl)}$ be the class of the proper transform of the unique quadric that contains all the points $p_1, \ldots, p_5$ and the lines $l_{ik}, l_{il}, l_{jk}, l_{jl}$:

$$Q_{(ij)(kl)} = 2H - \sum_i E_i - E_{ik} - E_{il} - E_{jk} - E_{jl}.$$

The divisor classes $Q_{(ij)(kl)}$ are exactly the divisors considered by Keel and Vermeire: for example, if one considers the map $\overline{M}_{0,6} \to \overline{M}_3$ given by identifying the pairs of points (12)(34)(56), then the class of the pull-back of the hyperelliptic locus in $\overline{M}_3$ is computed in [HT] to be the class of $Q_{(12)(34)}$. We call the divisors $Q_{(ij)(kl)}$ the *Keel-Vermeire divisors.* We prove the following:

Theorem 1.4. *The Cox ring of $\overline{M}_{0,6}$ is generated by the sections (unique up to scaling) corresponding to the boundary divisors (i.e., Λ_{ijk} and the exceptional divisors E_i and E_{ij}) and the Keel-Vermeire divisors $Q_{(ij)(kl)}$.*

The paper is divided as follows: Section 2 explains the strategy of proof; there are two main cases, the details of each are given in Section 3, respectively Section

4. The remaining sections contain auxiliary results needed in the proof. Section 5 contains proofs for some basic inequalities, while Section 6 contains some general multiplicity estimates needed for Case II. Section 7 contains the proof of Lemma 2.21 (needed for Case II) that states that the Cox ring of the blow-up of $\mathbb{P}^2$ in seven (non-general) points is generated by sections corresponding to -1 and -2 curves. Section 8 gives necessary and sufficient conditions for a divisor on X, the iterated blow-up of $\mathbb{P}^3$ in four general points and lines through them, to have sections. Finally, in Section 9 we compute the restrictions of an arbitrary divisor D to all the boundary divisors and Keel-Vermeire divisors on $\overline{M}$. Moreover, we derive some necessary conditions for these restrictions to be effective (an assumption in our main proof).

2. Plan of proof

Consider an arbitrary divisor class on $\overline{M}$:

$$D = dH - \sum_i m_i E_i - \sum_{i,j} m_{ij} E_{ij}.$$

In all that follows we assume $\mathrm{H}^0(\overline{M}, D) \neq 0$.

Notation 2.1. Let l be the class of the proper transform in $\overline{M}$ of a general line in $\mathbb{P}^3$. Let e_i be the class of a general line in E_i. Let C be the class of the proper transform of a general cubic that passes through $p_1, \ldots, p_5$:

$$C = 3l - e_1 - \ldots - e_5.$$

The curves with class C cover a dense set of $\overline{M}$; hence, $D.C \geq 0$ for any effective divisor D.

Definition 2.2. Let $x_i, x_{ij}, x_{ijk}, x_{(ij)(kl)}$ be the sections (unique up to scalar) corresponding to the divisors:

$$E_i, E_{ij}, \Lambda_{ijk}, Q_{(ij)(kl)}. \tag{2.1}$$

Definition 2.3. We call a section $s \in \mathrm{H}^0(\overline{M}, D)$ a *distinguished section* if

$$s = x_i^{n_i} x_{ij}^{n_{ij}} x_{ijk}^{n_{ijk}} x_{(ij)(kl)}^{n_{(ij)(kl)}},$$

where $n_i, n_{ij}, n_{ijk}, n_{(ij)(kl)}$ are non-negative integers.

To show that $\mathrm{H}^0(\overline{M}, D)$ is generated by distinguished sections, we do an induction on $D.C$. Note that we may assume that D contains none of the divisors (2.1) in its base locus, i.e., equivalently, if for E any of the divisors in (2.1), one has $\mathrm{H}^0(E, D_{|E}) \neq 0$. To see this, note that if E is an effective divisor, say E is the zero locus of a section $x_E \in \mathrm{H}^0(\overline{M}, E)$, then there is an exact sequence,

$$0 \to \mathrm{H}^0(\overline{M}, D - E) \to \mathrm{H}^0(\overline{M}, D) \to \mathrm{H}^0(E, D_{|E}).$$

If $\mathrm{H}^0(E, D_{|E}) = 0$, then any $s \in \mathrm{H}^0(\overline{M}, D)$ is of the form $x_E t$, where $t \in \mathrm{H}^0(\overline{M}, D - E)$. If in addition E is a divisor in (2.1), then we may replace D with $D - E$ and s with t. (Clearly, if t is generated by distinguished sections, then s is too.) Therefore, we may assume:

Assumption 2.4. *$H^0(E, D_{|E}) \neq 0$ for all divisors E in* (2.1).

Denote by r_E the restriction to E:

$$r_E : \mathrm{H}^0(\overline{M}, D) \to \mathrm{H}^0(E, D_{|E}).$$

To prove Theorem 1.4 it is enough to prove the following:

Main Claim. Let D be a divisor on $\overline{M}$:

$$D = dH - \sum_i m_i E_i - \sum_{i,j} m_{ij} E_{ij},$$

such that $\mathrm{H}^0(\overline{M}, D) \neq 0$ and that satisfies Assumption 2.4. Up to a renumbering, we may assume that $m_5 \leq m_i$, for $i = 1, \ldots, 4$. If $m_i = m_5$ for all $i = 1, \ldots, 4$, then we may assume that the maximum of the m_{ij}'s for all $i, j \in \{1, \ldots, 5\}$ is attained for m_{i5} for some $i = 1, \ldots, 4$. Let $E = E_5$. Then for any $s \in \mathrm{H}^0(D\overline{M}, D)$, there is $s' \in \mathrm{H}^0(\overline{M}, D)$, generated by distinguished sections, such that $r_E(s) = r_E(s')$.

To see how the Main Claim implies Theorem 1.4, note that the kernel of the restriction r_E is $\mathrm{H}^0(\overline{M}, D - E)$ and the map $\mathrm{H}^0(\overline{M}, D - E) \to \mathrm{H}^0(\overline{M}, D)$ is given by multiplication with x_E. If $r_E(s) = r_E(s')$, then $s - s' = x_E t$, where $t \in \mathrm{H}^0(\overline{M}, D - E)$. If s' is generated by distinguished sections, then to show that s is generated by distinguished sections is enough to show that $\mathrm{H}^0(\overline{M}, D - E)$ is generated by distinguished sections. We may replace D with $D - E$ and continue the procedure. Since E is always among the E_i's, note that $(D - E).C < D.C$ and $\mathrm{H}^0(\overline{M}, D - E)$ is generated by distinguished sections by induction. The process has to stop as $D.C \geq 0$ for any effective divisor D. (In particular, note that $D.C$ also decreases when we substract from D any of the divisors E in (2.1) for which $\mathrm{H}^0(E, D_{|E}) = 0$.)

Notation 2.5. Given any divisor D on $\overline{M}$ we denote by $\overline{D}$ the restriction $D_{|E_5}$ of D to E_5. By (9.3) one has

$$\overline{D} = m_5 \overline{H} - \sum_{i=1}^{4} m_{i5} \overline{E}_i.$$

Let $\rho_5 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ be the projection from p_5. Let $q_i = \rho_5(p_i)$ ($i \in \{1, \ldots, 4\}$). The divisor E_5 is isomorphic to the blow-up of $\mathbb{P}^2$ along the points $q_1, \ldots, q_4$ (as q_i determines the direction of the line l_{i5}). The divisors $\overline{H}$, respectively $\overline{E}_i$, are the hyperplane class, respectively the exceptional divisors on E_5 (see also Section 9.1). The map ρ_5 is resolved by the morphism $\pi_5 : \overline{M} \to \overline{M}_{0,5}$ that forgets the 5'th marking (which is also a retract for the inclusion $E_5 \subset \overline{M}$).

Notation 2.6. Let $\bar{l}_{ij}$ be the line $\overline{q_i q_j}$ in $\mathbb{P}^2$. Denote:

$$x = \bar{l}_{13} \cap \bar{l}_{24}, \quad y = \bar{l}_{14} \cap \bar{l}_{23}, \quad z = \bar{l}_{12} \cap \bar{l}_{34}.$$

Notation 2.7. Let L_x be the proper transform in $\overline{M}$ of the unique line in $\mathbb{P}^3$ that passes through p_5 and intersects the skew lines l_{13} and l_{24}. Similarly, let L_y (respectively L_z) be the unique line that passes through p_5 and intersects the skew lines l_{14} and l_{23} (respectively l_{12} and l_{34}).

Remark that $x = \rho_5(L_x)$, $y = \rho_5(L_y)$, $z = \rho_5(L_z)$.

In order to prove the Main Claim, we distinguish two cases.

Case I: Assume that $D.L_x \geq 0, \quad D.L_y \geq 0, \quad D.L_z \geq 0$.

Notation 2.8. Denote by s_{ij} the section on E_5 corresponding to the proper transform of the line $\overline{l}_{ij}$ in $\mathbb{P}^2$. Let s_i $(i = 1, \dots, 4)$ be the sections corresponding to the exceptional divisors $\overline{E}_i$.

Definition 2.9. We call a section $s \in \mathrm{H}^0(E_5, \overline{D})$ a *distinguished section on* E_5 if s can be written as a monomial in the sections s_{ij} and s_i.

Since $E_5 \cong \overline{M}_{0,5}$ is the blow-up of $\mathbb{P}^2$ along $q_1, \dots, q_4$, by Lemma 7.3 the Cox ring $\mathrm{Cox}(E_5)$ of E_5 is generated by distinguished sections. The Main Claim follows from the following:

Proposition 2.10. *Under the assumptions of the Main Claim and the assumptions in Case I, the restriction map*

$$r_{E_5} : H^0(\overline{M}, D) \to H^0(E_5, \overline{D})$$

is surjective and one may lift any distinguished section (hence, any section) in $H^0(E_5, \overline{D})$ *to a section generated by distinguished sections in* $H^0(\overline{M}, D)$.

The following is the main observation needed to prove Proposition 2.10:
Main Observation – Case I. Distinguished sections on E_5 may be lifted to distinguished sections on $\overline{M}$ using the following rules:

$$x_{ij5_{|E_5}} = s_{ij}, \quad x_{i5|E_5} = s_i. \tag{2.2}$$

This is because $\Lambda_{ij5_{|E_5}} = \overline{l}_{ij}$, $E_{i5|E_5} = \overline{E}_i$ (see Section 9, formula (9.3)).

Sketch of Proof of Proposition 2.10. We lift a distinguished section $\overline{s} \in \mathrm{H}^0(E_5, \overline{D})$ using the rules (2.2). Hence, there is a section t' belonging to some $\mathrm{H}^0(\overline{M}, D')$, where $\overline{D}' = \overline{D}$ and $r_{E_5}(t') = \overline{s}$.

Notation 2.11. Let $\Delta = D - D'$.

Notation 2.12. Denote by X the iterated blow-up of $\mathbb{P}^3$ in $p_1, \dots, p_4$ and proper transforms of lines l_{ij} $(i, j \in \{1, \dots, 4\})$.

Since D and D' have the same restriction to E_5, it follows from (9.3) that the divisor Δ is a divisor on X. Note, X is a toric variety. The following is a standard result:

Lemma 2.13. *The Cox ring of X is generated by sections x_i,x_{ij}, x_{ijk} corresponding to the exceptional divisors E_i,E_{ij} and proper transforms of hyperplanes Λ_{ijk} $(i, j, k \in \{1, \dots, 4\})$.*

Proposition 2.10 is now immediate if $\mathrm{H}^0(\Delta) \neq 0$: Since the points $p_1, \dots, p_5$ are general, the restriction to E_5 of any distinguished section in $\mathrm{Cox}(X)$ is non-zero. In particular, if t'' is any non-zero section in $\mathrm{H}^0(\Delta)$, then $t''_{|E_5} \in \mathrm{H}^0(E_5, \mathcal{O})$ is non-zero. Therefore, the section $s = t't''$ is a section in $\mathrm{H}^0(\overline{M}, D)$ that restricts to (a non-zero multiple of) $\overline{s}$ in $\mathrm{H}^0(E_5, \overline{D})$. Since t'' is a distinguished section, it follows that t is generated by distinguished sections.

Definition 2.14. We call a distinguished section $\overline{s}$ on E_5 *a section with straightforward lifting to* D if after lifting using the rules (2.2) we end up with a divisor D' for which $\Delta = D - D'$ has $\mathrm{H}^0(\Delta) \neq 0$.

The following claim (proof in Section 3) finishes the proof of Proposition 2.10.

Claim 2.15. Under the assumptions of Proposition 2.10, any distinguished section $\overline{s} \in \mathrm{H}^0(E_5, \overline{D})$ is a linear combination of distinguished sections with straightforward lifting to D.

Case II: Assume one of $D.L_x, D.L_y, D.L_z$ is negative.

Definition 2.16. Let:

$$m_x = \max\ \{0, -D.L_x\}, \quad m_y = \max\ \{0, -D.L_y\}, \quad m_z = \max\ \{0, -D.L_z\}.$$

Notation 2.17. Denote by Y the blow-up of $\mathbb{P}^2$ along $q_1, q_2, q_3, q_4, x, y, z$. Let $\overline{E}_i$,$\overline{E}_x$,$\overline{E}_y$,$\overline{E}_z$ be the corresponding exceptional divisors. For a given divisor D on $\overline{M}$ we consider the following divisor $\overline{D}^Y$ on Y:

$$\overline{D}^Y = \overline{D} - m_x\overline{E}_x - m_y\overline{E}_y - m_z\overline{E}_z.$$

Clearly, the linear system $\mathrm{H}^0(Y, \overline{D}^Y)$ is a subspace of the linear system $\mathrm{H}^0(E_5, \overline{D})$.

Claim 2.18. The restriction map r_{E_5} factors through $\mathrm{H}^0(Y, \overline{D}^Y)$.

Proof. Clearly, Claim 2.18 is non-trivial only when one of m_x,m_y,m_z is positive. Take for example the case when $m_x > 0$ (the other cases are identical). By Proposition 6.1, the line L_x is contained in D with multiplicity $m \geq m_x > 0$. It follows that for any $s \in \mathrm{H}^0(\overline{M}, D)$ the section $r_{E_5}(s)$ vanishes at x with multiplicity $\geq m$; hence, $r_{E_5}(s)$ lies in the subspace $\mathrm{H}^0(Y, \overline{D}^Y)$. $\square$

In Case II we follow the exact same steps as in Case I, with the only difference being that we work on Y instead of E_5.

Notation 2.19. Denote by s'_{ij} the section corresponding to the proper transform in Y of the line $\overline{l}_{ij}$. Similarly, let s_{xy}, s_{xz}, s_{yz} be the sections corresponding to the proper transforms of the lines $\overline{xy}, \overline{xz}, \overline{yz}$. Let s_i, s_x, s_y, s_z be the sections corresponding to the exceptional divisors E_i, E_x, E_y, E_z.

Note:

$$x \in \overline{l}_{13}, \overline{l}_{24}, \quad y \in \overline{l}_{14}, \overline{l}_{23}, \quad z \in \overline{l}_{12}, \overline{l}_{34}.$$

Hence, for example, s'_{13} is a section of the divisor $\overline{H} - \overline{E}_1 - \overline{E}_3 - \overline{E}_x$, and the section s_{13} (Notation 2.8) is given by $s_{13} = s'_{13}s_x$. Moreover, if we let

$$r_Y : \mathrm{H}^0(\overline{M}, D) \to \mathrm{H}^0(Y, \overline{D}^Y)$$

be the morphism of Claim 2.18, then $r_{E_5}(s) = r_Y(s)s_x^{m_x}s_y^{m_y}s_z^{m_z}$.

Definition 2.20. We call a section $s \in \mathrm{H}^0(Y, \overline{D}^Y)$ a *distinguished section on* Y if s can be written as a monomial in the sections s'_{ij},s_{xy},s_{xz},s_{yz}, s_i,s_x,s_y,s_z.

In Section 7 we prove the following:

Lemma 2.21. *The Cox ring* $\mathrm{Cox}(Y)$ *of* Y *is generated by distinguished sections.*

Note, by Lemma 2.21, the generators of $\mathrm{Cox}(Y)$ are given by the sections (unique up to scalar multiplication) corresponding to the (-1) and (-2) curves on Y. The Main Claim follows from:

Proposition 2.22. *Under the assumptions of the Main Claim, the restriction map*

$$r_Y : H^0(\overline{M}, D) \to H^0(Y, \overline{D}')$$

is surjective and one may lift any distinguished section (hence, any section) in $H^0(Y, \overline{D}^Y)$ *to a section generated by distinguished sections in* $H^0(\overline{M}, D)$.

The following is the main observation needed to prove Proposition 2.22:
Main Observation – Case II. Distinguished sections on Y may be lifted to distinguished sections on $\overline{M}$ using the following rules:

$$r_Y(x_{ij5}) = s'_{ij}, \quad r_Y(x_{i5}) = s_i, \tag{2.3}$$

$$r_Y(x_{(13)(24)}) = s_{yz}, \quad r_Y(x_{(14)(23)}) = s_{xz}, \quad r_Y(x_{(12)(34)}) = s_{xy}. \tag{2.4}$$

This is because when $D = \Lambda_{ij5}$, one has

$$\overline{D}^Y = \overline{H} - \overline{E}_i - \overline{E}_j - \overline{E}_\alpha,$$

where $\alpha = x$ if $ij \in \{13, 24\}$, $\alpha = y$ if $ij \in \{14, 23\}$, and $\alpha = z$ if $ij \in \{12, 34\}$. Similarly,

$$\overline{Q}^Y_{(13)(24)} = \overline{H} - \overline{E}_y - \overline{E}_z, \overline{Q}^Y_{(14)(23)} = \overline{H} - \overline{E}_x - \overline{E}_z, \overline{Q}^Y_{(12)(34)} = \overline{H} - \overline{E}_x - \overline{E}_y.$$

Sketch of Proof of Proposition 2.22. We lift a distinguished section $\overline{s} \in \mathrm{H}^0(E_5, \overline{D})$ using the rules (2.3) and (2.4). Hence, there is a section t' in some $\mathrm{H}^0(\overline{M}, D')$ and $r_Y(t') = \overline{s}$. As in Case I, we let $\Delta = D - D'$. The divisor Δ is a divisor on X (Notation 2.12). As in Case I, Proposition 2.22 follows from Lemma 2.13 if $\mathrm{H}^0(\Delta) \neq 0$.

Definition 2.23. We call a distinguished section $\overline{s}$ on Y *a section with straightforward lifting to* D if lifting using the rules (2.3) and (2.4) results in a divisor D' for which $\Delta = D - D'$ has $\mathrm{H}^0(\Delta) \neq 0$.

The following claim (proof in Section 4) finishes the proof of Proposition 2.22.

Claim 2.24. Under the assumptions of the Main Claim, any distinguished section $\overline{s} \in \mathrm{H}^0(Y, \overline{D}^Y)$ is a linear combination of distinguished sections with straightforward lifting to D.

3. Proof of Claim 2.15

The idea is that any distinguished section on E_5 can be rewritten, using the relations in $\mathrm{Cox}(E_5)$, as a linear combination of distinguished sections with straightforward lifting. To check that $\mathrm{H}^0(\Delta) \neq 0$ we use Lemma 8.2. Assumption 2.4 is equivalent to inequalities (9.4), (9.6), (9.7), (9.8) (for all permutations of indices).

We use the notation from Section 8. Recall that e_{ij} is the class of a fiber of the $\mathbb{P}^1$-bundle $E_{ij} \to \overline{l}_{ij}$. One has $D.l = d$, $D.e_i = m_i$, $D.e_{ij} = m_{ij}$ (see for example (9.1), (9.5)). The inequalities defining Case I are equivalent to

$$D.(l - e_5 - e_{ij} - e_{kl}) \geq 0, \quad \{i, j, k, l\} = \{1, 2, 3, 4\}. \tag{3.1}$$

Lemma 3.1. *Let $\overline{s}$ be a distinguished section on E_5:*

$$\overline{s} = \prod_{i,j} s_{ij}^{a_{ij}} \prod_{i} s_i^{l_i}, \tag{3.2}$$

where $a_{ij}, l_i \geq 0$. If $\overline{s}$ is a section $H^0(E_5, \overline{D})$, then $\overline{s}$ has straightforward lifting to D if and only if for all $\{i, j, k, l\} = \{1, 2, 3, 4\}$ one has

$$a_{ij} \leq D.(C_{k;l} - e_5), \tag{3.3}$$

where $C_{k;l} = 2l - e_{ki} - e_{kj} - e_l$.

Remark 3.2. By (9.8) one has $D.(C_{k;l} - e_5) \geq 0$ for all $k, l \in \{1, 2, 3, 4\}$.

Remark 3.3. The condition that $\overline{s}$ is in $\mathrm{H}^0(\overline{D})$ is equivalent to

$$\sum a_{ij} = D.e_5, \quad a_{ij} + a_{ik} + a_{il} - l_i = D.e_{i5} \tag{3.4}$$

(the coefficients of $\overline{H}$ and $\overline{E}_i$ in $\overline{D}$). It follows from (3.4) that

$$a_{kl} - a_{ij} - l_k - l_l = D.(e_{k5} + e_{l5} - e_5), \tag{3.5}$$

$$\sum_{i=1}^{4} l_i = D.(2e_5 - \sum_{i \neq 5} e_{i5}), \tag{3.6}$$

$$a_{jk} + a_{jl} + a_{kl} + l_i = D.(e_5 - e_{i5}). \tag{3.7}$$

Proof of Lemma 3.1. If $\overline{D} = 0$, $\overline{s} = 1$ (i.e., $a_{ij} = 0, l_i = 0$), then the lift D' is 0. Hence, $\Delta = D - D' = D$. Since $\mathrm{H}^0(D) \neq 0$, there is nothing to prove in this case.

Assume now $\overline{D} \neq 0$. Recall that $E_5 \subset \overline{M}$ has a retract $\pi : \overline{M} \to E_5 \cong \overline{M}_{0,5}$ given by the morphism that forgets the 5'th marking. One has

$$\pi^* \bar{l}_{ij} = \Lambda_{ij5} + E_{ij}, \quad \pi^* \overline{E}_i = E_{i5} + E_i. \tag{3.8}$$

(This is a general fact about the forgetful morphisms $\pi_i : \overline{M}_{0,n} \to \overline{M}_{0,n-1}$ that forget a marking i. If Δ_S is a boundary divisor in $\overline{M}_{0,n-1}$, corresponding to the partition $S \cup S^c$, then $\pi^* \Delta_S = \Delta_S + \Delta_{S \cup \{i\}}$.)

Since we lift $\overline{D}$ to D' by lifting $\bar{l}_{ij}$ to Λ_{ij5} and $\overline{E}_i$ to E_{i5}, it follows that

$$D' = \pi^* \overline{D} - \Delta_0,$$

where Δ_0 is the effective divisor on X given by

$$\Delta_0 = \sum_{i,j \in \{1,\ldots,4\}} a_{ij} E_{ij} + \sum_{i \in \{1,\ldots,4\}} l_i E_i.$$

Then $\Delta = D - D' = D - \pi^* \overline{D} + \Delta_0$.

Observation 3.4. If $(D - \pi^* \overline{D}).C \geq 0$ for some nef curve C on X, then $\Delta.C \geq 0$.

Below we show that $(D - \pi^* \overline{D}).C \geq 0$ for all the nef curves C in Lemma 8.2 giving inequalities (1)–(4). Hence, by Observation 3.4, $\Delta.C \geq 0$. For the remaining nef curves C in general it will not be true that $(D - \pi^* \overline{D}).C \geq 0$, but we show that we still have $\Delta.C \geq 0$ for the nef curves C giving inequalities (5), (7), (8), (9) and that for $C = C_{k;l}$ (inequality (6)) $\Delta.C \geq 0$ is equivalent to (3.3). Note that

$$(\pi^* \overline{D}).l = D.e_5, \quad (\pi^* \overline{D}).e_i = D.e_{i5}, \quad (\pi^* \overline{D}).e_{ij} = 0 \quad (i, j \neq 5). \tag{3.9}$$

(It is enough to check this when $D = H, E_i, E_{ij}$. For this, use the formulas (3.8).)

We check one by one the inequalities (1) – (9) in Lemma 8.2:

(1) $(D-\pi^*\overline{D}).l = D.(l-e_5) \geq 0$,

as $l-e_5$ is a nef curve on $\overline{M}$. Similarly:

(2) $(D-\pi^*\overline{D}).(l-e_i) = D.(l-e_i-e_5+e_{i5}) \geq 0$ by (9.6),

(3) $(D-\pi^*\overline{D}).(l-e_{ij}) = D.(l-e_5-e_{ij}) \geq 0$ by (9.7),

(4) $(D-\pi^*\overline{D}).(l-e_{ij}-e_{kl}) = D.(l-e_5-e_{ij}-e_{kl}) \geq 0$ by (3.1).

For inequality (5) (recall $C_{ij} = 2l-e_{ij}-e_k-e_l$),

$$(D-\pi^*\overline{D}).C_{ij} = D.(C_{ij}-2e_5+e_{k5}+e_{l5}),$$
$$\Delta_0.C_{ij} = a_{ij}+l_k+l_l,$$
$$\Delta.C_{ij} = D.(C_{ij}-e_5)+D.(e_{k5}+e_{l5}-e_5)+a_{ij}+l_k+l_l.$$

By (3.5), $\Delta.C_{ij} = D.(C_{ij}-e_5)+a_{kl}$. From (9.7) (and $a_{kl}\geq 0$), $\Delta.C_{ij}\geq 0$.

For inequality (7) (recall $C_i = 2l-e_{ij}-e_{ik}-e_{il}$),

$$(D-\pi^*\overline{D}).C_i = D.(C_i-2e_5)$$

and $\Delta_0.C_i = a_{ij}+a_{ik}+a_{il}$. Using (3.4), $\Delta_0.C_i = D.e_{i5}+l_i$. Therefore,

$$\Delta.C_i = D.(C_i-2e_5+e_{i5})+l_i = 2D.(l-e_i-e_5+e_{i5})+D.(2e_i-\sum_{u\neq i}e_{iu})+l_i.$$

It follows from (9.4) and (9.6) that $\Delta.C_i\geq 0$.

For inequality (8) (recall $B = 3l-\sum_{i=1}^4 e_i$),

$$(D-\pi^*\overline{D}).B = D.(B-3e_5+\sum_{i=1}^4 e_{i5})$$

and $\Delta_0.B = \sum_{i=1}^4 l_i$. It follows from (3.6) that $\Delta.B = D.(3l-\sum_{i=1}^5 e_i)\geq 0$.

For inequality (9) (recall $B_i = 3l-2e_i-e_{jk}-e_{jl}-e_{kl}$),

$$(D-\pi^*\overline{D}).B_i = D.(B_i-3e_5+2e_{i5})$$

and $\Delta_0.B_i = a_{jk}+a_{jl}+a_{kl}+2l_i$. From (3.7) one has $\Delta_0.B_i = D.(e_5-e_{i5})+l_i$,

$$\Delta.B_i = D.(B_i-2e_5+e_{i5}) = 2D.(l-e_i-e_5+e_{i5})+D.(l-e_{jk}-e_{jl}-e_{kl}-e_{i5}).$$

It follows by (9.6) and (9.7) that $D.B_i\geq 0$.

There is at least one strict inequality in (4): assume $\Delta.(l-e_{ij}-e_{kl}) = 0$, for all $\{i,j,k,l\} = \{1,2,3,4\}$. From the computation above for case (4), we have

$$(D-\pi^*\overline{D}).(l-e_{ij}-e_{kl}) = D.(l-e_5-e_{ij}-e_{kl})\geq 0.$$

As $\Delta_0.(l-e_{ij}-e_{kl}) = 0$ $(l-e_{ij}-e_{kl})$ is a nef curve) it follows that

$$(D-\pi^*\overline{D}).(l-e_{ij}-e_{kl}) = \Delta_0.(l-e_{ij}-e_{kl}) = 0.$$

Since $\Delta_0.(l-e_{ij}-e_{kl}) = a_{ij}+a_{kl}$ it follows that $a_{ij} = 0$ for all i,j. By (3.4), $D.e_5 = 0$ and $D.e_{i5} = 0$,$l_i = 0$ for all $i\neq 5$. Hence, $\overline{D} = 0, \overline{s} = 1$, which contradicts our assumption.

We now show that inequality (6) is equivalent to (3.3). One has

$$(D-\pi^*\overline{D}).C_{i;j} = D.(C_{i;j}-2e_5+e_{j5})$$

and $\Delta_0.C_{i;j} = a_{ik}+a_{il}+l_j$. From (3.7), $\Delta_0.C_{i;j} = D.(e_5-e_{j5})-a_{kl}$. Therefore,

$$\Delta.C_{i;j} = D.(C_{i;j}-e_5)-a_{kl}.$$

Hence, inequality (6) is equivalent to (3.3). □

3.1. **Proof of Claim 2.15.** Let $\overline{s}$ be a distinguished section in $\mathrm{H}^0(E_5, \overline{D})$ as in (3.2). If $a_{ij} \leq D.(C_{k;l} - e_5)$ for all $\{i,j,k,l\} = \{1,2,3,4\}$, then by Lemma 3.1 $\overline{s}$ has straightforward lifting to D. Assume now that $a_{ij} > D.(C_{k;l} - e_5)$ for some choice of i,j,k,l. Without loss of generality, we may assume $a_{12} > D.(C_{3;4} - e_5)$. Note that by Remark 3.2 it follows that $a_{12} > 0$.

Claim 3.5. If $a_{12} > D.(C_{3;4} - e_5)$, then either $a_{34} > 0$ or $l_1 + l_2 > 0$.

Proof. By (3.5) one has

$$a_{12} - a_{34} - l_1 - l_2 = D.(e_{15} + e_{25} - e_5). \tag{3.10}$$

Assume $a_{34} = l_1 = l_2 = 0$. It follows from (3.10) and $a_{12} > D.(C_{3;4} - e_5)$ that

$$a_{12} = D.(e_{15} + e_{25} - e_5) > D.(C_{3;4} - e_5).$$

This is a contradiction, as by (9.8) one has

$$D.(C_{3;4} - e_5) - D.(e_{15} + e_{25} - e_5) = D.(2l - e_4 - e_{13} - e_{23} - e_{15} - e_{25}) \geq 0.$$

□

3.2. **Algorithm for replacing $\overline{s}$.** We now give an algorithm for replacing $\overline{s}$ with another distinguished section $\overline{s}'$ for which $a_{12} - D.(C_{3;4} - e_5)$ is strictly smaller than for $\overline{s}$ and moreover, for all i,j for which $a_{ij} - D.(C_{k;l} - e_5)$ increases by this change, the section $\overline{s}'$ (still) satisfies $a_{ij} - D.(C_{k;l} - e_5) \leq 0$. We repeat the following two steps until $a_{12} \leq D.(C_{3;4} - e_5)$ (as by Claim 3.5 one of the two situations must happen if $a_{12} > D.(C_{3;4} - e_5)$). The same argument works for any a_{ij}.

<u>Step 1: If $l_1 + l_2 > 0$</u>: We may assume without loss of generality that $l_1 > 0$. Consider the following sections in the linear system $|\overline{H} - \overline{E}_2|$:

$$s_{12}s_1, \quad s_{23}s_3, \quad s_{24}s_4.$$

The linear system $|\overline{H} - \overline{E}_2|$ is 1-dimensional and any two of the above sections are linearly independent. Since $a_{12} > 0, l_1 > 0$, we may replace $s_{12}s_1$ in s with a linear combination of $s_{23}s_3$ and $s_{24}s_4$. The effect on the coefficients a_{ij} and l_i (of the corresponding two distinguished sections) is as follows: a_{12} and l_1 both decrease by 1, while either a_{23}, l_3 increase by 1, or a_{24}, l_4 increase by 1 (everything else stays the same). But by Lemma 3.6 one has

$$a_{2j} < D.(C_{k;l} - e_5), \qquad \text{for all } j \in \{3,4\}, \{j,k,l\} = \{1,3,4\}.$$

Therefore, after increasing a_{23} or a_{24} by 1 one still has $a_{2j} \leq D.(C_{k;l} - e_5)$.

<u>Step 2: If $a_{34} > 0$</u>: Consider the following sections in the linear system $|2\overline{H} - \overline{E}_1 - \ldots - \overline{E}_4|$:

$$s_{12}s_{34}, \quad s_{13}s_{24}, \quad s_{14}s_{23}.$$

The linear system $|2\overline{H} - \overline{E}_1 - \ldots - \overline{E}_4|$ is 1-dimensional, and any two of the above sections are linearly independent. Since $a_{12} > 0, a_{34} > 0$, we may replace $s_{12}s_{34}$ in s with a linear combination of $s_{13}s_{24}$ and $s_{14}s_{23}$. The effect on the coefficients a_{ij} is: a_{12} and a_{34} both decrease by 1, while either a_{13}, a_{24} increase by 1, or a_{14}, a_{23} increase by 1. By Lemma 3.6 one has

$$a_{ij} < D.(C_{k;l} - e_5), \qquad \text{for all } i \in \{1,2\}, j \in \{3,4\}, \{i,j,k,l\} = \{1,2,3,4\}.$$

Therefore after increasing $a_{13}, a_{14}, a_{23}, a_{24}$ by 1, each of them still satisfies its corresponding inequalities.

Lemma 3.6. *If $a_{ij} > D.(C_{k;l} - e_5)$, then $a_{iu} < D.(C_{v;w} - e_5)$ for all $\{u, v, w\} = \{j, k, l\}$ such that $u \in \{k, l\}$.*

Proof. Assume the contrary. Then $a_{ij} + a_{iu} > D.(C_{k;l} - e_5) + D.(C_{v;w} - e_5)$. But by (3.7) $a_{ij} + a_{iu} \leq D.(e_5 - e_{u'5})$, where $\{u', u\} = \{k, l\}$. This is a contradiction with Claim 3.7. □

Claim 3.7. $D.(C_{k;l} + C_{v;w} - 2e_5) \geq D.(e_5 - e_{u'5})$ for all v, w, u' such that $\{u', u\} = \{k, l\}$ and $\{v, w, u\} = \{j, k, l\}$ for some $u \in \{k, l\}$.

Proof. There are four cases:

Case (i): $v = j, w = l$ ($u = k$,$u' = l$). Using (9.6) and (9.7) one has

$$\begin{aligned} &D.(C_{k;l} + C_{j;l} - 2e_5) - D.(e_5 - e_{l5}) \\ &= 2D.(l - e_5 - e_l + e_{l5}) + D.(l - e_{ij} - e_{ik} - e_{jk} - e_{l5}) + D.(l - e_5 - e_{jk}) \geq 0. \end{aligned}$$

Case (ii): $v = l, w = j$ ($u = k$,$u' = l$). Using (9.6) and (9.7) one has

$$\begin{aligned} &D.(C_{k;l} + C_{l;j} - 2e_5) - D.(e_5 - e_{l5}) = D.(l - e_5 - e_l + e_{l5}) \\ &+D.(l - e_5 - e_j + e_{j5}) + D.(l - e_{iu} - e_{il} - e_{ul} - e_{j5}) + D.(l - e_5 - e_{jk}) \geq 0. \end{aligned}$$

Case (iii): $v = j, w = k$ ($u = l$,$u' = k$). This is symmetric to Case (ii).

Case (iv): $v = k, w = j$ ($u = l$,$u' = k$). Using (9.4), (9.6) and (9.7) one has

$$\begin{aligned} &D.(C_{k;l} + C_{k;j} - 2e_5) - D.(e_5 - e_{l5}) \\ &= 2D.(l - e_5 - e_k + e_{k5}) + D.(2l - e_5 - e_j - e_l - e_{ik}) + D.(2e_k - \sum_{\alpha \neq k} e_{k\alpha}) \geq 0. \end{aligned}$$

□

4. Proof of Claim 2.24

As in Section 3, we show that any distinguished section on Y can be rewritten, using the relations in $\mathrm{Cox}(Y)$, as a linear combination of distinguished sections with straightforward lifting. Assumption 2.4 is equivalent to the inequalities (9.4), (9.6), (9.7), (9.8) (for all permutations of indices). We use the notation from Section 8.

Notation 4.1. Let $\chi : \{12, 13, 14, 23, 24, 34\} \to \{x, y, z\}$ be the function

$$\chi(13) = \chi(24) = x, \quad \chi(14) = \chi(23) = y, \quad \chi(12) = \chi(34) = z.$$

Note that one has

$$L_{\chi(ij)} = l - e_5 - e_{ij} - e_{kl}, \qquad \text{for all } \{i, j, k, l\} = \{1, 2, 3, 4\}.$$

Remark 4.2. By Definition 2.16 one has $m_\alpha + D.L_\alpha \geq 0$ for all $\alpha \in \{x, y, z\}$, with equality if and only if $D.L_\alpha \leq 0$.

Lemma 4.3. *Let $\overline{s}$ be a distinguished section on Y:*

$$\overline{s} = \prod_{i,j} {s'}_{ij}^{a_{ij}} \prod_i s_i^{l_i} s_{xy}^{c_z} s_{xz}^{c_y} s_{yz}^{c_x} s_x^{l_x} s_y^{l_y} s_z^{l_z}, \tag{4.1}$$

where $a_{ij}, l_i, c_x, c_y, c_z, l_x, l_y, l_z \geq 0$. If $\overline{s}$ is a section $H^0(Y, \overline{D}^Y)$, then $\overline{s}$ has straightforward lifting to D if and only if for all $\{i,j,k,l\} = \{1,2,3,4\}$ and $\alpha \in \{x,y,z\}$,

$$\text{(i)} \qquad c_{\chi(ij)} - a_{ij} \leq D.(l - e_5 - e_{ij}),$$

$$\text{(ii)} \qquad c_\alpha - l_\alpha \leq m_\alpha + D.L_\alpha,$$

$$\text{(iii)} \qquad c_{\chi(ij)} - a_{ij} \leq D.(C_{kl} - e_5),$$

$$\text{(iv)} \qquad a_{ij} + \sum_{\alpha \neq \chi(ij)} c_\alpha \leq D.(C_{k;l} - e_5),$$

$$\text{(v)} \qquad c_x + c_y + c_z \leq D.C,$$

where $C_{kl} = 2l - e_i - e_j - e_{kl}$, $C_{k;l} = 2l - e_{ki} - e_{kj} - e_l$, $C = 3l - \sum_{i=1}^5 e_i$.

Remark 4.4. Note that the right sides of the inequalities in Lemma 4.3 are nonnegative due to (9.7) (for (i)), (iii)), (9.8) (for (iv)), Remark 4.2 (for (ii)) and because C is a nef curve on $\overline{M}$ (for (v)).

Remark 4.5. The condition that $\overline{s}$ is in $\mathrm{H}^0(Y, \overline{D}^Y)$ is equivalent to

$$\sum a_{ij} + (c_x + c_y + c_z) = D.e_5, \tag{4.2}$$

$$a_{ij} + a_{ik} + a_{il} - l_i = D.e_{i5} \tag{4.3}$$

(the coefficients of $\overline{H}$ and $\overline{E}_i$ in $\overline{D}^Y$),

$$a_{ij} + a_{kl} + \sum_{\alpha \neq \chi(ij)} c_\alpha - l_{\chi(ij)} = m_{\chi(ij)} \tag{4.4}$$

(the coefficient of $\overline{E}_\alpha$ in $\overline{D}^Y$ for $\alpha \in \{x,y,z\}$). From (4.2), (4.3) and (4.4) one has

$$\sum l_i + 2(c_x + c_y + c_z) = D.(2e_5 - \sum_{i \neq 5} e_{i5}), \tag{4.5}$$

$$a_{jk} + a_{jl} + a_{kl} + (c_x + c_y + c_z) + l_i = D.(e_5 - e_{i5}), \tag{4.6}$$

$$(c_x + c_y + c_z) - (l_x + l_y + l_z) = (m_x + m_y + m_z) - D.e_5. \tag{4.7}$$

Proof of Lemma 4.3. We lift $\overline{s}$ using the rules (2.3) and (2.4) (see also Remark 4.6) to a section of the divisor:

$$\begin{aligned} D' &= \sum a_{ij}\Lambda_{ij5} + \sum l_i E_{i5} + c_x Q_{(13)(24)} + c_y Q_{(14)(23)} + c_z Q_{(12)(34)} \\ &= (\sum a_{ij} + 2\sum c_\alpha)H - \sum_{i \neq 5}(a_{ij} + a_{ik} + a_{il} + \sum c_\alpha)E_i - (\sum a_{ij} + \sum c_\alpha)E_5 \\ &\quad - \sum_{i,j \neq 5}(a_{ij} + \sum_{\alpha \neq \chi(ij)} c_\alpha)E_{ij} - \sum_{i \neq 5}(a_{ij} + a_{ik} + a_{il} - l_i)E_{i5}. \end{aligned}$$

Using (4.2) and (4.3) one has

$$\begin{aligned} D' &= (D.e_5 + \sum c_\alpha)H - \sum_{i \neq 5}(D.e_{i5} + l_i + \sum c_\alpha)E_i - (D.e_5)E_5 \\ &\quad - \sum_{i,j \neq 5}(a_{ij} + \sum_{\alpha \neq \chi(ij)} c_\alpha)E_{ij} - \sum_{i \neq 5}(D.e_{i5})E_{i5}. \end{aligned}$$

Then $\Delta = D - D'$ is given by the following formula:

$$\Delta = (D.(l - e_5) - \sum c_\alpha)H - \sum_{i=1}^{4}(D.(e_i - e_{i5}) - l_i - \sum c_\alpha)E_i$$
$$- \sum_{i,j\neq 5}(D.e_{ij} - a_{ij} - \sum_{\alpha\neq\chi(ij)} c_\alpha)E_{ij}.$$

We show that $\Delta.C \geq 0$ for the nef curves C giving the inequalities (1), (2), (7), (9) in Lemma 8.2 and that for the nef curves C giving the remaining inequalities, $\Delta.C \geq 0$ is equivalent to (i), (ii), (iii), (iv), (v).

For inequality (1),

$$\Delta.l = D.(l - e_5) - \sum c_\alpha.$$

By (4.6), one has $\sum c_\alpha \leq D.(e_5 - e_{i5})$. By the assumption in the Main Claim $D.e_5 \leq D.e_i$. Then $\Delta.l \geq D.(l - e_5 - e_i + e_{i5})$. It follows from (9.6) that $\Delta.l \geq 0$.

For inequality (2),

$$\Delta.(l - e_i) = D.(l - e_5 - e_i + e_{i5}) + l_i.$$

It follows from (9.6) that $\Delta.(l - e_i) \geq 0$.

Inequality (3) is equivalent to (i) as one has

$$\Delta.(l - e_{ij}) = c_z - a_{12} \leq D.(l - e_5 - e_{ij}) + a_{ij} - c_{\chi(ij)}.$$

For inequality (4),

$$\Delta.(l - e_{ij} - e_{kl}) = D.(l - e_5 - e_{ij} - e_{kl}) + a_{ij} + a_{kl} + \sum_{\alpha\neq\chi(ij)} -c_{\chi(ij)}.$$

By using (4.4) to substitute $a_{ij} + a_{kl} + \sum_{\alpha\neq\chi(ij)}$ one has that $\Delta.(l - e_{ij} - e_{kl}) \geq 0$ is equivalent to (ii). Note that in Lemma 8.2 we require that at least one of the inequalities is strict. As Lemma 4.7 shows, this is automatically satisfied in this case.

For inequality (5),

$$\Delta.C_{kl} = D.(C_{kl} - 2e_5 + e_{i5} + e_{j5}) + a_{kl} + l_i + l_j - c_{\chi(kl)}.$$

Using (4.3) (to substitute l_i, l_j) and (4.2) $\Delta.C_{kl} \geq 0$ is equivalent to (iii).

For inequality (6),

$$\Delta.C_{k;l} = D.(C_{k;l} - 2e_5 + e_{l5}) + a_{ik} + a_{jk} + l_l + c_{\chi(ij)}.$$

By using (4.6) to substitute $a_{ik} + a_{jk} + l_l + c_{\chi(ij)}$, $\Delta.C_{k;l} \geq 0$ is equivalent to (iv).

For inequality (7) (recall that $C_i = 2l - e_{ij} - e_{ik} - e_{il}$),

$$\Delta.C_i = D.(C_i - 2e_5) + a_{ij} + a_{ik} + a_{il}.$$

By using (4.3) to substitute $a_{ij} + a_{ik} + a_{il}$, $\Delta.C_i = D.(C_i - 2e_5 + E_{i5}) + l_i$. But,

$$D.(C_i - 2e_5 + E_{i5}) = 2D.(l - e_i - e_5 + m_{i5}) + D.(2e_i - \sum_{j\neq i} e_{ij}).$$

From (9.6) and (9.4) it follows that $\Delta.C_i \geq 0$.

For inequality (8) (recall that $B = 3l - \sum_{i=1}^{4} e_i$),

$$\Delta.B = D.(B - 3e_5 + \sum_{i\neq 5} e_{i5}) + \sum l_i + \sum c_\alpha.$$

By using (4.5) to substitute $\sum l_i + 2\sum c_\alpha$, $\Delta.B \geq 0$ is equivalent to (v).

For inequality (9) (recall that $B = 3l - 2e_i - e_{jk} - e_{jl} - e_{kl}$),

$$\Delta.B_i = D.(B_i - 3e_5 + 2e_{i5}) + a_{jk} + a_{jl} + a_{kl} + 2l_i + \sum c_\alpha.$$

By using (4.6) to substitute $a_{jk} + a_{jl} + a_{kl} + l_i + \sum c_\alpha$, $\Delta.B_i = D.(B_i - 2e_5 + e_{i5})$. But one has

$$D.(B_i - 2e_5 + e_{i5}) = 2D.(l - e_i - e_5 + e_{i5}) + D.(l - e_{jk} - e_{jl} - e_{kl} - e_{i5}).$$

It follows from (9.6) and (9.7) that $\Delta.B_i \geq 0$. $\square$

Remark 4.6. In order to lift $\overline{s} \in \mathrm{H}^0(Y, \overline{D}^Y)$ we need to group s'_{ij} with $s_{\chi(ij)}$, such that we may lift $s_{ij} = s'_{ij} s_{\chi(ij)}$ to x_{ij5}, etc. (so in fact we lift $\overline{s} s_x^{m_x} s_y^{m_y} s_z^{m_z}$). For this we need to have enough sections s_x,s_y,s_z. Take the case of s_x: one needs exactly $a_{13} + a_{24} + c_y + c_z$ of them (to be distributed to s_{13},s_{24},s_y,s_z). Since the image of the restriction map r_Y in $\mathrm{H}^0(E_5, \overline{D})$ is

$$\mathrm{H}^0(Y, \overline{D}') s_x^{m_x} s_y^{m_y} s_z^{m_z},$$

the number of s_x's appearing in $\overline{s} s_x^{m_x} s_y^{m_y} s_z^{m_z}$ is $m_x + l_x$, and by (4.4) one has

$$m_x + l_x = a_{13} + a_{24} + c_y + c_z.$$

Lemma 4.7. *It is not possible to have $c_\alpha - l_\alpha \geq m_\alpha + D.L_\alpha$ for all $\alpha \in \{x, y, z\}$.*

Proof. Assume the contrary and add up the three inequalities. Then one has

$$\sum c_\alpha - \sum l_\alpha \geq \sum m_\alpha + \sum D.L_\alpha.$$

By (4.7), this is equivalent to $\sum D.L_\alpha \leq -D.e_5$, which contradicts Lemma 5.3. $\square$

4.1. **Proof of Claim 2.24.** Let $\overline{s}$ be a distinguished section in $\mathrm{H}^0(Y, \overline{D}^Y)$ as in (4.1). If inequalities (i)-(v) in Lemma 4.3 are satisfied, then by Lemma 4.3 $\overline{s}$ has straightforward lifting to D. Assume now that one of the inequalities (i)-(v) fails. We first show that we can keep replacing the section $\overline{s}$ with a linear combination of distinguished sections until we are in one of the following cases:

(A) $$c_x = c_y = c_z = 0,$$

(B) $$l_x = l_y = l_z = 0, c_x + c_y + c_z > 0,$$

(C) $$c_x = c_y = l_x = l_y = 0, c_z > 0, l_z > 0 \quad \text{(up to a permutation of } x, y, z).$$

This follows from:

Claim 4.8. If $l_\alpha > 0$ and $c_\beta > 0$ for $\alpha, \beta \in \{x, y, z\}$, $\beta \neq \alpha$, then we may replace $\overline{s}$ with a sum of distinguished sections $\overline{s}'$ for which both $c_x + c_y + c_z$ and $l_x + l_y + l_z$ decreased.

Proof. We may assume without loss of generality that $l_x > 0, c_z > 0$. Consider the following sections in the linear system $|\overline{H} - \overline{E}_y|$:

$$s_{xy} s_x, \quad s'_{14} s_1 s_4, \quad s'_{23} s_2 s_3.$$

The linear system $|\overline{H} - \overline{E}_y|$ is 1-dimensional, and any two of the above sections are linearly independent. Hence, we may replace $s_{xy} s_x$ with a linear combination of the sections $s'_{14} s_1 s_4, s'_{23} s_2 s_3$. The effect is: c_z, l_x decrease by 1, and either a_{14}, l_1, l_4 or a_{23}, l_2, l_3 increase by 1. Note that c_x, c_y, l_y, l_z stay the same. Hence, both $c_x + c_y + c_z$ and $l_x + l_y + l_z$ decreased by 1. $\square$

Note that while doing replacements as in Claim 4.8 we ignore how the changes affect inequalities in Lemma 4.3.

4.2. **Case (A): $c_x = c_y = c_z = 0$.** This case is very similar to Case I.

Lemma 4.9. *If $c_x = c_y = c_z = 0$, then $\overline{s}$ has straightforward lifting to D if and only if for all $\{i, j, k, l\} = \{1, 2, 3, 4\}$ one has*

$$a_{ij} \leq D.(C_{k;l} - e_5).$$

Proof. One may immediately see (use for example Remark 4.4) that the inequalities (i), (iii), (v) in Lemma 4.3 are satisfied. The inequality (ii) is satisfied (see Remark 4.2). Condition (iv) in Lemma 4.3 becomes $a_{ij} \leq D.(C_{k;l} - e_5)$ in Case (A). □

Algorithm for replacing s – Case (A). *If for all $\{i, j, k, l\} = \{1, 2, 3, 4\}$, one has $a_{ij} \leq D.(C_{k;l} - e_5)$, and by Lemma 4.9 $\overline{s}$ has straightforward lifting to D. If for some i, j, k, l one has $a_{ij} > D.(C_{k;l} - e_5)$, we will replace $\overline{s}$ with a sum of distinguished sections such that all the inequalities improve, while leaving $c_x = c_y = c_z = 0$. We do this in exactly the same way as we did in Case I, as Lemma 3.6, Claim 3.5, as well as the Algorithm 3.2 all apply word by word.*

4.3. **Case (B): $l_x = l_y = l_z = 0$, $c_x + c_y + c_z > 0$.** This is impossible because of (4.7) and Lemma 5.3.

4.4. **Case (C): $c_x = c_y = l_x = l_y = 0$, $c_z > 0, l_z > 0$.**

Remark 4.10. Under the assumptions of Case (C) the relations in Remark 4.5 become

$$\sum a_{ij} + c_z = D.e_5, \tag{4.8}$$

$$a_{13} + a_{24} + c_z = m_x, \quad a_{14} + a_{23} + c_z = m_y, \tag{4.9}$$

$$a_{12} + a_{34} - l_z = m_z, \tag{4.10}$$

$$c_z - l_z = m_x + m_y + m_z - D.e_5. \tag{4.11}$$

From (4.9) one has

$$0 < c_z \leq \min\{m_x, m_y\}. \tag{4.12}$$

From the definitions of m_x, m_y it follows that $m_x = -D.L_x$,$m_y = -D.L_y$. From (4.10) and (4.11) one has

$$a_{12} + a_{34} - c_z = m_5 - m_x - m_y = D.(2l - e_5 - e_{13} - e_{14} - e_{23} - e_{24}). \tag{4.13}$$

Lemma 4.11. *Under the assumptions of Case (C) $\overline{s}$ has straightforward lifting to D if and only if:*

$$a_{ij} \leq D.(C_{ij} - e_5) + D.(2l - e_5 - e_{13} - e_{14} - e_{23} - e_{24}), \tag{iii$'$}$$

$$a_{ij} \leq D.(C_{k;l} - e_5), \tag{iv$'$}$$

whenever either $ij = 12, kl = 34$ or $ij = 34, kl = 12$.

Remark 4.12. By (9.7) and (9.8) the right hand sides of (iii′), (iv′) are ≥ 0.

Proof of Lemma 4.11. We claim that in Lemma 4.3 the inequalities (i), (ii) and (v) are satisfied and that (iii), respectively (iv), are equivalent to (iii′) and (iv′).

Inequality (i): by Remark 4.4 the inequalities involving c_x, c_y are automatic. We claim that $c_z \leq D.(l - e_5 - e_{ij})$ whenever $ij = 12$ or 34: by (4.12) one has $c_z \leq m_x, m_y$, hence $c_z \leq (m_x + m_y)/2$, and the claim follows from Lemma 5.2.

Inequality (ii): this is clearly satisfied for $l_x - c_x = 0$, $l_y - c_y = 0$. From (4.11) and Lemma 5.3 it follows that $c_z - l_z \leq 0$, and we are done by Remark 4.12.

Inequality (iii): the inequalities involving c_x and c_y are automatically satisfied. The inequalities (iii) involving c_z are of the form (here $ij = 12$ or 34)

$$c_z - a_{ij} \leq D.(C_{kl} - e_5). \tag{4.14}$$

Using (4.13) to substitute $c_z - a_{ij}$ in (4.14), one obtains (iii′):

$$a_{kl} \leq D.(C_{kl} - e_5) + D.(2l - e_5 - e_{13} - e_{14} - e_{23} - e_{24}).$$

Inequality (iv): We claim that the inequalities involving $a_{13}, a_{14}, a_{23}, a_{24}$ are satisfied: this is because by (4.9) $a_{ij} + c_z \leq m_x$ whenever $ij \neq 12, 34$. By Lemma 5.1 $m_x \geq D.(C_{k;l} - e_5)$, and we are done. The inequalities (iv) involving a_{12}, a_{34} are exactly the inequalities (iv′).

Inequality (v): this follows from (4.12) and Lemma 5.2. □

4.5. **Algorithm for replacing s in Case (C).** If the inequalities in Lemma 4.11 are satisfied, then $\overline{s}$ has straightforward lifting to D. Assume one of (iii′) or (iv′) is not satisfied, say for a_{12} (the same argument applies for a_{34}). Then by Remark 4.12 one has $a_{12} > 0$. Then we make replacements to decrease a_{12} as follows. Consider the following sections in the linear system $|2\overline{H} - \overline{E}_1 - \overline{E}_2 - \overline{E}_x - \overline{E}_y|$:

$$s'_{12}s_{xy}s_z, \quad s'_{13}s_{23}s_3^2, \quad s'_{14}s_{24}s_4^2.$$

The linear system is 1-dimensional, and any two of the above sections are linearly independent. Since $a_{12}, c_z, l_z > 0$, we may replace $s'_{12}s_{xy}s_z$ in s with a linear combination of $s'_{13}s'_{23}s_3^2, s'_{14}s'_{24}s_4^2$. The effect is: a_{12}, c_z, l_z decrease by 1, while either a_{13}, a_{23} increase by 1 or a_{14}, a_{24} increase by 1. Note that besides the above changes and the changes affecting the l_i's (which we ignore, since they do not appear in (iii′), (iv′)) no other changes occur. In particular, we still have $c_x = c_y = l_x = l_y = 0$.

The inequalities involving a_{12} were improved (while the ones involving a_{34} remained the same). If after the replacement $c_z = 0$ or $l_z = 0$ we are in Case (A) or Case (B), we apply the procedure described for those cases. If after the replacement we still have $c_z > 0$ and $l_z > 0$, then we are in Case (C), and therefore all inequalities are satisfied, except perhaps (iii′), (iv′) for a_{12} or a_{34}.

5. Inequalities involving m_x,m_y,m_z

The assumptions in this section are the same as in the Main Claim. Recall:

$$L_{\chi(ij)} = L_{\chi(kl)} = l - e_5 - e_{ij} - e_{kl}.$$

Lemma 5.1. *For any* $\{i, j, k, l\} = \{1, 2, 3, 4\}$ *one has*

$$-D.L_{\chi(kl)} \leq D.(C_{k;l} - e_5),$$

where $C_{k;l} = 2l - e_l - e_{ik} - e_{jk}$.

Proof. One has

$$D.(C_{k;l} - e_5) + D.L_{\chi(kl)} = D.(l - e_{ik} - e_{jk} - e_{ij} - e_{l5})$$
$$+D.(l - e_5 - e_l + e_{l5}) + D.(l - e_5 - e_{kl}) \geq 0.$$

It follows from (9.6) and (9.7) that $D.(C_{k;l} - e_5) + D.L_{\chi(kl)} \geq 0$. □

Lemma 5.2. *For any $i, j \in \{1, 2, 3, 4\}$ one has*

$$-\frac{1}{2}D.(\sum_{\alpha \neq \chi(ij)} L_\alpha) \leq min\ \{d - m_5 - m_{ij}, 3d - \sum_{i=1}^{5} m_i\}.$$

Proof. Without loss of generality, we may assume $ij = 12$. One has

$$2D.(l - e_5 - e_{12}) + D.(L_x + L_y)$$
$$= 2D(l - e_5 - e_{12}) + D.(2l - 2e_5 - e_{13} - e_{14} - e_{23} - e_{24})$$
$$= D.(l - e_{12} - e_{13} - e_{23} - e_{45}) + (l - e_{12} - e_{14} - e_{24} - e_{35})$$
$$+D.(l - e_5 - e_3 + e_{35}) + D.(l - e_5 - e_4 + e_{45}) + D.(e_3 + e_4 - 2e_5).$$

The first inequality follows from (9.7) and the assumption $D.e_5 \leq D.e_i$. Moreover,

$$2D(3l - \sum_{i=1}^{5} e_i) + D.(L_x + L_y)$$
$$= 2(3l - \sum_{i=1}^{5} e_i) + D.(2l - 2e_5 - e_{13} - e_{14} - e_{23} - e_{24})$$
$$= D.(2l - e_1 - e_3 - e_5 - e_{24}) + D.(2l - e_1 - e_4 - e_5 - e_{23})$$
$$+D.(2l - e_2 - e_3 - e_5 - e_{14}) + D.(2l - e_2 - e_4 - e_5 - e_{13}).$$

The second inequality now follows from (9.7). □

Lemma 5.3. *One has $(m_x + m_y + m_z) \leq D.e_5$,$-D.(L_x + L_y + L_z) < D.e_5$.*

Proof. Note by definition of m_x that if $m_x > 0$, then $m_x = -D.L_x$ (similarly for y, z), if $m_x = m_y = m_z = 0$. The claim follows from (9.4).

Case 1) Assume just one of m_x, m_y, m_z is > 0, say $m_x > 0, m_y = m_z = 0$:

$$D.e_5 - (m_x + m_y + m_z) = D.(l - e_{13} - e_{24}).$$

However, $D.(l - e_{13} - e_{24}) \geq 0$ (see Lemma 8.2). The other cases are similar.

Case 2) Assume two of m_x, m_y, m_z is > 0, say $m_x, m_y > 0, m_z = 0$:

$$D.e_5 - (m_x + m_y + m_z) = D.(2l - e_5 - e_{13} - e_{14} - e_{23} - e_{24}).$$

By (9.8) $D.e_5 - (m_x + m_y + m_z) \geq 0$. The other cases are similar.

Case 3) Assume $m_x, m_y, m_z > 0$:

$$D.e_5 - (m_x + m_y + m_z) = D.(L_x + L_y + L_z + e_5) = D.(3l - 2e_5 - \sum_{i,j=1,\ldots,4} e_{ij})$$
$$= D.(2e_i - \sum_{j \neq i} e_{ij}) + 2D(l - e_5 - e_i + e_{i5}) + D.(l - e_{jk} - e_{kl} - e_{jl} - e_{i5}),$$

for any $\{i, j, k, l\} = \{1, 2, 3, 4\}$. By (9.4), (9.6), (9.7) $D.e_5 - (m_x + m_y + m_z) \geq 0$.

If $-D.(L_x + L_y + L_z) = D.e_5$, by the above computation one has (here for simplicity, we let $d = D.l, m_i = D.e_i, m_{ij} = D.e_{ij}$)

$$2m_i - \sum_{j \neq i} m_{ij} = 0, \quad d - m_5 - m_i + m_{i5} = 0, \quad m_{jk} + m_{kl} + m_{jl} + m_{i5} = d.$$

It follows that

$$m_{ij} + m_{ik} + m_{il} = d - m_5 + m_i, \tag{5.1}$$

$$m_{jk} + m_{kl} + m_{jl} = 2d - m_5 - m_i. \tag{5.2}$$

Adding up all relations (5.1) and (5.2), one has:

$$2 \sum_{i,j=1,\dots,4} m_{ij} = 4d - 4m_5 + \sum_{i=1}^{4} m_i, \quad 2 \sum_{i,j=1,\dots,4} m_{ij} = 8d - 4m_5 - \sum_{i=1}^{4} m_i.$$

It follows that $\sum_{i=1}^4 m_i = 2d$. However, by assumption $m_i \geq m_5$ for all i, hence $m_5 \leq d/2$. As $0 \leq m_{i5} = m_i + m_5 - d$ it follows that $m_i \geq d - m_5 \geq d/2$. Since $\sum_{i=1}^4 m_i = 2d$ it follows that $m_i = d/2$, $m_{i5} = 0$. Moreover, $m_{ij} + m_{ik} + m_{il} = d$. As $d > 0$ it follows that $m_{ij} > 0$ for some $i, j \in \{1, \dots, 4\}$. This contradicts the assumption in the Main Claim. $\square$

6. Multiplicity estimates

Let l be the unique line in $\mathbb{P}^3$ that passes through p_5 and intersects lines l_{13} and l_{24} (the other cases are similar). Let L be the proper transform of l in $\overline{M}$.

Proposition 6.1. *Let $D = dH - \sum m_i E_i - m_{ij} E_{ij}$ be an effective divisor on $\overline{M}$. Let m be the multiplicity of D along L. Then*

$$m \geq m_5 + m_{13} + m_{24} - d.$$

Proof. Let $\rho : X \to \overline{M}$ be the blow-up of $\overline{M}$ along L and let E be the exceptional divisor. Let $\tilde{D}$ be the proper transform of D. Then $\rho^* D = \tilde{D} + mE$. Restricting to E, one has

$$(\rho^* D)_{|E} = \tilde{D}_{|E} + mE_{|E}. \tag{6.1}$$

Let N be the normal bundle of L in $\overline{M}$ and let $N_{l|\mathbb{P}^3}$ be the normal bundle of l in $\mathbb{P}^3$. If l' is the proper transform of l in the blow-up X of $\mathbb{P}^3$ along $p_1, \dots, p_5$, let N' be the normal bundle of l' in $\overline{M}$. One has

$$N_{l|\mathbb{P}^3} = \mathcal{O}(-1) \oplus \mathcal{O}(-1), \quad N' = \pi^* N_{l|\mathbb{P}^3}(-E_5) = \mathcal{O} \oplus \mathcal{O}. \tag{6.2}$$

It is easy to see that $\deg(N) = \deg(N') - 2 = -2$. In fact we have the following:

Claim 6.2. $N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Proof. Note that one could obtain $\overline{M}$ by blowing up $\mathbb{P}^3$ first along the points $p_1, \dots, p_4$, then the proper transforms of the lines l_{13} and l_{24}, and then the point p_5 and the proper transforms of the lines l_{ij}, for all $ij \neq 13, 24$. Let Λ be the plane in $\mathbb{P}^2$ spanned by the line l and l_{13}. Then the proper transform $\tilde{\Lambda}$ of Λ in $\overline{M}$ is the blow-up of $\Lambda \cong \mathbb{P}^2$ along p_1, p_3, p_5, q, where $q = l_{24} \cap \Lambda$. If $N_{L|\tilde{\Lambda}}$ is the normal bundle of L in $\tilde{\Lambda}$ and $N_{\tilde{\Lambda}|\overline{M}}$ is the normal bundle of $\tilde{\Lambda}$ in $\overline{M}$, one has an exact sequence:

$$0 \to N_{L|\tilde{\Lambda}} \to N \to (N_{\tilde{\Lambda}|\overline{M}})_{|L} \to 0. \tag{6.3}$$

It is easy to see that $N_{L|\tilde{\Lambda}} = \mathcal{O}(-1)$. Since $\deg(N) = -2$ and $\mathcal{O}(-1)$ is a subbundle of N (the quotient is a line bundle), it follows that $N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. $\square$

Then $E = \mathbb{P}(N) \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let $p : E \to \bar{l} = \mathbb{P}^1$ be the restriction of ρ to E. Let $q : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the other projection. Then

$$E_{|E} = \mathcal{O}_E(-1) = q^*\mathcal{O}(-1) \otimes p^*\mathcal{O}(-1).$$

Note that $(\rho^*D)_{|E} = p^*(D_{|L})$ and $D_{|L} = \mathcal{O}(a)$, where we let $a = D.L$. One has

$$H.L = E_5.L = E_{13}.L = E_{24}.L = 1, E_i.L = E_{ij}.L = 0, \text{ for all other indices } i, j.$$

It follows that $a = d - m_5 - m_{13} - m_{24}$. From (6.1) one has

$$\tilde{D}_{|E} = p^*\mathcal{O}(a + m) \otimes q^*\mathcal{O}(m).$$

Since $\tilde{D}_{|E}$ is effective, it follows that $m \geq -a = m_5 + m_{13} + m_{24} - d$. $\square$

7. Proof of Lemma 2.21

Recall that Y is the blow-up of $\mathbb{P}^2$ along $q_1, \ldots, q_4, x, y, z$.

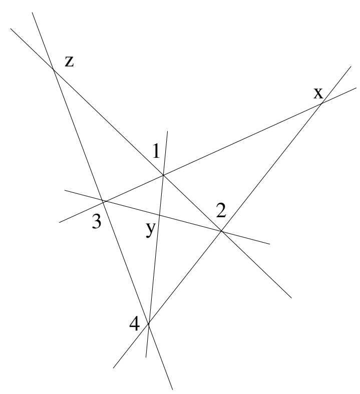

Figure 1. The configuration of the points $q_1, q_2, q_3, q_4, x, y, z$

Let

$$D = d\overline{H} - \sum_{i=1}^{4} m_i \overline{E}_i - m_x \overline{E}_x - m_y \overline{E}_y - m_z \overline{E}_z$$

be a divisor on Y. Assume D is effective and let s be a section in $\mathrm{H}^0(Y, D)$.

We show that s is generated by distinguished sections on Y by induction on d. Let $\bar{l}'_{ij}$ (respectively $\bar{l}_{xy}$,$\bar{l}_{yz}$,$\bar{l}_{xz}$) be the proper transforms in Y of the lines $\bar{l}_{ij}$ (respectively $\overline{xy}$, $\overline{yz}$,$\overline{xz}$). We may assume $D.C \geq 0$ for C among the classes:

$$\bar{l}'_{ij}, \bar{l}_{xy}, \bar{l}_{xz}, \bar{l}_{yz}, \overline{E}_i, \overline{E}_x, \overline{E}_y, \overline{E}_z.$$

This is because if $D.C < 0$, then $s = x_C s'$, where $s' \in \mathrm{H}^0(Y, D - C)$ and x_C is a generator of $\mathrm{H}^0(Y, C)$, and s' is generated by distinguished sections by induction. Hence, we assume

$$(*) \qquad d \geq m_i + m_j + m_{\chi(ij)}, \quad d \geq m_x + m_y, \quad d \geq m_x + m_z, \quad d \geq m_y + m_z,$$
$$d \geq m_i \geq 0, \quad d \geq m_x \geq 0, \quad d \geq m_y \geq 0, \quad d \geq m_z \geq 0.$$

If $d = 0$, then it follows by $(*)$ $D = 0$. Assume $d > 0$. We may assume without loss of generality that

$$m_4 \leq m_1 \leq m_2 \leq m_3.$$

Consider the restriction map:

$$r : \mathrm{H}^0(Y, D) \to \mathrm{H}^0(\overline{E}_4, D_{|\overline{E}_4}) = \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(m_4)).$$

It is enough to show that we may lift any $t \in \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(m_4))$ to a section in $\mathrm{H}^0(Y, D)$ generated by distinguished sections on Y. This is because by the same argument as in Section 2, if s, s' are sections in $\mathrm{H}^0(Y, D)$ are such that $r(s) = r(s')$, then $s - s'$ is in $\mathrm{H}^0(Y, D - \overline{E}_4)$ and we are done by induction.

Let t_i be the restriction in $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(1))$ of the section s_{i4} corresponding to $\overline{l}'_{i4}$. Any two of t_1, t_2, t_3 generate $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(1))$. In particular, it is enough to lift $t = {t_1}^k {t_3}^{m_4 - k}$ (for any $0 \leq k \leq m_4$) to a combination of distinguished sections. We lift t_i to s_{i4}, hence t to $s_{14}^k s_{34}^{m_4-k}$ (a section of $D' = k\overline{l}'_{14} + (m_4 - k)\overline{l}'_{34}$). Let:

$$\Delta = D - D' = (d - m_4)\overline{H} - (m_1 - k)\overline{E}_1 - m_2\overline{E}_2 - (m_3 - m_4 + k)\overline{E}_3$$
$$-m_x\overline{E}_x - (m_y - k)\overline{E}_y - (m_z - m_4 + k)\overline{E}_z.$$

Claim 7.1. There is a section $u \in \mathrm{H}^0(\Delta)$, generated by distinguished sections and such that $u_{|\overline{E}_4} \in \mathrm{H}^0(\overline{E}_4, \mathcal{O})$ is non-zero.

Assuming Claim 7.1, we lift t to $us_{14}^k s_{24}^{m_4-k}$ and we are done.

Proof of Claim 7.1. Let

$$\Delta' = \Delta - (m_1 - k)\overline{l}'_{12} = (d - m_4 - m_1 + k)\overline{H} - (m_2 - m_1 + k)\overline{E}_2$$
$$-(m_3 - m_4 + k)\overline{E}_3 - m_x\overline{E}_x - (m_y - k)\overline{E}_y - (m_z - m_1 - m_4 + 2k)\overline{E}_z.$$

Note that since $k \leq m_4 \leq m_1$ and since a section corresponding to $\overline{l}'_{12}$ has non-zero restriction to $\overline{E}_4$, it is enough to show that there is a section $u' \in \mathrm{H}^0(\Delta')$, generated by distinguished sections and such that $u'_{|\overline{E}_4} \in \mathrm{H}^0(\overline{E}_4, \mathcal{O})$ is non-zero. $\square$

Case when $m_y - k < 0$. Let

$$\Delta'' = \Delta + (m_y - k)\overline{E}_y = (d - m_4 - m_1 + k)\overline{H} - (m_2 - m_1 + k)\overline{E}_2$$
$$-(m_3 - m_4 + k)\overline{E}_3 - m_x\overline{E}_x - (m_z - m_1 - m_4 + 2k)\overline{E}_z.$$

It is enough to show that there is a section $u'' \in \mathrm{H}^0(\Delta'')$, generated by distinguished sections and such that $u''_{|\overline{E}_4} \in \mathrm{H}^0(\overline{E}_4, \mathcal{O})$ is non-zero. Since Δ'' is a divisor on the blow-up of $\mathbb{P}^2$ along the points q_2, q_3, x, z, it follows from Lemma 7.3 (a direct check shows that all inequalities (7.1) hold; use $k \leq m_4 \leq m_i$ and $(*)$) and Lemma 7.4 applied to the lines $\overline{q_2, x}$ and $\overline{q_3, z}$, that there is a section $u'' \in \mathrm{H}^0(\Delta'')$, generated by distinguished sections and not containing q_4 in its zero-locus.

Case when $m_y - k \geq 0$. Denote

$$N_1 = m_1 + m_4 + m_x + m_y - d - 2k,$$
$$N_2 = 2d - m_2 - m_3 - m_x - m_z - 2k.$$

Claim 7.2. $N_1 \leq N_2$, $0 \leq N_2$, $N_1 \leq m_y - k$.

Proof Claim 7.2. We have

$$N_2 - N_1 = (d - m_1 - m_2 - m_z) + (d - m_3 - m_4 - m_x) + (d - m_x - m_y) \geq 0,$$

using $(*)$ and $m_4 \leq m_i$. Similarly, as $0 \leq k \leq m_4$, we have $N_2 \geq 0$ and $N_1 \leq m_y - k$ (using $(*)$ and $m_4 \leq m_i$). □

By Claim 7.2, we may choose $\alpha, \beta \geq 0$ be integers such that $\alpha + \beta = m_y - k$ and $N_1 \leq \alpha \leq N_2$. Let

$$\begin{aligned}\Delta'' = \Delta - \alpha\bar{l}_{xy} - \beta\bar{l}'_{23} &= (d - m_1 - m_4 - m_y + 2k)\overline{H} - (m_2 - m_1 + k - \beta)\overline{E}_2\\ &-(m_3 - m_4 + k - \beta)\overline{E}_3 - (m_x - \alpha)\overline{E}_x - (m_z - m_1 - m_4 + 2k)\overline{E}_z.\end{aligned}$$

Since $\bar{l}_{xy}$ and $\bar{l}'_{23}$ have non-zero restriction to $\overline{E}_4$, it is enough to find $u \in \mathrm{H}^0(\Delta'')$ such that $u_{|\overline{E}_4} \neq 0$. As before, since Δ'' is a divisor on the blow-up of $\mathbb{P}^2$ along the points q_2, q_3, x, z, it follows from Lemma 7.3 and Lemma 7.4 applied to the lines $\overline{q_2, x}$ and $\overline{q_3, z}$, that there is a section $u'' \in \mathrm{H}^0(\Delta'')$, generated by distinguished sections and not containing q_4 in its zero-locus. All inequalities follow in a straightforward way from $(*)$ and $m_4 \leq m_i$, except for:

- $\Delta''. (\overline{H} - \overline{E}_x) \geq 0$, (equivalent to $\alpha \geq N_1$)
- $\Delta''. (\overline{H} - \overline{E}_3) \geq 0$, (use that $m_1 \leq m_2$)
- $\Delta''. (2\overline{H} - \overline{E}_2 - \overline{E}_3 - \overline{E}_x - \overline{E}_z) \geq 0$ (equivalent to $\alpha \leq N_2$). □

Lemma 7.3. *Let Z be the blow-up of $\mathbb{P}^2$ along points $q_1, \ldots, q_4$ (no three collinear). One has $H^0(D) \neq 0$ for a divisor $D = d\overline{H} - \sum_{i=1}^4 m_i\overline{E}_i$ if and only if*

$$d \geq 0, \quad d \geq m_i \quad 2d \geq \sum_{i=1}^{4} m_i. \tag{7.1}$$

The Cox ring $\mathrm{Cox}(Z)$ *is generated by sections corresponding to the lines $\bar{l}_{ij}$ and the exceptional divisors $\overline{E}_i$.*

Proof. It is a well known result that the Cox ring $\mathrm{Cox}(Z)$ is generated by sections corresponding to the lines $\bar{l}_{ij}$ and the exceptional divisors $\overline{E}_i$; see for example [BP].

If D is an effective divisor, then clearly, the inequalities (7.1) hold. Conversely, assume (7.1) hold. We write D as an effective combination of the classes of the lines $\bar{l}_{ij} = \overline{H} - \overline{E}_i - \overline{E}_j$ and the exceptional divisors $\overline{E}_i$. Consider the table with 2 rows and d columns filled with $\overline{E}_i$'s in the following way. Start in the upper left corner and write m_1 $\overline{E}_1$'s in the first row. Then write m_2 $\overline{E}_2$'s passing to the second row if necessary, and so on. Fill the remaining entries with zeros. For example, if $D = 5\overline{H} - 3\overline{E}_1 - 3\overline{E}_2 - 2\overline{E}_3 - \overline{E}_4$,

$$\begin{array}{ccccc}\overline{E}_1 & \overline{E}_1 & \overline{E}_1 & \overline{E}_2 & \overline{E}_2,\\ \overline{E}_2 & \overline{E}_3 & \overline{E}_3 & \overline{E}_4 & 0.\end{array}$$

Our conditions guarantee that all entries of a given column are different. Therefore D is the sum of classes $H - (\overline{E}_i + \overline{E}_j)$, one for each column, where $\overline{E}_i, \overline{E}_j$ are the entries of the column. In the example above,

$$D = (\overline{H} - \overline{E}_1 - \overline{E}_2) + (\overline{H} - \overline{E}_1 - \overline{E}_3) + (\overline{H} - \overline{E}_1 - \overline{E}_3) + (\overline{H} - \overline{E}_2 - \overline{E}_4) + (\overline{H} - \overline{E}_2).$$

□

Lemma 7.4. *In the notations of Lemma 7.3, let D be a divisor such that $H^0(D) \neq 0$. Let q be the intersection of the lines $\overline{q_1q_2}$ and $\overline{q_3q_4}$. The linear system $|D|$ does not contain q as a base point if and only if*

$$D.(\overline{H} - \overline{E}_1 - \overline{E}_2) \geq 0, \quad D.(\overline{H} - \overline{E}_3 - \overline{E}_4) \geq 0.$$

Proof. The conditions are clearly necessary. It is enough to show that D can be written as an effective combination of lines $\overline{l}_{ij}$ ($\overline{l}_{ij} \neq \overline{l}_{12}, \overline{l}_{34}$) and the exceptional divisors $\overline{E}_i$. Let

$$D = \sum k_{ij}\overline{l}_{ij} + \sum k_i\overline{E}_i, \quad k_{ij}, k_i \geq 0.$$

Assume $k_{12} > 0$. Note that the only generators E of $\mathrm{Cox}(Z)$ with the property that $E.\overline{l}_{12} > 0$ are $\overline{l}_{34}, \overline{E}_1, \overline{E}_2$. Since $D.\overline{l}_{12} \geq 0$, it follows that one of $k_{34}, k_1, k_2 > 0$. If $k_1 > 0$ we may replace $\overline{l}_{12} + \overline{E}_1$ with a divisor in the pencil $|\overline{H} - \overline{E}_2|$ that does not contain $\overline{l}_{12}$ (for example $\overline{l}_{23} + \overline{E}_3$). The case $k_2 > 0$ is similar. If $k_{34} > 0$, we replace $\overline{l}_{12} + \overline{l}_{34}$ with, for example, $\overline{l}_{13} + \overline{l}_{24}$. The case when $k_{34} > 0$ is similar. At the end of this process, we have $k_{12} = k_{34} = 0$. □

8. Inequalities for the effective cone of X

Let X be the iterated blow-up of $\mathbb{P}^3$ in points $p_1, \ldots, p_4$ (in a linearly general position) and proper transforms of lines l_{ij} ($i, j = 1, \ldots, 4$, $i \neq j$). Let E_i, E_{ij} be the exceptional divisors. Let l be the class on X of the proper transform of a general line in $\mathbb{P}^3$. Let e_i be the class of (the proper transform of) a general line in E_i. Let e_{ij} be the class of a fiber of the $\mathbb{P}^1$-bundle $E_{ij} \to l_{ij}$.

Notation 8.1. For $\{i, j, k, l\} = \{1, 2, 3, 4\}$ let

$$C_{ij} = 2l - e_{ij} - e_k - e_l,$$
$$C_{i;j} = 2l - e_{ik} - e_{il} - e_j,$$
$$C_i = 2l - e_{ij} - e_{ik} - e_{il},$$
$$B = 3l - \sum_{i=1}^{4} e_i,$$
$$B_i = 3l - 2e_i - e_{jk} - e_{jl} - e_{kl}.$$

Lemma 8.2. *Let D be a divisor on X. Then D is an effective sum (with integer, non-negative coefficients) of boundary divisors $\Lambda_{ijk}, E_{ij}, E_i$ (in particular, $H^0(D) \neq 0$) if $D.C \geq 0$ for all C in the list below (for all $\{i, j, k, l\} = \{1, 2, 3, 4\}$):* (1) *$l$;* (2) *$l - e_i$;* (3) *$l - e_{ij}$;* (4) *$l - e_{ij} - e_{kl}$;* (5) *$C_{ij}$;* (6) *$C_{i;j}$;* (7) *$C_i$;* (8) *$B$;* (9) *$B_i$, and moreover, if one has $D.(l - e_{ij} - e_{kl}) > 0$ for some i, j, k, l.*

It is easy to see that each of the classes C in (1)-(9) in Lemma 8.2 cover a dense set of X; hence, for any effective divisor D one has $D.C \geq 0$, i.e., C is a *nef* curve.

Remark 8.3. It is a standard fact that the divisor D is in the convex hull of the effective divisors $\Lambda_{ijk}, E_{ij}, E_i$ (where Λ_{ijk} is the proper transform of the plane $\overline{p_ip_jp_k}$) if and only if inequalities (1)-(9) hold. However, the extra condition of having at least one strict inequality in (4) is necessary for $\mathrm{H}^0(D) \neq 0$, as the following example shows: if $D = 2H - \sum_{i \neq j} E_{ij}$, then it is easy to see that $\mathrm{H}^0(D) = 0$ ($\mathrm{H}^0(2D) \neq 0$) and D satisfies all of (1)-(9).

Observation 8.4. If

$$D = dH - \sum m_i E_i - \sum m_{ij} E_{ij}$$

is such that $d \geq 0$, $d \geq m_i, m_{ij}$ (for all i, j) and there is an i such that $m_i \leq 0$ and $m_{ij} \leq 0$ for all $j \neq i$, then D is an effective sum of boundary, as one has

$$D = d\Lambda_{jkl} + \sum_{j \neq i} (d - m_j) E_j + \sum_{u,v \neq i} (d - m_{uv}) E_{uv} + (-m_i) E_i + \sum_{j \neq i} (-m_{ij}) E_{ij}.$$

Proof of Lemma 8.2. Let $D = dH - \sum m_i E_i - \sum m_{ij} E_{ij}$. One has

$$d = D.l, \quad m_i = D.e_i, \quad m_{ij} = D.e_{ij}.$$

We do an induction on d. If $d = 0$, then from (2) and (3) $m_i, m_{ij} \leq 0$ and we are done by Observation 8.4. Assume $d > 0$. We show that there are i, j, k such that $D' = D - \Lambda_{ijk}$ also satisfies (1)-(9), and hence, we are done by induction.

Note that $D' = D - \Lambda_{ijk}$ (for any i, j, k) always satisfies (1), (4), (5), (8), (9). Moreover, one has at least one strict inequality in (4). Inequality (2) fails for D' if and only if $m_l = d$, where $l \neq i, j, k$, and (3) fails for D' if and only if one of m_{il}, m_{jl}, m_{kl} equal d. Inequality (6) fails for D' if and only if one has

$$2d = m_{li} + m_{lj} + m_k$$

(or the similar inequalities for a permutation of indices i, j, k). Inequality (7) fails for D' if and only if $m_{il} + m_{jl} + m_{kl} \in \{2d - 1, 2d\}$.
Case I: $m_{ij} = d$ for some i, j. We may assume $d = m_{12}$. From (4), $m_{34} \leq 0$.
Case 1: $m_i = d$ for $i \in \{3, 4\}$. We may assume $m_4 = d$. Then by (5) one has that $m_3 \leq 0$ and by (6) one has $m_{13}, m_{23} \leq 0$, and we are done by Observation 8.4.
Case 2: Assume $m_3 < d, m_4 < d$.

We may assume that m_{13} is the largest among $m_{13}, m_{14}, m_{23}, m_{24}$.

Claim 8.5. One has $m_{14}, m_{24} < d$.

Proof. Assume $m_{i4} = d$ for $i = \{1, 2\}$. Since by assumption $m_{i4} \leq m_{13}$ and $m_{13} \leq d$, one has $m_{13} = d$. If $m_{14} = d$, since $m_{12} = d$, one has a contradiction with (7). If $m_{24} = d$ one contradicts (4). □

Claim 8.6. The divisor $D' = D - \Lambda_{123}$ satisfies (1)-(9).

Proof. Inequality (2) holds, as $m_4 < d$. Since by Claim 8.5 $m_{14}, m_{24} < d$ and since $m_{34} \leq 0$, inequality (3) holds. If (7) is not satisfied, i.e., $m_{14} + m_{24} + m_{34} = 2d - 1$ or $2d$, one has a contradiction, as $m_{34} \leq 0$ and (by Claim 8.5) $m_{14}, m_{24} < d$. If (6) is not satisfied for D', then one has

$$2d = m_{i4} + m_{j4} + m_k, \tag{8.1}$$

for some $\{i, j, k\} = \{1, 2, 3\}$. If $k = 3$, then by (6) one has $2d \geq m_{12} + m_{24} + m_3$. Since $m_{12} = d$ one has $d \geq m_{24} + m_3$ and hence, by (8.1), $m_{14} = d$. This contradicts Claim 8.5. If $k \in \{1, 2\}$ (say $i = 3$), one has $2d = m_{34} + m_{j4} + m_k$. Since $m_{34} \leq 0$, $m_{j4} < d$ (Claim 8.5) and $m_k \leq d$; this is a contradiction. □

Case II: $m_i = d$ for some i, $m_{ij} < d$ for all i, j. We may assume that $d = m_4$. We may also assume that $d \geq m_3 \geq m_2 \geq m_1$. By (9) and (5) one has

$$m_{12} + m_{13} + m_{23} \leq d, \tag{8.2}$$

$$m_{ij} + m_k \leq d, \tag{8.3}$$

where $\{i, j, k\} = \{1, 2, 3\}$.
Case 1: $m_{23} > 0$. By (8.2) one has

$$m_{12} + m_{13} < d. \tag{8.4}$$

Claim 8.7. The divisor $D' = D - \Lambda_{234}$ satisfies (1)-(9).

Proof. Inequality (3) is satisfied by the assumption $m_{ij} < d$. Inequality (2) is not satisfied if and only if one has $m_1 = d$. If $m_1 = d$, it follows from the assumptions that $m_2 = m_3 = d$. As $m_4 = d$, one has a contradiction with (8). If (7) is not satisfied, i.e., $m_{12}+m_{13}+m_{14} = \{2d-1, 2d\}$, one has a contradiction with $m_{14} < d$ and $m_{12} + m_{13} < d$ (8.4). If (6) is not satisfied, then one has

$$2d = m_{1i} + m_{1j} + m_k, \tag{8.5}$$

for some $\{i, j, k\} = \{2, 3, 4\}$. If $k = 4$, since $m_4 = d$ one has from (8.5) $d = m_{12} + m_{13}$, which contradicts (8.4). If $k \in \{2, 3\}$ (say $i = 4$) one has $2d = m_{14} + m_{1j} + m_k$ for $\{k, j\} = \{2, 3\}$. But $m_{14} < d$ and $m_{j1} + m_k \leq d$ (8.3). This is a contradiction. □

Case 2: $m_{23} \leq 0$. If $m_2 = d$, then it follows from the assumptions that $m_3 = d$. As $m_4 = d$, one has from (8) that $m_1 \leq 0$. It follows from (5) that $m_{1i} \leq 0$ for all $i = 2, 3, 4$. Then we are done by Observation 8.4. Hence, we may assume $m_2 < d$.

Claim 8.8. The divisor $D' = D - \Lambda_{134}$ satisfies (1)-(9).

Proof. Inequality (2) is satisfied as $m_2 < d$. Inequality (3) is satisfied by assumption. If (7) is not satisfied, i.e., $m_{12} + m_{23} + m_{24} \in \{2d - 1, 2d\}$, one has a contradiction with $m_{24} < d$ and $m_{12} + m_{23} \leq m_{12} < d$. If (6) is not satisfied, then

$$2d = m_{2i} + m_{2j} + m_k, \tag{8.6}$$

for some $\{i, j, k\} = \{1, 3, 4\}$. If $k = 4$, since $m_4 = d$ one has from (8.6) $d = m_{12} + m_{23}$. But $m_{12} < d$ and $m_{23} \leq 0$. This is a contradiction. If $k \in \{1, 3\}$ (say $i = 4$) one has $2d = m_{24} + m_{j2} + m_k$ for $\{k, j\} = \{1, 3\}$. But $m_{24} < d$ and $m_{j2} + m_k \leq d$ (8.3). This is a contradiction. □

Case III: $m_i < d$, $m_{ij} < d$ for all i, j.

By Claim 8.10 we may assume $D.C_{i;j} > 0$ for $i = 1, 2, 3$, and all $j \neq i$.

Claim 8.9. One of $D_1 = D - \Lambda_{234}$, $D_2 = D - \Lambda_{134}$, $D_3 = D - \Lambda_{124}$ satisfies all the inequalities $(1) - (9)$.

Proof. Inequalities (2), (3) follow from the assumptions. If (6) is not satisfied for D_i, then $D.C_{i;j}$, for some $j \neq i$, which we assume does not happen. Hence, (6) is satisfied for all D_i. If (7) fails for all D_i, then one has for all $i \in \{1, 2, 3\}$

$$m_{ij} + m_{ik} + m_{il} \geq 2d - 1. \tag{8.7}$$

Adding up (8.7) for $i = 1$ and $i = 2$ one has

$$2m_{12} + (m_{13} + m_{24}) + (m_{14} + m_{23}) \geq 4d - 2. \tag{8.8}$$

From (4) $d \geq m_{ij} + m_{kl}$. As $m_{12} < d$, it follows from (8.8) that $m_{13} + m_{24} = m_{14} + m_{23} = d$. Similarly, by adding (8.8) for $i = 1$ and $i = 3$ one has $m_{12} + m_{34} = d$. This contradicts our assumption that one of the inequalities in (4) is strict. □

Claim 8.10. There are at least three indices $i \in \{1,2,3,4\}$ such that $D.C_{i;j} > 0$ for all $j \neq i$.

Proof. Assume $D.C_{i;j} = 0$, for some i, j. We may assume without loss of generality that $D.C_{1;2} = 0$:

$$2d = m_{13} + m_{14} + m_2.$$

We claim that for all $i \in \{2,3,4\}$ one has $D.C_{i;j} > 0$ for all $j \neq i$. This follows from $D.(C_{1;2} + C_{i;j}) > 0$ for all $i \in \{2,3,4\}$, $j \neq i$. This is because

$$D.(C_{1;2} + C_{3;j}) = D.C_{3k} + D.(l - e_{23} - e_{14}) + D.e_{13} \quad (\{j,k\} = \{1,4\}).$$

It follows from (5), (4) and $m_{13} < d$ that $D.(C_{1;2} + C_{3;j}) > 0$. Similarly,

$$D.(C_{1;2} + C_{3;2}) = D.B_2 + D.e_{13}.$$

By (9) and $m_{13} < d$, $D.(C_{1;2} + C_{3;2}) > 0$. By symmetry, $D.C_{4;j} > 0$, for all $j \neq 4$.

If $\{j,k\} = \{3,4\}$, one has

$$D.(C_{1;2} + C_{2;j}) = D.C_{1k} + D.(l - e_{1j} - e_{2k}) + D.e_{12}.$$

From (5), (4) and $m_{12} < d$ one has $D.(C_{1;2} + C_{2;j}) > 0$. Similarly,

$$D.(C_{1;2} + C_{2;1}) = D.(2l - e_1 - e_2) + D.(l - e_{13} - e_{24}) + D.(l - e_{14} - e_{23}).$$

From (4) and $m_1, m_2 < d$ one has $D.(C_{1;2} + C_{2;1}) > 0$. □

9. Restrictions to generators

Let $\pi' : \overline{M}' \to \mathbb{P}^3$ be the blow-up along $p_1, \ldots, p_5$ and let E'_i be the corresponding exceptional divisors. Let $\pi : \overline{M} \to \overline{M}'$ be the blow-up of the proper transforms of the lines l_{ij}. In what follows, we compute the classes of the restrictions of an arbitrary divisor D on $\overline{M}$ to the divisors $E_i, E_{ij}, \Lambda_{ijk}, Q_{(ij)(kl)}$.

9.1. **Restrictions to E_i.** The divisor E_i is the inverse image $\pi^{-1}(E'_i)$. By Fact 9.1 the divisor E_i is the blow-up of $E'_i \cong \mathbb{P}^2$ along the 4 points corresponding to the directions of the lines l_{ij}, for $j \neq i$. Denote by $\overline{E}_j$ the corresponding exceptional divisors. Denote by $\overline{H}$ the hyperplane class on E_i. One may easily see the following:

$$H_{|E_i} = 0, \quad E_{i|E_i} = -\overline{H}, \quad E_{j|E_i} = 0, \quad E_{ij|E_i} = \overline{E}_j, \quad E_{jk|E_i} = 0, \tag{9.1}$$

where $j, k \neq i$, $j \neq k$. This is clear from Fact 9.1.

Fact 9.1 ([EH, Prop. IV.21, p. 167]). Let Y and Z be closed subschemes in a scheme X and let $\tilde{X}$ be the blow-up of X along Z. Let E be the exceptional divisor. The proper transform $\tilde{Y}$ of Y is the blow-up of Y along the scheme theoretic intersection $Y \cap Z$, and the exceptional divisor is $\tilde{Y} \cap E$. In particular, if Z is contained in Y, the scheme $\tilde{Y}$ is the blow-up of Y along Z.

Consider an arbitrary divisor D on $\overline{M}$:

$$D = dH - \sum_i m_i E_i - \sum_{i,j} m_{ij} E_{ij}, \text{ where } d, m_i, m_{ij} \in \mathbb{Z}. \tag{9.2}$$

It follows from (9.1) that the restriction of D to E_i is given by

$$D_{|E_i} = m_i\overline{H} - \sum_{j\neq i} m_{ij}\overline{E}_j. \tag{9.3}$$

Lemma 9.2. *The divisor $D_{|E_i}$ is an effective divisor if and only if*

$$m_i \geq 0, \quad m_i \geq m_{ij} \quad (j\neq i), \quad 2m_i \geq \sum_{j\neq i} m_{ij}. \tag{9.4}$$

Proof. This is Lemma 7.3 applied to (9.3). □

9.2. **Restrictions to E_{ij}.** The normal bundle N of the proper transform of the line l_{ij} in $\overline{M}'$ is given by

$$N = (\pi^* N_{l_{ij}|\mathbb{P}^3})(-E_i - E_j) = \mathcal{O}(-1)\oplus\mathcal{O}(-1).$$

The divisor $E_{ij} = \mathbb{P}(N)$ is isomorphic to $\mathbb{P}(\mathcal{O}\oplus\mathcal{O}) = \mathbb{P}^1\times\mathbb{P}^1$. Let $p_1 : \mathbb{P}^1\times\mathbb{P}^1 \to \mathbb{P}^1$ be the projection map given by the blow-up map $E_{ij}\to l_{ij} = \mathbb{P}^1$ and let p_2 be the other projection. Since $\mathcal{O}(E_{ij})_{|E_{ij}} = \mathcal{O}_{\mathbb{P}(N)|\mathbb{P}^1}(-1)$ and

$$\mathcal{O}_{\mathbb{P}(N)|\mathbb{P}^1}(-1) \cong \mathcal{O}_{\mathbb{P}(\mathcal{O}\oplus\mathcal{O})|\mathbb{P}^1}(-1)\otimes p_1^*\mathcal{O}(-1),$$

it follows that

$$E_{ij}{}_{|E_{ij}} = p_1^*\mathcal{O}(-1)\otimes p_2^*\mathcal{O}(-1).$$

Moreover, one may easily see, for all distinct i, j, k, l,

$$H_{|E_{ij}} = E_{i|E_{ij}} = p_1^*\mathcal{O}(1), \quad E_{k|E_{ij}} = 0, \quad E_{kl|E_{ij}} = E_{ik|E_{ij}} = 0. \tag{9.5}$$

It follows from (9.5) that the restriction of D in (9.2) to E_{ij} is given by

$$D_{|E_{ij}} = p_1^*\mathcal{O}(d - m_i - m_j + m_{ij})\otimes p_2^*\mathcal{O}(m_{ij}).$$

Clearly, the divisor $D_{|E_{ij}}$ is an effective divisor if and only if

$$m_{ij}\geq 0, \quad d - m_i - m_j + m_{ij} \geq 0. \tag{9.6}$$

9.3. **Restrictions to Λ_{ijk}.** Take the case of Λ_{123} (the other cases are similar). Let Λ be the plane $\overline{p_1p_2p_3}$. Then Λ_{123} is the proper transform $\tilde{\Lambda}$ of Λ in $\overline{M}'$. Denote by Λ' the proper transform of Λ in $\overline{M}'$. Let q be the point $l_{45}\cap\Lambda$. Note that by Fact 9.1, Λ' is the blow-up $\Lambda = \mathbb{P}^2$ along p_1, p_2, p_3 and $\tilde{\Lambda}$ is isomorphic to the blow-up of Λ' in q, i.e., $\tilde{\Lambda}$ is isomorphic to the blow-up of $\mathbb{P}^2$ along p_1, p_2, p_3, q. Let $\overline{E}_1, \overline{E}_2, \overline{E}_3, \overline{E}_q$ be the exceptional divisors and $\overline{H}$ the hyperplane class. One may easily see that

$$H_{|\tilde{\Lambda}} = \overline{H}, \quad E_{i|\tilde{\Lambda}} = 0 \quad (i = 4,5), \quad E_{ij|\tilde{\Lambda}} = 0 \quad (ij\neq 12, 13, 23, 45).$$

Using Fact 9.1, one has that

$$E_{i|\tilde{\Lambda}} = \overline{E}_i \quad (i = 1,2,3), \quad E_{45|\tilde{\Lambda}} = \overline{E}_q, \quad E_{ij|\tilde{\Lambda}} = \overline{H} - \overline{E}_i - \overline{E}_j \quad (ij\in\{12, 13, 23\}).$$

It follows that the restriction of D in (9.2) to Λ_{123} is given by

$$D_{|\Lambda_{123}} = (d - m_{12} - m_{13} - m_{23})\overline{H} - \sum_{\{i,j,k\}=\{1,2,3\}} (m_i - m_{ij} - m_{ik})\overline{E}_i - m_{45}\overline{E}_q.$$

By permuting indices and applying Lemma 7.3, one has the following:

Lemma 9.3. *If the divisor $D_{|\Lambda_{ijk}}$ is effective and $\{i, j, k, u, v\} = \{1, 2, 3, 4, 5\}$, then*

$$d\geq m_i + m_{jk}, \quad d\geq m_{ij} + m_{ik} + m_{jk} + m_{uv}, \quad 2d \geq m_i + m_j + m_j + m_{uv}. \tag{9.7}$$

9.4. **Restrictions to the Keel-Vermeire divisors $Q_{(ij)(kl)}$.** Take the case of $Q_{(12)(34)}$. There is a unique (smooth) quadric Q in $\mathbb{P}^3$ that contains the points $p_1, \ldots, p_5$ and the lines $l_{13}, l_{14}, l_{23}, l_{24}$. Since $Q_{(12)(34)}$ has class,

$$Q_{(12)(34)} = 2H - \sum_i E_i - E_{13} - E_{14} - E_{23} - E_{24},$$

it follows that $Q_{(12)(34)}$ is the proper transform $\tilde{Q}$ of Q in $\overline{M}$. Denote by Q' the proper transform of Q in $\overline{M}'$. By Fact 9.1 it follows that Q' is the blow-up of $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ along the points $p_1, \ldots, p_5$. Moreover $\tilde{Q} \cong Q'$.

Let F_1, respectively F_2, be the class of the lines in the ruling of $\mathbb{P}^1 \times \mathbb{P}^1$ that contains l_{13} and l_{24}, respectively l_{14} and l_{23}. Let $\overline{E}_1, \ldots, \overline{E}_5$ be the exceptional divisors on $\tilde{Q}$, considered as a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ along $p_1, \ldots, p_5$. By Fact 9.1:

$$H_{|\tilde{Q}} = F_1 + F_2, \quad E_{i|\tilde{Q}} = \overline{E}_i,$$
$$E_{ij|\tilde{Q}} = F_1 - \overline{E}_i - \overline{E}_j \quad (ij = 13, 24), \quad E_{ij|\tilde{Q}} = F_2 - \overline{E}_i - \overline{E}_j \quad (ij = 14, 23),$$
$$E_{ij|\tilde{Q}} = 0 \quad \text{for all other cases.}$$

It follows that restriction $D_{|\tilde{Q}}$ of the divisor D in (9.2) to $\tilde{Q}$ is given by

$$D_{|\tilde{Q}} = (d - m_{13} - m_{24})F_1 + (d - m_{14} - m_{23})F_2 - (m_1 - m_{13} - m_{14})\overline{E}_1$$
$$-(m_2 - m_{23} - m_{24})\overline{E}_2 - (m_3 - m_{13} - m_{23})\overline{E}_3 - (m_4 - m_{14} - m_{24})\overline{E}_4 - m_5\overline{E}_5.$$

Alternative description of $\tilde{Q}$. Let $\rho : \mathbb{P}^3 \setminus \{p_5\} \to \mathbb{P}^2$ be the projection from p_5 and let $q_i = \rho(p_i)$ $(i = 1, \ldots, 4)$. Let l_1 (respectively l_2) be the unique line through p_5 in the ruling of F_1 (respectively F_2).

Let y (respectively x) be the image l_1 (respectively l_2). The blow-up of $Q = \mathbb{P}^1 \times \mathbb{P}^1$ in p_5 is isomorphic to the blow-up of $\mathbb{P}^2$ in x, y. Hence, $\tilde{Q}$ is isomorphic to the blow-up of $\mathbb{P}^2$ along $p_1, \ldots, p_4, x, y$. Denote by $\overline{E}'_1, \ldots, \overline{E}'_4, \overline{E}_x, \overline{E}_y$ the exceptional divisors corresponding to the points $p_1, \ldots, p_4, x, y$ and let $\overline{H}$ be the hyperplane class. One may immediately see

$$\overline{H} = \rho^* \mathcal{O}(1) = F_1 + F_2 - \overline{E}_5.$$

Note that lines in the ruling F_2 (respectively F_1) intersect l_1 (respectively l_2); therefore their images in $\mathbb{P}^2$ all pass through y (respectively x). In particular, the lines $\overline{q_1 q_3}$ and $\overline{q_2 q_4}$ intersect in x, while the lines $\overline{q_1 q_4}$ and $\overline{q_2 q_3}$ intersect in y. Moreover, one has

$$F_1 = \overline{H} - \overline{E}_x, \quad F_2 = \overline{H} - \overline{E}_y.$$

It follows that

$$\overline{E}_5 = \overline{H} - \overline{E}_x - \overline{E}_y, \quad \overline{E}_i = \overline{E}'_i \quad (i = 1, \ldots, 4).$$

Hence, the restriction $D_{|\tilde{Q}}$ of the divisor D in (9.2) to $\tilde{Q}$ is given by

$$D_{|\tilde{Q}} = (2d - m_5 - m_{13} - m_{14} - m_{23} - m_{24})\overline{H} - (m_1 - m_{13} - m_{14})\overline{E}_1$$
$$-(m_2 - m_{23} - m_{24})\overline{E}_2 - (m_3 - m_{13} - m_{23})\overline{E}_3 - (m_4 - m_{14} - m_{24})\overline{E}_4$$
$$-(d - m_5 - m_{13} - m_{24})\overline{E}_x - (d - m_5 - m_{14} - m_{23})\overline{E}_y.$$

Lemma 9.4. *If the divisor $D_{|\tilde{Q}}$ is effective, then*

$$2d \geq m_5 + m_{13} + m_{14} + m_{23} + m_{24}, \quad 2d \geq m_1 + m_5 + m_{23} + m_{24}. \tag{9.8}$$

Proof. This follows from Lemma 7.3. □

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References

[BCHM] Birkar, C., Cascini, P., Hacon, C., McKernan, J., *Existence of minimal models for varieties of log general type*, preprint (2006)

[BP] Batyrev, V., Popov, O., *The Cox ring of a del Pezzo surface*, Arithmetic of higher-dimensional algebraic varieties, Palo Alto, CA, 2002, 149–173, Progr. Math., **226**, Birkhäuser Boston, Boston, MA, 2004 MR2029863 (2005h:14091)

[CT] Castravet, A.-M., Tevelev, J., *Hilbert's 14'th Problem and Cox Rings*, (2005), Compositio Math., Vol. **142** (2006), 1479–1498 MR2278756 (2007i:14044)

[EH] Eisenbud, D., Harris, J., *The Geometry of Schemes*, Graduate Texts in Mathematics, Vol. **197**, Springer-Verlag, New York, 2000 MR1730819 (2001d:14002)

[GKM] Gibney, A., Keel, S., Morrison, I., *Towards the ample cone of* $\overline{M}_{g,n}$, J. Amer. Math. Soc., Vol. **15**, No. 2 (2001), 273–294 MR1887636 (2003c:14029)

[HT] Hassett, B., Tschinkel, Y., *On the effective cone of the moduli space of pointed rational curves*, Topology and geometry: commemorating SISTAG, 83–96, Contemp. Math., **314**, Amer. Math. Soc., Providence, RI, 2002 MR1941624 (2004d:14028)

[HK] Hu, Y., Keel, S., *Mori Dream Spaces and GIT*, Michigan Math. J., Vol. **48** (2000), 331-348 MR1786494 (2001i:14059)

[KM] Keel, S., McKernan, J., *Contractible extremal rays of* $\overline{M}_{0,n}$, preprint (1997), arxiv:alg-geom/9707016

[V] Vermeire, P., *A counterexample to Fulton's Conjecture on* $\overline{M}_{0,n}$, J. of Algebra, Vol. **248**, (2002), 780–784 MR1882122 (2002k:14043)

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