TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 361, Number 7, July 2009, Pages 3721–3746
S 0002-9947(09)04667-4
Article electronically published on February 10, 2009

COISOTROPIC EMBEDDINGS IN POISSON MANIFOLDS

A. S. CATTANEO AND M. ZAMBON

Abstract. We consider existence and uniqueness of two kinds of coisotropic embeddings and deduce the existence of deformation quantizations of certain Poisson algebras of basic functions. First we show that any submanifold of a Poisson manifold satisfying a certain constant rank condition, already considered by Calvo and Falceto (2004), sits coisotropically inside some larger cosymplectic submanifold, which is naturally endowed with a Poisson structure. Then we give conditions under which a Dirac manifold can be embedded coisotropically in a Poisson manifold, extending a classical theorem of Gotay.

1. Introduction

The following two results in symplectic geometry are well known. First: a submanifold C of a symplectic manifold (M,Ω) is contained coisotropically in some symplectic submanifold of M iff the pullback of Ω to C has constant rank; see Marle's work [17]. Second: a manifold endowed with a closed 2-form ω can be embedded coisotropically into a symplectic manifold (M,Ω) so that $i^*\Omega=\omega$ (where i is the embedding) iff ω has constant rank; see Gotay's work [15].

In this paper we extend these results to the setting of Poisson geometry. Recall that P is a Poisson manifold if it is endowed with a bivector field $\Pi\in\Gamma(\wedge^2TP)$ satisfying the Schouten-bracket condition $[\Pi,\Pi]=0$. A submanifold C of (P,Π) is coisotropic if $\sharp N^*C\subset TC$, where the conormal bundle N^*C is defined as the annihilator of TC in $TP|_C$ and $\sharp\colon T^*P\to TP$ is the contraction with the bivector Π. Coisotropic submanifolds appear naturally; for instance the graph of any Poisson map is coisotropic, and for any Lie subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ the annihilator $\mathfrak{h}^\circ$ is a coisotropic submanifold of the Poisson manifold $\mathfrak{g}^*$. Further coisotropic submanifolds C are interesting for a variety of reasons, one being that the distribution $\sharp N^*C$ is a (usually singular) integrable distribution whose leaf space, if smooth, is a Poisson manifold.

To give a Poisson analogue of Marle's result we consider *pre-Poisson* submanifolds, i.e. submanifolds C for which $TC+\sharp N^*C$ has constant rank (or equivalently $pr_{NC}\circ\sharp\colon N^*C\to TP|_C\to NC:=TP|_C/TC$ has constant rank). Natural classes of pre-Poisson submanifolds are given by affine subspaces $\mathfrak{h}^\circ+\lambda$ of $\mathfrak{g}^*$, where $\mathfrak{h}$ is a Lie subalgebra of the Lie algebra $\mathfrak{g}^*$ and λ is any element of $\mathfrak{g}^*$, and of course by coisotropic submanifolds and by points. More details are given in [12], where it is also shown that pre-Poisson submanifolds satisfy some functorial properties. This can be used to show that on a Poisson-Lie group G the graph of L_h (the left

Received by the editors May 25, 2007.
2000 *Mathematics Subject Classification.* Primary 53D17; Secondary 53D55.

translation by some fixed $h \in G$, which clearly is not a Poisson map) is a pre-Poisson submanifold, giving rise to a natural constant rank distribution D_h on G that leads to interesting constructions. For instance, if the Poisson structure on G comes from an r-matrix and the point h is chosen appropriately, G/D_h (when smooth) inherits a Poisson structure from G, and $[L_h] : G \to G/D_h$ is a Poisson map which is moreover equivariant w.r.t. the natural Poisson actions of G.

In the following table we characterize submanifolds of a symplectic or Poisson manifold in terms of the bundle map $\rho := pr_{NC} \circ \sharp \colon N^*C \to NC$:

	P symplectic	P Poisson
$Im(\rho) = 0$	C coisotropic	C coisotropic
$Im(\rho) = NC$	C symplectic	C cosymplectic
$Rk(\rho)$ =const	C presymplectic	C pre-Poisson

In the first part of this paper (sections 3- 6) we consider the Poisson analogue of Marle's result; i.e., we ask the following question:

> Given an arbitrary submanifold C of a Poisson manifold (P, Π), under what conditions does there exist some submanifold $\tilde{P} \subset P$ such that
> a) $\tilde{P}$ has a Poisson structure induced from Π,
> b) C is a coisotropic submanifold of $\tilde{P}$?
> When the submanifold $\tilde{P}$ exists, is it unique up to neighborhood equivalence (i.e. up to a Poisson diffeomorphism on a tubular neighborhood which fixes C)?

We show in section 3 that for any pre-Poisson submanifold C of a Poisson manifold P there is a submanifold $\tilde{P}$ which is cosymplectic (and hence has a canonically induced Poisson structure) such that C lies coisotropically in $\tilde{P}$. Further (section 4) this cosymplectic submanifold is unique up to neighborhood equivalence; to the best of our knowledge, this uniqueness result is new even in the symplectic setting. In section 5 we give sufficient conditions and necessary conditions for the existence of a submanifold $\tilde{P}$ as in the above question and we provide examples. Then in section 6 we deduce statements about the algebra $C^\infty_{bas}(C)$ of functions on C which are basic (invariant), meaning that their differentials annihilate the distribution $\sharp N^*C \cap TC$, and about its deformation quantization. We show that if C is a pre-Poisson submanifold so that the first and second Lie algebroid cohomology of $N^*C \cap \sharp^{-1}TC$ vanish, then the Poisson algebra of basic functions on C admits a deformation quantization. Finally in section 7, assuming that the symplectic groupoid $\Gamma_s(P)$ of P exists, we describe two subgroupoids (an isotropic and a presymplectic one) naturally associated to a pre-Poisson submanifold C of P.

The second part of this paper (sections 8 and 9) deals with a different embedding problem, where we start with an abstract manifold instead of a submanifold of some Poisson manifold. This is the Poisson-analogue of Gotay's result. The question we ask is:

> Let (M, L) be a Dirac manifold. Is there an embedding $i \colon (M, L) \to (P, \Pi)$ into a Poisson manifold such that
> a) $i(M)$ is a coisotropic submanifold of P,
> b) the Dirac structure L is induced by the Poisson structure Π?
> Is such an embedding unique up to neighborhood equivalence?

In the symplectic setting both existence and uniqueness hold [15]. One motivation for this question is the deformation quantization of the Poisson algebra of so-called admissible functions on (M, L), for a coisotropic embedding as above allows one to reduce the problem to [10], i.e. to the deformation quantization of the basic functions on a coisotropic submanifold of a Poisson manifold.

It turns out (section 8) that the above question admits a positive answer iff the distribution $L \cap TM$ on the Dirac manifold M is regular. In that case one expects the Poisson manifold $\tilde{P}$ to be unique (up to a Poisson diffeomorphism fixing M), provided $\tilde{P}$ has minimal dimension. We are not able to prove this global uniqueness; we can just show in section 9 that the Poisson vector bundle $T\tilde{P}|_M$ is unique (an infinitesimal statement along M) and that around each point of M a small neighborhood of $\tilde{P}$ is unique (a local statement). We remark that A. Wade [20] has been considering a similar question too. Our result about deformation quantization is the following (Thm. 8.5): let (M, L) be a Dirac manifold such that $L \cap TM$ has constant rank, and denote by $\mathcal{F}$ the regular foliation integrating $L \cap TM$. If the first and second foliated de Rham cohomologies of the foliation $\mathcal{F}$ vanish, then the Poisson algebra of admissible functions on (M, L) has a deformation quantization. In Prop. 8.6 we also notice that the foliated de Rham cohomology $\Omega^{\bullet}_{\mathcal{F}}(M)$ admits the structure of an L_∞-algebra (canonically up to L_∞-isomorphism), generalizing a result of Oh and Park in the presymplectic setting (Thm. 9.4 of [18]).

We end this introduction by describing one of our motivations for the first question above, namely an application of the Poisson sigma model to quantization problems. The Poisson sigma model is a topological field theory, whose fields are bundle maps from $T\Sigma$ (for Σ a surface) to the cotangent bundle T^*P of a Poisson manifold (P, Π). It was used by Felder and the first author [8] to derive and interpret Kontsevich's formality theorem and his star product on the Poisson manifold P. The Poisson sigma model with boundary conditions on a coisotropic submanifold C, when suitable assumptions on C are satisfied and P is assumed to be an open subset of $\mathbb{R}^n$, provides [9] a deformation quantization of the Poisson algebra of basic (invariant) functions $C^\infty_{bas}(C)$ on C. This result was globalized using a supergeometric version of Kontsevich's formality theorem [10]: when the first and second cohomology of the Lie algebroid N^*C vanish, for C a coisotropic submanifold of any Poisson manifold P, the Poisson algebra $C^\infty_{bas}(C)$ admits a deformation quantization. Notice that the quotient of C by the distribution $\sharp N^*C$ is usually not a smooth manifold. Hence $C^\infty_{bas}(C)$ is usually not the algebra of functions on any Poisson manifold, and one cannot apply Kontsevich's theorem [16] on deformation quantization of Poisson manifolds directly.

Calvo and Falceto observed that the most general boundary conditions for the Poisson sigma model are given by pre-Poisson submanifolds of (P, Π) (which they referred to as "strongly regular submanifolds"). They show [4] that when P is an open subset of $\mathbb{R}^n$ the problem of deformation quantizing the Poisson algebra of basic functions on C can be reduced to the results of [9]. The computations in [4] are carried out by choosing local coordinates on P adapted to C. The strong regularity condition allows one to choose local constraints for C such that the number of first class constraints (X^μs whose Poisson brackets with all other constrains vanish on C) and second class constraints (the remaining constraints X^A, which automatically satisfy $\det\{X^A, X^B\} \neq 0$ on C) be constant along C. Setting the second class constraints X^A to zero locally gives a submanifold with an induced

Poisson structure, and the fact that only first class constraints are left means that C lies in it as a coisotropic submanifold. Our first question above can be seen as a globalization of Calvo and Falceto's results.

Conventions. We use the term "presymplectic manifold" to denote a manifold endowed with a closed 2-form of *constant rank*, i.e. such that its kernel has constant rank. However we stick to the denominations "presymplectic groupoid" coined in [3] and "presymplectic leaves" (of a Dirac manifold) despite the fact that the 2-forms on these objects do not have constant rank, for these denominations seem to be established in the literature.

2. Basic definitions

We will use some notions from Dirac linear algebra ([13], [2]). A Dirac structure on a vector space P is a subspace $L \subset P \oplus P^*$ which is maximal isotropic w.r.t. the natural symmetric inner product on $P \oplus P^*$ (i.e. L is isotropic and has the same dimension as P). A Dirac structure L specifies a subspace $\mathcal{O}$, defined as the image of L under the projection $P \oplus P^* \to P$, and a skew-symmetric bilinear form ω on $\mathcal{O}$, given by $\omega(X_1, X_2) = \langle \xi_1, X_2 \rangle$ where ξ_1 is any element of P^* such that $(X_1, \xi_1) \in L$. The kernel of ω (which in terms of L is given as $L \cap P$) is called the *characteristic subspace*. Conversely, any choice of bilinear form defined on a subspace of P determines a Dirac structure on P. Given this equivalence, we will sometimes work with the bilinear form ω on $\mathcal{O}$ instead of working with L.

We now consider Poisson vector spaces (P, Π) (i.e. $\Pi \in \wedge^2 P$; we denote by $\sharp\colon P^* \to P$ the map induced by contraction with Π). The Poisson structure on P is encoded by the Dirac structure $L_P = \{(\sharp\xi, \xi)\colon \xi \in P^*\}$. The image of L_P under the projection onto the first factor is $\mathcal{O} = \sharp P^*$, and the bilinear form ω is non-degenerate.

Remark 2.1. We recall that any subspace W of a Dirac vector space (P, L) has an induced Dirac structure L_W; the bilinear form characterizing L_W is just the pullback of ω (hence it is defined on $W \cap \mathcal{O}$). When (P, Π) is actually a Poisson vector space, one shows that the symplectic orthogonal of $W \cap \mathcal{O}$ in $(\mathcal{O}, \omega)$ is $\sharp W^\circ$. Hence $\sharp W^\circ \cap W$ is the kernel of the restriction of ω to $W \cap \mathcal{O}$, i.e. it is the characteristic subspace of the Dirac structure L_W, and we will refer to it as the *characteristic subspace of* W. Notice that pulling back Dirac structure is functorial [2] (i.e. if W is contained in some other subspace W' of P, pulling back L first to W' and then to W gives the Dirac structure L_W), hence L_W, along with the corresponding bilinear form and characteristic subspace, is *intrinsic* to W.

Let W be a subspace of the Poisson vector space (P, Π). W is called a *coisotropic* if $\sharp W^\circ \subset W$, which by the above means that $W \cap \mathcal{O}$ is coisotropic in $(\mathcal{O}, \omega)$.

W is called a *Poisson-Dirac* subspace [14] when $\sharp W^\circ \cap W = \{0\}$; equivalent conditions are that $W \cap \mathcal{O}$ be a symplectic subspace of $(\mathcal{O}, \omega)$ or that the pullback Dirac structure L_P correspond to a Poisson bivector on W. The Poisson bivector on W is described as follows [14]: its sharp map $\sharp_W\colon W^* \to W$ is given by $\sharp_W \tilde{\xi} = \sharp\xi$, where $\xi \in P^*$ is any extension of $\tilde{\xi}$ which annihilates $\sharp W^\circ$.

W is called a *cosymplectic* subspace if $\sharp W^\circ \oplus W = P$, or equivalently if the pushforward of Π via the projection $P \to P/W$ is an invertible bivector. Notice that if W is cosymplectic, then it has a canonical complement $\sharp W^\circ$ which is a symplectic subspace of $(\mathcal{O}, \omega)$. Clearly a cosymplectic subspace is automatically a Poisson-Dirac subspace.

Now we pass to the global definitions. A Dirac structure on P is a maximal isotropic subbundle $L \subset TP \oplus T^*P$ which is integrable, in the sense that its sections are closed under the so-called Courant bracket (see [13]). The image of L under the projection onto the first factor is an integrable singular distribution, whose leaves (which are called presymplectic leaves) are endowed with closed 2-forms. A Poisson structure on P is a bivector Π such that $[\Pi, \Pi] = 0$.

Coisotropic and cosymplectic submanifolds of a Poisson manifold are defined exactly as in the linear case; a Poisson-Dirac submanifold additionally requires that the bivector induced on the submanifold by the point-wise condition be smooth [14]. Cosymplectic submanifolds are automatically Poisson-Dirac submanifolds (the smoothness of the induced bivector is ensured because $L_P \cap (\{0\} \oplus N^*\tilde{P})$ has constant rank zero). The Poisson bracket on a Poisson-Dirac submanifold $\tilde{P}$ of (P, Π) is computed as follows: $\{\tilde{f}_2, \tilde{f}_2\}_{\tilde{P}}$ is the restriction to $\tilde{P}$ of $\{f_1, f_2\}$, where the f_i are extensions of $\tilde{f}_i$ to P such that $df_i|_{\sharp N^*\tilde{P}} = 0$ (for at least one of the two functions). We will also need a definition which does not have a linear algebra counterpart:

Definition 2.2. A submanifold C of a Poisson manifold (P, Π) is called *pre-Poisson* if the rank of $TC + \sharp N^*C$ is constant along C.

Remark 2.3. An alternative characterization of pre-Poisson submanifolds is the requirement that $\Pi|_{\wedge^2 N^*C}$ (or equivalently the corresponding sharp map $pr_{NC} \circ \sharp \colon N^*C \to TP|_C \to NC := TP|_C/TC$) have constant rank. Indeed the kernel of $N^*C \to NC$ is $N^*C \cap \sharp^{-1}TC$, which is the annihilator of $TC + \sharp N^*C$. The map $N^*C \to NC$ is identically zero iff C is coisotropic and is an isomorphism iff C is cosymplectic.

Calvo and Falceto already considered ([5], [4]) such submanifolds and called them "strongly regular submanifolds." We prefer to call them "pre-Poisson" because when P is a symplectic manifold they reduce to presymplectic submanifolds[1]. See Section 5 for several examples.

3. Existence of coisotropic embeddings for pre-Poisson submanifolds

In this section we consider the problem of embedding a submanifold of a Poisson manifold coisotropically in a Poisson-Dirac submanifold, and show that this can always be done for pre-Poisson submanifolds.

We start with some linear algebra.

Lemma 3.1. *Let (P, Π) be a Poisson vector space and C a subspace. The Poisson-Dirac subspaces of P in which C sits coisotropically are exactly the subspaces W satisfying*

$$W + \sharp C^\circ \supset \mathcal{O}, \tag{1}$$

$$W \cap (C + \sharp C^\circ) = C, \tag{2}$$

where $\mathcal{O} = \sharp P^*$. Among the Poisson-Dirac subspaces above the cosymplectic ones are exactly those of maximal dimension, i.e. those for which $W + \sharp C^\circ = P$.

[1]Further reasons are the following: the subgroupoid associated to a pre-Poisson manifold, when it exists, is presymplectic (see Prop. 7.5). The Hamiltonian version of the Poisson Sigma Model with boundary conditions on P (at $t = 0$) and on a submanifold C (at $t = 1$) delivers a space of solutions which is presymplectic iff C is pre-Poisson.

Remark 3.2. It is often more convenient to work with the following characterization of the Poisson-Dirac subspaces W containing coisotropically C: $W = R \oplus C$, where the subspace R satisfies

$$R \oplus (C + \sharp C^\circ) \supset \mathcal{O}. \tag{3}$$

Among these, the cosymplectic subspaces are those for which R satisfies the stronger condition $R \oplus (C + \sharp C^\circ) = P$. When Π corresponds to a linear symplectic form ω, both conditions become $R \oplus (C + C^\omega) = P$.

Proof. The condition that W be a Poisson-Dirac subspace is

$$W \cap \sharp W^\circ = 0. \tag{4}$$

Let us denote by $\sharp_W$ the sharp map of the induced bivector on W. The condition that C is contained in W coisotropically is $\sharp_W \tilde{\xi} \in C$ for all elements $\tilde{\xi} \in W^*$ annihilating C. $\sharp_W \tilde{\xi}$ is obtained by extending $\tilde{\xi}$ to some $\xi \in (\sharp W^\circ)^\circ = \sharp^{-1} W$ and applying $\sharp$. Hence the condition that C is contained in W coisotropically can be phrased as

$$W \cap \sharp C^\circ \subset C \subset W. \tag{5}$$

We now show that conditions (4) and (5) are equivalent to conditions (1) and (2).

We have (5) $\Rightarrow$ (2), because due to $C \subset W$ we have $W \cap (C + \sharp C^\circ) = C + (W \cap \sharp C^\circ)$. The implication (5) $\Leftarrow$ (2) is immediate.

Now assume that either of (5) or (2) holds true. Applying $\sharp(\bullet)^\circ$ we see that condition (1) is equivalent to $C \cap \sharp W^\circ = \{0\}$. Since, applying condition (5), we have

$$W \cap \sharp W^\circ = (W \cap \sharp C^\circ) \cap \sharp W^\circ \subset C \cap \sharp W^\circ \subset W \cap \sharp W^\circ,$$

the equivalence of conditions (4) and (1) is proven.

To prove the last statement of the lemma let W satisfy (1) and (2); in particular W is Poisson-Dirac. By dimension counting W is cosymplectic iff the restriction of $\sharp$ to W° is injective, i.e. iff $W^\circ \cap \mathcal{O}^\circ = \{0\}$ or $W + \mathcal{O} = P$. Using (1) this is seen to be equivalent to $W + \sharp C^\circ = P$. □

Now we pass from linear algebra to global geometry. Given a submanifold C of a Poisson manifold P, one might try to construct a Poisson-Dirac submanifold in which C embeds coisotropically by applying the corresponding symplectic construction "leaf by leaf" in a smooth way. It would then be natural to require that the characteristic "distribution" $TC \cap \sharp N^*C$ of C have constant rank. However this approach generally does not work, because even when it has constant rank $TC \cap \sharp N^*C$ might not be smooth (see Example 5.7). The right condition to ask instead is that $TC + \sharp N^*C$ have constant rank:

Theorem 3.3. *Let C be a pre-Poisson submanifold of a Poisson manifold (P, Π). Then there exists a cosymplectic submanifold $\tilde{P}$ containing C such that C is coisotropic in $\tilde{P}$.*

Proof. Because of the rank condition on C we can choose a smooth subbundle R of $TP|_C$ which is a complement to $TC + \sharp N^*C$. Then by Lemma 3.1 at every point p of C we have that $T_pC \oplus R_p$ is a cosymplectic subspace of T_pP in which T_pC sits coisotropically. "Thicken" C to a smooth submanifold $\tilde{P}$ of P satisfying $T\tilde{P}|_C = TC \oplus R$. Since $T_p\tilde{P} \oplus \sharp N_p^*\tilde{P} = T_pP$ is an open condition that holds at every point p of C, it holds at points in a tubular neighborhood of C in $\tilde{P}$. Hence, shrinking $\tilde{P}$ if necessary, we obtain a cosymplectic submanifold of P containing coisotropically C. $\square$

Remark 3.4. The above proposition says that if C is a pre-Poisson submanifold, then we can choose a subbundle R over C with fibers as in (3) and "extend" C in the direction of R to obtain a Poisson-Dirac submanifold of P containing C coisotropically. If C is not a pre-Poisson submanifold of (P, Π), we might still be able to find a smooth bundle R over C consisting of subspaces as in (3). However "extending" C in the direction of this subbundle will usually not give a submanifold with a smooth Poisson-Dirac structure; see Example 5.4 below.

Now we deduce consequences about Lie algebroids. See section 7 for the corresponding "integrated" statements.

Lemma 3.5. *Let C be a subspace of a Poisson vector space (P, Π) and W a cosymplectic subspace containing C as a coisotropic subspace. Then $C + \sharp C^\circ = C \oplus \sharp W^\circ$.*

Proof. The inclusion "$\supset$" holds because $C \subset W$. The other inclusion follows by this argument: write any $\xi \in C^\circ$ uniquely as $\xi_1 + \xi_2$ where ξ_1 annihilates $\sharp W^\circ$ and ξ_2 annihilates W. Then $\sharp \xi_1 = \sharp_W(\xi_1|_W) \in C$, where $\sharp_W$ denotes the sharp map of W, since C is coisotropic in W. Hence $\sharp\xi = \sharp\xi_1 + \sharp\xi_2 \in C + \sharp W^\circ$. Finally, we have a direct sum in $C \oplus \sharp W^\circ$ because $\sharp W^\circ \cap W = \{0\}$ and $C \subset W$. $\square$

Proposition 3.6. *Let C be a submanifold of a Poisson manifold (P, Π). Then $N^*C \cap \sharp^{-1}TC$ is a Lie subalgebroid of T^*P iff C is pre-Poisson. In that case, for any cosymplectic submanifold $\tilde{P}$ in which C sits coisotropically, $N^*C \cap \sharp^{-1}TC$ is isomorphic as a Lie algebroid to the annihilator of C in $\tilde{P}$.*

Proof. At every point, $N^*C \cap \sharp^{-1}TC$ is the annihilator of $TC + \sharp N^*C$, so it is a vector bundle iff C is pre-Poisson. So assume that C be pre-Poisson. For any cosymplectic submanifold $\tilde{P}$ the embedding $T^*\tilde{P} \to T^*P$, obtained by extending a covector in $T^*\tilde{P}$ so that it annihilates $\sharp N^*\tilde{P}$, is a Lie algebroid morphism (Cor. 2.11 and Thm. 2.3 of [22]). If C lies coisotropically in $\tilde{P}$, by Lemma 3.5, $TC + \sharp N^*C = TC \oplus \sharp N^*\tilde{P}|_C$. Hence $N^*_{\tilde{P}}C$, the conormal bundle of C in $\tilde{P}$, is mapped *isomorphically* onto $(TC \oplus \sharp N^*\tilde{P})^\circ = (TC + \sharp N^*C)^\circ = N^*C \cap \sharp^{-1}TC$. Since $N^*_{\tilde{P}}C$ is a Lie subalgebroid of $T^*\tilde{P}$ [7], we are done. $\square$

Remark 3.7. The fact that $N^*C \cap \sharp^{-1}TC$ is a Lie algebroid if C is pre-Poisson can also be deduced as follows. The Lie algebra $(\mathcal{F} \cap I)/I^2$ forms a Lie-Rinehart algebra over the commutative algebra $C^\infty(P)/I$, where I is the vanishing ideal of C and $\mathcal{F}$ its Poisson-normalizer in $C^\infty(P)$. Lemma 1 of [5] states that C being pre-Poisson is equivalent to $N^*C \cap \sharp^{-1}TC$ being spanned by differentials of functions in $\mathcal{F} \cap I$. From this one deduces easily that $(\mathcal{F} \cap I)/I^2$ is identified with the sections of $N^*C \cap \sharp^{-1}TC$, and since $C^\infty(P)/I$ are just the smooth functions on C we deduce that $N^*C \cap \sharp^{-1}TC$ is a Lie algebroid over C.

4. Uniqueness of coisotropic embeddings for pre-Poisson submanifolds

Given a submanifold C of a Poisson manifold (P, Π) in this section we investigate the uniqueness (up Poisson diffeomorphisms fixing C) of *cosymplectic* submanifolds in which C is embedded coisotropically.

This lemma tells us that we need to consider only the case in which C is pre-Poisson and the construction of Thm. 3.3:

Lemma 4.1. *A submanifold C of a Poisson manifold (P, Π) can be embedded coisotropically in a cosymplectic submanifold $\tilde{P}$ iff it is pre-Poisson. In this case all such $\tilde{P}$ are constructed (in a neighborhood of C) as in Thm.* 3.3.

Proof. In Thm. 3.3 we saw that given any pre-Poisson submanifold C, choosing a smooth subbundle R with $R \oplus (TC + \sharp N^*C) = TP|_C$ and "thickening" C in the direction of R gives a submanifold $\tilde{P}$ with the required properties.

Now let C be any submanifold embedded coisotropically in a cosymplectic submanifold $\tilde{P}$. By Remark 3.2, for any complement R of TC in $T\tilde{P}|_C$ we have $R \oplus (TC + \sharp N^*C) = TP|_C$. This has two consequences: first the rank of $TC + \sharp N^*C$ must be constant, concluding the proof of the "iff" statement of the lemma. Second, it proves the final statement of the lemma. □

When C is a point $\{p\}$, then $\tilde{P}$ as above is a slice transverse to the symplectic leaf through p (see Example 5.1) and $\tilde{P}$ is unique up Poisson diffeomorphism by Weinstein's splitting theorem (Lemma 2.2 in [21]; see also Thm. 2.16 in [19]). A generalization of its proof gives

Proposition 4.2. *Let $\tilde{P}_0$ be a cosymplectic submanifold of a Poisson manifold P and $\pi \colon U \to \tilde{P}_0$ a projection of some tubular neighborhood of $\tilde{P}_0$ onto $\tilde{P}_0$. Let $\tilde{P}_t$, $t \in \mathbb{R}$, be a smooth family of cosymplectic submanifolds such that all $\tilde{P}_t$ are images of sections of π. Then, for t close enough to zero, there are Poisson diffeomorphisms ϕ_t mapping open sets of $\tilde{P}_0$ to open sets of $\tilde{P}_t$. The ϕ_t's can be chosen so that the curves $t \mapsto \phi_t(y)$ (for $y \in \tilde{P}_0$) are tangent to $\sharp N^*\tilde{P}_t$ at time t.*

Proof. We will use the following fact, whose straightforward proof we omit: let $\tilde{P}_t$, $t \in \mathbb{R}$, be a smooth family of submanifolds of a manifold U, and Y_t a time-dependent vector field on U. Then $Y + \frac{\partial}{\partial t}$ (considered as a vector field on $U \times \mathbb{R}$) is tangent to the submanifold $\bigcup_{t\in\mathbb{R}}(\tilde{P}_t, t)$ iff for each $\bar{t}$ and each integral curve γ of Y_t in U with $\gamma(\bar{t}) \in \tilde{P}_{\bar{t}}$ we have $\gamma(t) \in \tilde{P}_t$ (at all times where γ is defined).

Denote by s_t the section of π whose image is $\tilde{P}_t$. We are interested in time-dependent vector fields Y_t on U such that for all $\bar{t}$ and $y \in \tilde{P}_{\bar{t}}$,

$$Y_{\bar{t}}(y) = s_{\bar{t}*}(\pi_* Y_y) + \frac{d}{dt}|_{\bar{t}} s_t(\pi(y)). \tag{6}$$

We claim that, for such a vector field, $(Y + \frac{\partial}{\partial t})$ is tangent to $\bigcup_{t\in\mathbb{R}}(\tilde{P}_t, t)$. Indeed

$$(Y + \frac{\partial}{\partial t})(y, \bar{t}) = Y_{\bar{t}}(y) + \frac{\partial}{\partial t} \tag{7}$$

$$= s_{\bar{t}*}(\pi_* Y_y) + \frac{d}{dt}|_{\bar{t}} s_t(\pi(y)) + \frac{\partial}{\partial t}. \tag{8}$$

Since $s_{\bar{t}*}(\pi_* Y_y)$ is tangent to $(\tilde{P}_{\bar{t}}, \bar{t})$, and $\frac{d}{dt}|_{\bar{t}} s_t(\pi(y)) + \frac{\partial}{\partial t}$ is the velocity at time $\bar{t}$ of the curve $(s_t(\pi(y)), t)$, the claimed tangency follows. Hence by the fact recalled in the first paragraph we deduce that the flow ϕ_t of Y_t takes points y of $\tilde{P}_0$ to $\tilde{P}_{\bar{t}}$ (if $\phi_t(y)$ is defined until time $\bar{t}$).

So we are done if we realize such Y_t as the Hamiltonian vector fields of a smooth family of functions H_t on U. For each fixed $\bar{t}$, eq. (6) for $Y_{\bar{t}}$ is just a condition on the second component of $Y_{\bar{t}} \in T_yP = T_y\tilde{P}_{\bar{t}} \oplus \ker_y \pi_*$ for all $y \in \tilde{P}_{\bar{t}}$, and the second component is determined exactly by the action of $Y_{\bar{t}}$ on functions f vanishing on $\tilde{P}_{\bar{t}}$. We have

$$Y_{\bar{t}}(f) = X_{H_{\bar{t}}}(f) = -dH_{\bar{t}}(\sharp df),$$

and the restriction of $\sharp$ to $N^*\tilde{P}_{\bar{t}}$ is injective because $\tilde{P}_{\bar{t}}$ is cosymplectic. Together we obtain that specifying the vertical component of $X_{H_{\bar{t}}}$ at points of $\tilde{P}_{\bar{t}}$ is equivalent to specifying the derivative of $H_{\bar{t}}$ in direction of $\sharp N^*\tilde{P}_{\bar{t}}$, which is transverse to $\tilde{P}_{\bar{t}}$. We can clearly find a function $H_{\bar{t}}$ satisfying the required conditions on its derivative along $\tilde{P}_{\bar{t}}$, i.e. so that $X_{H_{\bar{t}}}$ satisfies (6). Choosing H_t smoothly for every t we conclude that the flow ϕ_t of X_{H_t}, which obviously consists of Poisson diffeomorphisms, will take $\tilde{P}_0$ (or rather any subset of it on which the flow is defined up to time $\bar{t}$) to $\tilde{P}_{\bar{t}}$.

Choosing each H_t so that it vanishes on $\tilde{P}_t$ delivers a flow ϕ_t "tangent" to the $\sharp N^*\tilde{P}_t$'s. □

Now we are ready to prove the uniqueness of $\tilde{P}$:

Theorem 4.3. *Let C be a pre-Poisson submanifold (P,Π), and $\tilde{P}_0$, $\tilde{P}_1$ cosymplectic submanifolds that contain C as a coisotropic submanifold. Then, shrinking $\tilde{P}_0$ and $\tilde{P}_1$ to a smaller tubular neighborhood of C if necessary, there is a Poisson diffeomorphism Φ from $\tilde{P}_0$ to $\tilde{P}_1$ which is the identity on C.*

Proof. In a neighborhood U of $\tilde{P}_0$ take a projection $\pi\colon U \to \tilde{P}_0$; choose it so that at points of $C \subset \tilde{P}_0$ the fibers of π are tangent to $\sharp N^*\tilde{P}_0|_C$. For $i = 0,1$ make some choices of maximal dimensional subbundles R_i satisfying (3) to write $T\tilde{P}_i|_C = TC \oplus R_i$, and choose a smooth curve of subbundles R_t satisfying (3) and agreeing with R_0 and R_1 at $t = 0,1$ (there is no topological obstruction to this because R_0 and R_1 are both complements to the same subbundle $TC + \sharp N^*C$). By Thm. 3.3 we obtain a curve of cosymplectic submanifolds $\tilde{P}_t$, which moreover by Lemma 3.5 at points of C are all transverse to $\sharp N^*\tilde{P}_0|_C$, i.e. to the fibers of π.

Hence we are in the situation of Prop. 4.2, which allows us to construct a Poisson diffeomorphism from $\tilde{P}_0$ to $\tilde{P}_t$ for small t. Since $C \subset \tilde{P}_t$ for all t, in the proof of Prop. 4.2 we have that the sections s_t are trivial on C; hence by (6) the second component of $X_{H_t} \in T_y\tilde{P}_t \oplus \ker_y \pi_*$ at points y of $C \subset \tilde{P}_t$ is zero. Choosing H_t to vanish on $\tilde{P}_t$ we obtain $X_{H_t} = 0$ at points of $C \subset \tilde{P}_t$. From this we deduce two things: in a tubular neighborhood of C the flow ϕ_t of X_{H_t} is defined for all $t \in [0,1]$, and each ϕ_t keeps points of C fixed. Now just let $\Phi := \phi_1$. □

The derivative at points of C of the Poisson diffeomorphism Φ constructed in Thm. 4.3 gives an isomorphism of Poisson vector bundles $T\tilde{P}_0|_C \to T\tilde{P}_1|_C$ which is the identity on TC. The construction of Φ involves many choices; we now wish to give a *canonical* construction for such a vector bundle isomorphism. We first need a linear algebra lemma.

Lemma 4.4. *Let C be a subspace of a Poisson vector space (P,Π) and V,W two cosymplectic subspaces containing C as a coisotropic subspace. There exists a* canonical *isomorphism of Poisson vector spaces $\varphi\colon V \to W$ which is the identity on C.*

Proof. Notice that V and W have the same dimension by Lemma 3.1. First we consider

$$A\colon V \to \sharp V^\circ$$

determined by the requirement that $W = \{v + Av : v \in V\}$. A is well defined since $\sharp V^\circ$ is a complement in P both to V (because V is cosymplectic) and to W (because $W \cap (C + \sharp C^\circ) = C$ by Lemma 3.1 and $C + \sharp C^\circ = C \oplus \sharp V^\circ$ by Lemma 3.5). Notice that, since C lies in both V and W, the restriction of A to C is zero.

Now we mimic a construction in symplectic linear algebra [6] where one deforms canonically a complement of a coisotropic subspace C to obtain an isotropic complement. We deform $A + Id$ by adding

$$B \colon V \to C \cap \sharp C^\circ, v \mapsto \frac{1}{2}\sharp_V(\Omega(Av, A\bullet)).$$

Here $\sharp_V$ is the sharp map of the cosymplectic submanifold V and Ω denotes the symplectic form on $\mathcal{O} := \sharp P^*$. B is well defined because the element $\Omega(Av, A\bullet)$ of V^* annihilates C (recall that $A|_C = 0$) and because C is coisotropic in V. Further it is clear that the restriction of B to C is zero.

At this point we are ready to define

$$\varphi \colon V \to W, v \mapsto v + Av + Bv.$$

This is well defined (since $C \cap \sharp C^\circ \subset W$) and is an isomorphism: if $v + Bv + Av = 0$, then $v + Bv = 0$ and $Av = 0$ (because V is transversal to $\sharp V^\circ$); from $Av = 0$ we deduce $Bv = 0$ and hence $v = 0$. To show that φ matches the linear Poisson structures on V and W we notice that φ restricts to a map from $V \cap \mathcal{O}$ to $W \cap \mathcal{O}$ (because the images of A and B lie in $\mathcal{O}$). This restriction is an isomorphism because the source and target have the same dimension (they both contain $C \cap \mathcal{O}$ as a coisotropic subspace); we show that it is a linear symplectomorphism. If $v_1, v_2 \in V \cap \mathcal{O}$ we have $\Omega(\varphi v_1, \varphi v_2) = \Omega(v_1 + Bv_1, v_2 + Bv_2) + \Omega(Av_1, Av_2)$, for the cross terms vanish since A takes values in $\sharp V^\circ$. Now $\Omega(Bv_1, \bullet)|_{V \cap \mathcal{O}} = -\frac{1}{2}\Omega(Av_1, A\bullet)|_{V\cap\mathcal{O}}$ using the fact that $\Omega(\sharp\xi, \bullet) = -\xi|_{\mathcal{O}}$ for any covector ξ of P. Further $\Omega(Bv_1, Bv_2)$ vanishes because B takes values in $C \cap \sharp C^\circ$. So altogether we obtain $\Omega(\varphi v_1, \varphi v_2) = \Omega(v_1, v_2)$ as desired. □

Proposition 4.5. *Let C be a pre-Poisson submanifold (P, Π), and $\tilde{P}$, $\hat{P}$ cosymplectic submanifolds that contain C as a coisotropic submanifold. Then there is a* canonical *isomorphism of Poisson vector bundles $\varphi \colon T\tilde{P}|_C \to T\hat{P}|_C$ which is the identity on TC.*

Proof. At each point $p \in C$ we construct φ_p by applying Lemma 4.4 to $V = T_p\tilde{P}$ and $W = T_p\hat{P}$. We want to check that the resulting map $\varphi \colon T\tilde{P}|_C \to T\hat{P}|_C$ is smooth (this is not clear a priori because the construction of Lemma 4.4 involves the symplectic leaves $\mathcal{O}$ of P, which may be of different dimensions). It is enough to check that if X is a smooth section of $\sharp N^*\tilde{P}|_C$, then $\Omega(X, \bullet)|_{\sharp N^*\tilde{P}} : \sharp N^*\tilde{P} \to \mathbb{R}$ is smooth. This follows from the fact that $\tilde{P}$ is cosymplectic: since $\sharp \colon N^*\tilde{P} \to \sharp N^*\tilde{P}$ is bijective, there is a smooth section ξ of $N^*\tilde{P}$ with $\sharp\xi = X$, and $\Omega(X, \bullet)|_{\sharp N^*\tilde{P}} = \xi|_{\sharp N^*\tilde{P}}$. Altogether we obtain that φ is a smooth, canonical isomorphism of Poisson vector bundles. □

Remark 4.6. The isomorphism $\varphi \colon T\tilde{P}|_C \to T\hat{P}|_C$ constructed in Prop. 4.5 can be extended to a Poisson vector bundle automorphism of $TP|_C$, by applying the following at each point of C.

The linear isomorphism $\varphi\colon V \to W$ of Lemma 4.4 (using the notation of the lemma) can be extended to a Poisson automorphism of P as follows: define

$$(\varphi, pr)\colon V \oplus \sharp V^\circ \to W \oplus \sharp W^\circ,$$

where pr denotes the projection of $\sharp V^\circ$ onto $\sharp W^\circ$ along C (recall from Lemma 3.5 that $C \oplus \sharp V^\circ = C \oplus \sharp W^\circ$). (φ, pr) restricts to a linear automorphism of $\mathcal{O} = (V \cap \mathcal{O}) \oplus \sharp V^\circ$ which preserves the symplectic form: the only non-trivial check is $\Omega(pr(v_1), pr(v_2)) = \Omega(v_1, v_2)$ for $v_i \in \sharp V^\circ$, which follows because $pr(v_i) - v_i \in C \cap \sharp C^\circ$.

Remark 4.7. The isomorphism φ constructed in Prop. 4.5 can be extended to a Poisson vector bundle automorphism of $TP|_C$ as follows: define

$$(\varphi, pr)\colon T\tilde{P} \oplus \sharp N^*\tilde{P} \to T\hat{P} \oplus \sharp N^*\hat{P},$$

where pr denotes the projection of $N^*\tilde{P}$ onto $N^*\hat{P}$ along TC (recall from Lemma 3.5 that $TC \oplus N^*\tilde{P} = TC \oplus N^*\hat{P}$). (φ, pr) restricts to a linear automorphism of $T\mathcal{O} = (T\tilde{P} \cap T\mathcal{O}) \oplus \sharp N^*\tilde{P}$ which preserves the symplectic form: the only non-trivial check is $\Omega(pr(v_1), pr(v_2)) = \Omega(v_1, v_2)$ for $v_i \in \sharp N^*\tilde{P}$, which follows because $pr(v_1) - v_1 \in TC \cap \sharp N^*C$.

5. Conditions and examples

Let C be as usual a submanifold of the Poisson manifold (P, Π); in section 3 we considered the question of existence of a Poisson-Dirac submanifold $\tilde{P}$ of P in which C is contained coisotropically. In Thm. 3.3 we showed that a *sufficient* condition is that C be pre-Poisson, which by Prop. 3.6 is equivalent to requiring that $N^*C \cap \sharp^{-1}TC$ be a Lie algebroid.

A *necessary* condition is that the (intrinsically defined) characteristic distribution $TC \cap \sharp N^*C$ of C be the distribution associated to a Lie algebroid over C; in particular its rank locally can only increase. This is a necessary condition since the concept of characteristic distribution is an intrinsic one (see Remark 2.1), and the characteristic distribution of a coisotropic submanifold of a Poisson manifold is the image of the anchor of its conormal bundle, which is a Lie algebroid.

The submanifolds C which are not covered by the above conditions are those for which $N^*C \cap \sharp^{-1}TC$ is not a Lie algebroid but its image $TC \cap \sharp N^*C$ under $\sharp$ is the image of the anchor of some Lie algebroid over C. Diagrammatically:

$$\{C \text{ s.t. } N^*C \cap \sharp^{-1}TC \text{ is a Lie algebroid, i.e. } C \text{ is pre-Poisson}\} \subset$$

$$\{C \text{ sitting coisotropically in some Poisson-Dirac submanifold } \tilde{P} \text{ of } P\} \subset$$

$$\{C \text{ s.t. } TC \cap \sharp N^*C \text{ is the distribution of some Lie algebroid over } C\}.$$

In the remainder of this section we present examples of the above situations. We start with basic examples of pre-Poisson submanifolds; we refer the reader to Section 6 of [12] for examples in which the Poisson manifold P is the dual of a Lie algebra and C is an affine subspace.

Example 5.1. An obvious example is when C is a coisotropic submanifold of P, and in this case the construction of Thm. 3.3 delivers $\tilde{P} = P$ (or more precisely, a tubular neighborhood of C in P).

Another obvious example is when C is just a point p: then the construction of Thm. 3.3 delivers as $\tilde{P}$ any slice through x transverse to the symplectic leaf $\mathcal{O}_p$.

Now if $C_1 \subset P_1$ and $C_2 \subset P_2$ are pre-Poisson submanifolds of Poisson manifolds, the cartesian product $C_1 \times C_2 \subset P_1 \times P_2$ also is, and if the construction of Thm. 3.3 gives cosymplectic submanifolds $\tilde{P}_1 \subset P_1$ and $\tilde{P}_2 \subset P_2$, the same construction applied to $C_1 \times C_2$ (upon suitable choices of complementary subbundles) delivers the cosymplectic submanifold $\tilde{P}_1 \times \tilde{P}_2$ of $P_1 \times P_2$. In particular, if C_1 is coisotropic and C_2 just a point p, then $C_1 \times \{p\}$ is pre-Poisson.

The *sufficient* condition above is not necessary (i.e., the first inclusion in the diagram above is strict), as either of the following simple examples shows.

Example 5.2. Take C to be the vertical line $\{x = y = 0\}$ in the Poisson manifold $(P, \Pi) = (\mathbb{R}^3, f(z)\partial_x \wedge \partial_y)$, where f is any function with at least one zero. Then C is a Poisson-Dirac submanifold (with zero as induced Poisson structure); hence taking $\tilde{P} := C$ we obtain a Poisson-Dirac submanifold in which C embeds coisotropically. The sufficient conditions here are not satisfied, for the rank of $TC + \sharp N^*C$ at $(0, 0, z)$ is 3 at points where f does not vanish and 1 at points where f vanishes.

Example 5.3. Consider the Poisson manifold $(P, \Pi) = (\mathbb{R}^4, x^2\partial_x \wedge \partial_y + z\partial_z \wedge \partial_w)$ as in Example 6 of [14] and the submanifold $C = \{(z^2, 0, z, 0) : z \in \mathbb{R}\}$. The rank of $TC + \sharp N^*C$ is 3 away from the origin (because there C is an isotropic submanifold in an open symplectic leaf of P) and 1 at the origin (since Π vanishes there). The submanifold $\tilde{P} = \{(z^2, 0, z, w) : z, w \in \mathbb{R}\}$ is Poisson-Dirac and it clearly contains C as a coisotropic submanifold.

The *necessary* condition above is not sufficient (i.e., the second inclusion in the diagram above is strict):

Example 5.4. In Example 3 in Section 8.2 of [14] the authors consider the manifold $P = \mathbb{C}^3$ with complex coordinates x, y, z. They specify a Poisson structure on it by declaring the symplectic leaves to be the complex lines given by $dy = 0, dz - ydx = 0$, the symplectic forms being the restrictions of the canonical symplectic form on $\mathbb{C}^3$. They consider as submanifold C the complex plane $\{z = 0\}$ and show that C is point-wise Poisson-Dirac (i.e., $TC \cap \sharp N^*C = \{0\}$ at every point), but that the induced bivector field is not smooth. Being point-wise Poisson-Dirac, C satisfies the necessary condition above. However there exists no Poisson-Dirac submanifold $\tilde{P}$ of P in which C embeds coisotropically. Indeed at points p of C where $y \neq 0$ we have $T_pC \oplus T_p\mathcal{O} = TP$ (where as usual $\mathcal{O}$ is a symplectic leaf of P through p), from which it follows that $\sharp|_{N_p^*C}$ is injective and $T_pC \oplus \sharp N_p^*C = TP$. From Lemma 3.1 (notice that the subspace R there must have trivial intersection with $T_pC \oplus \sharp N_p^*C$, so R must be the zero subbundle over C) it follows that the only candidate for $\tilde{P}$ is C itself. However, as we have seen, the Poisson bivector induced on C is not smooth. (More generally, examples are provided by any submanifold C of a Poisson manifold P which is point-wise Poisson-Dirac but not Poisson-Dirac and for which there exists a point p at which $T_pC \oplus T_p\mathcal{O} = TP$.)

Notice that this provides an example for the claim made in Remark 3.4, because the zero subbundle R over C satisfies equation (3) at every point of C and is obviously a smooth subbundle.

We end with two examples of submanifolds C which do not satisfy the necessary condition above. In particular they cannot be imbedded coisotropically in any Poisson-Dirac submanifold.

Example 5.5. The submanifold $C = \{(x_1, x_2, x_2^2, x_1^2)\}$ of the symplectic manifold $(P, \omega) = (\mathbb{R}^4, dx_1 \wedge dx_3 + dx_2 \wedge dx_4)$ has characteristic distribution of rank 2 on the points with $x_1 = x_2$ and rank zero on the rest of C. The rank of the characteristic distribution locally decreases; hence C does not satisfies the necessary condition above.

Remark 5.6. If C is a submanifold of a symplectic manifold (P, ω), then the necessary and the sufficient conditions coincide, both being equivalent to saying that the characteristic distribution of C (which can be described as $\ker(i_C^*\omega)$ for i_C the inclusion) have constant rank, i.e. that C be presymplectic.

Example 5.7. Consider the Poisson manifold $(\mathbb{R}^6, x_1\partial_{x_2} \wedge \partial_{x_4} + (\partial_{x_3} + x_1\partial_{x_5}) \wedge \partial_{x_6})$. Let C be the three-dimensional subspace given by setting $x_4 = x_5 = x_6 = 0$. The characteristic subspaces are all one-dimensional, spanned by ∂_{x_3} at points of C where $x_1 = 0$ and by ∂_{x_2} on the rest of C. Hence the characteristic subspaces do not form a smooth distribution and cannot be the image of the anchor map of any Lie algebroid over C. Therefore C does not satisfies the necessary condition above.

6. Reduction of submanifolds and deformation quantization of pre-Poisson submanifolds

In this section we consider the set of basic functions on a submanifold of a Poisson manifold, and show that in certain cases it is a Poisson algebra and that it can be deformation quantized.

Given any submanifold C of a Poisson manifold (P, Π), it is natural to consider the characteristic "distribution" $\sharp N^*C \cap TC$ (which by Remark 2.1 consists of the kernels of the restrictions to C of the symplectic forms on the symplectic leaves of P) and the set of basic functions on C,

$$C^\infty_{bas}(C) = \{f \in C^\infty(C) : df|_{\sharp N^*C \cap TC} = 0\}.$$

$\sharp N^*C \cap TC$ usually does not have a constant rank and may not be smooth; if it is and the quotient $\underline{C}$ is a smooth manifold, then $C^\infty_{bas}(C)$ consists exactly of pullbacks of functions on $\underline{C}$.

Let us endow C with the (possibly non-smooth) point-wise Dirac structure i^*L_P, where $i\colon C \to P$ is the inclusion and L_P is the Dirac structure corresponding to Π. Then, since $\sharp N^*C \cap TC = i^*L_P \cap TC$, $C^\infty_{bas}(C)$ is exactly the set of basic functions of (C, i^*L_P) in the sense of Dirac geometry. Given basic functions f, g the expression

$$\{f, g\}_C(p) := Y(g)$$

is well defined. Here Y is any element of T_pC such that $(Y, df_p) \in i^*L_P$, and it exists because the annihilator of $i^*L_P \cap TC$ is the projection onto T^*C of i^*L_P. Notice that $C^\infty_{bas}(C)$ and $\{\bullet, \bullet\}_C$ are *intrinsic* to C in the following sense: they depend only on the point-wise Dirac structure i^*L_P on C, and if $\bar{P}$ is a submanifold of (P, Π) containing C, $L_{\bar{P}}$ the point-wise Dirac structure on $\bar{P}$ induced by P and $\bar{i} : C \to \bar{P}$ the inclusion, then $\bar{i}^*L_{\bar{P}} = i^*L_P$ by the functoriality of pullback.

The expression $\{f, g\}_C(p)$ does not usually vary smoothly with p, so we can *not* conclude that $C^\infty_{bas}(C)$ with $\{\bullet, \bullet\}_C$ is a Poisson algebra. There is however a Poisson algebra that C inherits from P [5], namely $\mathcal{F}/(\mathcal{F}\cap\mathcal{I})$, where $\mathcal{I}$ denotes the set of functions on P that vanish on C and $\mathcal{F} := \{\hat{f} \in C^\infty(P) : \{\hat{f},\mathcal{I}\} \subset \mathcal{I}\}$ (the so-called first class functions). $\mathcal{F}/(\mathcal{F}\cap\mathcal{I})$ is exactly the subset of functions f on C which admits an extension to some function $\hat{f}$ on P whose differential annihilates $\sharp N^*C$ (or equivalently $X_{\hat{f}}|_C \subset TC$). The bracket of $\mathcal{F}/(\mathcal{F}\cap\mathcal{I})$ is computed as follows:

$$\{f,g\} = \{\hat{f},\hat{g}\}_P|_C = X_{\hat{f}}(g)|_C$$

for extensions as above. Notice that $\mathcal{F}/(\mathcal{F}\cap\mathcal{I}) \subset C^\infty_{bas}(C)$, and that the Poisson bracket $\{\bullet,\bullet\}$ on $\mathcal{F}/(\mathcal{F}\cap\mathcal{I})$ coincides with $\{\bullet,\bullet\}_C$: if f,g belong to $\mathcal{F}/(\mathcal{F}\cap\mathcal{I})$ we can compute $\{f,g\}_C$ by choosing $Y = X_{\hat{f}}$ for some extension $\hat{f} \in \mathcal{F}$.

Proposition 6.1. *Let C be any submanifold of a Poisson manifold (P,Π). If there exists a Poisson-Dirac submanifold $\tilde{P}$ of P in which C is contained coisotropically, then the set of basic functions on C has an intrinsic Poisson algebra structure, and $(\mathcal{F}/(\mathcal{F}\cap\mathcal{I}),\{\bullet,\bullet\})$ is a Poisson subalgebra.*

Proof. We add a tilde in the notation introduced above when we view C as a submanifold of the Poisson manifold $\tilde{P}$ instead of P. By the last paragraph before the statement of this proposition, since $\tilde{\sharp}N^*C \subset TC$, it follows that $\tilde{\mathcal{F}}/\tilde{\mathcal{I}} = C^\infty_{bas}(C)$. So $(C^\infty_{bas}(C),\{\bullet,\bullet\}_C)$ is a Poisson algebra structure intrinsically associated to C, and it contains $\mathcal{F}/(\mathcal{F}\cap\mathcal{I})$ as a Poisson subalgebra. ☐

By Thm. 3.3, pre-Poisson submanifolds C satisfy the assumption of Prop. 6.1; hence they admit a Poisson algebra structure on their space of basic functions. This fact was already established in Theorem 3 of [5], where furthermore it is shown that $\mathcal{F}/(\mathcal{F}\cap\mathcal{I})$ is the whole space of basic functions. Now we state our result about deformation quantization:

Theorem 6.2. *Let C be a pre-Poisson submanifold, and assume that the first and second Lie algebroid cohomology of $N^*C\cap\sharp^{-1}TC$ vanish. Then the Poisson algebra $C^\infty_{bas}(C)$, endowed with the bracket inherited from P, admits a deformation quantization.*

Proof. By Thm. 3.3 we can embed C coisotropically in some cosymplectic submanifold $\tilde{P}$. We invoke Corollary 3.3 of [10]: if the first and second Lie algebroid cohomology of the conormal bundle of a coisotropic submanifold vanish, then the Poisson algebra of basic functions on the coisotropic submanifold (with the bracket inherited from the ambient Poisson manifold, which in our case is $\tilde{P}$) admits a deformation quantization. Now by Prop. 6.1 the Poisson bracket on $C^\infty_{bas}(C)$ induced by P agrees with the one induced by the embedding in $\tilde{P}$. Further the conditions in Corollary 3.3 of [10] translate into the conditions stated in the proposition because the conormal bundle of C in $\tilde{P}$ is isomorphic to $N^*C\cap\sharp^{-1}TC$ as a Lie algebroid; see Prop. 3.6. ☐

7. Subgroupoids associated to pre-Poisson submanifolds

Let C be a pre-Poisson submanifold of a Poisson manifold (P,Π). In Prop. 3.6 we showed that $N^*C\cap\sharp^{-1}TC$ is a Lie subalgebroid of T^*P. When $\sharp N^*C$ has constant rank there is another Lie subalgebroid associated to C; it is obtained by taking the pre-image of TC under the anchor map; i.e., it is $\sharp^{-1}TC = (\sharp N^*C)^\circ$. Now we assume that T^*P is an integrable Lie algebroid, i.e. that the source simply connected (s.s.c.) symplectic groupoid $(\Gamma_s(P), \Omega)$ of (P, Π) exists. In this section we study the (in general only immersed) subgroupoids of $\Gamma_s(P)$ integrating $N^*C \cap \sharp^{-1}TC$ and $\sharp^{-1}TC$. Here, for any Lie subalgebroid A of T^*P integrating to an s.s.c. Lie groupoid G, we take "subgroupoid" to mean the (usually just immersed) image of the (usually not injective) morphism $G \to \Gamma_s(P)$ induced from the inclusion $A \to T^*P$.

By Thm. 3.3 we can find a cosymplectic submanifold $\tilde{P}$ in which C lies coisotropically. We first make a few remarks on the subgroupoid corresponding to $\tilde{P}$.

Lemma 7.1. *The subgroupoid of $\Gamma_s(P)$ integrating $\sharp^{-1}T\tilde{P}$ is $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$ and is a symplectic subgroupoid. Its source (target) map is a Poisson (anti-Poisson) map onto $\tilde{P}$, where the latter is endowed with the Poisson structure induced by (P, Π).*

Proof. According to Thm. 3.7 of [22] the subgroupoid[2] of $\Gamma_s(P)$ corresponding to $\tilde{P}$, i.e. the one integrating $(\sharp N^*\tilde{P})^\circ$, is a symplectic subgroupoid of $\Gamma_s(P)$. It is given by $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$, because $(\sharp N^*\tilde{P})^\circ = \sharp^{-1}T\tilde{P}$.

To show that the maps $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P}) \to \tilde{P}$ given by the source and target maps of $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$ are Poisson (anti-Poisson) maps proceed as follows. Take a function $\tilde{f}$ on $\tilde{P}$, and extend it to a function f on P so that df annihilates $\sharp N^*\tilde{P}$, i.e. so that X_f is tangent to $\tilde{P}$ along $\tilde{P}$. Since $s : \Gamma_s(P) \to P$ is a Poisson map and s-fibers are symplectic orthogonal to t-fibers we know that the vector field X_{s^*f} on $\Gamma_s(P)$ is tangent to $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$, i.e. that $d(\mathbf{s}^*f)$ annihilates $T(\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P}))^\Omega$. Hence, denoting by $\tilde{s}$ the source map of $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$, we have

$$\tilde{s}^*\{\tilde{f}_1, \tilde{f}_2\} = \tilde{s}^*(\{f_1, f_2\}|_{\tilde{P}}) = \{s^*f_1, s^*f_2\}|_{\mathbf{s}^{-1}(\tilde{P})\cap\mathbf{t}^{-1}(\tilde{P})} = \{\tilde{s}^*f_1, \tilde{s}^*f_1\};$$

i.e. $\tilde{s}$ is a Poisson map. A similar reasoning holds for $\tilde{t}$. □

Now we describe the subgroupoid integrating $N^*C \cap \sharp^{-1}TC$:

Proposition 7.2. *Let C be a pre-Poisson submanifold of (P, Π). Then the subgroupoid of $\Gamma_s(P)$ integrating $N^*C \cap \sharp^{-1}TC$ is an isotropic subgroupoid of $\Gamma_s(P)$.*

Proof. The canonical vector bundle isomorphism $i : T^*\tilde{P} \cong (\sharp N^*\tilde{P})^\circ$ is a Lie algebroid isomorphism, where $T^*\tilde{P}$ is endowed with the cotangent algebroid structure coming from the Poisson structure on $\tilde{P}$ (Cor. 2.11 and Thm. 2.3 of [22]). Integrating this algebroid isomorphism we obtain a Lie groupoid morphism from $\Gamma_s(\tilde{P})$, the s.s.c. Lie groupoid integrating $T^*\tilde{P}$, to $\Gamma_s(P)$, and the image of this morphism is $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$. Since by Lemma 7.1 the symplectic form on $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$ is multiplicative, symplectic and the source map is a Poisson map, pulling back the symplectic form on $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$ endows $\Gamma_s(\tilde{P})$ with the structure of the s.s.c. symplectic groupoid of $\tilde{P}$. The subgroupoid of $\Gamma_s(\tilde{P})$ integrating $N^*_{\tilde{P}}C$, the annihilator of C in $\tilde{P}$, is Lagrangian ([7], Prop. 5.5). Hence $i(N^*_{\tilde{P}}C)$, which by Prop. 3.6 is equal to $N^*C \cap \sharp^{-1}TC$, integrates to a Lagrangian subgroupoid of $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$, which therefore is an isotropic subgroupoid of $\Gamma_s(P)$. □

---

[2]In [22] this is claimed only when the subgroupoid integrating $(\sharp N^*\tilde{P})^\circ$ is an embedded subgroupoid; however the proof there is valid for immersed subgroupoids too.

Now we consider $\sharp^{-1}TC$. For any submanifold N, $\sharp^{-1}TN$ has constant rank iff it is a Lie subalgebroid of T^*P, integrating to the subgroupoid $\mathbf{s}^{-1}(N) \cap \mathbf{t}^{-1}(N)$ of $\Gamma_s(P)$. So the constant rank condition on $\sharp^{-1}TN$ corresponds to a smoothness condition on $\mathbf{s}^{-1}(N) \cap \mathbf{t}^{-1}(N)$.

Remark 7.3. 1) If $\sharp^{-1}TN$ has constant rank it follows that the Poisson structure on P pulls back to a smooth Dirac structure on N, and that $\mathbf{s}^{-1}(N) \cap \mathbf{t}^{-1}(N)$ is an over-pre-symplectic groupoid inducing the same Dirac structure on N (Ex. 6.7 of [3]). Recall from Def. 4.6 of [3] that an over-pre-symplectic groupoid is a Lie groupoid G over a base M equipped with a closed multiplicative 2-form ω such that $\ker \omega_x \cap \ker(d\mathbf{s})_x \cap \ker(d\mathbf{t})_x$ has rank $\dim G - 2\dim M$ at all $x \in M$. Further, $\mathbf{s}^{-1}(N) \cap \mathbf{t}^{-1}(N)$ has dimension equal to $2\dim N + rk(N^*N \cap N^*\mathcal{O})$, where $\mathcal{O}$ denotes any symplectic leaf of P intersecting N.

2) For a pre-Poisson submanifold C, the condition that $\sharp^{-1}TC$ have constant rank is equivalent to the characteristic distribution $TC \cap \sharp N^*C$ having constant rank. This follows trivially from $rk(\sharp N^*C + TC) = rk(\sharp N^*C) + \dim C - rk(TC \cap \sharp N^*C)$.

Proposition 7.4. *Let C be a pre-Poisson submanifold with constant-rank characteristic distribution. Then for any cosymplectic submanifold $\tilde{P}$ in which C embeds coisotropically,* $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$ *is a coisotropic subgroupoid of* $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$.

Proof. By the comments above we know that $\sharp^{-1}TC$ is a Lie subalgebroid; hence $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$ is a (smooth) subgroupoid of $\Gamma_s(P)$, and it is clearly contained in $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$. We saw in Lemma 7.1 that $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$ is endowed with a symplectic form for which its source and target maps are (anti-)Poisson maps onto $\tilde{P}$. Further its source and target fibers are symplectic orthogonals of each other. Since $C \subset \tilde{P}$ is coisotropic, this implies that $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$ is coisotropic in $\mathbf{s}^{-1}(\tilde{P}) \cap \mathbf{t}^{-1}(\tilde{P})$. □

We now describe the subgroupoids corresponding to pre-Poisson manifolds.

Proposition 7.5. *Let C be any submanifold of P. Then* $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$ *is a (immersed) presymplectic submanifold of $\Gamma_s(P)$ iff C is pre-Poisson and its characteristic distribution has constant rank. In this case the characteristic distribution of* $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$ *has rank $2rk(\sharp N^*C \cap TC) + rk(N^*C \cap N^*\mathcal{O})$, where $\mathcal{O}$ denotes the symplectic leaves of P intersecting C.*

Proof. Assume that $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$ is a (immersed) presymplectic submanifold of $\Gamma_s(P)$. We apply the same proof as in Prop. 8 of [14]: there is an isomorphism of vector bundles $T\Gamma_s(P)|_P \cong TP \oplus T^*P$, under which the non-degenerate bilinear form $\Omega|_P$ corresponds to $(X_1 \oplus \xi_1, X_2 \oplus \xi_2) := \langle \xi_1, X_2 \rangle - \langle \xi_2, X_1 \rangle + \Pi(\xi_1, \xi_2)$. Under the above isomorphism $T(\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C))$ corresponds to $TC \oplus \sharp^{-1}TC$, and a short computation shows that the restriction of $(\bullet, \bullet)$ to $TC \oplus \sharp^{-1}TC$ has kernel $(TC \cap \sharp N^*C) \oplus (\sharp^{-1}TC \cap N^*C)$, which therefore has constant rank. From the smoothness of $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$ it follows that $(\sharp N^*C)^\circ = \sharp^{-1}TC$ has constant rank. This has two consequences: first by Remark 7.3, C has characteristic distribution of constant rank. Second, the above kernel is a direct sum of two intersections of smooth subbundles, so $\sharp^{-1}TC \cap N^*C$ has constant rank, i.e. (taking annihilators) C is pre-Poisson.

The other direction follows from Prop. 7.4. □

Remark 7.6. One can wonder whether any subgroupoid of a symplectic groupoid $(\Gamma_s(P), \Omega)$ which is a presymplectic submanifold (i.e. Ω pulls back to a constant rank 2-form) is contained coisotropically in some symplectic subgroupoid of $\Gamma_s(P)$. This would be exactly the "groupoid" version of Thm. 3.3. The above Prop. 7.4 and Prop. 7.5 together tell us that this is the case when the subgroupoid has the form $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$, where $C \subset P$ is its base. In general the answer to the above question is negative, as the following counterexample shows.

Let (P, ω) be some simply connected symplectic manifold, so that $\Gamma_s(P) = (P \times P, \omega_1 - \omega_2)$ and the units are embedded diagonally. Take C to be any 1-dimensional closed submanifold of P. $C \rightrightarrows C$ is clearly a subgroupoid and a presymplectic submanifold; since $\omega_1 - \omega_2$ there pulls back to zero, any subgroupoid G of $P \times P$ in which $C \rightrightarrows C$ embeds coisotropically must have dimension 2. If the base of G has dimension 2, then G is contained in the identity section of $P \times P$, which is Lagrangian. So let us assume that the base of G is C. Then G must be contained in $C \times C$, on which $\omega_1 - \omega_2$ vanishes because $C \subset P$ is isotropic. So we conclude that there is no symplectic subgroupoid of $P \times P$ containing $C \rightrightarrows C$ as a coisotropic submanifold.

8. Existence of coisotropic embeddings of Dirac manifolds in Poisson manifolds

Let (M, L) be a smooth Dirac manifold. We ask when (M, L) can be embedded coisotropically in some Poisson manifold (P, Π), i.e. when there exists an embedding i such that $i^*L_P = L$ and $i(M)$ is a coisotropic submanifold of P. Notice that for arbitrary coisotropic embeddings i^*L_P is usually not even continuous: for example the x-axis in $(\mathbb{R}^2, x\partial_x \wedge \partial_y)$ is coisotropic, but the pullback structure is not continuous at the origin.

When M consists of exactly one leaf, i.e. when M is a manifold endowed with a closed 2-form ω, the existence and uniqueness of coisotropic embeddings in symplectic manifolds was considered by Gotay in the short paper [15]: the coisotropic embedding exists iff $\ker \omega$ has constant rank, and in that case one has uniqueness up to neighborhood equivalence. Our strategy will be to check if we can apply Gotay's arguments "leaf by leaf" *smoothly* over M. Recall that $L \cap TM$ is the kernel of the 2-forms on the presymplectic leaves of (M, L).

Theorem 8.1. *(M, L) can be embedded coisotropically in a Poisson manifold iff $L \cap TM$ has constant rank.*

Proof. Suppose that an embedding $M \to P$ as above exists. Then $L \cap TM$ is equal to $\sharp N^*C$ (where N^*C is the conormal bundle of C in P), the image of a vector bundle under a smooth bundle map; hence its rank can locally only increase. On the other hand the rank of $L \cap TM$, which is the intersection of two smooth bundles, can locally only decrease. Hence the rank of $L \cap TM$ must be constant on M.

Conversely, assume that the rank of $E := L \cap TM$ is constant and define P to be the total space of the vector bundle $\pi \colon E^* \to M$. We define the Poisson structure on P as follows. First take the pullback Dirac structure π^*L (which is smooth and integrable since π is a submersion). Then choose a smooth distribution V such that $E \oplus V = TM$. This choice gives an embedding $i \colon E^* \to T^*M$, which we can use to pull back the canonical symplectic form ω_{T^*M}. Our Poisson structure is $L_{E^*} := \tau_{i^*\omega_{T^*M}} \pi^*L$; i.e., it is obtained by applying to π^*L the gauge transformation[3] by the closed 2-form $i^*\omega_{T^*M}$. It is clear that L_{E^*} is a smooth Dirac structure; we still have to show that it is actually Poisson, and that the zero section is coisotropic. In more concrete terms (E^*, L_{E^*}) can be described as follows: the leaves are all of the form $\pi^{-1}(F_\alpha)$ for $(F_\alpha, \omega_\alpha)$ a presymplectic leaf of M. The 2-form on the leaf is given by adding to $(\pi|_{\pi^{-1}(F_\alpha)})^*\omega_\alpha$ the 2-form $i_\alpha^*\omega_{T^*F_\alpha}$. The latter is defined by considering the distribution $V \cap TF_\alpha$ transverse to $E|_{F_\alpha}$ in TF_α, the induced embedding $i_\alpha\colon \pi^{-1}(F_\alpha) = E^*|_{F_\alpha} \to T^*F_\alpha$, and pulling back the canonical symplectic form. (One can check that $i_\alpha^*\omega_{T^*F_\alpha}$ is the pullback of $i^*\omega_{T^*M}$ via the inclusion of the leaf in E^*.) But this is exactly Gotay's recipe to endow (an open subset of) $\pi^{-1}(F_\alpha)$ with a symplectic form so that F_α is embedded as a coisotropic submanifold. Hence we conclude that a neighborhood of the zero section of E^*, with the above Dirac structure, is actually a Poisson manifold and that M is embedded as a coisotropic submanifold. □

We comment on how choices affect the construction of Thm. 8.1. We need the following version of Moser's theorem for Poisson structures (see Section 3.3. of [1]): suppose we are given Poisson structures Π_t on some manifold P, $t \in [0,1]$. Assume that each Π_t is related to Π_0 via the gauge transformation by some closed 2-form B_t, i.e. $\Pi_t = \tau_{B_t}\Pi_0$. This means that the symplectic foliations agree and on each symplectic leaf $\mathcal{O}$ we have $\Omega_t = \Omega_0 + i_{\mathcal{O}}^* B_t$, where Ω_0, Ω_t are the symplectic forms on the leaf $\mathcal{O}$ and $i_{\mathcal{O}}$ the inclusion. Assume further that each $\frac{d}{dt}B_t$ be exact, and let α_t be a smooth family of primitives vanishing on some submanifold M. Then the time-1 flow of the Moser vector field $\sharp_t\alpha_t$ is defined in a tubular neighborhood of M, it fixes M and maps Π_0 to Π_1. Here $\sharp_t$ denotes the map $T^*P \to TP$ induced by Π_t.

Proposition 8.2. *Different choices of splitting V in the construction of Thm.* 8.1 *yield isomorphic Poisson structures on E^*. Hence, given a Dirac manifold (M, L) for which $L \cap TM$ has constant rank, there is a canonical (up to neighborhood equivalence) Poisson manifold in which M embeds coisotropically.*

Proof. Let V_0, V_1 be two different splittings as in Thm. 8.1, i.e. $E \oplus V_i = TM$ for $i = 0, 1$. We can interpolate between them by defining the graphs $V_t := \{v + tAv : v \in V_0\}$ for $t \in [0,1]$, where $A\colon V_0 \to E$ is determined by requiring that its graph be V_1. Obviously each V_t also gives a splitting $E \oplus V_t = TM$; denote by $i_t\colon E^* \to T^*M$ the corresponding embedding. We obtain Dirac structures $\tau_{i_t^*\omega_{T^*M}}\pi^*L$ on the total space of $\pi\colon E^* \to M$; by Thm. 8.1 they correspond to Poisson bivectors, which we denote by Π_t. These Poisson structures are related by a gauge transformation: $\Pi_t = \tau_{B_t}\Pi_0$ for $B_t := i_t^*\omega_{T^*M} - i_0^*\omega_{T^*M}$. A primitive of $\frac{d}{dt}B_t$ is given by $\frac{d}{dt}i_t^*\alpha_{T^*M}$; notice that this primitive vanishes at points of M, because the canonical 1-form α_{T^*M} on T^*M vanishes along the zero section. Hence the time-1 flow of $\sharp_t(\frac{d}{dt}i_t^*\alpha_{T^*M})$ fixes M and maps Π_0 to Π_1. □

Assuming that (M, L) is integrable we describe the symplectic groupoid of (E^*, L_{E^*}), the Poisson manifold constructed in Thm. 8.1 with a choice of distribution V. It is $\pi^*(\Gamma_s(M))$, the pullback via $\pi\colon E^* \to M$ of the presymplectic groupoid of M, endowed with the following symplectic form: the pullback via

[3] Given a Dirac structure L on a vector space W, the gauge transformation of L by a bilinear form $B \in \wedge^2 W^*$ is $\tau_B L := \{(X, \xi + i_X B) : (X, \xi) \in L\}$. Given a Dirac structure L on a manifold, the gauge transformation $\tau_B L$ by a closed 2-form B is again a Dirac structure (i.e. $\tau_B L$ is again closed under the Courant bracket).

$\pi^*(\Gamma_s(M)) \to \Gamma_s(M)$ of the presymplectic form on the groupoid $\Gamma_s(M)$, plus $\mathbf{s}^*(i^*\omega_{T^*M}) - \mathbf{t}^*(i^*\omega_{T^*M})$, where $i\colon E^* \to T^*M$ is the inclusion given by the choice of distribution V, ω_{T^*M} is the canonical symplectic form, and $\mathbf{s}, \mathbf{t}$ are the source and target maps of $\pi^*(\Gamma_s(M))$. This follows easily from Examples 6.3 and 6.6 in [3]. Notice that this groupoid is source simply connected when $\pi^*(\Gamma_s(M))$ is.

Now we can give an affirmative answer to the possibility raised in [14] (Remark (e) in Section 8.2), although we prove it "working backwards"; this is the "groupoid" version of Gotay's embedding theorem. Recall that a presymplectic groupoid is a Lie groupoid G over M with $\dim(G) = 2\dim(M)$ equipped with a closed multiplicative 2-form ω such that $\ker \omega_x \cap \ker(d\mathbf{s})_x \cap \ker(d\mathbf{t})_x = 0$ at all $x \in M$ (Def. 2.1 of [3]).

Proposition 8.3. *Any presymplectic groupoid with constant rank characteristic distribution can be embedded coisotropically as a Lie subgroupoid in a symplectic groupoid.*

Proof. By Cor. 4.8 iv), v) of [3], a presymplectic groupoid $\Gamma_s(M)$ has characteristic distribution (the kernel of the multiplicative 2-form) of constant rank iff the Dirac structure L induced on its base M does. We can embed (M, L) coisotropically in the Poisson manifold (E^*, L_{E^*}) constructed in Thm. 8.1; we just showed that $\pi^*(\Gamma_s(M))$ is a symplectic groupoid for E^*. $\Gamma_s(M)$ embeds in $\pi^*(\Gamma_s(M))$ as $\mathbf{s}^{-1}(M) \cap \mathbf{t}^{-1}(M)$, and this embedding preserves both the groupoid structures and the 2-forms. $\mathbf{s}^{-1}(M) \cap \mathbf{t}^{-1}(M)$ is a coisotropic subgroupoid of $\pi^*(\Gamma_s(M))$ because M lies coisotropically in E^* and $\mathbf{s}, \mathbf{t}$ are (anti)Poisson maps. $\square$

Remark 8.4. A partial converse to this proposition is given as follows: if $\mathbf{s}^{-1}(M) \cap \mathbf{t}^{-1}(M)$ is a coisotropic subgroupoid of a symplectic groupoid $\Gamma_s(P)$, then M is a coisotropic submanifold of the Poisson manifold P, it has a smooth Dirac structure (induced from P) with characteristic distribution of constant rank, and $\mathbf{s}^{-1}(M) \cap \mathbf{t}^{-1}(M)$ is an *over*-pre-symplectic groupoid over M inducing the same Dirac structure. This follows from our arguments in section 7.

Now we draw the conclusions about deformation quantization. Recall that for any Dirac manifold (M, L) the set of admissible functions

$$C^\infty_{adm}(M) = \{f \in C^\infty(M)\colon \text{ there exists a smooth vector field } X_f \text{ s.t. } (X_f, df) \subset L\} \tag{9}$$

is naturally a Poisson algebra [13], with bracket $\{f, g\}_M = X_f(g)$.

Theorem 8.5. *Let (M, L) be a Dirac manifold such that $L \cap TM$ has constant rank, and denote by $\mathcal{F}$ the regular foliation integrating $L \cap TM$. If the first and second foliated de Rham cohomologies of the foliation $\mathcal{F}$ vanish, then the Poisson algebra of admissible functions on (M, L) admits a deformation quantization.*

Proof. By Thm. 8.1 we can embed (M, L) coisotropically in a Poisson manifold P; hence we can apply Corollary 3.3 of [10]: if the first and second Lie algebroid cohomology of the conormal bundle of a coisotropic submanifold vanish, then the Poisson algebra of basic functions on the coisotropic submanifold admits a deformation quantization. Since $L \cap TM$ has constant rank, the image of L under $TM \oplus T^*M \to T^*M$ has constant rank, so the inclusion $C^\infty_{adm}(M) \subset C^\infty_{bas}(M)$ is an equality. Further the Poisson algebra structure $\{\bullet, \bullet\}_M$ on $C^\infty_{bas}(M)$ coming from (M, L) coincides with the one induced by M as a coisotropic submanifold of P, as follows from Prop. 6.1 and $i^*L_P = L$. So when the assumptions are satisfied we really deformation quantize $(C^\infty_{adm}(M), \{\bullet,\bullet\}_M)$.

Notice that in Thm. 8.1 we constructed a Poisson manifold P of minimal dimension, i.e. of dimension $\dim M + rk(L \cap TM)$. The anchor map $\sharp$ of the Lie algebroid N^*C is injective; hence the Lie algebroids N^*C and $L \cap TM$ are isomorphic. This allows us to state the assumptions of Corollary 3.3 of [10] in terms of the foliation $\mathcal{F}$ on M. $\square$

Proposition 8.6. *Let (M, L) be a Dirac manifold such that $L \cap TM$ has constant rank, and denote by $\mathcal{F}$ the regular foliation integrating $L \cap TM$. Then the foliated de Rham complex $\Omega^\bullet_{\mathcal{F}}(M)$ admits the structure of an L_∞-algebra*[4] *$\{\lambda_n\}_{n\geq 1}$, the differential λ_1 being the foliated de Rham differential and the bracket λ_2 inducing on $H^0_{\lambda_1} = C^\infty_{bas}(M)$ the natural bracket $\{\bullet,\bullet\}_M$. This L_∞-structure is canonical up to L_∞-isomorphism.*

Proof. By the proof of Thm. 8.1 we know that M can be embedded coisotropically in a Poisson manifold P so that the Lie algebroids N^*M and $L \cap TM$ are isomorphic. After choosing an embedding of $NM := TP|_M/TM$ in a tubular neighborhood of M in P, Thm. 2.2 of [10] gives the desired L_∞-structure. By Prop. 8.2 the Poisson manifold P is canonical up to neighborhood equivalence, so the L_∞-structure depends only on the choice of embedding of NM in P; the first author and Schätz showed in [11] that different embeddings give the same structure up to L_∞-isomorphism. $\square$

9. Uniqueness of coisotropic embeddings of Dirac manifolds

The coisotropic embedding of Gotay [15] is unique up to neighborhood equivalence; i.e., any two coisotropic embeddings of a fixed presymplectic manifold in symplectic manifolds are intertwined by a symplectomorphism which is the identity on the coisotropic submanifold. It is natural to ask whether, given a Dirac manifold (M, L) such that $L \cap TM$ has constant rank, the coisotropic embedding constructed in Thm. 8.1 is the only one up to neighborhood equivalence. In general the answer will be negative: for example the origin is a coisotropic submanifold in $\mathbb{R}^2$ endowed either with the zero Poisson structure or with the Poisson structure $(x^2+y^2)\partial_x \wedge \partial_y$, and the two Poisson structures are clearly not equivalent.

As Aissa Wade pointed out to us, it is necessary to require that the Poisson manifold in which we embed be of minimal dimension, i.e. of dimension $\dim M + rk(L \cap TM)$. Before presenting some partial results on the uniqueness problem we need a simple lemma.

Lemma 9.1. *Let M be a coisotropic subspace of a Poisson vector space (P, Π). Then* $\operatorname{codim}(M) = \dim(\sharp M^\circ)$ *iff $\sharp|_{M^\circ}$ is an injective map iff M intersects transversely $\mathcal{O} := \sharp P^*$.*

Proof. The first equivalence is obvious by dimension reasons. For the second one notice that $\sharp|_{M^\circ}$ is injective iff $M^\circ \cap \mathcal{O}^\circ = \{0\}$, which taking annihilators is exactly the transversality statement. $\square$

[4] The λ_n are derivations w.r.t. the wedge product, so one actually obtains what in [10] is called a P_∞ algebra.

9.1. **Infinitesimal uniqueness and global issues.** We apply the construction of Gotay's uniqueness proof [15] on each presymplectic leaf of the Dirac manifold M; then we show that under certain assumptions the resulting diffeomorphism varies smoothly from leaf to leaf.

We start by establishing infinitesimal uniqueness, for which we need a Poisson linear algebra lemma.

Lemma 9.2. *Let (P,Π) be a Poisson vector space and M a coisotropic subspace for which $\dim(\sharp M^\circ) = \mathrm{codim}(M)$. Let V be a complement to $E := \sharp M^\circ$ in M.*

There exists an isomorphism of Poisson vector spaces

$$P \cong V \oplus E \oplus E^*$$

fixing M, where the Poisson structure on the r.h.s. is such that the induced symplectic vector space is $((V \cap \mathcal{O}) \oplus (E \oplus E^*), \Omega|_{V\cap\mathcal{O}} \oplus \omega_E)$. Here $(\mathcal{O}, \Omega)$ is the symplectic subspace corresponding to (P,Π) and ω_E is the antisymmetric pairing on $E \oplus E^*$.

Proof. We claim first that $V \oplus \sharp V^\circ = P$: indeed $V \cap \mathcal{O}$ is a symplectic subspace of $(\mathcal{O}, \Omega)$, being transverse to $E = \ker(\Omega|_{\mathcal{O}\cap M})$. Hence $(V \cap \mathcal{O})^\Omega$, which by section 2 is equal to $\sharp V^\circ$, is a complement to $V \cap \mathcal{O}$ in $\mathcal{O}$, so $V \oplus \sharp V^\circ = V + \mathcal{O}$, which equals P by Lemma 9.1.

Now we mimic the construction of Gotay's uniqueness proof [15]: since E is Lagrangian in the symplectic subspace $\sharp V^\circ$, by choosing a complementary Lagrangian we can find a linear symplectomorphism $(\sharp V^\circ, \Omega|_{\sharp V^\circ}) \cong (E \oplus E^*, \omega_E)$, which is the identity on E. Adding to this Id_V we obtain an isomorphism

$$P = V \oplus \sharp V^\circ \cong V \oplus E \oplus E^*,$$

which preserves the Poisson bivectors because it restricts to an isomorphism $\mathcal{O} \cong (V \cap \mathcal{O}) \oplus (E \oplus E^*)$ which matches the symplectic forms Ω and $\Omega|_{V\cap\mathcal{O}} \oplus \omega_E$. $\square$

Proposition 9.3. *Suppose we are given a Dirac manifold (M, L) for which $L \cap TM$ has constant rank k, and let (P_1, Π_1) and (P_2, Π_2) be Poisson manifolds of dimension $\dim M + k$ in which (M, L) embeds coisotropically. Then there is an isomorphism of Poisson vector bundles $\Phi\colon TP_1|_M \to TP_2|_M$ which is the identity on TM.*

Proof. We choose a smooth distribution V on M complementary to $E := L \cap TM$. For $i = 1, 2$, at every point $x \in M$ we apply the construction of Lemma 9.2 to the coisotropic subspace T_xC of T_xP_i, obtaining smooth isomorphisms of Poisson vector bundles

$$TP_1|_M \cong V \oplus E \oplus E^* \cong TP_2|_M.$$

Notice that the middle Poisson vector bundle depends only on (M, L) and V, because for any symplectic leaf $(\mathcal{O}, \Omega)$ of P_1 or P_2 the bilinear form $\Omega|_{V_x\cap T_x\mathcal{O}}$ is determined by the presymplectic form on the presymplectic leaf $\mathcal{O}\cap M$ of (M, L). $\square$

Making a regularity assumption we can extend the infinitesimal uniqueness of Prop. 9.3 to a global statement.

Proposition 9.4. *Let M, P_1 and P_2 be as in Proposition 9.3, and assume additionally that the presymplectic leaves of (M, L) have constant dimension. Then P_1 and P_2 are neighborhood equivalent.*

Proof. The symplectic leaves of each P_i have constant dimension in a tubular neighborhood of P_i, because they are transverse to M by Lemma 9.1 and because of the assumption on the presymplectic leaves of (M, L). By choosing normal bundles $N_i \subset TP_i|_M$ tangent to the symplectic leaves of P_i we can find identifications ϕ_i between the normal bundles N_i and tubular neighborhoods of M in P_i which, for every presymplectic leaf F of M, identify $N_i|_F$ and the corresponding symplectic leaf of P_i.

Using the Poisson vector bundle isomorphism $\Phi\colon TP_1|_M \to TP_2|_M$ of Proposition 9.3 we obtain an identification $\phi_2 \circ \Phi \circ \phi_1^{-1}$ between tubular neighborhoods of M in P_1 and P_2. Using this identification, we can view Π_2 as a Poisson structure on $P := P_1$ with two properties: it induces exactly the same foliation as Π_1, and it coincides with Π_1 on $TP|_M$. We want to show that there is a diffeomorphism near M, fixing M, which maps Π_1 to Π_2.

To this aim we apply Moser's theorem on each symplectic leaf $\mathcal{O}$ of P (Thm. 7.1 of [6]), in a way that varies smoothly with $\mathcal{O}$. Denote by Ω_i the symplectic form given by Π_i on a leaf $\mathcal{O}$. The convex linear combination $(1-t)\Omega_1 + t\Omega_2$ is symplectic (because Ω_1 and Ω_2 coincide at points of M). Let $F := M \cap \mathcal{O}$, identify $N|_F$ with a neighborhood in $\mathcal{O}$ via ϕ_1, and consider the retraction $\rho_t\colon N|_F \to N|_F, v \mapsto tv$ where $t \in [0, 1]$. Let Q be the homotopy operator given by the retraction ρ_t (see Chapter 6 of [6]); it satisfies $dQ - Qd = \rho_1^* - \rho_0^*$. So $\mu := Q(\Omega_2 - \Omega_1)$ is a primitive for $\Omega_2 - \Omega_1$; furthermore Q can be chosen so that μ vanishes at points of F. Consider the Moser vector field, obtained by inverting via $(1-t)\Omega_1 + t\Omega_2$ the 1-form μ. Following from time 1 to time 0 the flow of the Moser vector field gives a diffeomorphism ψ of $\mathcal{O}$ fixing F such that $\psi^*\Omega_2 = \Omega_1$.

Notice that since the symplectic foliation of P is regular near M this construction varies smoothly from leaf to leaf. Hence we obtain a diffeomorphism ψ of a tubular neighborhood of M in P, fixing M, which maps Π_1 to Π_2. □

Since local uniqueness holds (see subsection 9.2) and since by Proposition 9.3 there is no topological obstruction, it seems that the global uniqueness statement of Prop. 9.4 should hold in full generality (i.e., without the assumption on the presymplectic foliation of (M, L)); however we are not able to prove this.

The argument from [1] just before our Prop. 8.2 shows that the uniqueness of (minimal dimensional) coisotropic embeddings of a given Dirac manifold (M, L) is equivalent to the following: whenever (P_1, Π_1) and (P_2, Π_2) are minimal Poisson manifolds in which (M, L) embeds coisotropically there exists a diffeomorphism $\phi\colon P_1 \to P_2$ near M so that Π_2 and $\phi_*\Pi_1$ differ by the gauge transformation by a closed 2-form B vanishing on M. One could hope that if $\phi\colon P_1 \to P_2$ is chosen to match symplectic leaves and to match $\Pi_1|_M$ and $\Pi_2|_M$, then a 2-form B as above automatically exists. This is not the case, as the following example shows.

Example 9.5. Take $M = \mathbb{R}^3$ with Dirac structure

$$L = span\{(-x_1^2\partial_{x_2}, dx_1), (x_1^2\partial_{x_1}, dx_2), (\partial_{x_3}, 0)\}.$$

There are two open presymplectic leaves $(\mathbb{R}_\pm \times \mathbb{R}^2, \frac{1}{x_1^2} dx_1 \wedge dx_2)$ and 1-dimensional presymplectic leaves $\{0\} \times \{c\} \times \mathbb{R}$ with zero presymplectic form (for every real number c); hence our Dirac structure is a product of the Poisson structure $x_1^2\partial_{x_1} \wedge \partial_{x_2}$ and of the zero presymplectic form on the x_3-axis. The characteristic distribution

$L \cap TM$ is always $span\partial_{x_3}$. Clearly the construction of Thm. 8.1 gives

$$P_1 := (\mathbb{R}^4, x_1^2\partial_{x_1} \wedge \partial_{x_2} + \partial_{x_3} \wedge \partial_{y_3}),$$

where y_3 is the coordinate on the fibers of $P_1 \to M$.

Another Poisson structure on $\mathbb{R}^4$ with the same foliation as Π_1 and which coincides with Π_1 along M is the following:

$$\Pi_2 := x_1^2\partial_{x_1} \wedge \partial_{x_2} + \partial_{x_3} \wedge \partial_{y_3} + x_1 y_3 \partial_{x_2} \wedge \partial_{x_3}.$$

On each of the two open symplectic leaves $\mathbb{R}_\pm \times \mathbb{R}^3$ the symplectic form corresponding to Π_1 is $\Omega_1 = \frac{1}{x_1^2} dx_1 \wedge dx_2 + dx_3 \wedge dy_3$, whereas the one corresponding to Π_2 is $\Omega_2 = \Omega_1 + \frac{y_3}{x_1} dx_1 \wedge dy_3$. Clearly the difference $\Omega_1 - \Omega_2$ does not extend to smooth a 2-form on the whole of $\mathbb{R}^4$. Hence there is no smooth 2-form on $\mathbb{R}^4$ relating Π_1 and Π_2.

Nevertheless Π_1 and Π_2 are Poisson diffeomorphic: an explicit Poisson diffeomorphism is given by the global coordinate change that transforms x_2 into $x_2 + \frac{y_3^2}{2} x_1$ and leaves the other coordinates untouched.

9.2. **Local uniqueness.** While we are not able to prove a global uniqueness statement in the general case, we prove in this subsection that local uniqueness holds. We start with a normal form statement.

Proposition 9.6. *Let M^m be a coisotopic submanifold of a Poisson manifold P such that $k := \mathrm{codim}(M)$ equals $rk(\sharp N^*M)$. Then about any $x \in M$ there is a neighborhood $U \subset P$ and coordinates $\{q_1, \ldots, q_k, p_1, \ldots, p_k, y_1, \ldots, y_{m-k}\}$ defined on U such that locally M is given by the constraints $p_1 = 0, \ldots, p_k = 0$ and*

$$\Pi = \sum_{I=1}^{k} \partial_{q_I} \wedge \partial_{p_I} + \sum_{i,j=1}^{m-k} \varphi_{ij}(y) \partial_{y_i} \wedge \partial_{y_j} \tag{10}$$

for functions $\varphi_{ij} \colon \mathbb{R}^{m-k} \to \mathbb{R}$.

Remark 9.7. The existence of coordinates in which Π has the above split form is guaranteed by Weinstein's Splitting Theorem [21]; the point in the above proposition is that one can choose the coordinates (q, p, y) so that M is given by the constraint $p = 0$.

Proof. We adapt the proof of Weinstein's Splitting Theorem [21] to our setting. To simplify the notation we will often write P in place of U and M in place of $M \cap U$. We proceed by induction over k; for $k = 0$ there is nothing to prove, so let $k > 0$.

Choose a function q_1 on P near x such that dq_1 does not annihilate $\sharp N^*M$. Then $X_{q_1}|_M$ is transverse to M, because there is a $\xi \in N^*M$ with $0 \neq \langle \sharp\xi, dq_1 \rangle = -\langle \xi, X_{q_1} \rangle$. Choose a hypersurface in P containing M and transverse to $X_{q_1}|_M$, and determine the function p_1 by requiring that it vanishes on the hypersurface and $dp_1(X_{q_1}) = -1$. Since $[X_{q_1}, X_{p_1}] = X_1 = 0$ the span of X_{p_1} and X_{q_1} is an integrable distribution giving rise to a foliation of P by surfaces. This foliation is transverse to P_1, which we define as the codimension two submanifold where p_1 and q_1 vanish. $M_1 := P_1 \cap M$ is a clean intersection and is a codimension one submanifold of M. To proceed inductively we need

Lemma 9.8. *P_1 has an induced Poisson structure Π_1, $M_1 \subset P_1$ is a coisotropic submanifold of codimension $k - 1$, and the sharp-map $\sharp_1$ of P_1 is injective on the conormal bundle to M_1.*

Proof. P_1 is cosymplectic because $\sharp N^*P_1$ is spanned by X_{q_1} and X_{p_1}, which are transverse to P_1. Hence it has an induced Poisson structure Π_1. Recall from section 2 that if $\xi_1 \in T^*_xP_1$, then $\sharp_1\xi_1 \in TP_1$ is given as follows: extend ξ_1 to a covector ξ of P by asking that it annihilate $\sharp N^*_xP_1$ and apply $\sharp$ to it. Now in particular let $x \in M_1$ and ξ_1 be an element of the conormal bundle of M_1 in P_1. We have $T_xM = T_xM_1 \oplus \mathbb{R}X_{p_1}(x) \subset T_xM_1 + \sharp N^*_xP_1$, so $\xi \in N^*_xM$, and since M is coisotropic in P we have $\sharp\xi \in T_xM$. Hence $\sharp_1\xi_1 \in T_xP_1 \cap T_xM = T_xM_1$, which shows the claimed coisotropicity. The injectivity of $\sharp_1$ on the conormal bundle follows by the above together with the injectivity of $\sharp|_{N^*M}$, which holds by Lemma 9.1. □

By the induction assumption there are coordinates on P_1 so that

$$\Pi_1 = \sum_{I=2}^{k} \partial_{q_I} \wedge \partial_{p_I} + \sum_{i,j=1}^{m-k} \varphi_{ij}(y)\partial_{y_i} \wedge \partial_{y_j}$$

and $M_1 \subset P_1$ is given by the constraints $p_2 = 0, \dots, p_k = 0$. We extend the coordinates on P_1 to the whole of P so that they are constant along the surfaces tangent to $span\{X_{q_1}, X_{p_1}\}$. We denote collectively by x_α the resulting functions on P, which together with q_1 and p_1 form a coordinate system on P. We have $\{x_\alpha, q_1\} = 0$ and $\{x_\alpha, p_1\} = 0$, and using the Jacobi identity one sees that $\{x_\alpha, x_\beta\}$ Poisson commutes with q_1 and p_1, and hence it is a function of the x_α's only. Further $\{x_\alpha, x_\beta\}|_{P_1} = \{x_\alpha|_{P_1}, x_\beta|_{P_1}\}_1$ since x_α, x_β annihilate $\sharp N^*P_1$. Hence formula (10) for the Poisson bivector Π follows.

To show that M is given by the constraints $p_1 = \dots = p_k = 0$ we notice the following. p_1 was chosen to vanish on M. The functions $p_2, \dots, p_k$ on P_1 were chosen to vanish on M_1, and since $TM|_{M_1} = TM_1 \oplus \mathbb{R}X_{p_1}|_{M_1}$ it follows that their extensions vanish on the whole of M. This concludes the proof of Prop. 9.6. □

Using the normal forms derived above we can prove local uniqueness:

Proposition 9.9. *Suppose we are given a Dirac manifold (M, L) for which $L \cap TM$ has constant rank k, and let (P, Π) be a Poisson manifold of dimension $\dim M + k$ in which (M, L) embeds coisotropically. Then about each $x \in M$ there is a neighborhood $U \subset P$ which is Poisson diffeomorphic to an open set in the canonical Poisson manifold associated to (M, L) in Prop. 8.2.*

Proof. By Prop. 9.6 there are coordinates $\{q_I, p_I, y_i\}$ on U such that locally M is given by $p_I = 0$ and

$$\Pi = \sum_{I=1}^{k} \partial_{q_I} \wedge \partial_{p_I} + \sum_{i,j=1}^{m-k} \varphi_{ij}(y)\partial_{y_i} \wedge \partial_{y_j}.$$

We want to apply the construction of Thm. 8.1 to (M, L). To do so we need to make a choice of complement to $E := L \cap TM = span\{\partial_{q_I}|_M\}$; our choice is $V := span\{\partial_{y_i}|_M\}$. Since by assumption L is the pullback of $graph(\Pi)$ to M, L is spanned by sections $(\partial_{q_I}|_M \oplus 0)$ and $(\sum_j \varphi_{ij}(y)\partial_{y_j}|_M \oplus dy_i|_M)$. Hence the pullback of L to the total space of the vector bundle $\pi : E^* \to M$ is spanned by $(\partial_{q_I} \oplus 0)$, $(\partial_{p_I} \oplus 0)$, and $(\sum_j \varphi_{ij}(y)\partial_{y_j} \oplus dy_i)$. Next we consider the embedding $E^* \to T^*M$ induced by the splitting $TM = E \oplus V$ and pull back the canonical 2-form on T^*M. In the coordinates (q_I, y_i) on M the pullback 2-form is simply $\sum_{I=1}^k dp_I \wedge dq_I$ (see eq. (6.7) in [18]), where with p_I we denote the linear coordinates on the fibers of E^* dual to the q_I. Transforming π^*L by this 2-form gives exactly $graph(\Pi)$. Hence we conclude that, nearby $x \in M$, the Poisson manifold (P, Π) is obtained by the construction of Thm. 8.1 (with the above choice of distribution V). $\square$

Acknowledgements

When the second author was a graduate student, Marius Crainic first called to his attention some of the questions discussed in section 8, and some of the existence results obtained in the same section arose from discussions between the second author and Alan Weinstein, who at the time was his thesis advisor and whom he gratefully thanks. He also would like to thank Aissa Wade for pointing out the necessity of a minimal dimension condition mentioned in section 9 and Eva Miranda for showing him the reference [1]. We also thank Rui Loja Fernandes for comments and the referees for suggesting valuable improvements to a previous version of this manuscript. The first author acknowledges partial support of SNF Grant No. 20-113439. This work has been partially supported by the European Union through the FP6 Marie Curie RTN ENIGMA (Contract number MRTN-CT-2004-5652) and by the European Science Foundation through the MISGAM program. The second author acknowledges support from the Forschungskredit 2006 of the University of Zürich.

References

1. A. Alekseev and E. Meinrenken, *Ginzburg-Weinstein via Gelfand-Zeitlin*, J. Differential Geom. **76** (2007), no. 1, 1–34. MR2312047
2. H. Bursztyn and O. Radko, *Gauge equivalence of Dirac structures and symplectic groupoids*, Ann. Inst. Fourier (Grenoble) **53** (2003), no. 1, 309–337. MR1973074 (2004d:53101)
3. Henrique Bursztyn, Marius Crainic, Alan Weinstein, and Chenchang Zhu, *Integration of twisted Dirac brackets*, Duke Math. J. **123** (2004), no. 3, 549–607. MR2068969 (2005e:53128)
4. I. Calvo and F. Falceto, *Star products and branes in Poisson-Sigma models*, Commun. Math. Phys. **268** (2006), no. 3, 607–620. MR2259208 (2007h:81188)
5. Iván Calvo and Fernando Falceto, *Poisson reduction and branes in Poisson-sigma models*, Lett. Math. Phys. **70** (2004), no. 3, 231–247. MR2128952 (2006b:81283)
6. Ana Cannas da Silva, *Lectures on symplectic geometry*, Lecture Notes in Mathematics, vol. 1764, Springer-Verlag, Berlin, 2001. MR1853077 (2002i:53105)
7. Alberto S. Cattaneo, *On the integration of Poisson manifolds, Lie algebroids, and coisotropic submanifolds*, Lett. Math. Phys. **67** (2004), no. 1, 33–48. MR2063018 (2005e:53129)
8. Alberto S. Cattaneo and Giovanni Felder, *A path integral approach to the Kontsevich quantization formula*, Comm. Math. Phys. **212** (2000), no. 3, 591–611. MR1779159 (2002b:53141)
9. ———, *Coisotropic submanifolds in Poisson geometry and branes in the Poisson sigma model*, Lett. Math. Phys. **69** (2004), 157–175. MR2104442 (2005m:81285)
10. Alberto S. Cattaneo and Giovanni Felder, *Relative formality theorem and quantisation of coisotropic submanifolds*, Adv. in Math. **208** (2007), 521–548. MR2304327 (2008b:53119)
11. Alberto S. Cattaneo and Florian Schätz, *Equivalences of higher derived brackets*, 2007.
12. Alberto S. Cattaneo and Marco Zambon, *Pre-Poisson submanifolds*, 2006.
13. Theodore James Courant, *Dirac manifolds*, Trans. Amer. Math. Soc. **319** (1990), no. 2, 631–661. MR998124 (90m:58065)
14. Marius Crainic and Rui Loja Fernandes, *Integrability of Poisson brackets*, J. Differential Geom. **66** (2004), no. 1, 71–137. MR2128714
15. Mark J. Gotay, *On coisotropic imbeddings of presymplectic manifolds*, Proc. Amer. Math. Soc. **84** (1982), no. 1, 111–114. MR633290 (83j:53028)
16. Maxim Kontsevich, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. **66** (2003), no. 3, 157–216. MR2062626 (2005i:53122)
17. Charles-Michel Marle, *Sous-variétés de rang constant d'une variété symplectique*, Third Schnepfenried geometry conference, Vol. 1 (Schnepfenried, 1982), Astérisque, vol. 107, Soc. Math. France, Paris, 1983, pp. 69–86. MR753130 (85k:53037)

18. Yong-Geun Oh and Jae-Suk Park, *Deformations of coisotropic submanifolds and strong homotopy Lie algebroids*, Invent. Math. **161** (2005), no. 2, 287–360. MR2180451 (2006g:53152)
19. Izu Vaisman, *Lectures on the geometry of Poisson manifolds*, Progress in Mathematics, vol. 118, Birkhäuser Verlag, Basel, 1994. MR1269545 (95h:58057)
20. Aissa Wade, *Poisson thickenings of Dirac manifolds*, in preparation.
21. Alan Weinstein, *The local structure of Poisson manifolds*, J. Differential Geom. **18** (1983), no. 3, 523–557. MR723816 (86i:58059)
22. Ping Xu, *Dirac submanifolds and Poisson involutions*, Ann. Sci. École Norm. Sup. (4) **36** (2003), no. 3, 403–430. MR1977824 (2004e:53125)

Institut für Mathematik, Universität Zürich-Irchel, Winterthurerstr. 190, CH-8057 Zürich, Switzerland

E-mail address: alberto.cattaneo@math.unizh.ch

Departamentos de Matematica Pura, Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal

E-mail address: mzambon@fc.up.pt