

RECONSTRUCTION OF PATH ALGEBRAS FROM THEIR POSETS OF TILTING MODULES

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ABSTRACT. Let $\Lambda = k\overrightarrow{\Delta}$ be the path algebra of a finite quiver without oriented cycles. The set of isomorphism classes of multiplicity free tilting modules is in a natural way a partially ordered set. We will show here that \mathcal{T}_Λ uniquely determines $\overrightarrow{\Delta}$ if $\overrightarrow{\Delta}$ has no multiple arrows and no isolated vertices.

Let Λ be a basic, connected finite dimensional algebra over an algebraically closed field k and let $\text{mod } \Lambda$ be the category of finitely generated left Λ -modules. For a module $M \in \text{mod } \Lambda$ we denote by $\text{pd}_\Lambda M$ the projective dimension of M and by $\text{gl.dim } \Lambda$ the global dimension of Λ . A module $T \in \text{mod } \Lambda$ is called a tilting module if the following three conditions are satisfied: (i) $\text{pd}_\Lambda T < \infty$, (ii) $\text{Ext}_\Lambda^i(T, T) = 0$ for all $i > 0$ and (iii) there exists an exact sequence $0 \rightarrow \Lambda \rightarrow T^0 \rightarrow \cdots \rightarrow T^r \rightarrow 0$ with $T^i \in \text{add } T$ for all $0 \leq i \leq r$, where $\text{add } T$ is the full subcategory of $\text{mod } \Lambda$ whose objects are direct sums of direct summands of T . We will say that a tilting module is basic or multiplicity free if in a direct sum decomposition of T the indecomposable direct summands of T occur with multiplicity one. Unless stated otherwise all tilting modules considered here will be assumed to be basic.

Tilting theory usually has two aspects. Originally it was used, which one might call the external aspect, to compare $\text{mod } \Lambda$ to $\text{mod } \text{End}_\Lambda T$ for a Λ -tilting module T . The internal aspect, and this is the one we are interested in here, is to study tilting modules T for a fixed Λ , and try to gather information about $\text{mod } \Lambda$.

Following [AR] we consider for a tilting module $T \in \text{mod } \Lambda$ the right perpendicular category

$$T^\perp = \{X \in \text{mod } \Lambda \mid \text{Ext}_\Lambda^i(T, X) = 0 \text{ for all } i > 0\}.$$

We consider the set \mathcal{T}_Λ of all tilting modules over Λ up to isomorphism. Note that \mathcal{T}_Λ is always countable (compare for example [H2]). In [HU1] the following partial order \leq on \mathcal{T}_Λ was investigated. For $T, T' \in \mathcal{T}_\Lambda$ we set $T \leq T'$ provided $T^\perp \subset T'^\perp$. One of the main results in [HU1] was that the Hasse quiver $\overrightarrow{\mathcal{K}}_\Lambda$ (i.e. the quiver of minimal inclusions) of this poset allows the following description: The vertices of $\overrightarrow{\mathcal{K}}_\Lambda$ are the elements of \mathcal{T}_Λ and there is an arrow $T' \rightarrow T$ in $\overrightarrow{\mathcal{K}}_\Lambda$ if $T' = M \oplus X$ and

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$T = M \oplus Y$ with X, Y indecomposable, and there exists a non-split short exact sequence $0 \rightarrow X \rightarrow \tilde{M} \rightarrow Y \rightarrow 0$ with $\tilde{M} \in \text{add } M$.

The main aim of this article is to investigate the following natural question. In which way does $\vec{\mathcal{K}}_\Lambda$ or \mathcal{T}_Λ determine the isomorphism class of Λ ? In general the answer to this question is negative. For example if Λ is selfinjective, then $\vec{\mathcal{K}}_\Lambda$ consists of just a single vertex. In case Λ is a hereditary algebra the question is more interesting. Since neither \mathcal{T}_Λ nor $\vec{\mathcal{K}}_\Lambda$ will tell us whether or not Λ has simple modules which are both projective and injective in case there are $n \geq 2$ simple modules (see section 3), we will assume that $\Lambda \simeq k\vec{\Delta}$ for a finite quiver without oriented cycles and without isolated vertices. This is trivially satisfied if $\vec{\Delta}$ is connected and has $n \geq 2$ vertices. The case that $\vec{\Delta} = \circ$ is of course trivial.

We will show in section 6 as a main result of this article that \mathcal{T}_Λ uniquely determines $\vec{\Delta}$ if $\vec{\Delta}$ has no multiple arrows and no isolated vertices. Thus $\vec{\mathcal{K}}_\Lambda$ uniquely determines $\vec{\Delta}$ in case $\vec{\Delta}$ is a Dynkin quiver without isolated vertices. Note that in general it will be impossible to recover the precise number of arrows between two vertices in case of more than two arrows. Indeed, if $\vec{\Delta}_r$ is the two point quiver with $r \geq 2$ arrows and Λ_r the path algebra of $\vec{\Delta}_r$ over k , then the partially ordered sets $\mathcal{T}_{\Lambda_r} \simeq \mathcal{T}_{\Lambda_s}$ for all $r, s \geq 2$. But we will show in section 6 that \mathcal{T}_Λ contains the information as to whether or not there are multiple arrows in $\vec{\Delta}$ between a pair of vertices.

The article is organized as follows. In section 1 we recall from [H2] and [U1] some preliminary material about tilting modules. Section 2 deals with orthogonal categories in the sense of [GL]. There we will show that for a not necessarily hereditary algebra Λ the poset \mathcal{T}_Λ has an inductive property when forming orthogonal categories with respect to simple Λ -modules S with $\text{pd}_\Lambda S \leq 1$ and $\text{id}_\Lambda S \leq 1$, where $\text{id}_\Lambda M$ denotes the injective dimension of $M \in \text{mod } \Lambda$. Although this result will not be needed in the proof of the main theorem, we still think it is worthwhile to present here. In section 3 we will show how to determine the number of simple Λ -modules from $\vec{\mathcal{K}}_\Lambda$ for a connected hereditary algebra Λ . In section 4 we will study more closely the situation of Dynkin quivers and show that $\vec{\mathcal{K}}_\Lambda$ contains the information on the number of isomorphism classes of indecomposable modules. This easily allows us to deduce from $\vec{\mathcal{K}}_\Lambda$ the underlying Dynkin diagram. Section 5 contains some relevant information about the neighbors of ${}_\Lambda \Lambda$ and of $D\Lambda_\Lambda$ in $\vec{\mathcal{K}}_\Lambda$. Section 6 then deals with the actual reconstruction of the orientation and the detection of multiple arrows of $\vec{\Delta}$.

We denote the composition of morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in a given category \mathcal{K} by fg .

The notation and terminology introduced here will be fixed throughout this article. For unexplained representation-theoretic terminology we refer to [R] or [ARS].

1. PRELIMINARIES

We will briefly recall in this section some of the technical details and notation frequently used in the subsequent sections.

Let Λ be a finite dimensional k -algebra over an algebraically closed field k . We denote by n the number of simple Λ -modules and label them S_1, \dots, S_n . Their

projective covers (resp. injective envelopes) will be denoted by $P(1), \dots, P(n)$ (resp. $I(1), \dots, I(n)$). By D we denote the standard duality with respect to k , so $D\Lambda_\Lambda = \bigoplus_{i=1}^n I(i)$.

We say that a module M is a partial tilting module if M is a direct summand of a tilting module. A partial tilting module M with $n - 1$ non-isomorphic indecomposable direct summands is called an almost complete partial tilting module. We point out that for a hereditary algebra Λ any Λ -module N with $\text{Ext}_\Lambda^1(N, N) = 0$ is a partial tilting module (compare 4.1 in [R]). This fact will be frequently used in the subsequent sections. Given a partial tilting module M we say that C is a complement to M if $M \oplus C$ is a tilting module and $\text{add } M \cap \text{add } C = 0$. Note that a complement to an almost complete partial tilting module is indecomposable. If T is a multiplicity free tilting module, we have the direct sum decomposition $T = \bigoplus_{i=1}^n T_i$. For $1 \leq i \leq n$ we denote it by $T[i] = \bigoplus_{j \neq i} T_j$. So $T[i]$ is an almost complete partial tilting module for $1 \leq i \leq n$. The following lemma (compare [H2]) is important

Lemma 1.1. *Let $1 \leq i \leq n$. If $T[i]$ is not faithful, then T_i is the unique (up to isomorphism) complement to $T[i]$. If $T[i]$ is faithful, then T_i is either generated or cogenerated by $T[i]$.*

- (1) *If T_i is generated by $T[i]$ there exists a unique (up to isomorphism) indecomposable complement X not isomorphic to T_i and an exact sequence $0 \rightarrow X \rightarrow \widetilde{T[i]} \rightarrow T_i \rightarrow 0$ with $\widetilde{T[i]} \in \text{add } T[i]$ such that $\text{Ext}_\Lambda^i(X, T_i) = 0$ for $i > 0$ and $\text{Ext}_\Lambda^i(T_i, X) = 0$ for $i > 1$.*
- (2) *If T_i is cogenerated by $T[i]$ there exists a unique (up to isomorphism) indecomposable complement Y not isomorphic to T_i and an exact sequence $0 \rightarrow T_i \rightarrow \widetilde{T[i]} \rightarrow Y \rightarrow 0$ with $\widetilde{T[i]} \in \text{add } T[i]$ such that $\text{Ext}_\Lambda^i(Y, T_i) = 0$ for $i > 1$ and $\text{Ext}_\Lambda^i(T_i, Y) = 0$ for $i > 0$.*
- (3) *If Λ is hereditary, then the non-vanishing Ext-spaces in (2) and (3) are one dimensional.*
- (4) *If Λ is hereditary, then T_i is never both generated and cogenerated by $T[i]$.*

The exact sequences occurring in (1) and (2) will be called the exact sequences connecting the complements. We will also say that in the situation of (1) T_i is replaced by the module X or simply T_i is replaced.

Note that the assertions (3) and (4) will fail if we drop the assumption that Λ is hereditary.

We also point out the following trivial but important consequence of Lemma 1.1(4) which we will use frequently.

Corollary 1.2. *Let Λ be hereditary. If $w : T \rightarrow T' \rightarrow T''$ is a path in $\vec{\mathcal{K}}_\Lambda$, with $T = M \oplus X$ and $T' = M \oplus Y$, then Y is an indecomposable direct summand of T'' .*

For a Λ -module M we denote by $\text{fac } M$ the full subcategory of $\text{mod } \Lambda$ containing the epimorphic images of $\text{add } M$. We recall from [H2] the following alternative in the defining property (iii) of a tilting module which will turn out to be important in the next section.

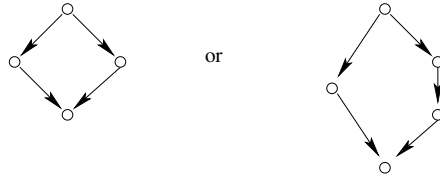
Proposition 1.3. *The following are equivalent for a Λ -module M with $pd_\Lambda M < \infty$ and $Ext_\Lambda^i(M, M) = 0$ for $i > 0$:*

- (1) M is tilting module,
- (2) $M^\perp \subset \text{fac } M$.

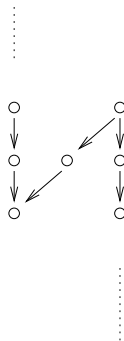
We recall from [U1] the structure of links of dimension one in case Λ is hereditary. For this let M be a multiplicity free partial tilting module with $n-2$ non-isomorphic indecomposable direct summands. Let $\vec{\mathcal{K}}_\Lambda(M)$ be the full subquiver of $\vec{\mathcal{K}}_\Lambda$ with vertices T such that M is a direct summand of T . Following [U1] and [U2] we call the quiver $\vec{\mathcal{K}}_\Lambda(M)$ a link of dimension one. This result will turn out to be quite useful in the latter sections.

Theorem 1.4. *Let Λ be hereditary with $n \geq 3$ simple modules and let M be a multiplicity free partial tilting module with $n-2$ non-isomorphic indecomposable direct summands.*

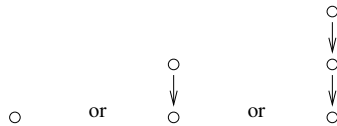
- (a) *If M is faithful and*
 - (1) *if $\vec{\mathcal{K}}_\Lambda(M)$ is finite, then $\vec{\mathcal{K}}_\Lambda(M)$ is of the form*



- (2) *if $\vec{\mathcal{K}}_\Lambda(M)$ is infinite, then $\vec{\mathcal{K}}_\Lambda(M)$ is of the form*



- (b) *If M is not faithful and*
 - (1) *if $\vec{\mathcal{K}}_\Lambda(M)$ is finite, then $\vec{\mathcal{K}}_\Lambda(M)$ is of the form*



(2) if $\vec{\mathcal{K}}_\Lambda(M)$ is infinite, then $\vec{\mathcal{K}}_\Lambda(M)$ is of the form



2. AN INDUCTIVE PROPERTY

In section 5 we will use orthogonal categories in the sense of [GL] (see also [H1]). For the convenience of the reader we recall some of the relevant details.

Although we will use this approach only for hereditary algebras, we will present a more general result which generalizes a result in [AHT].

For this let Λ be a finite dimensional k -algebra over an algebraically closed field k . Let $X \in \text{mod } \Lambda$ with $\text{pd}_\Lambda X \leq 1$, $\text{Ext}_\Lambda^1(X, X) = 0$ and $\text{End}_\Lambda X \simeq k$. We define the right orthogonal category X^{perp} to be the full subcategory of $\text{mod } \Lambda$ containing the Λ -modules Y such that $\text{Hom}_\Lambda(X, Y) = 0$ and $\text{Ext}_\Lambda^1(X, Y) = 0$. In accordance with [GL] (see also [H1]) we have the following.

Theorem 2.1. *The orthogonal category X^{perp} contains a projective generator Q , such that $X^{\text{perp}} \simeq \text{mod } \Gamma$, where $\Gamma = \text{End}_\Lambda Q$. The number of simple Γ -modules equals $n - 1$. If $\text{Ext}_\Lambda^i(Q, Q) = 0$ for $i \geq 2$, then the following holds:*

- (1) $\text{Ext}_\Lambda^i(Y, Z) \simeq \text{Ext}_\Gamma^i(Y, Z)$ for all $Y, Z \in X^{\text{perp}}$ and all $i \geq 0$.
- (2) Let $Z \in X^{\text{perp}}$ with $\text{pd}_\Lambda Z < \infty$, then $\text{pd}_\Gamma Z \leq \text{pd}_\Lambda Z$.
- (3) $\text{gl.dim } \Gamma \leq \text{gl.dim } \Lambda$.

Note that there are examples where $\text{Ext}_\Lambda^i(Q, Q) = 0$ for $i \geq 2$ is not always satisfied (compare for example [H1]).

We also recall from [H1] the following:

Lemma 2.2. *Let S be a simple Λ -module with $\text{pd}_\Lambda S \leq 1$. Then the projective generator Q of S^{perp} satisfies $\text{Ext}_\Lambda^i(Q, Q) = 0$ for $i \geq 2$ and $\text{pd}_\Lambda Q \leq 2$. Moreover for all $Z \in S^{\text{perp}} = \text{mod } \Gamma$ we have that $\text{pd}_\Lambda Z \leq \text{pd}_\Gamma Z + 2$.*

In subsequent sections we will need the following very useful fact. The first part is well-known. For the convenience of the reader we indicate the argument.

Lemma 2.3. *Let Λ be a hereditary algebra and X an indecomposable non-projective Λ -module. Let P be a minimal projective generator of X^{perp} . Then the following hold:*

- (1) $T^\alpha(X) = X \oplus P \in \mathcal{T}_\Lambda$.
- (2) $\text{Hom}_\Lambda(T^\alpha(X), Y) \neq 0$ and $\text{Ext}_\Lambda^1(T^\alpha(X), Y) = 0$ for all modules $Y \in X^{\text{perp}}$.

- (3) If $T' \in \mathcal{T}_\Lambda$ with $\text{Ext}_\Lambda^1(X, T') = 0$, then $T' \leq T^\alpha(X)$ in \mathcal{T}_Λ .
- (4) $T^\alpha(X)$ has a unique immediate predecessor in $\vec{\mathcal{K}}_\Lambda$.
- (5) X is a direct summand of each immediate successor T' of $T^\alpha(X)$ in $\vec{\mathcal{K}}_\Lambda$.

Proof. We recall the construction of P . Let $s = \dim \text{Ext}_\Lambda^1(X, \Lambda)$ and let

$$(*) \quad 0 \rightarrow \Lambda \rightarrow Q \rightarrow X^s \rightarrow 0$$

be the universal extension. Let P be multiplicity free with $\text{add } P = \text{add } Q$. It is well-known that $T^\alpha(X) = X \oplus P$ is a tilting module (compare 4.1 in [R]). By the construction of the extension $(*)$ we have that the connecting homomorphism $\theta : \text{Hom}_\Lambda(X, X^s) \rightarrow \text{Ext}_\Lambda^1(X, \Lambda)$ is surjective. Since $\text{Hom}_\Lambda(X, X) = k$ we see that θ is an isomorphism. Since X is not projective, $\text{Hom}_\Lambda(X, \Lambda) = 0$, so we see that $Q \in X^{\text{perp}}$. Thus also $P \in X^{\text{perp}}$. Let $Y \in X^{\text{perp}}$. Applying $\text{Hom}_\Lambda(-, Y)$ to $(*)$ shows that Q and thus also P is projective in X^{perp} . It follows from $(*)$ that Q is a generator of X^{perp} , but then P is a minimal projective generator of X^{perp} . Thus $\text{Hom}_\Lambda(T^\alpha(X), Y) \neq 0$ and $\text{Ext}_\Lambda^1(T^\alpha(X), Y) = 0$ for all $Y \in X^{\text{perp}}$, which shows (1) and (2).

For (3) let $T' \in \mathcal{T}_\Lambda$. Applying $\text{Hom}_\Lambda(-, T')$ to the exact sequence $(*)$ and using $\text{Ext}_\Lambda^1(X, T') = 0$ shows that $\text{Ext}_\Lambda^1(T^\alpha(X), T') = 0$; hence $T' \leq T^\alpha(X)$ by [HU3].

It follows from $(*)$ that X is generated by P . So by Lemma 1.1 we see that $T^\alpha(X)$ has an immediate predecessor T'' in $\vec{\mathcal{K}}_\Lambda$. Let $T''' \in \mathcal{T}_\Lambda$ with $T''' \neq T''$ and $T''' \rightarrow T^\alpha(X)$ in $\vec{\mathcal{K}}_\Lambda$. So X is a direct summand of T''' . Let $T''' = X \oplus P' \oplus Y$ and $T^\alpha(X) = X \oplus P' \oplus P_0$. Since $\text{Hom}_\Lambda(X, P) = 0$, the sequence connecting the complements P_0 and Y of $X \oplus P'$ implies that P_0 is generated by P' ; but P_0 is projective in X^{perp} , a contradiction, thus showing (4).

Let $T^\alpha(X) \rightarrow T'$ be in $\vec{\mathcal{K}}_\Lambda$. Then $T^\alpha(X) = M \oplus Y$ such that Y is cogenerated by M . Since $\text{Hom}_\Lambda(X, P) = 0$ this clearly implies that $X \not\leq Y$, hence X is a direct summand of T' , so (5) holds.

We will now study certain subposets of \mathcal{T}_Λ . Let X be a partial tilting module. We define $\mathcal{T}(X) = \{T \in \mathcal{T}_\Lambda \mid X \text{ is a direct summand of } T\}$.

Lemma 2.4. $\mathcal{T}(X)$ is a convex subposet of \mathcal{T}_Λ .

Proof. For this let $T^1, T^2 \in \mathcal{T}(X)$ and $T \in \mathcal{T}_\Lambda$ with $T^1 \leq T \leq T^2$ in \mathcal{T}_Λ . Let $T^1 = M^1 \oplus X$ and $T^2 = M^2 \oplus X$. By definition $T^{1\perp} \subset T^\perp$, so $X \in T^\perp$. Also $T^\perp \subset T^{2\perp}$ shows that $\text{Ext}_\Lambda^i(X, T^\perp) = 0$ for $i > 0$, hence X is T^\perp -projective, or equivalently X is a direct summand of T . Thus $T \in \mathcal{T}(X)$.

Corollary 2.5. Let $\vec{\mathcal{K}}_\Lambda(X)$ be the full subquiver of $\vec{\mathcal{K}}_\Lambda$ with vertices in $\mathcal{T}(X)$. Then $\vec{\mathcal{K}}_\Lambda(X)$ is a convex subquiver of $\vec{\mathcal{K}}_\Lambda$.

Now suppose that X is an indecomposable partial tilting module with $\text{pd}_\Lambda X \leq 1$. Let $X^{\text{perp}} = \text{mod } \Gamma$. Note that Γ has $n - 1$ simple modules. It is easy to construct examples showing that $\mathcal{T}(X)$ is not isomorphic to \mathcal{T}_Γ . In fact, let

$$\Lambda = k(\circ \rightarrow \circ \rightarrow \cdots \rightarrow \circ \rightarrow \circ)$$

and let X be the projective-injective indecomposable Λ -module. Then $\vec{\mathcal{K}}_\Lambda(X) = \vec{\mathcal{K}}_\Lambda$, since X is a direct summand of each tilting module. We will show below that $\mathcal{T}(X)$ is isomorphic to \mathcal{T}_Γ if X is simple with $\text{id}_\Lambda X \leq 1$ and $\text{pd}_\Lambda X \leq 1$. Note

that this applies to the situation investigated in [AHT]. We refer to section 5 for a combinatorial description of $\mathcal{T}(X)$ inside \mathcal{T}_Λ for hereditary algebras.

So we will assume that S is a simple Λ -module with $\text{pd}_\Lambda S \leq 1$ and $\text{id}_\Lambda S \leq 1$. Let $S^{\text{perp}} = \text{mod } \Gamma$. In this situation we will define a map $\varphi_S : \mathcal{T}(S) \rightarrow \mathcal{T}_\Gamma$. For this let $T \in \mathcal{T}(S)$, so $T = M \oplus S$. Let f_1, \dots, f_t be a basis of $\text{Hom}_\Lambda(S, M)$. Consider $f : S^t \rightarrow M$ with components given by the f_i . Since S is simple, we have that f is injective. So consider the exact sequence

$$(*) \quad 0 \longrightarrow S^t \xrightarrow{f} M \longrightarrow N \longrightarrow 0.$$

By construction we see that $N \in S^{\text{perp}}$. It is easy to see that N is multiplicity free. In fact, if $M = \bigoplus_{i=1}^{n-1} M_i$, then $N_i = M_i$, if $\text{Hom}_\Lambda(S, M_i) = 0$, and if $\text{Hom}_\Lambda(S, M_i) \neq 0$ we have an exact sequence

$$(*)_i \quad 0 \longrightarrow S^{t_i} \xrightarrow{g_i} M_i \longrightarrow N_i \longrightarrow 0$$

constructed as $(*)$. Note that we then have that $N = \bigoplus_{i=1}^{n-1} N_i$. Let

$$\psi_i \in \text{Hom}_\Lambda(N_i, N_i).$$

Since $\text{Ext}_\Lambda^1(M_i, S) = 0$ we obtain the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^{t_i} & \xrightarrow{g_i} & M_i & \xrightarrow{h_i} & N_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \phi_i & & \downarrow \psi_i & & \\ 0 & \longrightarrow & S^{t_i} & \xrightarrow{g_i} & M_i & \xrightarrow{h_i} & N_i & \longrightarrow & 0 \end{array}$$

So for each $j \geq 1$ we have that $\phi_i^j h_i = h_i \psi_i^j$. Since M_i is indecomposable, or equivalently $\text{End}_\Lambda M_i$ is local, this clearly implies that N_i is indecomposable. Next we show that for $i \neq j$ we have that $N_i \not\cong N_j$. Assume that $\phi : N_i \rightarrow N_j$ is an isomorphism. Since $\text{Ext}_\Lambda^1(M, S) = 0$ we obtain the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^{t_i} & \xrightarrow{g_i} & M_i & \xrightarrow{h_i} & N_i & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow \phi & & \\ 0 & \longrightarrow & S^{t_j} & \xrightarrow{g_j} & M_j & \xrightarrow{h_j} & N_j & \longrightarrow & 0 \end{array}$$

Since the lower sequence does not split, we have that $f \neq 0$, hence $g \neq 0$. Now $\ker g = \ker f \in \text{add } S$ and $\text{coker } g = \text{coker } f \in \text{add } S$. So $\text{Ext}_\Lambda^2(\text{coker } g, \ker g) = 0$. But then it follows that g is either mono or epi (compare 4.1 in [HR]). Since $S \oplus M$ is a tilting module, this implies that g is an isomorphism, so $i = j$. Note that the lemma below gives a different proof of the last assertion.

We point out that it is straightforward to check that the isomorphism class of N does not depend on the choice of the basis in $\text{Hom}_\Lambda(S, M)$. Indeed, if $f'_1, \dots, f'_t \in \text{Hom}_\Lambda(S, M)$ is another basis, there exists an invertible $t \times t$ -matrix λ over k such that $f' = \lambda f$, where f' is the map from S^t to M with components given by the f'_i . We may consider $\lambda \in \text{Hom}_\Lambda(S^t, S^t)$. Thus we obtain a commutative diagram of

exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & S^t & \xrightarrow{f} & M & \longrightarrow & N \longrightarrow 0 \\
 & & \downarrow \lambda & & \parallel & & \downarrow h \\
 0 & \longrightarrow & S^t & \xrightarrow{f'} & M & \longrightarrow & N' \longrightarrow 0
 \end{array}$$

Since λ is an isomorphism we infer that h is an isomorphism.

We point out that one could use a functorial approach in this situation. Since we want an explicit description of this correspondence we prefer the approach given here.

Lemma 2.6. *N is a Γ -tilting module.*

Proof. Since $\text{pd}_\Lambda T < \infty$ then also $\text{pd}_\Lambda N < \infty$. By Lemma 2.2 we may apply Theorem 2.1(2). So we have that $\text{pd}_\Gamma N < \infty$. Applying $\text{Hom}_\Lambda(M, -)$ to $(*)$ shows that $\text{Ext}_\Lambda^i(M, N) = 0$ for $i > 0$. Applying $\text{Hom}_\Lambda(-, N)$ to $(*)$ and using Theorem 2.1 shows that $\text{Ext}_\Gamma^i(N, N) = \text{Ext}_\Lambda^i(N, N) = 0$ for $i > 0$. Let $X \in N^\perp \subset \text{mod } \Gamma \simeq S^{\text{perp}} \subset \text{mod } \Lambda$. Applying $\text{Hom}_\Lambda(-, X)$ to $(*)$ shows that $X \in T^\perp$. Since T is a tilting module we know that $T^\perp \subset \text{fac } T$, where $\text{fac } T$ is the full subcategory of $\text{mod } \Lambda$ containing the epimorphic images of $\text{add } T$. So $X \in \text{fac } T$. Since $X \in S^{\text{perp}}$ it follows that $X \in \text{fac } M$. But then $(*)$ shows that $X \in \text{fac } N$, hence N is a Γ -tilting module by Proposition 1.3.

So the lemma gives the desired map $\varphi_S : \mathcal{T}(S) \rightarrow \mathcal{T}_\Gamma$, where $\varphi_S(T) = N$ with the notation above.

Theorem 2.7. *φ_S is a bijection of posets.*

Proof. We show first that φ_S is injective. For this let $T^1 = M^1 \oplus S$ and $T^2 = M^2 \oplus S$ be in $\mathcal{T}(S)$ such that $\varphi_S(T^1) = \varphi_S(T^2)$. By the definition of φ_S we have two exact sequences,

$$0 \rightarrow S^{t_1} \rightarrow M^1 \rightarrow N^1 \rightarrow 0$$

and

$$0 \rightarrow S^{t_2} \rightarrow M^2 \rightarrow N^2 \rightarrow 0.$$

Since $\varphi_S(T^1) = \varphi_S(T^2)$ we have $\text{add } N^1 = \text{add } N^2$. This trivially implies that $\text{Ext}_\Lambda^i(M^1, N^2) = \text{Ext}_\Lambda^i(M^2, N^1) = 0$ for $i > 0$. Thus also $\text{Ext}_\Lambda^i(M^1, M^2) = 0$ and $\text{Ext}_\Lambda^i(M^2, M^1) = 0$ for $i > 0$, hence $\text{Ext}_\Lambda^i(T^1 \oplus T^2, T^1 \oplus T^2) = 0$ for $i > 0$, in particular $T^1 = T^2$.

Next we show that φ_S is surjective. For this let $N \in \mathcal{T}_\Gamma$ and consider the universal extension

$$(+)\quad 0 \longrightarrow S^r \xrightarrow{g} M \longrightarrow N \longrightarrow 0,$$

so $r = \dim \text{Ext}_\Lambda^1(N, S)$.

It is easy to see that M is multiplicity free. In fact, let $N = \bigoplus_{i=1}^{n-1} N_i$; then $(+)$ is the direct sum of the universal extensions

$$(+)_i\quad 0 \longrightarrow S^{r_i} \longrightarrow M_i \longrightarrow N_i \longrightarrow 0$$

for $1 \leq i \leq n - 1$. So $r_i = \dim \text{Ext}_\Lambda^1(N_i, S)$. Thus $M_i = N_i$ if $r_i = 0$. If $r_i \neq 0$ the exact sequence $(+)_i$ is non-split. As above let $\phi_i \in \text{Hom}_\Lambda(M_i, M_i)$. Since

$\text{Hom}_\Lambda(S, N_i) = 0$ we obtain the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^{r_i} & \xrightarrow{g_i} & M_i & \xrightarrow{h_i} & N_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \phi_i & & \downarrow \psi_i & & \\ 0 & \longrightarrow & S^{r_i} & \xrightarrow{g_i} & M_i & \xrightarrow{h_i} & N_i & \longrightarrow & 0 \end{array}$$

So for each $j \geq 1$ we have that $\phi_i^j h_i = h_i \psi_i^j$. Since N_i is indecomposable, or equivalently $\text{End}_\Lambda N_i$ is local, this clearly implies that M_i is indecomposable. Next we show that for $i \neq j$ we have that $M_i \not\cong M_j$. Assume that $\phi : M_i \rightarrow M_j$ is an isomorphism. Since $\text{Hom}_\Lambda(S, N) = 0$ we obtain the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^{r_i} & \xrightarrow{g_i} & M_i & \xrightarrow{h_i} & N_i & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow \phi & & \downarrow g & & \\ 0 & \longrightarrow & S^{r_j} & \xrightarrow{g_j} & M_j & \xrightarrow{h_j} & N_j & \longrightarrow & 0 \end{array}$$

So g is epi, f is mono and $\text{cok } f = \ker g$. Now $\text{cok } f \in \text{add } S$ and $\text{Hom}_\Lambda(S, N) = 0$ implies that both f and g are isomorphisms, so $i = j$.

We will now show that $T = M \oplus S$ is a Λ -tilting module. Since $\text{pd}_\Gamma N < \infty$ we have by Lemma 2.2 that also $\text{pd}_\Lambda N < \infty$. Since $\text{pd}_\Lambda S \leq 1$ we conclude from (+) that $\text{pd}_\Lambda T < \infty$. Applying $\text{Hom}_\Lambda(S, -)$ to (+) shows that $\text{Ext}_\Lambda^i(S, M) = 0$ for $i > 0$. Applying $\text{Hom}_\Lambda(-, S)$ to (+) and using the universality of (+) together with $\text{id}_\Lambda S \leq 1$ shows that $\text{Ext}_\Lambda^i(M, S) = 0$ for $i > 0$. Applying $\text{Hom}_\Lambda(-, N)$ to (+) and using the fact that $0 = \text{Ext}_\Gamma^i(N, N) = \text{Ext}_\Lambda^i(N, N)$ for $i > 0$ also shows that $\text{Ext}_\Lambda^i(M, N) = 0$ for $i > 0$. Finally applying $\text{Hom}_\Lambda(M, -)$ to (+) shows that $\text{Ext}_\Lambda^i(M, M) = 0$ for $i > 0$, hence $\text{Ext}_\Lambda^i(M \oplus S, M \oplus S) = 0$ for $i > 0$. Next let $X \in (M \oplus S)^\perp$ and consider

$$(++) \quad 0 \longrightarrow S^m \xrightarrow{\alpha} X \xrightarrow{\beta} Y \longrightarrow 0$$

as in (*), so $Y \in S^{\text{perp}}$ and also $Y \in (M \oplus S)^\perp$. Applying $\text{Hom}_\Lambda(-, Y)$ to (+) then shows that $Y \in N^\perp$ and thus $Y \in \text{fac } N$, since N is a Γ -tilting module. So there is $\pi : \tilde{N} \rightarrow Y$ surjective with $\tilde{N} \in \text{add } N$. Since N is generated by M there is also $\tilde{\pi} : \tilde{M} \rightarrow Y$ surjective with $\tilde{M} \in \text{add } M$. Since $\text{Ext}_\Lambda^1(\tilde{M}, S^r) = 0$, we obtain $\gamma : \tilde{M} \rightarrow X$ such that $\gamma\beta = \tilde{\pi}$. In particular we obtain the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^m & \xrightarrow{(1 \ 0)} & S^m \oplus \tilde{M} & \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} & M & \longrightarrow & 0 \\ & & \parallel & & \downarrow \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} & & \downarrow \tilde{\pi} & & \\ 0 & \longrightarrow & S^m & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & Y & \longrightarrow & 0 \end{array}$$

Since the outer maps are surjective, so is $\begin{pmatrix} \alpha \\ \gamma \end{pmatrix}$; thus $X \in \text{fac } (M \oplus S)$, so $T = M \oplus S$ is a Λ -tilting module by Proposition 1.3.

We have shown above that M is multiplicity free, and clearly we have that $\dim \text{Hom}_\Lambda(S, M) = r$. Applying $\text{Hom}_\Lambda(S, -)$ to (+) shows that $\psi : \text{Hom}_\Lambda(S, S^r) \rightarrow \text{Hom}(S, M)$ is an isomorphism. Let g_1, \dots, g_r be the components of g . Then for

$\lambda = (\lambda_1, \dots, \lambda_r) \in \text{Hom}_\Lambda(S, S^r) \simeq k^r$ we obtain that $\psi(\lambda) = \sum_{i=1}^r \lambda_i g_i$. So if $0 = \sum_{i=1}^r \lambda_i g_i = \psi(\lambda)$, we have $\lambda = 0$, since ψ is an isomorphism. Hence g_1, \dots, g_r are linearly independent in $\text{Hom}_\Lambda(S, M)$, and so form a basis of $\text{Hom}_\Lambda(S, M)$, since $r = \dim \text{Hom}_\Lambda(S, M)$. Thus $\varphi_S(T) = N$, hence φ_S is surjective.

It remains to be seen that both φ_S and φ_S^{-1} are maps of posets.

For this let $T^1 = M^1 \oplus S$ and $T^2 = M^2 \oplus S$ be in $\mathcal{T}(S)$ with $T^1 \leq T^2$ in \mathcal{T}_Λ . By definition of φ_S we have two exact sequences,

$$(-) \quad 0 \longrightarrow S^{t_1} \longrightarrow M^1 \longrightarrow N^1 \longrightarrow 0$$

and

$$(- -) \quad 0 \longrightarrow S^{t_2} \longrightarrow M^2 \longrightarrow N^2 \longrightarrow 0.$$

So $\varphi_S(T^1) = N^1$ and $\varphi_S(T^2) = N^2$. We claim that $N^1 \leq N^2$ in \mathcal{T}_Γ . Let $X \in N^{1\perp}$. Applying $\text{Hom}_\Lambda(-, X)$ to $(-)$ shows that $X \in T^{1\perp}$, so $X \in T^{2\perp}$. Applying $\text{Hom}_\Lambda(-, X)$ to $(- -)$ and using the fact that $X \in S^{\text{perp}}$ shows that $X \in N^{2\perp}$; thus φ_S is a map of posets.

Next we investigate φ_S^{-1} . For this let $N^1, N^2 \in \mathcal{T}_\Gamma$ with $N^1 \leq N^2$. As in the proof of the surjectivity we obtain two exact sequences:

$$(1) \quad 0 \longrightarrow S^{t_1} \longrightarrow M^1 \longrightarrow N^1 \longrightarrow 0$$

and

$$(2) \quad 0 \longrightarrow S^{t_2} \longrightarrow M^2 \longrightarrow N^2 \longrightarrow 0.$$

So $\varphi_S^{-1}(N^1) = M^1 \oplus S$ and $\varphi_S^{-1}(N^2) = M^2 \oplus S$. Let $X \in (M^1 \oplus S)^\perp$ and consider

$$(++) \quad 0 \longrightarrow S^r \xrightarrow{\alpha} X \xrightarrow{\beta} Y \longrightarrow 0$$

as in $(*)$, so $Y \in S^{\text{perp}}$ and also $Y \in M^{1\perp}$. Thus $Y \in N^{1\perp}$ by applying $\text{Hom}_\Lambda(-, Y)$ to (1) . Thus $Y \in N^{2\perp}$, since $N^1 \leq N^2$ in \mathcal{T}_Γ . Applying $\text{Hom}_\Lambda(-, Y)$ to (2) then shows that $Y \in (M^2 \oplus S)^\perp$. But then also $X \in (M^2 \oplus S)^\perp$, since $(M^2 \oplus S)^\perp$ is closed under extensions. Thus $T^1 = M^1 \oplus S \leq T^2 = M^2 \oplus S$ in \mathcal{T}_Λ , which finishes the proof of the theorem.

Corollary 2.8. φ_S induces an isomorphism $\vec{\mathcal{K}}_\Lambda(S) \rightarrow \vec{\mathcal{K}}_\Gamma$.

Proof. Clearly it is enough to show that minimal inclusions are sent to minimal inclusions under φ_S . So let $T^1 \rightarrow T^2$ be an arrow in $\vec{\mathcal{K}}_\Lambda(S)$. Let $\varphi_S(T^2) \leq T \leq \varphi_S(T^1)$ for some $T \in \mathcal{T}_\Gamma$. Let φ_S^{-1} be the inverse to φ_S . By the previous theorem φ_S^{-1} is a map of posets. So $T^2 \leq \varphi_S^{-1}(T) \leq T^1$ shows the assertion.

We refer to Corollary 5.9 for an alternative description of $\vec{\mathcal{K}}_\Lambda(S)$ for a hereditary algebra Λ and S a simple Λ -module.

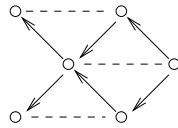
Note that in the proof of Theorem 2.7 the assumption $\text{id}_\Lambda S \leq 1$ is only used to get the surjectivity of φ_S . So in general we just obtain an embedding $\varphi_S : \mathcal{T}(S) \rightarrow \mathcal{T}_\Gamma$ of posets. The following example shows that if we drop this assumption the map φ_S need not be surjective. Indeed let Λ be given over a field k by the following quiver

$$\circ \xleftarrow{\alpha} \circ \xleftarrow{\beta} \circ$$

with relation $\beta\alpha = 0$. Let S be the unique simple projective Λ -module. Then $\text{id}_\Lambda S = 2$. Also ${}_\Lambda\Lambda$ is the unique Λ -tilting module which has S as a direct summand. So $\mathcal{T}(S) = \circ$. Now $S^{\text{perp}} = \text{mod } k(\circ \rightarrow \circ) = \text{mod } \Gamma$. So \mathcal{T}_Γ contains two elements. The map φ_S sends ${}_\Lambda\Lambda$ to ${}_\Gamma\Gamma$.

We point out that if S is simple injective and $\text{pd}_\Lambda S \leq 1$, we are in the situation investigated in [AHT].

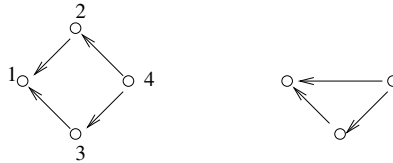
Trivially, if Λ is hereditary, then each simple Λ -module S satisfies our assumptions. However there are even tilted algebras with no simple Λ -module S satisfying these assumptions. In fact, let Λ be given over a field k by the following quiver with relations:



The following properties are easily verified. Λ is a tilted algebra of type \mathbb{E}_6 and there is no simple Λ -module S with both $\text{pd}_\Lambda S \leq 1$ and $\text{id}_\Lambda S \leq 1$.

We also point out that for $\Lambda_n = k(\circ \rightarrow \circ \cdots \circ \rightarrow \circ)$ with n vertices each tilting module T has a simple direct summand S . So in this case $\vec{\mathcal{K}}_\Lambda = \bigcup_S \vec{\mathcal{K}}_{\Gamma(S)}$, where $\text{mod } \Gamma(S) = S^{\text{perp}}$. Note that $\Gamma(S) \simeq \Lambda_{n-1}$ for each S , but the embeddings of S^{perp} into $\text{mod } \Lambda_n$ vary for the different choices of S . The union above is clearly not a disjoint union.

We also give the following example which shows that there is $\vec{\mathcal{K}}_\Gamma \subset \vec{\mathcal{K}}_\Lambda$ for $\text{mod } \Gamma \simeq S^{\text{perp}}$ and S a simple Λ -module such that $\vec{\mathcal{K}}_\Gamma$ is not connected while $\vec{\mathcal{K}}_\Lambda$ is connected. Indeed, consider the following two quivers $\vec{\Delta}$ and $\vec{\Delta}'$



Let $\Lambda = k\vec{\Delta}$ and let $S = S_2$ be the simple Λ -module associated with the vertex of $\vec{\Delta}$ labelled 2. Then $S^{\text{perp}} \simeq \text{mod } \Gamma$ for $\Gamma = k\vec{\Delta}'$. For this note that the projective generator ${}_\Lambda Q$ of S^{perp} is given by

$${}_\Lambda Q = {}_\Lambda P(2) \oplus \tau_\Lambda^- P(1) \oplus \tau_\Lambda^- P(2).$$

It is well-known that $\vec{\mathcal{K}}_\Lambda$ is connected and $\vec{\mathcal{K}}_\Gamma$ has exactly two connected components [U2].

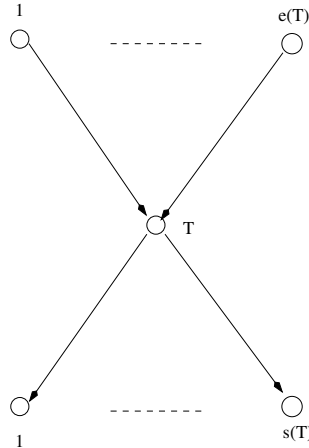
We will also need orthogonal categories for Λ -modules which may be decomposable. For this let Λ be a hereditary algebra and let M be a Λ -module with $\text{Ext}_\Lambda^1(M, M) = 0$. We define the right orthogonal category M^{perp} to be the full subcategory of $\text{mod } \Lambda$ containing the Λ -modules X such that $\text{Hom}_\Lambda(M, X) = 0$ and $\text{Ext}_\Lambda^1(M, X) = 0$. Note that the indecomposable direct summands M_i of M can be ordered such that $\text{Hom}_\Lambda(M_i, M_j) = 0$ if $i > j$, so this definition is compatible with the definition given above. We assume that M has t non-isomorphic indecomposable direct summands. Then it is shown in [GL] that $M^{\text{perp}} \simeq \text{mod } \Gamma$, where

Γ is a finite dimensional hereditary algebra with $n - t$ simple modules. Moreover the inclusion $M^{\text{perp}} \rightarrow \text{mod } \Lambda$ is an exact embedding. One situation will be of particular importance for us. Let M be a faithful projective Λ -module with $n - 2$ non-isomorphic indecomposable summands, say $M = \bigoplus_{i=1}^{n-2} P(i)$. We will assume that $n \geq 3$. Clearly $\text{mod } \Gamma = M^{\text{perp}}$ is equivalent to the subcategory of Λ -modules with composition factors of the form $S(n - 1)$ and $S(n)$. Let $T = M \oplus X \oplus Y$ be a tilting module such that X and Y are not projective. Then by the Auslander-Reiten formula (compare for example 2.4 in [R]) we see that $\tau_{\Lambda}(X \oplus Y) \in M^{\text{perp}}$ is a Γ -tilting module. Moreover we obtain an embedding of quivers $\vec{\mathcal{K}}_{\Gamma} \rightarrow \vec{\mathcal{K}}_{\Lambda}$ such that the unique sink $D\Gamma_{\Gamma}$ of $\vec{\mathcal{K}}_{\Gamma}$ is mapped to the unique sink of $\vec{\mathcal{K}}_{\Lambda}(M)$. In this way Theorem 1.4 can be interpreted such that the first case corresponds to the case where Γ is a product of two copies of k , that the second case corresponds to the case where Γ is of type \mathbb{A}_2 , and the last case corresponds to the case where Γ is isomorphic to a representation infinite hereditary algebra with two simple modules. Of course there are infinitely many such examples.

3. DETECTION OF SIMPLES

In the introduction we have defined the partially ordered set $(\mathcal{T}_{\Lambda}, \leq)$ and the quiver $\vec{\mathcal{K}}_{\Lambda}$ of tilting modules. Since we are mainly interested in the structure of $\vec{\mathcal{K}}_{\Lambda}$ for hereditary algebras, we will assume in this section that Λ is a finite dimensional hereditary k -algebra for an algebraically closed field k . Thus $\Lambda \simeq k\vec{\Delta}$, where $k\vec{\Delta}$ denotes the path algebra over k of a finite quiver without oriented cycles. We denote by n the number of vertices of $\vec{\Delta}$. The main aim of this section is to determine this number n from $\vec{\mathcal{K}}_{\Lambda}$ in case Λ is connected. A more general result will be shown in section 5.

Let $T \in \vec{\mathcal{K}}_{\Lambda}$. We denote by $s(T)$ (resp. $e(T)$) the number of arrows starting (resp. ending) at T in $\vec{\mathcal{K}}_{\Lambda}$.



Denote by $K_0(\Lambda)$ the Grothendieck group of Λ . We recall from [HU2].

Lemma 3.1. $\delta(T) = s(T) + e(T) \leq rk K_0(\Lambda) = n$.

We say that T is saturated if equality holds. Note that the proof of Lemma 3.1 shows that T is saturated if and only if $T[i]$ is faithful or equivalently sincere for

each $1 \leq i \leq n$. Here we will investigate the situation more closely. We denote it by $\delta(\Lambda) = \max \delta(T)$. So $\delta(\Lambda)$ is the degree of $\vec{\mathcal{K}}_\Lambda$.

For a Λ -module X we denote by $\mathbf{dim} X \in K_0(\Lambda)$ its dimension vector which we will identify with the corresponding element in \mathbb{Z}^n . The following is an easy generalization of 3.2 in [HU2]. We also refer to [HU2] for explicit ways to compute $e(T)$ and $s(T)$ independently.

Proposition 3.2. *If $T \in \text{mod} k\vec{\Delta}$ is a tilting module, then the degree $\delta(T) = n - |\{i | (\mathbf{dim} T)_i = 1\}|$.*

Proof. We consider the almost complete partial tilting modules $T[1], \dots, T[n]$. We may assume that $T[1], \dots, T[r]$ are faithful and that $T[r + 1], \dots, T[n]$ are not faithful. By Lemma 1.1 and the definition of $\vec{\mathcal{K}}_\Lambda$ we clearly have that $\delta(T) = r$.

Let $r + 1 \leq j \leq n$, so there is i_j such that $\text{Hom}_\Lambda(T[j], I(i_j)) = 0$. If $r + 1 \leq j' \leq n$ and $j \neq j'$, then $i_j \neq i_{j'}$ for $\text{add } T = \text{add } T[j] \oplus T[j']$ and T is faithful.

Let $r + 1 \leq j \leq n$. Consider $\Gamma = \text{End}_\Lambda T$. By the tilting theory we have that $\text{Hom}_\Lambda(T, I(i_j))$ is an indecomposable Γ -module. Set $P_j = \text{Hom}_\Lambda(T, T_j)$ and $S_j = \text{top } P_j$. Now

$$\text{Hom}_\Lambda(T, I(i_j)) \simeq S_j^{\dim \text{Hom}_\Lambda(T, I(i_j))} \simeq S_j^{\dim \text{Hom}_\Lambda(T_j, I(i_j))}.$$

$\text{Hom}_\Lambda(T, I(i_j))$ indecomposable implies that $\dim \text{Hom}_\Lambda(T, I(i_j)) = 1$. So there is $1 \leq i \leq n$ with $(\mathbf{dim} T)_i = 1$, and by the first remark we see that $|\{i | (\mathbf{dim} T)_i = 1\}| \geq n - r = n - \delta(T)$.

For the other inequality let $1 \leq i \leq n$ with $(\mathbf{dim} T)_i = 1$. Then there is $1 \leq j_i \leq n$ such that $T = T[j_i] \oplus T_{j_i}$ with $(\mathbf{dim} T_{j_i})_i = 1$. Then $(\mathbf{dim} T[j_i])_i = 0$, so $T[j_i]$ is not faithful, hence $r + 1 \leq j_i \leq n$. This finishes the proof.

The following is an immediate consequence of Proposition 3.2.

Corollary 3.3. *Let $\Lambda = k\vec{\Delta}$. Then $s(\Lambda) = n - |\text{sources in } \vec{\Delta}|$ and $e(D\Lambda) = n - |\text{sinks in } \vec{\Delta}|$. So $s(\Lambda) + e(D\Lambda) \geq n$ with equality if and only if there is no path of length two in $\vec{\Delta}$.*

Proof. Clearly we have that $i \in \vec{\Delta}$ is a source if and only if $(\mathbf{dim} \Lambda)_i = 1$ and $i \in \vec{\Delta}$ is sink if and only if $(\mathbf{dim} D\Lambda)_i = 1$. The result now follows from Proposition 3.2.

Next we will show how to deduce the number n of simple Λ -modules from $\vec{\mathcal{K}}_\Lambda$. Recall that we have denoted by $\delta(\Lambda)$ the degree of $\vec{\mathcal{K}}_\Lambda$.

Proposition 3.4. *Let Λ be a connected finite dimensional hereditary k -algebra. Then $\delta(\Lambda) = n$ if and only if $\Lambda \not\cong k(\circ \rightarrow \circ \cdots \circ \rightarrow \circ)$ with n vertices. Moreover, if $\Lambda \simeq k(\circ \rightarrow \circ \cdots \circ \rightarrow \circ)$ with n vertices, then $\delta(\Lambda) = s(\Lambda) = n - 1$.*

Proof. If $\Lambda \simeq k(\circ \rightarrow \circ \cdots \circ \rightarrow \circ)$, denote by P the unique indecomposable projective-injective Λ -module. If T is a tilting module, then P is an indecomposable direct summand of T , so $T = P \oplus T'$. Since P is neither generated nor cogenerated by T' , we infer that $\delta(T) < n$.

Conversely, assume that $\Lambda \not\cong k(\circ \rightarrow \circ \cdots \circ \rightarrow \circ)$. If Λ is of infinite representation type it is well-known that there is $t \in \mathbb{N}$ such $(\mathbf{dim} \tau_\Lambda^{-t} \Lambda)_i \geq 2$ for all i . Here of course we use the fact that Λ is connected. So by the proposition we see that $\delta(\Lambda) = n$. If Λ is of finite representation type, let X be the indecomposable Λ -module

corresponding to the maximal root. Since $\Lambda \not\cong k(\circ \rightarrow \circ \cdots \circ \rightarrow \circ)$ and Λ is connected, we have that X is not both projective and injective. We assume that X is not injective. The other case can be dealt with in a dual way. Consider the Auslander-Reiten sequence $0 \rightarrow X \rightarrow Y \rightarrow \tau^-X \rightarrow 0$ starting in X . Since X is sincere, we infer that Y is sincere. Clearly $X \oplus Y$ is a partial tilting module, since Λ is hereditary. Let $T = X \oplus Y \oplus C$ be a tilting module. Then clearly $(\mathbf{dim} T)_i \geq 2$ for all i , hence $\delta(\Lambda) = n$.

Now let $\Lambda = k(\circ \rightarrow \circ \cdots \circ \rightarrow \circ)$ with n vertices. We have seen above that $\delta(\Lambda) \leq n - 1$. Since there is a unique source in Λ we have by Corollary 3.3 that $n - 1 = s(\Lambda\Lambda) = \delta(\Lambda\Lambda) \leq \delta(\Lambda)$, hence $\delta(\Lambda) = s(\Lambda\Lambda) = n - 1$.

Corollary 3.5. *Let Λ be a connected hereditary algebra. Then $n - 1 \leq \delta(\Lambda) \leq n$.*

Proof. This follows immediately from the previous proposition.

We summarize these results in the following corollary, which also shows how to detect the number of simple Λ -modules from $\vec{\mathcal{K}}_\Lambda$.

Corollary 3.6. *Let Λ be a connected hereditary algebra. If $s(\Lambda\Lambda) = \delta(\Lambda)$, then $n = \delta(\Lambda) + 1$. Otherwise $n = \delta(\Lambda)$.*

Proof. By Corollary 3.3 we have that $s(\Lambda\Lambda) \leq n - 1$, since there is at least one source. So if $s(\Lambda\Lambda) = \delta(\Lambda)$, we infer by Corollary 3.5 that $\delta(\Lambda) = n - 1$, so by Proposition 3.4 we have that $\Lambda \simeq k(\circ \rightarrow \circ \cdots \circ \rightarrow \circ)$ with n vertices, hence $n = \delta(\Lambda) + 1$. Otherwise we have that $s(\Lambda\Lambda) \neq \delta(\Lambda)$. So we conclude from Proposition 3.4 that $\Lambda \not\cong k(\circ \rightarrow \circ \cdots \circ \rightarrow \circ)$. Using Proposition 3.4 again we finally see that $n = \delta(\Lambda)$.

We remark that the proof of Proposition 3.4 actually shows that we can read off the degree $\delta(\Lambda)$ by just considering the connected component of $\vec{\mathcal{K}}_\Lambda$ containing $\Lambda\Lambda$.

4. THE DYNKIN CASE

In this section we investigate more closely the situation of representation finite hereditary algebras. We assume that Λ is connected. Thus $\Lambda \simeq k\vec{\Delta}$, where Δ is a Dynkin diagram of the form A_n, D_n, E_6, E_7 , or E_8 (see for example [ARS]). In this situation $\vec{\mathcal{K}}_\Lambda$ is finite, so we can read off the number of tilting modules and by the result in the previous section the number of simple modules. In this section we will show how to determine the number of indecomposable Λ -modules (up to isomorphism). As an easy consequence we will see that this already determines Δ from $\vec{\mathcal{K}}_\Lambda$. The precise reconstruction (i.e. the orientation) will be done in section 6.

Since $\vec{\mathcal{K}}_\Lambda$ is finite, we may define $\ell(\vec{\mathcal{K}}_\Lambda) = \max\{l(w) \mid w \text{ path in } \vec{\mathcal{K}}_\Lambda\}$, where for a path $w \in \vec{\mathcal{K}}_\Lambda$ we denote by $l(w)$ the number of arrows occurring in w . By $m(\Delta)$ we denote the number of positive roots in the root system associated with Δ . So $m(\Delta)$ coincides with the number of isomorphism classes of indecomposable $k\vec{\Delta}$ -modules.

The following theorem is an easy application of the representation theory of Dynkin quivers, but it still seems to be worthwhile mentioning.

Theorem 4.1. $\ell(\vec{\mathcal{K}}_\Lambda) = m(\Delta) - n$.

Proof. Let $\Lambda = k\vec{\Delta}$ for a Dynkin diagram Δ . If $T \rightarrow T'$ is an arrow in $\vec{\mathcal{K}}_\Lambda$ we have that $T'^\perp \subset T^\perp$. Let $T = M \oplus X$ and $T' = M \oplus Y$. Then the non-split

exact sequence connecting the complements (compare Lemma 1.1) shows that the inclusion is proper. In fact, $X \in T^\perp$, but $X \notin T'^\perp$. We denote by $m(T)$ the number of indecomposable Λ -modules (up to isomorphism) in T^\perp . Since $D\Lambda_\Lambda \in T^\perp$ for all T , we clearly have that $m(\Delta) = m({}_\Lambda\Lambda) \geq m(T) \geq m(D\Lambda_\Lambda) = n$ for all $T \in \vec{\mathcal{K}}_\Lambda$. Thus $\ell(\vec{\mathcal{K}}_\Lambda) \leq m(\Delta) - n$.

For the converse inequality we will now construct a path in $\vec{\mathcal{K}}_\Lambda$ of the form

$$T_0 = {}_\Lambda\Lambda \rightarrow T_1 \rightarrow \dots \rightarrow T_{r-1} \rightarrow T_r = D\Lambda_\Lambda$$

such that $m(T_i) = m(T_{i+1}) + 1$ for each $0 \leq i \leq r - 1$, and for each $0 \leq i \leq r$ we have that $\text{End } T_i$ is hereditary. We assume that $T_i \neq D\Lambda_\Lambda$ is constructed and show how to construct T_{i+1} with the desired properties. Since $T_i \neq D\Lambda_\Lambda$, there is an indecomposable direct summand X of $T_i = M \oplus X$ such that $\text{Hom}_\Lambda(M, X) = 0$ and X is not injective. Since $\text{End } T_i$ is hereditary we know that the indecomposable direct summands of T_i form a complete slice (compare 4.2 in [R]). Let $0 \rightarrow X \rightarrow E \rightarrow \tau^-X \rightarrow 0$ be the Auslander-Reiten sequence starting in X . Then E is a direct summand of T_i . Set $T_{i+1} = M \oplus \tau^-X$. It is straightforward to check that T_{i+1} is a tilting module such that $m(T_i) = m(T_{i+1}) + 1$ and that $\text{End } T_{i+1}$ is hereditary. Since $\vec{\mathcal{K}}_\Lambda$ is finite this process has to stop, and this is precisely the case when we reach $D\Lambda_\Lambda$, which finishes the proof of the theorem.

The following is an immediate consequence of the theorem above.

Corollary 4.2. *Let $\vec{\Delta}$ be a Dynkin quiver and $\Lambda = k\vec{\Delta}$. Then $\ell(\vec{\mathcal{K}}_\Lambda)$ is independent of the orientation.*

We note, however, that a corresponding result for the length of a shortest path from the source to the sink of $\vec{\mathcal{K}}_\Lambda$ will fail, as the following example shows. Let $\Lambda = k(\circ \rightarrow \circ \rightarrow \circ)$ and $\Lambda' = k(\circ \rightarrow \circ \leftarrow \circ)$. In the first case a shortest path from the source to the sink of $\vec{\mathcal{K}}_\Lambda$ will have length two, while in the second case it will have length three, which can be seen from the quivers $\vec{\mathcal{K}}_\Lambda$ and $\vec{\mathcal{K}}_{\Lambda'}$ depicted below.



In the following corollary we characterize the tilting modules lying on maximal paths in $\vec{\mathcal{K}}_\Lambda$.

Corollary 4.3. *Let T be a vertex of $\vec{\mathcal{K}}_\Lambda$. The indecomposable direct summands of T form a complete slice if and only if T lies on a path of maximal length. In particular, the number of complete slices coincides with the number of different vertices on paths of maximal length.*

Proof. If the indecomposable direct summands of T form a complete slice, we may construct as in the proof of the theorem a path from T to $D\Lambda_\Lambda$ and dually a path from ${}_\Lambda\Lambda$ to T . The concatenation of these two paths yields a path of maximal length.

Conversely, let T be a vertex on path w of maximal length. As pointed out above this path has to start in ${}_{\Lambda}\Lambda$ and terminate in $D\Lambda_{\Lambda}$. Moreover we have for each arrow $T \rightarrow T'$ on w that $m(T) = m(T') + 1$. We may assume that the indecomposable direct summands of T form a complete slice, and we will show that T' is of the form constructed as in the proof of the theorem. Since $T \rightarrow T'$ in $\vec{\mathcal{K}}_{\Lambda}$ we have that $T = M \oplus X$ and that $T' = M \oplus Y$. Since $m(T) = m(T') + 1$, we have that X is the unique indecomposable module in T^{\perp} not contained in T'^{\perp} . Since X is cogenerated by M we have by Lemma 1.1(3) that X is not generated by M . Since Y is generated by M this shows that X is not generated by T' . If $\text{Hom}(M, X) \neq 0$ we have by tilting theory a non-split exact sequence $0 \rightarrow t(X) \rightarrow X \rightarrow X/t(X) \rightarrow 0$ with $t(X) \in T'^{\perp}$. Clearly, $X/t(X) \in T^{\perp}$ and $X/t(X) \notin T'^{\perp}$. Since by assumption $t(X) \neq 0$, there is an indecomposable direct summand Z of $X/t(X)$ with $Z \in T^{\perp}$ and $Z \notin T'^{\perp}$. Since $Z \not\cong X$ this gives a contradiction to $m(T) = m(T') + 1$. Thus $\text{Hom}(M, X) = 0$. Trivially X is not injective, as the non-split exact sequence connecting the complements induced by the arrow $T \rightarrow T'$ in $\vec{\mathcal{K}}_{\Lambda}$ shows. Since the indecomposable direct summands of T form a complete slice, we conclude that the exact sequence connecting the complements is the Auslander-Reiten sequence starting in X ; thus $T' = M \oplus \tau^{-}X$ and $\text{End}_{\Lambda}T'$ is hereditary.

It is well-known that $m(\mathbb{A}_n) = \frac{n(n+1)}{2}$, $m(\mathbb{D}_n) = n(n-1)$, $m(\mathbb{E}_6) = 36$, $m(\mathbb{E}_7) = 63$ and $m(\mathbb{E}_8) = 120$. It is easily checked that for a fixed n these numbers are pairwise different. By Corollary 3.6 we can compute n from $\vec{\mathcal{K}}_{\Lambda}$, and thus by Theorem 4.1 we can compute $m(\Delta)$ from $\vec{\mathcal{K}}_{\Lambda}$; hence we have the following corollary.

Corollary 4.4. *If $\vec{\mathcal{K}}_{\Lambda}$ is finite, then Δ is uniquely determined (up to isomorphism) by $\vec{\mathcal{K}}_{\Lambda}$.*

A different way of determining $m(\Delta)$ is given in the following proposition.

Proposition 4.5. $m(\Delta) = |\{T \in \mathcal{T}_{\Lambda} \mid e(T) = 1\}| + n$.

Proof. Let $\vec{\Delta}$ be a Dynkin quiver. Clearly it is enough to construct a bijection between the isomorphism classes of indecomposable non-projective $k\vec{\Delta}$ -modules and the set $\{T \in \mathcal{T}_{\Lambda} \mid e(T) = 1\}$. Let X be an indecomposable non-projective $k\vec{\Delta}$ -module. We consider the category X^{perp} . Let $P \in X^{\text{perp}}$ be a minimal projective generator. By Lemma 2.3 we have that $T = X \oplus P$ is a tilting module and that $e(T) = 1$. This defines a map between the two sets above which is injective, since for $X \not\cong X'$ indecomposable and non-projective $X \oplus P = X' \oplus P'$ implies that X is a direct summand of P' , so in X'^{perp} and that X' is a direct of summand of P . By 4.1 in [R] each indecomposable direct summand P_i of P satisfies $\text{Hom}_{\Lambda}(P_i, X) \neq 0$, a contradiction. So the map defined above is injective. Next we show that it is also surjective. For this let $T \in \mathcal{T}_{\Lambda}$ with $e(T) = 1$. Let $T = M \oplus X$, where X is the unique indecomposable direct summand of T which gives rise to the arrow in $\vec{\mathcal{K}}_{\Lambda}$ to T . We claim that $M \in X^{\text{perp}}$ is projective. If $\text{Hom}_{\Lambda}(X, M) \neq 0$, there is an indecomposable direct summand M_i of M and $0 \neq f_i : X \rightarrow M_i$. Now f_i is either mono or epi [HR]. Since X is generated by M it is not cogenerated by M , so f_i is not mono, and since $e(T) = 1$ we have that f_i is not epi. So $M \in X^{\text{perp}}$. Clearly, M is a tilting module in X^{perp} . If M is not projective in X^{perp} , there is an indecomposable direct summand Y of $M = N \oplus Y$ such that Y is generated by N . But then Y is also generated by $N \oplus X$, in contrast to $e(T) = 1$.

5. TILTING MODULES ASSOCIATED WITH SIMPLE MODULES

In this section we will investigate more closely the neighbors of ${}_{\Lambda}\Lambda$ and $D\Lambda_{\Lambda}$ in $\vec{\mathcal{K}}_{\Lambda}$. First we introduce tilting modules associated with certain simple modules.

Lemma 5.1. *Let $\Lambda = k\vec{\Delta}$ be a hereditary algebra. Let S_i be a simple Λ -module which is not injective. Then $T_{S_i} = {}_{\Lambda}\Lambda[i] \oplus \tau_{\Lambda}^{-} S_i$ is a tilting module. In this way we obtain all immediate successors of ${}_{\Lambda}\Lambda$ in $\vec{\mathcal{K}}_{\Lambda}$.*

Proof. Since Λ is hereditary and T_{S_i} has the right number of non-isomorphic indecomposable direct summands, it is enough to show that $\text{Ext}_{\Lambda}^1(T_{S_i}, T_{S_i}) = 0$. In fact, apply the Auslander-Reiten formula (compare for example 2.4 in [R]) to obtain the following chain of isomorphisms:

$$\begin{aligned} \text{Ext}_{\Lambda}^1(T_{S_i}, T_{S_i}) &\simeq D\text{Hom}_{\Lambda}(T_{S_i}, \tau_{\Lambda} T_{S_i}) = D\text{Hom}_{\Lambda}({}_{\Lambda}\Lambda[i] \oplus \tau_{\Lambda}^{-} S_i, S_i) \\ &= D\text{Hom}_{\Lambda}(\tau_{\Lambda}^{-} S_i, S_i) \simeq \text{Ext}_{\Lambda}^1(S_i, S_i) = 0. \end{aligned}$$

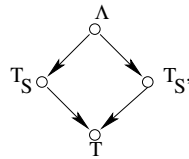
Thus T_{S_i} is a tilting module.

By Corollary 3.3 we obtain in this way all immediate successors of ${}_{\Lambda}\Lambda$ in $\vec{\mathcal{K}}_{\Lambda}$.

Dually we obtain for each simple Λ -module S_i which is not projective a tilting module $T^{S_i} = (D\Lambda_{\Lambda})[i] \oplus \tau S_i$ yielding all immediate predecessors of $D\Lambda_{\Lambda}$ in $\vec{\mathcal{K}}_{\Lambda}$.

Given a quiver $\vec{\Delta}$ we will identify the simple $k\vec{\Delta}$ -modules with the vertices of $\vec{\Delta}$.

Lemma 5.2. *Let $\Lambda = k\vec{\Delta}$ be a hereditary algebra and let S, S' be different simple Λ -modules which are not injective. Then there is no edge in $\vec{\Delta}$ between S and S' if and only if T_S and $T_{S'}$ are contained in $\vec{\mathcal{K}}_{\Lambda}$ in a diamond of the form*



Proof. Let $S = S_i$ and $S' = S_j$, so $i \neq j$. There is no edge in $\vec{\Delta}$ between S_i and S_j if and only if $\text{Ext}_{\Lambda}^1(S_i \oplus S_j, S_i \oplus S_j) = 0$, or equivalently that the module $T = {}_{\Lambda}\Lambda[i, j] \oplus \tau_{\Lambda}^{-} S_i \oplus \tau^{-} S_j$ is a tilting module. Thus if there is no edge in $\vec{\Delta}$ between S_i and S_j , we obtain that T_{S_i} and T_{S_j} are contained in a diamond. Conversely, if T_{S_i} and T_{S_j} are contained in a diamond, then trivially $T = {}_{\Lambda}\Lambda[i, j] \oplus \tau_{\Lambda}^{-} S_i \oplus \tau^{-} S_j$ by using Corollary 1.2; thus T is a tilting module, and hence there is no edge in $\vec{\Delta}$ between S_i and S_j .

Note that we have the dual statement for different simple Λ -modules S, S' , which are not projective, involving the tilting modules T^S and $T^{S'}$. Also note that in this way we can determine for an arbitrary $\vec{\Delta}$ whether or not there are edges in $\vec{\Delta}$ between non-injective simples and also between non-projective simples.

Given a tilting module $T = \bigoplus_{t=1}^n T_t$ and $i \neq j$ with $1 \leq i, j \leq n$ we denote it by $T[i, j] = \bigoplus_{t \neq i, j} T_t$.

Lemma 5.3. For $1 \leq i, j \leq n$ let $M = {}_{\Lambda}\Lambda[i, j]$ and assume that M is faithful. Let $T = M \oplus X \oplus Y$ be the unique sink in $\text{lk}(M)$. Then:

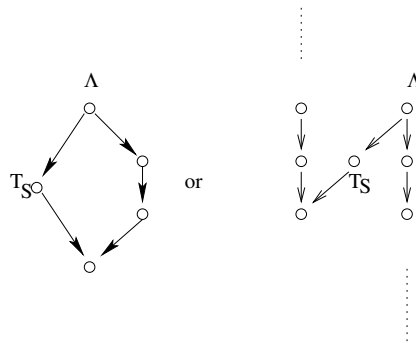
- (1) X and Y are not Λ -projective.
- (2) $\tau_{\Lambda}(X \oplus Y)$ is a tilting module in M^{perp} which is injective in M^{perp} .
- (3) $T' \in \text{lk}(M)$ if and only if $T' \geq T$ in \mathcal{T}_{Λ} .

Proof. By Theorem 1.4 we have that there is a path of length two from ${}_{\Lambda}\Lambda$ to T in $\vec{\mathcal{K}}_{\Lambda}$. By Lemma 1.1 we then have that X and Y are both not projective, thus showing (1). Since M is projective and $M \oplus X \oplus Y$ is a tilting module, we have that $\tau_{\Lambda}(X \oplus Y) \in M^{\text{perp}}$. If $\tau_{\Lambda}(X \oplus Y)$ is not injective in M^{perp} , then there is a connecting sequence in M^{perp} of the form $0 \rightarrow \tau_{\Lambda}X \rightarrow \tau_{\Lambda}Y \rightarrow Z \rightarrow 0$ with $\tau_{\Lambda}Y \in \text{add } \tau_{\Lambda}Y$. Since M is faithful Z is not Λ -injective. But then $M \oplus Y \oplus \tau_{\Lambda}^{-1}Z \in \text{lk}(M)$ is a successor of T , in contrast to T being the sink in $\text{lk}(M)$, thus showing (2). Finally let $T' = M \oplus X_1 \oplus Y_1 \in \text{lk}(M)$; then as above $\tau_{\Lambda}(X_1 \oplus Y_1) \in M^{\text{perp}}$. So by (2) we infer that $\text{Ext}_{\Lambda}^1(\tau_{\Lambda}(X_1 \oplus Y_2), \tau_{\Lambda}(X \oplus Y)) = 0$. But this clearly implies that $T' \geq T$. Conversely let $T' \in \mathcal{T}_{\Lambda}$ with $T' \geq T$. Then $\text{Ext}_{\Lambda}^1(T', T) = 0$ shows that $\text{Ext}_{\Lambda}^1(T' \oplus M, T' \oplus M) = 0$; thus M is a direct summand of T' . So $T' \in \text{lk } M$.

We point out that in the situations where we apply Lemma 5.3 the unique sink is combinatorially given. Then Lemma 5.3(3) describes the link inside \mathcal{T}_{Λ} combinatorially.

Next we will determine which of the neighbors T_s of ${}_{\Lambda}\Lambda$ in $\vec{\mathcal{K}}_{\Lambda}$ correspond to simple S which are not projective.

Lemma 5.4. Let Λ be a hereditary algebra and let S be a simple Λ -module which is not injective. Then S is not projective if and only if T_S occurs in a subquiver of $\vec{\mathcal{K}}_{\Lambda}$ of the form



Proof. If $S_i = S$ is not projective there is a simple Λ -module S_j with $S_i \rightarrow S_j$ in $\vec{\Delta}$. Let $M = {}_{\Lambda}\Lambda[i, j]$. Since S_i, S_j are not injective we have that M is faithful. By Theorem 1.4 we know the shape of $\vec{\mathcal{K}}_{\Lambda}(M)$. By Lemma 5.2 we know that the diamond cannot occur. Let $T = M \oplus X \oplus Y$ be the unique sink in $\vec{\mathcal{K}}_{\Lambda}(M)$. We have by Lemma 5.3 that neither X nor Y is a projective Λ -module and that $\tau_{\Lambda}(X \oplus Y)$ is a tilting module in M^{perp} which is injective in M^{perp} . Since $S_i \in M^{\text{perp}}$ is injective we may assume that $\tau_{\Lambda}X = S_i$, or $X = \tau_{\Lambda}^{-1}S_i$. Let T' be the tilting module which lies on the path of length two in $\vec{\mathcal{K}}_{\Lambda}(M)$ from ${}_{\Lambda}\Lambda$ to T . Then T' is of the form T_{S_m}

for some m . By Corollary 1.2 we infer that τ^-S_m is a direct summand of T . Since $\text{Ext}_\Lambda^1(\tau^-S_m, M) = 0$ we see that $m = i$ or $m = j$. But $m = j$ is impossible since τ^-S_i is a direct summand of T and there is an arrow $S_i \rightarrow S_j$ in $\vec{\Delta}$. So $m = i$, and thus $T_S = T_{S_i}$ is in the position as indicated.

Conversely, assume that $T_S = T_{S_i}$ is contained in a subquiver of $\vec{\mathcal{K}}_\Lambda$ as indicated. The path of length two from ${}_\Lambda\Lambda$ to the unique sink T in the subquiver shows that T is of the form $T = {}_\Lambda\Lambda[i, j] \oplus \tau_\Lambda^-S_i \oplus X$ for some $j \neq i$ and a non-projective indecomposable Λ -module X . The position of T in the subquiver shows that T has two predecessors in $\text{lk}({}_\Lambda\Lambda[i, j])$. By Theorem 1.4 we see that T is the unique sink in $\text{lk}({}_\Lambda\Lambda[i, j])$ and that ${}_\Lambda\Lambda[i, j]$ is faithful. So we have by Lemma 5.3 that $S_i \oplus \tau_\Lambda X$ is injective in ${}_\Lambda\Lambda[i, j]^{\text{perp}}$. The subquiver shows that there is at least one more tilting module $T' = {}_\Lambda\Lambda[i, j] \oplus Z$ such that $\tau_\Lambda Z \in M^{\text{perp}}$. Indeed let T' be the immediate predecessor of T different from T_S . The arrow $T' \rightarrow T$ in $\vec{\mathcal{K}}_\Lambda$ shows that ${}_\Lambda\Lambda[i, j]$ is a direct summand of T , so $T' = {}_\Lambda\Lambda[i, j] \oplus Z$. If Z has an indecomposable projective direct summand, then T' is of the form T_{S_m} for some m , but T' is not an immediate successor of ${}_\Lambda\Lambda$. Thus $\tau_\Lambda Z$ is a tilting module in ${}_\Lambda\Lambda[i, j]^{\text{perp}}$ not isomorphic to $S \oplus \tau_\Lambda X$. This shows that M^{perp} is connected, and hence there is an arrow from $S_i \rightarrow S_j$ in ${}_\Lambda\Lambda[i, j]^{\text{perp}}$, since S_i is injective in ${}_\Lambda\Lambda[i, j]^{\text{perp}}$. But then there is an arrow from $S_i \rightarrow S_j$ in $\vec{\Delta}$, hence $S = S_i$ is not projective.

Note that in a dual way we may decide which neighbors of $D\Lambda_\Lambda$ in $\vec{\mathcal{K}}_\Lambda$ correspond to simple injective modules.

If $\vec{\Delta}$ is a Dynkin quiver there is a different way to detect in $\vec{\mathcal{K}}_\Lambda$ which neighbors of ${}_\Lambda\Lambda$ correspond to simple projective Λ -modules.

Proposition 5.5. *Let $\Lambda = k\vec{\Delta}$ for a Dynkin quiver $\vec{\Delta}$. Let S be a simple Λ -module which is not injective. Then S is projective if and only if T_S lies on a path of maximal length in $\vec{\mathcal{K}}_\Lambda$.*

Proof. If S is simple projective, then the indecomposable direct summands of T_S form a complete slice, so T_S lies on a path of maximal length in $\vec{\mathcal{K}}_\Lambda$ by Corollary 4.3. Conversely, assume that T_S lies on a path of maximal length in $\vec{\mathcal{K}}_\Lambda$ for a simple Λ -module $S_i = S$. Then the sequence connecting the two complements $P(i), \tau^-S_i$ of ${}_\Lambda\Lambda[i]$ is the Auslander-Reiten sequence starting at $P(i)$; hence $P(i) = \tau(\tau^-S_i) = S_i$, so S_i is simple projective.

Clearly there is a dual statement for the neighbors of $D\Lambda_\Lambda$ in $\vec{\mathcal{K}}_\Lambda$, in case $\vec{\mathcal{K}}_\Lambda$ is finite.

Let Λ be a finite dimensional k -algebra and $\Lambda' = \Lambda \times k$. Let S be the simple Λ' -module corresponding to the factor k in Λ' . Then $\varphi : \mathcal{T}_\Lambda \rightarrow \mathcal{T}_{\Lambda'}$ defined by $\varphi(T) = T \times S$ is an isomorphism of posets. Thus it is impossible to detect the number of simples unless we assume that Λ has no simple projective which is injective. In our situation we equivalently assume that $\vec{\Delta}$ does not have isolated vertices. In Corollary 3.6 we have shown how to determine the number of simple modules in case $\vec{\Delta}$ is connected. We now present a more general result.

Theorem 5.6. *Let $\vec{\Delta}$ be a quiver without isolated vertices and $\Lambda = k\vec{\Delta}$. Then the number of simple Λ -modules is uniquely determined by the neighbors of ${}_\Lambda\Lambda$ and by the neighbors of $D\Lambda_\Lambda$ in $\vec{\mathcal{K}}_\Lambda$.*

Proof. By Corollary 3.3 we have that $s(\Lambda\Lambda) = n - |\text{sources in } \overrightarrow{\Delta}|$. Let i be a source of $\overrightarrow{\Delta}$. By assumption i is not a sink of $\overrightarrow{\Delta}$, so by Lemma 5.1 the tilting module T^{S_i} is defined and a neighbor of $D\Lambda_\Lambda$ is in $\overrightarrow{\mathcal{K}}_\Lambda$. By the dual of Lemma 5.4 we can determine the number of neighbors T^S of $D\Lambda_\Lambda$ corresponding to simple injectives S , so we can determine the number of sources and hence we can determine n .

Let S be a simple Λ -module which is not injective. We will consider the subposet defined by $\mathcal{T}_S = \{T \in \mathcal{T}_\Lambda \mid T \leq T_S\}$. If S is simple injective we set $\mathcal{T}_S = \emptyset$. Let S be a simple Λ -module which is not projective. We define dually a subposet $\mathcal{T}^S = \{T \in \mathcal{T}_\Lambda \mid T \geq T^S\}$. If S is projective we set $\mathcal{T}^S = \emptyset$. If S is a simple Λ -module we denote the complement of \mathcal{T}_S inside \mathcal{T}_Λ by $\mathcal{C}(\mathcal{T}_S) = \{T \in \mathcal{T}_\Lambda \mid T \notin \mathcal{T}_S\}$. If S' is a simple Λ -module we denote the complement of $\mathcal{T}^{S'}$ by $\mathcal{C}(\mathcal{T}^{S'}) = \{T \in \mathcal{T}_\Lambda \mid T \notin \mathcal{T}^{S'}\}$. Recall from Lemma 2.3 that for an indecomposable Λ -module X which is not projective we have defined $T^\alpha(X) \in \mathcal{T}_\Lambda$. Using left orthogonal categories we can dually define $T_\omega(X)$ in case X is not injective.

Proposition 5.7. *Let S, S' be simple Λ -modules. Then we have the following for $T \in \mathcal{T}_\Lambda$:*

- (1) $T \in \mathcal{C}(\mathcal{T}_S)$ if and only if $\text{Hom}_\Lambda(T, S) \neq 0$.
- (2) $T \in \mathcal{C}(\mathcal{T}^S)$ if and only if $\text{Hom}_\Lambda(S, T) \neq 0$.
- (3) $\mathcal{C}(\mathcal{T}_S)$ has a unique minimal element. If S is not injective it is of the form $T_\omega(S)$.
- (4) $\mathcal{C}(\mathcal{T}^S)$ has a unique maximal element. If S is not projective it is of the form $T^\alpha(S)$.
- (5) $\mathcal{C}(\mathcal{T}_S) \cap \mathcal{C}(\mathcal{T}^S) = \{T \in \mathcal{T}_\Lambda \mid S \text{ is a direct summand of } T\}$.
- (6) $\mathcal{C}(\mathcal{T}_S) \cap \mathcal{C}(\mathcal{T}^{S'}) \neq \emptyset$ if and only if $\text{Ext}_\Lambda^1(S', S) = 0$.

Proof. We first deal with the case that S is injective. Then by definition $\mathcal{C}(\mathcal{T}_S) = \mathcal{T}_\Lambda$. So $\text{Hom}_\Lambda(T, S) \neq 0$ for $T \in \mathcal{T}_\Lambda$ implies $T \in \mathcal{C}(\mathcal{T}_S)$. Conversely let $T \in \mathcal{T}_\Lambda$. Since T is sincere we have that $\text{Hom}_\Lambda(T, S) \neq 0$. So we may assume that S is not injective. Let $T \in \mathcal{C}(\mathcal{T}_S)$. Then we have that $T_S \not\geq T$, thus $\text{Ext}_\Lambda^1(T_S, T) \neq 0$; hence by the Auslander-Reiten formula also $\text{Hom}_\Lambda(T, S) \neq 0$. Conversely, $\text{Hom}_\Lambda(T, S) \neq 0$ implies that $\text{Ext}_\Lambda^1(T_S, T) \neq 0$, thus $T_S \not\geq T$, and so $T \in \mathcal{C}(\mathcal{T}_S)$. This shows (1). Clearly (2) is dual to (1).

We will show (4). Clearly (3) is dual to (4). If S is projective, then $\mathcal{C}(\mathcal{T}^S) = \mathcal{T}_\Lambda$, so ${}_\Lambda\Lambda \in \mathcal{C}(\mathcal{T}^S)$; hence $\mathcal{C}(\mathcal{T}^S)$ has a unique maximal element. If S is not projective, then it follows from (2) that $T^\alpha(S) \in \mathcal{C}(\mathcal{T}^S)$, since S is a direct summand of $T^\alpha(S)$. Let $T \in \mathcal{C}(\mathcal{T}^S)$; then by (2) we have that $\text{Hom}_\Lambda(S, T) \neq 0$. Since S is simple we have that S is cogenerated by T . But then $\text{Ext}_\Lambda^1(S, T) = 0$. The assertion now follows from Lemma 2.3.

Let $T \in \mathcal{C}(\mathcal{T}_S) \cap \mathcal{C}(\mathcal{T}^S)$ be a tilting module. Thus by (1) and (2) we have that $\text{Hom}_\Lambda(T, S) \neq 0$ and that $\text{Hom}_\Lambda(S, T) \neq 0$. So S is both generated and cogenerated by T . But then $\text{Ext}_\Lambda^1(T, S) = 0$ and $\text{Ext}_\Lambda^1(S, T) = 0$ show that S is a direct summand of T . Conversely let $T \in \mathcal{T}_\Lambda$ with S a direct summand of T . Then $\text{Hom}_\Lambda(T, S) \neq 0$ and $\text{Hom}_\Lambda(S, T) \neq 0$. So $T \in \mathcal{C}(\mathcal{T}_S) \cap \mathcal{C}(\mathcal{T}^S)$ by (1) and (2). This shows (5).

Let $T \in \mathcal{C}(\mathcal{T}_S) \cap \mathcal{C}(\mathcal{T}^{S'})$. By (1) and (2) we have that $\text{Hom}_\Lambda(T, S) \neq 0$ and that $\text{Hom}_\Lambda(S', T) \neq 0$. So S is generated and S' is cogenerated by T . So we obtain a surjection $T' \rightarrow S$ and an injection $S' \rightarrow T''$ with $T', T'' \in \text{add } T$. In particular we see that $\text{Ext}_\Lambda^1(T, S) = 0$. This implies that $\text{Ext}_\Lambda^1(S', S) = 0$. Conversely assume that

$\text{Ext}_\Lambda^1(S', S) = 0$. If $S = S'$, then trivially $\mathcal{C}(\mathcal{T}_S) \cap \mathcal{C}(\mathcal{T}^S) \neq \emptyset$ by (4) and the fact that S is a partial tilting module. If $S \neq S'$, then $S \in S'^{\text{perp}}$. If S' is projective, then ${}_\Lambda\Lambda \in \mathcal{C}(\mathcal{T}_S) \cap \mathcal{C}(\mathcal{T}^{S'})$. So we may assume that S' is not projective. We consider $T^\alpha(S') = S' \oplus P$, where P is a minimal projective generator in S'^{perp} . Then $T^\alpha(S') \in \mathcal{C}(\mathcal{T}^{S'})$ by (4). Since $S \in S'^{\text{perp}}$, we clearly have that $\text{Hom}_\Lambda(P, S) \neq 0$; hence $T^\alpha(S') \in \mathcal{C}(\mathcal{T}_S)$, thus showing (6).

We point out that Proposition 5.7 also allows us to compute for a given $T \in \mathcal{T}_\Lambda$ the number of simple direct summands of T .

Corollary 5.8. *Let $T \in \mathcal{T}_\Lambda$ and S_1, \dots, S_n be the simple Λ -modules. Then the following are equivalent for $I \subset \{1, \dots, n\}$:*

- (1) $T \in (\bigcap_{i \in I} \mathcal{C}(\mathcal{T}_{S_i})) \cap (\bigcap_{i \in I} \mathcal{C}(\mathcal{T}^{S_i}))$.
- (2) S_i is a direct summand of T for all $i \in I$.

If we choose $I(T) \subset \{1, \dots, r\}$ maximal with respect to (1) the cardinality of $I(T)$ gives the number of simple direct summands of T .

In connection with the results in section 2 we obtain the promised combinatorial description of the orthogonal categories with respect to simple modules.

Corollary 5.9. *Let Λ be a hereditary algebra and let S be a simple Λ -module and $\text{mod } \Gamma = S^{\text{perp}}$. Then the partially ordered sets $\mathcal{C}(\mathcal{T}_S) \cap \mathcal{C}(\mathcal{T}^S)$ and \mathcal{T}_Γ are isomorphic.*

If S is a simple Λ -module which is neither projective nor injective, we have the two vertices T_S and T^S in $\vec{\mathcal{K}}_\Lambda$. By Lemma 5.4 and its dual we have identified as a set those neighbors of ${}_\Lambda\Lambda$ in $\vec{\mathcal{K}}_\Lambda$ of the form T_S and those neighbors of $D\Lambda_\Lambda$ of $\vec{\mathcal{K}}_\Lambda$ of the form T^S . We denote by S_{r+1}, \dots, S_t the simple Λ -modules which are neither projective nor injective. Next we determine a combinatorial criterion which provides us with the natural bijection between $\{T_{S_i} \mid r+1 \leq i \leq t\}$ and $\{T^{S_i} \mid r+1 \leq i \leq t\}$. Let $S = S_{r+1}$ and S_j for some $r+1 \leq j \leq t$. We consider $\mathcal{U}_j = \mathcal{C}(\mathcal{T}_S) \cap \mathcal{C}(\mathcal{T}^{S_j})$ for all $r+1 \leq j \leq t$. If $\mathcal{U}_j = \emptyset$, then by Proposition 5.7 we clearly have that $j \neq r+1$. So we may assume that $\mathcal{U}_j \neq \emptyset$. Thus $\text{Ext}_\Lambda^1(S_j, S) = 0$. As in Lemma 2.3 we consider the tilting module $T^\alpha(S_j) = S_j \oplus P_j$, where P_j is a minimal projective generator of S_j^{perp} . By the proof of Proposition 5.7(6) we have that $T^\alpha(S_j) \in \mathcal{U}_j$. The following theorem provides us with a combinatorial criterion to determine the natural bijection aimed at the above.

Theorem 5.10. *Using the notation above we have that:*

- (1) $T^\alpha(S_j)$ is the unique maximal element in \mathcal{U}_j .
- (2) $j = r+1$ if and only if every immediate successor T of $T^\alpha(S_j)$ in $\vec{\mathcal{K}}_\Lambda$ is contained in $\mathcal{C}(\mathcal{T}_S)$.

Proof. (1) follows from Proposition 5.7(4).

Assume that $j = r+1$. So $S_j = S$. Let T be an immediate successor of $T^\alpha(S)$ in $\vec{\mathcal{K}}_\Lambda$; then S is a direct summand of T by Lemma 2.3. So $\text{Hom}_\Lambda(T, S) \neq 0$, thus $T \in \mathcal{C}(\mathcal{T}_S)$ by Proposition 5.7.

Conversely assume that $j \neq r+1$. Since $\mathcal{U}_j \neq \emptyset$ we have by Proposition 5.7 that $\text{Ext}_\Lambda^1(S_j, S) = 0$, hence $S \in S_j^{\text{perp}}$. We first show that S is not injective in S_j^{perp} . Since S is not injective in $\text{mod } \Lambda$ there is a simple S' with $\text{Ext}_\Lambda^1(S', S) \neq 0$. Clearly $S' \neq S_j$. If $S' \in S_j^{\text{perp}}$, then this trivially implies that S is not injective in S_j^{perp} . Otherwise $\text{Ext}_\Lambda^1(S_j, S') \neq 0$. Let $0 \rightarrow S' \rightarrow Z \rightarrow S_j^\ell \rightarrow 0$ be the universal

extension with $\ell = \dim \text{Ext}_\Lambda^1(S_j, S')$. Applying $\text{Hom}_\Lambda(S_j, -)$ to this exact sequence shows that $Z \in S_j^{\text{perp}}$. Applying $\text{Hom}_\Lambda(-, S)$ to this exact sequence shows that $\text{Ext}_\Lambda^1(Z, S) \simeq \text{Ext}_\Lambda^1(S', S)$. Thus $\text{Ext}_\Lambda^1(Z, S) \neq 0$, and hence S is not injective in S_j^{perp} .

As in the proof of Lemma 2.3 we consider the universal extension

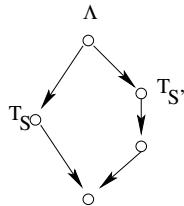
$$(*) \quad 0 \rightarrow {}_\Lambda \Lambda \rightarrow Q \rightarrow S_j^\ell \rightarrow 0.$$

Applying $\text{Hom}_\Lambda(-, S)$ to the exact sequence $(*)$ shows that $\text{Hom}_\Lambda({}_\Lambda \Lambda, S) \simeq \text{Hom}_\Lambda(Q, S)$. Clearly $\text{Hom}_\Lambda({}_\Lambda \Lambda, S) \simeq k$. Recall that $T^\alpha(S_j) = S_j \oplus P_j$, where P_j was multiplicity free with $\text{add } P_j = \text{add } Q$. Now $\dim \text{Hom}_\Lambda(Q, S) = 1$ implies that there exists a unique indecomposable direct summand P_S of P_j such that $T^\alpha(S_j) = S_j \oplus P_S \oplus P'$ with $\text{Hom}_\Lambda(P_S, S) \simeq k$ and $\text{Hom}_\Lambda(S_j \oplus P', S) = 0$. Note that P_S is the projective cover of S in S_j^{perp} . Since S is not injective in S_j^{perp} , we see that P' is faithful in S_j^{perp} ; thus by Lemma 1.1 we have that P_S is co-generated by P' in S_j^{perp} . Then P_S is co-generated by $M = P' \oplus S_j$ in $\text{mod } \Lambda$. By construction we have that $\text{Hom}_\Lambda(M, S) = 0$. Let $0 \rightarrow P_S \rightarrow \tilde{M} \rightarrow Y \rightarrow 0$ with $\tilde{M} \in \text{add } M$ be the exact sequence connecting the complements P_S and Y of M . Then $T = M \oplus Y \in \mathcal{T}_\Lambda$ is an immediate successor of $T^\alpha(S_j)$. Applying $\text{Hom}_\Lambda(-, S)$ to this sequence shows that $\text{Hom}_\Lambda(Y, S) = 0$. Thus $\text{Hom}_\Lambda(T, S) = 0$, and so $T \notin \mathcal{C}(\mathcal{T}_S)$ by Proposition 5.7(1).

6. THE RECONSTRUCTION

This section contains the proof the main theorem of this article. The first two lemmas show how to obtain arrows between certain simple Λ -modules. In the first lemma we will determine the arrows between the simple non-injective Λ -modules.

Lemma 6.1. *Let $\Lambda = k\overrightarrow{\Delta}$ be a hereditary algebra. Let S, S' be simple Λ -modules which are not injective. Then there is exactly one arrow $S \rightarrow S'$ in $\overrightarrow{\Delta}$ if and only if $T_S, T_{S'}$ occur in a subquiver of $\overrightarrow{\mathcal{K}}_\Lambda$ of the form*



Proof. Let $S_i = S$ and $S_j = S'$. We may assume that $i \neq j$. First assume that T_S and $T_{S'}$ occur in a subquiver of $\overrightarrow{\mathcal{K}}_\Lambda$ as indicated. The unique sink T of the subquiver has to be of the form $T = {}_\Lambda \Lambda[i, t] \oplus \tau_\Lambda^- S_i \oplus Y$ for some $t \neq i$ and for some non-projective indecomposable Λ -module Y . By Corollary 1.2 we see that the immediate successor T' of $T_{S'}$ in this subquiver has to be of the form $T' = {}_\Lambda \Lambda[t', j] \oplus \tau_\Lambda^- S_j \oplus Z$ for some $t' \neq j$ and for some non-projective indecomposable Λ -module Z . Since there is an arrow $T' \rightarrow T$ in $\overrightarrow{\mathcal{K}}_\Lambda$ the projective direct summand, ${}_\Lambda \Lambda[i, t]$ of T must be a direct summand of T' , hence $i = t'$ and $t = j$. Moreover the arrow $T' \rightarrow T$ in $\overrightarrow{\mathcal{K}}_\Lambda$ yields that the tilting modules T, T' must have a common direct summand which is an almost complete partial tilting module. Since $S_i \not\cong S_j$

we infer that $Y \simeq Z$. Let $X = Y \simeq Z$. Then it follows that $\tau_{\Lambda}^{-}S_j$ and $\tau_{\Lambda}^{-}S_i$ are the complements to ${}_{\Lambda}\Lambda[i, j] \oplus X$ and $\dim \text{Ext}_{\Lambda}^1(\tau_{\Lambda}^{-}S_i, \tau_{\Lambda}^{-}S_j) = 1$ by Lemma 1.1, hence there is exactly one arrow from S to S' in $\overrightarrow{\Delta}$.

Conversely, assume that there is exactly one arrow $S_i = S \rightarrow S_j = S'$ in $\overrightarrow{\Delta}$. Let $M = {}_{\Lambda}\Lambda[i, j]$. Since S_i, S_j are not injective we have that M is faithful. By Theorem 1.4 we know the shape of $\overrightarrow{\mathcal{K}}_{\Lambda}(M)$. By Lemma 5.2 we know that the diamond cannot occur. Since there is exactly one arrow we see that $\overrightarrow{\mathcal{K}}_{\Lambda}(M)$ is finite. Finally we determine the position of T_{S_i} and T_{S_j} in $\overrightarrow{\mathcal{K}}_{\Lambda}(M)$. Assume to the contrary that T_{S_j} lies on the path of length two from the source ${}_{\Lambda}\Lambda$ to the sink T in $\overrightarrow{\mathcal{K}}_{\Lambda}(M)$. By Corollary 1.2 we have that $\tau^{-}S_j$ is an indecomposable direct summand of T . Now $T \leq T_{S_i}$ in \mathcal{T}_{Λ} , thus $\text{Ext}_{\Lambda}^1(T_{S_i}, T) = 0$. In particular we have that $0 = \text{Ext}_{\Lambda}^1(\tau^{-}S_i, \tau^{-}S_j) \simeq \text{Ext}_{\Lambda}^1(S_i, S_j)$. But $\text{Ext}_{\Lambda}^1(S_i, S_j) \neq 0$, since there is an arrow $S_i \rightarrow S_j$, a contradiction. So T_{S_i} and T_{S_j} are in the position as indicated.

Dually we obtain the arrows between the simple Λ -modules which are not projective.

Next we determine whether or not there is an arrow between a simple injective S and a simple projective S' . Note that the result shown above only used the information given by $\overrightarrow{\mathcal{K}}_{\Lambda}$. If Λ is of infinite representation type or equivalently \mathcal{T}_{Λ} is infinite, we will use for the next result the full structure given by \mathcal{T}_{Λ} . The following trivial lemma shows that the existence of arrows from S' to S in $\overrightarrow{\Delta}$ is deducible from \mathcal{T}_{Λ} , hence is an invariant of the isomorphism class of \mathcal{T}_{Λ} .

Lemma 6.2. *Let $\Lambda = k\overrightarrow{\Delta}$ be a hereditary algebra. Let S be a simple projective Λ -module and let S' be a simple injective Λ -module. Then there is an arrow $S' \rightarrow S$ in $\overrightarrow{\Delta}$ if and only if $\mathcal{C}(\mathcal{T}_S) \cap \mathcal{C}(\mathcal{T}^{S'}) = \emptyset$.*

Proof. This is a reformulation of Proposition 5.7.

In the proof of the main theorem we have to show whether or not there are multiple arrows between a pair of simple Λ -modules S', S . While Proposition 5.7, Lemma 6.1 and its dual give a criterion in case S is not projective and S' is not injective, the situation seems to be more complicated if S' is injective and S is projective. The following lemma will be quite helpful.

Lemma 6.3. *Let $\Lambda = k\overrightarrow{\Delta}$ be a hereditary algebra. Let S be a simple projective Λ -module and let S' be a simple injective Λ -module with $\mathcal{C}(\mathcal{T}_S) \cap \mathcal{C}(\mathcal{T}^{S'}) = \emptyset$. Then the following hold:*

- (1) $\delta(\mathcal{T}_S) \geq \delta({}_{\Lambda}\Lambda) = t$.
- (2) If $\delta(\mathcal{T}_S) = \delta({}_{\Lambda}\Lambda)$, then there is a unique arrow from S' to S in $\overrightarrow{\Delta}$.
- (3) If $\delta(\mathcal{T}_S) = t + s$ for some $s > 0$, then there exists a set of neighbors $\{M_1, \dots, M_s\}$ of \mathcal{T}_S such that $M_i \in k_{\Lambda}\Lambda[1, j_i]$, where $S = S_1$ and S_{j_i} is simple injective with $\text{Ext}_{\Lambda}^1(S_{j_i}, S) \neq 0$. We denote by I_S the set of j_i occurring in M_i for $1 \leq i \leq s$.
- (4) $I_S = \{1 \leq j \leq n \mid S_j \text{ is injective and } \mathcal{C}(\mathcal{T}_S) \cap \mathcal{C}(\mathcal{T}^{S_j}) = \emptyset\}$, if $\delta(\mathcal{T}_S) > t$.
- (5) The set $\{M_1, \dots, M_s\}$ can be combinatorially identified in \mathcal{T}_{Λ} .

Proof. Since $e(\mathcal{T}_S) = 1$ it is enough to show that $s(\mathcal{T}_S) \geq t - 1$. Let $S = S_1$ and let $\mathcal{T}_S = T_{S_1}, T_{S_2}, \dots, T_{S_t}$ be the neighbors of ${}_{\Lambda}\Lambda$ in $\overrightarrow{\mathcal{K}}_{\Lambda}$. By Lemma 5.1 S_i is not injective for $1 \leq i \leq t$ and all non-injective simple injective Λ -modules occur. So

${}_{\Lambda}\Lambda[1, i]$ is sincere for $2 \leq i \leq t$, since the projective covers of the simple injective Λ -modules are direct summands of ${}_{\Lambda}\Lambda[1, i]$; hence ${}_{\Lambda}\Lambda[1, i] \oplus \tau_{\Lambda}^{-}S$ is sincere. Thus ${}_{\Lambda}\Lambda[1, i] \oplus \tau_{\Lambda}^{-}S$ has two complements, yielding neighbors T_2, \dots, T_t of T_S in $\vec{\mathcal{K}}_{\Lambda}$. This shows (1).

Assume that $\delta(T_S) = \delta({}_{\Lambda}\Lambda) = t$. Let $S_{t+1} = S'$. We consider the almost complete partial tilting module ${}_{\Lambda}\Lambda[1, t + 1] \oplus \tau_{\Lambda}^{-}S$. Then $\delta(T_S) = t$ and (1) imply that ${}_{\Lambda}\Lambda[1, t + 1] \oplus \tau_{\Lambda}^{-}S$ has a unique complement $P(S')$, and so is not sincere by Lemma 1.1. Let $0 \rightarrow S \rightarrow P \rightarrow \tau_{\Lambda}^{-}S \rightarrow 0$ be the Auslander-Reiten sequence starting at S . Then P is projective. By assumption $\mathcal{C}(T_S) \cap \mathcal{C}(T^{S'}) = \emptyset$, so $\text{Ext}_{\Lambda}^1(S', S) \neq 0$ by Proposition 5.7. But then $\text{Hom}_{\Lambda}(\tau_{\Lambda}^{-}S, S') \neq 0$. So $\text{Hom}_{\Lambda}(P, S') \neq 0$, and therefore $P(S')$ is a direct summand of P ; hence $\text{Hom}_{\Lambda}(P(S'), \tau_{\Lambda}^{-}S) \neq 0$. If S'' is a simple Λ -module with $S'' \neq S$, then we also have that $\text{Hom}_{\Lambda}(P(S''), {}_{\Lambda}\Lambda[1, t + 1] \oplus \tau_{\Lambda}^{-}S) \neq 0$. Thus if ${}_{\Lambda}\Lambda[1, t + 1] \oplus \tau_{\Lambda}^{-}S$ is not sincere, then $\text{Hom}_{\Lambda}(S, {}_{\Lambda}\Lambda[1, t + 1] \oplus \tau_{\Lambda}^{-}S) = 0$. But this implies that P is indecomposable. So $P(S') = P$. So there is exactly one arrow from S' to S in $\vec{\Delta}$. This shows (2).

If $\delta(T_S) = t + s$ for some $s > 0$ we have the neighbors T_2, \dots, T_t constructed in (1). Let M_1, \dots, M_s be the different neighbors. Since there is a path of length two from ${}_{\Lambda}\Lambda$ to M_i , there is a simple S_{j_i} such that $M_i = {}_{\Lambda}\Lambda[1, j_i] \oplus \tau_{\Lambda}^{-}S \oplus X_i$ for some X_i non-projective. Since $M_i \notin \{T_2, \dots, T_t\}$ we clearly have that S_{j_i} is injective. Since ${}_{\Lambda}\Lambda[1, j_i] \oplus \tau_{\Lambda}^{-}S$ has the two complements $P(j_i)$ and X_i , we see that ${}_{\Lambda}\Lambda[1, j_i] \oplus \tau_{\Lambda}^{-}S$ is sincere. In particular we have that $\text{Hom}_{\Lambda}(P(S_{j_i}), {}_{\Lambda}\Lambda[1, j_i] \oplus \tau_{\Lambda}^{-}S) \neq 0$. Since $\text{Hom}_{\Lambda}(P(S_{j_i}), {}_{\Lambda}\Lambda[1, j_i]) = 0$, we have that $\text{Hom}_{\Lambda}(P(S_{j_i}), \tau_{\Lambda}^{-}S) \neq 0$. But then $\text{Hom}_{\Lambda}(P(S_{j_i}), P) \neq 0$, where P is the middle term of the Auslander-Reiten sequence starting in S . Since S_{j_i} is injective this implies that $P(S_{j_i})$ is a direct summand of P , thus $\text{Hom}_{\Lambda}(\tau_{\Lambda}^{-}S, S_{j_i}) \neq 0$; hence $\text{Ext}_{\Lambda}^1(S_{j_i}, S) \neq 0$. This shows (3).

It follows from (3) in combination with Proposition 5.7(6) that we have the inclusion $I_S \subset \{1 \leq j \leq n \mid S_j \text{ is injective and } \mathcal{C}(T_S) \cap \mathcal{C}(T^{S_j}) = \emptyset\}$.

Conversely, let j be such that $\text{Ext}_{\Lambda}^1(S_j, S) \neq 0$ and that S_j is injective. Assume that $j \notin I_S$. We consider the almost complete partial tilting module $N = {}_{\Lambda}\Lambda[1, j] \oplus \tau_{\Lambda}^{-}S$. Since $j \notin I_S$ we have that N is not sincere. Clearly $\text{Hom}_{\Lambda}(P(S'), N) \neq 0$ for $S' \neq S$ and $S' \neq S_j$. Since $\text{Ext}_{\Lambda}^1(S_j, S) \neq 0$, we have that $P(S_j)$ is a direct summand of the middle term P of the Auslander-Reiten sequence starting at S ; hence $\text{Hom}_{\Lambda}(P(S_j), \tau_{\Lambda}^{-}S) \neq 0$. So $\text{Hom}_{\Lambda}(S, N) = 0$. But then $P = P(S_j)$ is indecomposable. By (3) and the assumption $\delta(T_S) > t$ we have that $I_S \neq \emptyset$, and the proof of (3) shows that each element I_S gives rise to a direct summand of P , a contradiction. So $j \in I_S$. This shows (4).

Finally we show assertion (5). Clearly we may assume that $\delta(T_S) = t + s$ for some $s > 0$. In (1) and (3) we have constructed two sets of neighbors, namely $\{T_2, \dots, T_t\}$ and $\{M_1, \dots, M_s\}$. The tilting modules T_i for $2 \leq i \leq t$ correspond to the neighbors T_{S_i} of ${}_{\Lambda}\Lambda$ different from T_S . Now ${}_{\Lambda}\Lambda[1, i]$ is sincere for $2 \leq i \leq t$. Let K_i be the sink in $\text{lk}_{\Lambda}\Lambda[1, i]$. We claim that K_i is the unique neighbor of T_{S_i} such that $T_S \geq K_i$. Indeed, let K be a neighbor of T_{S_i} ; then $K = {}_{\Lambda}\Lambda[i, j] \oplus \tau_{\Lambda}^{-}S_i \oplus X$ for some j and some X . Then $T_S \geq K$ if and only if $\text{Ext}_{\Lambda}^1(T_S, K) = 0$, which is equivalent to $\text{Hom}_{\Lambda}(K, S) = 0$. But $\text{Hom}_{\Lambda}(K, S) = 0$ is equivalent to $j = 1$. In this way we can identify the tilting modules K_i as distinguished neighbors of T_{S_i} . We have shown in Lemma 5.3 that $T' \geq K_i$ for each $T' \in \text{lk}_{\Lambda}\Lambda[1, i]$. Since $T_i \in \text{lk}_{\Lambda}\Lambda[1, i]$ we see that $T_i \geq K_i$. Let $1 \leq j \leq s$ and $2 \leq i \leq t$. We claim that

$M_j \not\cong K_i$. By (3) we have that $M_j = {}_{\Lambda}\Lambda[1, i_j] \oplus \tau_{\Lambda}^{-} S \oplus X_j$ for some simple injective S_{i_j} . Thus $\text{Ext}_{\Lambda}^1(M_j, P(S_{i_j})) \neq 0$. But $P(S_{i_j})$ is a direct summand of K_i , thus $M_j \not\cong K_i$. By the previous considerations we have described the set $\{K_2, \dots, K_t\}$ combinatorially. So we obtain from the last two assertions that $\{M_1, \dots, M_s\} = \{T \mid T_S \rightarrow T \text{ in } \vec{\mathcal{K}}_{\Lambda} \text{ and } T \not\cong K_i \text{ for } 2 \leq i \leq t\}$.

We now present the reconstruction from \mathcal{T}_{Λ} . To simplify the terminology we define for a quiver $\vec{\Delta}$ the decorated underlying simply-laced quiver $\vec{\Delta}_{\text{dec}}$ as follows. The vertices of both quivers are the same. If for a pair of vertices (i, j) of $\vec{\Delta}$ there is exactly one arrow from i to j in $\vec{\Delta}$, then there is also exactly one arrow from i to j in $\vec{\Delta}_{\text{dec}}$. If for a pair of vertices (i, j) of $\vec{\Delta}$ there are $r \geq 2$ arrows from i to j in $\vec{\Delta}$, then there is exactly one decorated arrow $i \xrightarrow{*} j$ from i to j in $\vec{\Delta}_{\text{dec}}$. So $\vec{\Delta}_{\text{dec}}$ does only carry the information as to whether there are multiple arrows in $\vec{\Delta}$ between two vertices. Trivially, $\vec{\Delta} = \vec{\Delta}_{\text{dec}}$ in the case where there are no multiple arrows in $\vec{\Delta}$.

Theorem 6.4. *Let $\vec{\Delta}, \vec{\Delta}'$ be two finite quivers without oriented cycles and isolated vertices. Let $\Lambda = k\vec{\Delta}$ and $\Lambda' = k\vec{\Delta}'$. If $\mathcal{T}_{\Lambda} \simeq \mathcal{T}_{\Lambda'}$, then $\vec{\Delta}_{\text{dec}} \simeq \vec{\Delta}'_{\text{dec}}$.*

Proof. We will show that we can reconstruct $\vec{\Delta}_{\text{dec}}$ from \mathcal{T}_{Λ} . We will assume that the number of vertices of $\vec{\Delta}$ is $n \geq 2$. By Theorem 5.6 we can determine this number n from $\vec{\mathcal{K}}_{\Lambda}$. So we know from Corollary 3.3 the number of sources and the number of sinks in $\vec{\Delta}$. We label the simple Λ -modules $S_1, \dots, S_r, S_{r+1}, \dots, S_t, S_{t+1}, \dots, S_n$ such that S_i is projective for $1 \leq i \leq r$, S_i is injective for $t+1 \leq i \leq n$ and S_i is neither projective nor injective for $r+1 \leq i \leq t$. By Lemma 5.4 we know which of the neighbors of ${}_{\Lambda}\Lambda$ in $\vec{\mathcal{K}}_{\Lambda}$ correspond to the simple projective Λ -modules. Dually we can determine the neighbors of $D\Lambda_{\Lambda}$ which correspond to the simple injective Λ -modules. By Theorem 5.10 we can combinatorially obtain the natural bijection between the neighbors T_{S_i} of ${}_{\Lambda}\Lambda$ and T^{S_i} of $D\Lambda_{\Lambda}$ for $r+1 \leq i \leq t$.

To reconstruct $\vec{\Delta}_{\text{dec}}$ from \mathcal{T}_{Λ} we proceed in several steps.

(0) The vertices $\{1, \dots, n\}$ of $\vec{\Delta}_{\text{dec}}$ correspond bijectively to all neighbors of ${}_{\Lambda}\Lambda$ in $\vec{\mathcal{K}}_{\Lambda}$ and certain neighbors of $D\Lambda_{\Lambda}$ in $\vec{\mathcal{K}}_{\Lambda}$, namely to the set $\{T_{S_i} \mid 1 \leq i \leq t\} \cup \{T^{S_j} \mid t+1 \leq j \leq n\}$, or equivalently by Theorem 5.10 they correspond bijectively to all neighbors of $D\Lambda_{\Lambda}$ in $\vec{\mathcal{K}}_{\Lambda}$ and certain neighbors of ${}_{\Lambda}\Lambda$ in $\vec{\mathcal{K}}_{\Lambda}$, namely to the set $\{T^{S_i} \mid r+1 \leq i \leq n\} \cup \{T_{S_j} \mid 1 \leq j \leq r\}$.

For the determination of the arrows in $\vec{\Delta}_{\text{dec}}$ we distinguish the following three cases.

(1) Let S_i, S_j be simple Λ -modules with $1 \leq i, j \leq t$. We are interested in arrows from j to i . Thus we can assume that $r+1 \leq j \leq t$. We apply Proposition 5.7. If $\mathcal{C}(\mathcal{T}_{S_i}) \cap \mathcal{C}(\mathcal{T}^{S_j}) \neq \emptyset$, then there is no arrow from S_j to S_i . If $\mathcal{C}(\mathcal{T}_{S_i}) \cap \mathcal{C}(\mathcal{T}^{S_j}) = \emptyset$, then there are arrows from j to i . There is exactly one arrow in $\vec{\Delta}_{\text{dec}}$ from j to i if the situation in Lemma 6.1 applies; otherwise we obtain $j \xrightarrow{*} i$ in $\vec{\Delta}_{\text{dec}}$.

(2) Let S_i, S_j be simple Λ -modules with $r+1 \leq i, j \leq n$. We are interested in arrows from j to i . Thus we can assume that $r+1 \leq i \leq t$. This is clearly dual to (1).

(3) Let S_i, S_j be simple Λ -modules with $1 \leq i \leq r$ and $t+1 \leq j \leq n$. We are interested in arrows from j to i . Trivially there are no arrows from i to j . We

apply Lemma 6.2. If $\mathcal{C}(\mathcal{T}_{S_i}) \cap \mathcal{C}(\mathcal{T}^{S_j}) \neq \emptyset$, then there is no arrow from S_j to S_i . If $\mathcal{C}(\mathcal{T}_{S_i}) \cap \mathcal{C}(\mathcal{T}^{S_j}) = \emptyset$, then there are arrows from j to i . To determine multiple arrows we will use Lemma 6.3.

To simplify the notation we set $S = S_i$ and $S' = S_j$. By (0) both T_S and $T^{S'}$ are identified in \mathcal{T}_Λ . We denote it by $m = \dim \text{Ext}_\Lambda^1(S', S)$ and assume that $m > 0$.

If $\delta(T_S) = \delta(\Lambda) = t$, then by Lemma 6.3(2) there is exactly one arrow from S' to S in $\vec{\Delta}_{\text{dec}}$.

If $\delta(T_S) = t + s$ for some $s > 0$, then we obtain from Lemmas 6.3(3) and 6.3(5) a set of neighbors $\{M_1, \dots, M_s\}$ of T_S with $M_q = {}_\Lambda \Lambda[i, j_q] \oplus \tau_\Lambda^- S \oplus X_q$ for some $t + 1 \leq j_q \leq n$ with $\text{Ext}_\Lambda^1(S_{j_q}, S) \neq 0$ and $1 \leq q \leq s$. Since there exists a path of length two from ${}_\Lambda \Lambda$ to M_q in $\vec{\mathcal{K}}_\Lambda$, we see that X_q is not projective. Let I_S be the set of vertices defined and characterized in Lemma 6.3. By Lemma 6.2 we can then determine the set $I_S = \{j_q \mid 1 \leq q \leq s\}$. Note that $j \in I_S$. We also note that $S \oplus \tau_\Lambda X_q \in {}_\Lambda \Lambda[i, j_q]^{\text{perp}}$ is a tilting module. Now ${}_\Lambda \Lambda[i, j_q]^{\text{perp}}$ is equivalent to the subcategory of $\text{mod } \Lambda$ containing those modules which have only composition factors S and S_{j_q} . In particular ${}_\Lambda \Lambda[i, j_q]^{\text{perp}}$ is closed under submodules. Since S is projective we infer that also $\tau_\Lambda X_q$ is projective in ${}_\Lambda \Lambda[i, j_q]^{\text{perp}}$, since $S \oplus \tau_\Lambda X_q$ is a tilting module in ${}_\Lambda \Lambda[i, j_q]^{\text{perp}}$. Since $\text{Ext}_\Lambda^1(S_{j_q}, S) \neq 0$ we have that ${}_\Lambda \Lambda[i, j_q]^{\text{perp}}$ is connected and so $\text{Hom}_\Lambda(S, \tau_\Lambda X_q) \neq 0$.

For each $1 \leq q \leq s$ we now consider the subposet \mathcal{T}_q of \mathcal{T}_Λ defined by $\mathcal{T}_q = \{T \in \mathcal{T}_\Lambda \mid T \leq M_q\}$. Since $\mathcal{T}_q \subset \mathcal{T}_S$ we see that $\mathcal{C}(\mathcal{T}_S) \subset \mathcal{C}(\mathcal{T}_q)$. We denote by \mathcal{C}_q the subposet of \mathcal{T}_Λ defined by $\mathcal{C}_q = \mathcal{C}(\mathcal{T}_q) \setminus \mathcal{C}(\mathcal{T}_S)$. Clearly $\mathcal{C}_q = \{T \in \mathcal{T}_\Lambda \mid \text{Ext}_\Lambda^1(M_q, T) \neq 0 \text{ and } S \text{ is not a direct summand of } T\}$ by Proposition 5.7. Thus $T \in \mathcal{C}_q$ if and only if $\text{Hom}_\Lambda(T, \tau_\Lambda M_q) \neq 0$ and S is not direct summand of T . Since S is projective, we have that $\text{Hom}_\Lambda(T, S) = 0$ if and only if S is not a direct summand of T . Thus $T \in \mathcal{C}_q$ is equivalent to $\text{Hom}_\Lambda(T, \tau_\Lambda X_q) \neq 0$ and S is not a direct summand of T . We claim that any non-zero map $\phi : T \rightarrow \tau_\Lambda X_q$ is surjective for $T \in \mathcal{C}_q$. Indeed, let $0 \neq \phi : T \rightarrow \tau_\Lambda X_q$. If ϕ is not surjective, we have that $\text{im } \phi$ is a proper submodule of $\tau_\Lambda X_q$, and so is contained in ${}_\Lambda \Lambda[i, j_q]^{\text{perp}}$. Since $\tau_\Lambda X_q$ is projective in ${}_\Lambda \Lambda[i, j_q]^{\text{perp}}$ we infer that $\text{im } \phi \in \text{add } S$. But then S is a direct summand of T , in contrast to $T \in \mathcal{C}_q$. Thus ϕ is surjective. This implies that $\text{Ext}_\Lambda^1(T, \tau_\Lambda X_q) = 0$ for each $T \in \mathcal{C}_q$.

We now consider for each $1 \leq q \leq s$ the subposet $\mathcal{U}_q = \mathcal{C}_q \cap \mathcal{C}(\mathcal{T}^{S'})$. We claim that $\mathcal{U}_q = \emptyset$ for all $1 \leq q \leq s$ if and only if $m \geq 2$.

Indeed, assume first that there is a tilting module $T \in \mathcal{U}_q$ for some q . By the previous considerations we have that $\text{Ext}_\Lambda^1(T, \tau_\Lambda X_q) = 0$, since $T \in \mathcal{C}_q$, and by Proposition 5.7 that S' is a direct summand of T , since $T \in \mathcal{C}(\mathcal{T}^{S'})$. Thus also $\text{Ext}_\Lambda^1(S', \tau_\Lambda X_q) = 0$. Consider the projective resolution

$$(*) \quad 0 \rightarrow K' \rightarrow P(S') \rightarrow S' \rightarrow 0 \text{ of } S'.$$

Since $\text{Ext}_\Lambda^1(S', S) \neq 0$ we have that S is a direct summand of K' . If $S_{j_q} \neq S'$, then S' is not a composition factor of $\tau_\Lambda X_q$, so $\text{Hom}_\Lambda(P(S'), \tau_\Lambda X_q) = 0$. This yields by applying $\text{Hom}_\Lambda(-, \tau_\Lambda X_q)$ to $(*)$ that $\text{Hom}_\Lambda(K', \tau_\Lambda X_q) \simeq \text{Ext}_\Lambda^1(S', \tau_\Lambda X_q)$. Since S is a direct summand of K' and $\text{Hom}_\Lambda(S, \tau_\Lambda X_q) \neq 0$, we see that $\text{Ext}_\Lambda^1(S', \tau_\Lambda X_q) \neq 0$, a contradiction. Thus $S_{j_q} = S'$. In particular, $S' \in {}_\Lambda \Lambda[i, j_q]^{\text{perp}}$. Since $\text{Ext}_\Lambda^1(S', \tau_\Lambda X_q) = 0$ this clearly implies that $m = 1$.

Conversely, assume that $m = 1$. We have $j \in I_S$ for $S' = S_j$ by Lemma 6.3(4) and our assumption that $\text{Ext}_\Lambda^1(S', S) \neq 0$. So $j = j_q$ for some $1 \leq q \leq s$. Let

$M_q = {}_{\Lambda}\Lambda[i, j] \oplus \tau_{\Lambda}^{-}S \oplus X_q$. Since $m = 1$, we have that S' and $\tau_{\Lambda}X_q$ are both injective in ${}_{\Lambda}\Lambda[i, j]^{\text{perp}}$, so $S' \oplus \tau_{\Lambda}X_q$ is a partial tilting module in $\text{mod } \Lambda$. So there is a tilting module $T = S' \oplus \tau_{\Lambda}X_q \oplus C$ for some C . Trivially $T \in \mathcal{C}(\mathcal{T}^{S'})$. Since $\text{Ext}_{\Lambda}^1(S', S) \neq 0$ we see that S is not a direct summand of T . Since $\tau_{\Lambda}X_q$ is a direct summand of T , we then see that $T \in \mathcal{C}_q$; thus $T \in \mathcal{U}_q$. So there is q such that $\mathcal{U}_q \neq \emptyset$.

The last assertion now enables us to determine if $m \geq 2$. If for each $1 \leq q \leq s$ the subposet $\mathcal{U}_q = \emptyset$, then we obtain in $\vec{\Delta}_{\text{dec}}$ a decorated arrow $S' \xrightarrow{*} S$. Otherwise there is exactly one arrow from S' to S in $\vec{\Delta}_{\text{dec}}$.

This finishes the proof of the theorem.

Corollary 6.5. *Let $\vec{\Delta}$ be a quiver without oriented cycles. If $\vec{\Delta}$ does not contain multiple arrows and isolated vertices, then $\Lambda = k\vec{\Delta}$ is uniquely determined (up to isomorphism) by \mathcal{T}_{Λ} .*

Corollary 6.6. *Let $\vec{\Delta}$ be a Dynkin quiver without isolated vertices. Then $\vec{\Delta}$ is uniquely determined (up to isomorphism) by $\vec{\mathcal{K}}_{\Lambda}$.*

Proof. Since $\vec{\mathcal{K}}_{\Lambda}$ is finite the partially ordered set \mathcal{T}_{Λ} is determined by $\vec{\mathcal{K}}_{\Lambda}$, since it is easily seen that $T \leq T'$ in \mathcal{T}_{Λ} implies the existence of a path from T to T' in $\vec{\mathcal{K}}_{\Lambda}$ by applying 2.2 of [HU2]. Thus the result follows from Theorem 6.4.

We also mention the following corollary.

Corollary 6.7. *Let $\vec{\Delta}$ be a finite quiver without oriented cycles and isolated vertices. Then \mathcal{T}_{Λ} determines whether or not $\vec{\Delta}$ is connected.*

Proof. Indeed, we can reconstruct $\vec{\Delta}_{\text{dec}}$ from \mathcal{T}_{Λ} . But $\vec{\Delta}_{\text{dec}}$ is connected if and only if $\vec{\Delta}$ is connected.

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