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DEGENERATIONS OF RATIONALLY CONNECTED VARIETIES

AMIT HOGADI AND CHENYANG XU

ABSTRACT. We prove that a degeneration of rationally connected varieties over a field of characteristic zero always contains a geometrically irreducible subvariety which is rationally connected.

1. INTRODUCTION

Throughout this paper, unless otherwise mentioned, k will denote a field of characteristic zero. A k-scheme X of finite type is called *rationally chain connected* if for an uncountable algebraically closed field K containing k, any two K-points of X can be connected by a chain of rational curves defined over K. A k-scheme X is called *rationally connected* if any two K-points of X can be connected by an irreducible rational curve defined over K. Thus a rationally connected variety is always geometrically irreducible. See [Kol96] for a general background on rational connectedness and rational chain connectedness. The notions of rational connectedness and rational chain connectedness are known to coincide when X/k is smooth and proper. A result of Campana and Kollár-Miyaoka-Mori says that all smooth Fano varieties are rationally connected (see [Ca92], [KMM92a] or [Kol96]). More generally, log $\mathbb{Q}$-Fano varieties are rationally connected ([Zh06] or [HM07]). A degeneration of a proper rationally chain connected variety is rationally chain connected, but a degeneration of a proper rationally connected variety may not be rationally connected. For example a cone over an elliptic curve is not rationally connected, but it is a degeneration of smooth cubic surfaces which are rationally connected. This paper has been motivated by the following question raised in [Kol05a].

Question 1.1 (Kollár)**.** Suppose we have a family of smooth proper rationally connected varieties over a field k. Does the degeneration X_0 contain a rationally connected (in particular geometrically irreducible) subvariety?

In [Kol05a] it was proved that a degeneration of Fano varieties over a smooth curve always contains a geometrically irreducible subvariety. A recent paper by Jason Starr [St06] shows that if the field k (of any characteristic) contains an algebraically closed field, then a degeneration of smooth proper separably rationally connected varieties contains a geometrically irreducible subvariety. In fact, A.J. de Jong proved, under the same assumption as in [St06], that if $\mathrm{char}(k) = 0$, then the degeneration contains a rationally connected subvariety. As observed in

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[Kol05a], the existence of a geometrically irreducible subvariety can fail even for a degeneration of elliptic curves.

The following result, which is the main result of this paper, gives an affirmative answer to Question 1.1.

Theorem 1.2. *Let $\pi : X \to C$ be a dominant proper morphism of k varieties such that*

(1) *C is klt, and*
(2) *the generic fibre of π is a rationally connected variety.*

Then for any point $0 \in C$, $X_0 = \pi^{-1}(0)$ contains a subvariety (defined over $k(0)$) which is geometrically irreducible and rationally connected.

Our approach to Question 1.1 uses the minimal model program, particularly as developed in [BCHM06]. After running the minimal model program for a suitable birational model for X we obtain a Fano fibration. This is then used to reduce Theorem 1.2 to the case when the generic fibre of π is a log Fano variety (Theorem 3.1). We first prove Theorem 3.1 in the special case when C is a smooth curve and $0 \in C$ is a k-point. The general case of an arbitrary *klt* base C is then reduced to this case using the following result.

Theorem 1.3. *Let (X, Δ) be a klt pair with Δ effective. Let $\pi : X' \to X$ be a proper birational morphism. Then for any point $p \in X$, there exists a rationally connected subvariety of $\pi^{-1}(p)$ (defined over $k(p)$).*

The following definition was suggested by J. Kollár.

Definition 1.4. A field k is called *geometrically-C_1* if every smooth proper rationally connected variety over k has a k-point.

Following are some of the consequences of Theorem 1.2.

Corollary 1.5. *Every geometrically-C_1 field (of characteristic 0) is C_1.*

Proof. To show that k is C_1 we have to show that every hypersurface H_d of degree d in $\mathbb{P}^n_k$ has a k-point if $n \geq d$. If H_d is smooth, it is Fano and hence rationally connected and has a k-point by the given hypothesis. Even if H_d is not smooth, it can be expressed as a degeneration of smooth Fano hypersurfaces and has a rationally connected subvariety Z by Theorem 1.2. Then Z has a k-point since its resolution Z' is smooth proper and rationally connected and has a k-point by hypothesis. □

Corollary 1.6. *Let k be a C_1 field. Then every degeneration of a proper rationally connected surface defined over k has a k-point.*

Remark 1.7. (In this remark, k will denote any field, not necessarily of characteristic zero.) The converse of Corollary 1.5 is a well known problem. It is known that a smooth proper rationally connected variety X/k has a k-point if

(1) k is any C_1 field and X is a geometrically rational surface (see for example ([Kol96], IV.6.8)),
(2) k is a field of transcendence degree one over an algebraically closed field of characteristic zero [GHS03] (in the positive characteristic case one needs to assume X is separably rationally connected [dJS03]), or
(3) k is a finite field [Es03].

However, the converse of Corollary 1.5 is unknown, even for specific C_1 fields like the maximal unramified extension of local fields.

In view of Theorem 1.2, if a field k is geometrically-C_1, then not only do smooth proper rationally connected varieties over k have k-points, but one also gets k-points on the degenerations of such varieties. In fact, besides Corollary 1.6, the following was already known:

(1) A degeneration of smooth rationally connected varieties over a finite field has a rational point ([FR05], [EX07]).
(2) A degeneration of a separably rationally connected variety over a field of transcendence degree one over an algebraically closed field has a rational point (this result has been mentioned in [St06] and can be derived by using the properness of Kontsevich's moduli space of stable maps).

A stronger version of Question 1.1 was asked in [Kol05a].

Question 1.8 (Kollár). For a family of proper varieties over a smooth k-curve whose general fiber is smooth and rationally connected, does every simple normal crossing (*snc*) fiber X_0 have an irreducible component (defined over k) which is rationally connected (and thus geometrically irreducible)?

Remark 1.9. Unlike Question 1.1, the answer to this question is unknown even if k is algebraically closed. Over an algebraically closed field $\overline{k}$ one can ask the following more general question: Does an *snc* rationally chain connected variety $X_0/\overline{k}$ always contain an irreducible component which is rationally connected? The answer to this question is indeed true if $\dim(X_0) \leq 2$ but is false in higher dimensions (see Example 5.14).

We show that Question 1.8 has an affirmative answer in the case that the generic fiber satisfies $\dim(X_t) \leq 3$.

Theorem 1.10. *Let $\pi : X/k \to C/k$ be a dominant proper morphism such that*

(1) *C is a smooth curve with a k-point* 0*,*
(2) *π is smooth outside* 0*,*
(3) *the generic fibre of π is geometrically rationally connected and* $\dim(X_t) \leq 3$*, and*
(4) *X is smooth, and $X_0 = \pi^{-1}(0)$ is snc (over k).*

Then there exists an irreducible component of X_0/k which is rationally connected.

In Section 3 we prove Theorem 1.2 in the case of degenerations of log Fano varieties, assuming Theorem 1.3. In Section 4 we prove Theorem 1.2 using the results of the minimal model program as proved in [BCHM06]. The proof uses results of Section 3 and some results on Fano fibrations which are proved in Section 5. The stronger result, Theorem 1.10, for a degeneration of rationally connected threefolds (or lower dimension) is proved in Section 5. Finally, in Section 6 we prove Theorem 1.3.

2. Summary of known results

In this section we first recall the related results from [HM07] and [Kol05a]. In our proof, their ideas are heavily used. Then we state another main ingredient from the minimal model program (cf. [KM98]) proved in [BCHM06].

The following is the main theorem of [HM07], stated in slightly less generality.

Theorem 2.1 ([HM07])**.** *Let (X, Δ) be a klt pair. Let $g : Y \to X$ be a birational morphism such that $-K_X$ is relatively big and $\mathcal{O}_X(-m(K_X + \Delta))$ is relatively generated for some $m > 0$. Then every fibre of the the composite morphism $\pi : Y \to S$ is rationally chain connected.*

Specialising to the case when g is the identity morphism, (X, Δ) is rationally chain connected and $Y \to X$ is a resolution of singularities, one gets the following important corollary.

Corollary 2.2 ([HM07])**.** *Let (X, Δ) be a klt pair which is rationally chain connected. Then X is rationally connected.*

Recall that there exist singular varieties (for example the cone over an elliptic curve) which are rationally chain connected but not rationally connected. However, the above corollary says that such examples necessarily have singularities worse than *klt*.

In Section 3, we consider degenerations of Fano varieties (see Theorem 3.1), which has also been considered in [HM07]. Part of ([HM07], 5.1) guarantees the existence of an irreducible component which is rationally chain connected modulo the intersection of some other components. But this component may be neither geometrically irreducible nor rationally connected. Although the statement of [HM07], 5.1 does not imply Theorem 3.1, a modification of methods used in [HM07] will be enough for it to be proved. We recall the following technical result, 4.1 from [HM07], which we state in a weaker form sufficient for our purpose.

Proposition 2.3 ([HM07], 4.1)**.** *Let (F, Δ) be a projective log pair, and let $t : F \dashrightarrow Z$ be a dominant rational map, respectively, where F and Z are projective, with the following properties:*

(1) *the locus of log canonical singularities of $(K_F + \Delta)$ does not dominate Z;*
(2) *$F + \Delta$ has Kodaira dimension at least zero on the general fibre of $F \dashrightarrow Z$ (i.e., if $g : F' \to F$ resolves the indeterminacy of $F \dashrightarrow Z$, then $g^*(K_F + \Delta)$ has Kodaira dimension at least zero on the general fibre of the induced morphism $F' \to Z$);*
(3) *$K_F + \Delta$ has Kodaira dimension at most zero; and*
(4) *there is an ample divisor A on F such that $A \leq \Delta$.*

Then either Z is a point, or it is uniruled.

The main application of this proposition is when t is the rational map from F to its maximal rationally connected fibration (cf. [Kol96], IV Theorem 5.4). In fact, with the help of the following lemma, one obtains a quite useful criterion to show the rational connectedness of varieties.

Lemma 2.4 ([HM07], 4.2(i))**.** *A normal variety F is rationally connected if and only if, for every nonconstant dominant rational map $t : F \dashrightarrow Z$, Z is uniruled.*

Next we recall the main result of [Kol05a].

Theorem 2.5 ([Kol05a])**.** *Let k be a field of characteristic zero, C be a smooth curve, Z an irreducible projective variety and $g : Z \to C$ a morphism. Assume that* *the general fibres F_{gen} are*

(1) *smooth,*
(2) *geometrically connected, and*
(3) *Fano.*

Then for every point c of C, the fibre $g^{-1}(c)$ contains a $k(c)$-subvariety which is geometrically irreducible.

In our proof, as a special case, and also as a middle step, we improve Kollár's result by showing that the subvariety which he looks at is not only geometrically irreducible but also rationally connected (see Lemma 3.4). Theorem 1.2 further improves the result by allowing C to be any *klt* variety.

Finally, we recall a result from the very important paper [BCHM06], in which a 'large part' of the minimal model program is proved. In particular, we state the following theorem which is a special case of their results. This will be the main tool in the proof of Theorem 1.2.

Theorem 2.6 ([BCHM06])**.** *Let (X, Δ) be a klt pair, projective over a base S. Assume $K_X + \Delta$ is not pseudoeffective. Then one can run a relative minimal model program for (X, Δ) to obtain a Fano fibration; i.e. there exists a sequence of maps*

$$X = X_0 \overset{\phi_1}{\dashrightarrow} X_1 \overset{\phi_2}{\dashrightarrow} \cdots \overset{\phi_r}{\dashrightarrow} X_r \xrightarrow{h} Y$$

where each ϕ_i is either a divisorial contraction or a flip and where h is a Fano fibration of relative Picard number one.

The important case in which the above hypothesis is satisfied is when X is a smooth rationally connected variety and Δ is empty. Indeed, in this case, the canonical divisor K_X has a negative intersection with any free rational curve and hence it cannot be pseudoeffective.

3. Degenerations of Fano varieties

In this section we prove Theorem 3.1, which is Theorem 1.2 in the special case when the generic fibre of $\pi : X \to C$ is a log Fano variety. Recall that a proper variety X/k is called *log Fano* (or *log $\mathbb{Q}$-Fano*) if there exists an effective divisor D on X such that (X, D) is *klt* and $-(K_X + D)$ is nef and big. It is known that a log Fano variety is always rationally connected (cf. [Zh06], [HM07]).

Theorem 3.1. *Let $\pi : X \to C$ be a dominant proper morphism of k varieties such that*

(1) *C is klt, and*
(2) *the generic fibre of π is a log Fano variety.*

Then for any point $0 \in C$, $X_0 = \pi^{-1}(0)$ contains a subvariety (defined over $k(0)$) which is geometrically irreducible and rationally connected (over $k(0)$). In other words, Theorem 1.2 is true in the case when the generic fibre of π is a log Fano variety.

The extension theorem ([HM06], 3.17) lies at the heart of the proof. We first recall the following terminology.

Definition 3.2 ([HM06, 3.7])**.** Let $\pi : (X, \Delta) \to S$ be a relative log pair. A Cartier divisor D on X will be called *π-transverse* to (X, Δ) if the natural map

$$\pi^*\pi_*(\mathcal{O}_X(D)) \to \mathcal{O}_X(D)$$

is surjective at the generic point of every log canonical centre of (X, Δ). Obviously, the notion only depends on the divisor class of D. Similarly, a $\mathbb{Q}$-divisor D will be called π-$\mathbb{Q}$-*transverse* to (X, Δ) if mD is π-transverse for some $m > 0$.

The proof of Theorem 3.1 relies on the following extension theorem proved in [HM06]. The statement given below is less general than the one in [HM06].

Theorem 3.3 ([HM06], 3.17)**.** *Let $\pi : X \to S$ be a projective morphism where X/k is a smooth variety of dimension n. Let $F \subset X$ be a smooth divisor. Let B be an effective $\mathbb{Q}$-divisor on X such that*

(1) *$B + F$ is snc,*
(2) *the support of B and F have no common components,*
(3) *$\lfloor B \rfloor = 0$,*
(4) *$K_F + B_{|F}$ is effective, and*
(5) *$K_X + B + F$ is π-$\mathbb{Q}$-transverse to $(X, \lceil B \rceil + F)$.*

For every sufficiently π-ample divisor H on X and for every sufficiently divisible positive integer m, the image of the morphism

$$\pi_*(\mathcal{O}_X(m(K_X + B + F) + (n+2)H)) \to \pi_*(\mathcal{O}_F(m(K_F + B_{|F}) + (n+2)H_{|F}))$$

contains the image of

$$\pi_*(\mathcal{O}_F(m(K_F + B_{|F}) + H_{|F})) \xrightarrow{(n+1)H_{|F}} \pi_*(\mathcal{O}_F(m(K_F + B_{|F}) + (n+2)H_{|F})).$$

In the situation of our interest, condition (4) above will be checked as follows. There will be an effective $\mathbb{Q}$-divisor A such that

(i) $K_X + F + B \sim A$, and
(ii) A and $B + F$ have no common components, and $A + B + F$ is *snc*.

Since $\lceil B \rceil + F$ is an *snc* divisor where each irreducible component occurs with coefficient one, the log canonical centres of $(X, \lceil B \rceil + F)$ are exactly all possible intersections of the components of $\lceil B \rceil + F$. But since $A + B + F$ is *snc* and A and $B + F$ have no common components, $\mathcal{O}_X(A)$ is generated by its global sections at the generic point of every log canonical center of $(X, \lceil B \rceil + F)$.

As we mentioned before, the proof is given first when C is a smooth curve.

Lemma 3.4. *When C is a smooth curve, Theorem* 3.1 *is true.*

Proof. By extending the base field, we can assume $0 \in C$ is a k-point. By assumption, the generic fibre X_g of $\pi : X \to C$ is a log Fano variety. Thus there exists an effective divisor D_g on X_g such that (X_g, D_g) is *klt* and $-(K_{X_g} + D_g)$ is nef and big. Thus there exists an ample divisor H_g and an effective divisor E_g on X_g such that $-(K_{X_g} + D_g) \sim H_g + E_g$. Furthermore, since $-(K_{X_g} + D_g)$ is nef and big, E_g and H_g can be chosen such that $(X_g, D_g + E_g)$ is *klt* (cf. [KM98], (2.61)). By replacing D_g by $D_g + E_g$ we may assume $-(K_{X_g} + D_g) \sim H_g$ is ample. Let D be an effective divisor on X such that $D_{|X_g} = D_g$. By shrinking C we may assume that C is affine and D does not contain any vertical divisor outside 0.

After successively blowing up X with suitable centers and shrinking C one can find a birational morphism $h : Y \to X$ such that

(1) Y is smooth, and the natural projection $\pi'(= \pi \circ h) : Y \to C$ is smooth outside 0, and
(2) $\mathrm{Supp}(h^{-1}D + Y_0) + \mathrm{Ex}(h)$ is *snc* (over k) in Y. Here $Y_0 = h^{-1}(0)$.

Then there exist effective $\mathbb{Q}$-Cartier divisors G and R on Y having no common components in their support and such that $K_{Y/X}+G-R=h^*D$. Moreover, G and R satisfy

(1) any irreducible component of G which is not supported in Y_0 appears in G with coefficient strictly less than one (this is a consequence of the fact that (X_g, D_g) is *klt*), and
(2) R is h-exceptional.

Note that the restriction of $-(K_Y+G-R)$ to the generic fibre Y_g of Y over C is the pull back of an ample divisor on X_g. Thus for some sufficiently small h-exceptional effective divisor E on Y, $-(K_Y+G-R+E)$ is ample on the generic fibre. If necessary blow up Y along smooth centers in the fibre Y_0 so that there exists a vertical $\mathbb{Q}$-Cartier divisor $\widetilde{\Delta}$, supported in Y_0, such that $-(K_Y+G-R+E+\widetilde{\Delta})$ is relatively ample over C (see [Kol05a], page 5).

Let $\widetilde{\Delta}=\sum d_i\widetilde{\Delta}_i$, where the divisors $\widetilde{\Delta}_i$ are the irreducible components of the support $\widetilde{\Delta}$. The only irreducible components of $G-R+E+\widetilde{\Delta}$ which appear with coefficient ≥ 1 are supported in Y_0. Thus by perturbing the coefficients d_i slightly (by small rational numbers) and by changing $\widetilde{\Delta}\to\widetilde{\Delta}+rY_0$ for a suitable $r\in\mathbb{Q}$ we may assume

(1) $-(K_Y+G-R+E+\widetilde{\Delta})$ is π-ample, $\mathrm{Supp}(\widetilde{\Delta})\subset X_0$,
(2) there exists an irreducible divisor F (over k) in Y, an effective $\mathbb{Q}$-Cartier divisor A, and a boundary $\widetilde{B}$ in Y such that $G-R+E+\widetilde{\Delta}=F+\widetilde{B}-A$ and such that F and the supports of A and $\widetilde{B}$ have no pairwise-common components, and
(3) there exist effective $\mathbb{Q}$-Cartier divisors A_0 and R_0 in Y such that $A=A_0+R_0$, $\mathrm{Supp}(A_0)\subset Y_0$ and $\mathrm{Supp}(R_0)$ is in the exceptional set of h and contains no component of Y_0.

The necessary perturbations as well as the proof of the existence of F, $\widetilde{B}$ and A can be found in [Kol05a]. Notice that $A+\widetilde{B}+F$ is *snc* since it is contained in $\mathrm{Supp}(h^{-1}D+Y_0+\mathrm{Ex}(h))$. In ([Kol05a], Remark 8) it was proved that F is geometrically irreducible. If A were equal to 0, then ([HM07], 5.1) would imply that F is rationally connected. But in general A can be nonzero. Fortunately, as we will show below, we are saved in this situation by fact (3).

Let m be a sufficiently large and divisible integer and let L_m be the Cartier divisor of a general section of $\mathcal{O}_X(m(-K_Y-F-\widetilde{B}+A))$. Let $L=\frac{1}{m}L_m$. Let $B=\widetilde{B}+L$ and $\Delta=F+B-A$. Thus we are finally in the following situation:

(i) $-(K_Y+\Delta)\sim_{\pi'} 0$,
(ii) $\Delta=F+B-A$, where $A+B+F$ is *snc*,
(iii) F is geometrically irreducible $\mathrm{Supp}(F)\subset\mathrm{Supp}(Y_0)$,
(iv) A, B are effective $\mathbb{Q}$-divisors, and $\lfloor B\rfloor=0$,
(v) $A=A_0+R_0$ such that $\mathrm{Supp}(A_0)\subset\mathrm{Supp}(Y_0)$, R_0 is h-exceptional, and
(vi) there exists a π'-ample divisor L such that $L\leq B$.

To apply Lemma 2.4, we now make the following two claims.

Claim (1): Let $p: F'\to F$ be a proper birational morphism such that the induced rational map $f': F'\to Z$ is a morphism. Then $p^*(K_F+B_{|F})$ has Kodaira dimension at least zero on the general fibre of f'.

Claim (2): $\kappa(K_F+B_{|F})=0$.

Notice that $(F, B_{|F})$ is *klt* since $B+F$ is *snc*, B and F do not have common components and $\lfloor B \rfloor = 0$. Also, the support of B contains the support of an ample divisor L. Hence once we prove the above two claims it will follow that the hypotheses of Proposition 2.3 are implied by our hypotheses. In fact, the first hypothesis of Proposition 2.3 follows from the conclusion that $(F, B|_F)$ is klt, Claim (1) implies the second hypothesis, Claim (2) implies the third hypothesis and the fourth hypothesis follows easily from the fact that $B+F$ is *snc*. So we can conclude that Z is uniruled, thus proving the rational connectedness of F.

By restricting $K_Y + F + B \sim_{\pi'} A$ on F, we have $K_F + B_{|F} \sim A_F$. But A and F have no common components, and hence A_F is effective. Thus $p^*(K_F + B_{|F})$ is effective. Hence it is also effective when restricted to the general fibre of f'. This proves Claim (1). To prove Claim (2) we use Theorem 3.3. Let M be a sufficiently π'-ample divisor. To prove $\kappa(K_F + B_{|F}) = 0$, it is enough to show that for m large and divisible, $h^0(F, m(K_F + B_{|F}) + M) = \mathit{length}\left(\pi'_*(\mathcal{O}_F(m(K_F + B_{|F}) + M_{|F}))\right)$ is bounded independent of m. By Theorem 3.3 it is enough to bound the rank of the torsion free sheaf $\pi'_*(\mathcal{O}_Y(m(K_Y + B + F) + (n+2)M))$. $K_X + B + F \sim_{\pi'} A$, and hence we have to bound the rank of $\pi'_*(\mathcal{O}_Y(mA + (n+2)M))$. Since $A = A_0 + R_0$, where R_0 is effective and h-exceptional, this sheaf is isomorphic to a subsheaf of $\pi_*(\mathcal{O}_X(mh_*(A_0) + h_*((n+2)M)))$. But since $\mathrm{Supp}(h_*(A_0)) \subset \mathrm{Supp}(X_0)$, $\pi_*(\mathcal{O}_X(mh_*(A_0) + h_*((n+2)M)))$ is isomorphic to $\pi_*(\mathcal{O}_X(h_*((n+2)M)))$ on $C - \{0\}$, and hence it has constant rank. □

Proof of Theorem 3.1 *in the general case.* To prove Theorem 3.1 for any *klt* variety C and an arbitrary point $0 \in C$, we need Theorem 1.3 in the following way. Let $h : C' \to C$ be a resolution of C; then by Theorem 1.3, there exists a rationally connected subvariety $Z \subset h^{-1}(0)$. Let L be the function field of Z. Since Z is itself rationally connected, to find a rationally connected subvariety of X_0, it suffices to find a rationally connected subvariety of $X_0 \times_k L$ by [GHS03]. Thus by a base extending to L we may assume there is a k-point $0' \in C'$ such that $h(0') = 0$. Now let $D \subset C'$ be a general smooth curve passing through $0'$. Note that it is possible to find such a curve since C' is smooth. Let $X_D = X \times_C D$. Since D is general, the generic fibre of $X_D \to D$ is a log Fano variety. Thus Theorem 3.1 now follows from the above case. □

4. Degenerations of rationally connected varieties and the minimal model program

In this section we prove Theorem 1.2. It suffices to prove Theorem 1.2 after replacing X by any other higher birational model $X' \to X$. Thus by resolution of singularities, we may assume X is smooth, and $X_0 = \pi^{-1}(0)$ is *snc* (over k). The proof of Theorem 1.2 will proceed by studying the relative minimal model program for the pair (X, Δ), where Δ denotes the sum of all irreducible components of X_0, each appearing with coefficient one. In this situation X is smooth and Δ is *snc*.

Lemma 4.1. *Let $\phi : (X, \Delta) \dashrightarrow (X', \Delta')$ be a birational morphism obtained by running the relative minimal model program for (X, Δ) (i.e. ϕ is a composition of extremal divisorial contractions and flips). Then*

(i) *every irreducible component Δ'_j of $\Delta'(= \sum_{j=1}^m \Delta'_j)$ is of the form $\phi_*\Delta_i$ for some i, and*

(ii) *every k-irreducible component of Δ' is klt. In particular, after running the relative minimal model program, the* $\mathrm{Gal}(\bar{k}/k)$*-conjugate components of* $\Delta \times_k \overline{k}$ *do not intersect.*

Example 4.2. Let $Y = (x^2 + y^2 + tz^2 = 0) \subset \mathbb{P}^2_{\mathbb{R}} \times \mathbb{A}^1_{\mathbb{R}}$, where t is the coordinate on the affine line $C := \mathbb{A}^1_{\mathbb{R}}$. Let X be the blow up of Y at $([0,0,1],0)$, and E be the exceptional divisor. Then the projection $f : X \to C$ gives us a family which is defined over $\mathbb{R}$, whose generic fibres are smooth conics. Look at the point $t = 0$ on C; the fibre X_0 consists of E and the birational transform of $Y_0 = (x^2 + y^2 = 0)$, which has 2 components E_1, E_2 over $\mathbb{C}$, and are conjugate to each other under the Galois action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. So $NE(X/C)(\mathbb{R})$ is generated by E and $E_1 + E_2$. We have $(K_X + E + E_1 + E_2) \cdot E = 0$ and $(K_X + E + E_1 + E_2) \cdot (E_1 + E_2) = -2$, so to run the relative program for $f : (X, E + E_1 + E_2) \to C$ over $\mathbb{R}$, we can only contract $E_1 + E_2$.

Proof of Lemma 4.1. (i) For arbitrary ϕ as in Theorem 3.1, it is a morphism away from codimension 2 subsets of both the domain and the target; hence (i) is obvious.

(ii) We divide the argument into several steps.

Step (1) Let $\Delta' = \sum_{i=1}^{r} \Delta'_i$, where $\Delta'_i = \phi_*\Delta_i$ and $\phi_*\Delta_i = 0 \ \forall\ r+1 \le i \le m$. By Inversion of Adjunction, to prove Δ'_1 is *klt* it is enough to prove (X', Δ'_1) is *plt* ([KM98], 5.4). For $I \subset \{1, ..., m\}$ let $\Delta_I = \bigcap_{i\in I} \Delta_i$. Since X is smooth and $\Delta \subset X$ is *snc*, (X, Δ) is log canonical, and the only log canonical centers of (X, Δ) are connected components of closed subsets Δ_I. Because during each step of the minimal model program discrepancy never decreases ([KM98], 3.38), (X', Δ') is also log canonical.

Step (2) Let E be an exceptional divisor of X' with $a(E, X', \Delta') = -1$. We need to show $a(E, X', \Delta'_1) > -1$. We claim that $center_X E = \Delta_I$ for some I. Applying the nondecreasing of the discrepancy again, we conclude $a(E, X, \Delta) = -1$. But as remarked earlier the only log canonical centers of (X, Δ) are Δ_I. Hence $center_X E = \Delta_I$ for some I.

Step (3) We claim that for every $i \in I$, $\phi_*\Delta_i \ne 0$; i.e. none of the components of Δ which contain the generic point of $center_X E$ can be contracted to a lower dimension subvariety by ϕ. This is a consequence of ([KM98], 3.38), since if ϕ is not an isomorphism at the generic point of $center_X E$, then

$$a(E, X', \Delta') > a(E, X, \Delta) \ge -1,$$

which is contradictory to our assumption. Since we have assumed E is an exceptional divisor of X', this also shows that $center_X E$ is not a divisor. Thus I contains at least two elements. Without loss of generality we assume $2 \in I$.

Step (4) Now we have $a(E, X', \Delta'_1 + \Delta'_2) \ge a(E, X', \Delta') = -1$. But both Δ'_1 and Δ'_2 contain $center_{X'} E$, hence

$$a(E, X', \Delta'_1) > a(E, X', \Delta'_1 + \Delta'_2) \ge -1.$$

□

The following lemma is useful in proving Theorem 1.2 because it allows us to replace X by any other *klt* birational model.

Lemma 4.3. *Let $f : X' \dashrightarrow X$ be a birational map of varieties over C such that both X' and X are klt. Then for $0 \in C$, X'_0 has a rationally connected subvariety if and only if the same is true for X_0.*

Proof. Suppose X_0 contains a rationally connected subvariety Z. It is enough to prove X_0' contains a rationally connected subvariety after replacing X' by a higher birational model. Thus we may assume f is in fact a morphism. Let $k(Z)$ be the generic point of Z. By Theorem 1.3, there exists a rationally connected subvariety in the fibre of X' over $k(Z)$. Take its closure Z' in X'. By [GHS03] Z' is rationally connected and is clearly contained in X_0'. $\square$

Proof of Theorem 1.2. Let $\{\Delta_i\}$ be the set of irreducible components of X_0. Let $\Delta = \sum_i \Delta_i$. We now want to run a relative minimal model program for the pair (X, Δ) over C and apply the main theorem (see Theorem 2.6) of [BCHM06]. However, (X, Δ) is not *klt* but only log canonical. Nevertheless, for sufficiently small and positive ϵ, $(X, \Delta - \epsilon\pi^*(0))$ is *klt*. Also, for any curve class η in $N_1(X/C)$, $\pi^*(0) \cdot \eta = 0$, running the relative minimal model program for $(X, \Delta - \epsilon\pi^*(0))$, it is equivalent to running the relative minimal model program for (X, Δ). Thus in order to run the minimal model program resulting in a Mori fibration we only need to check that $K_X + \Delta$ is not π-pseudo-effective. But in our case the generic fibre, X_g, of $X \to C$ is a smooth rationally connected variety. The intersection number of K_{X_g} with any free rational curve in X_g is negative. Thus K_{X_g} is not pseudo-effective. Since Δ is vertical, this implies that $K_X + \Delta$ restricted to the generic fibre is not pseudo-effective. In particular $K_X + \Delta$ itself cannot be pseudo-effective. After running the program one arrives at the following situation:

$$\begin{tikzcd}
(X, \Delta) \arrow[r, dashed, "\phi"] \arrow[d, "\pi"'] & (X', \Delta') \arrow[r, "h"] \arrow[d, "\pi'"] & Y \arrow[dl, "f"] \\
C \arrow[r, equal] & C &
\end{tikzcd}$$

where

(i) relative Picard number of h is one,
(ii) X and X' are birational, and
(iii) $h : X' \to Y$ is a Fano contraction, i.e. the generic fibre of h is a log Fano variety.

The generic fibre of $Y \to C$ is also a rationally connected since it is dominated by the generic fibre of $X' \to C$. Thus by induction on dimension, $Y_0 = f^{-1}(0)$ has a rationally connected subvariety Z/k. Let p be the generic point of Z. By Proposition 5.5, Y is *klt*. The generic fibre of $X' \to Y$ is a log Fano variety. Thus by Theorem 3.1 there exists a rationally connected subvariety of $h^{-1}(p)$ defined over $k(p)$. Let Z' be its closure in X'. By [GHS03], Z' is a rationally connected subvariety of X_0'. Since both X' and X are *klt*, an application of Lemma 4.3 completes the proof. $\square$

5. Fano fibrations

In this section we prove some results (of independent interest) on Fano fibrations obtained by contraction of an extremal ray. These results will then be used to prove Theorem 1.10.

Let $\pi : (X, \Delta) \to S$ be a relative log canonical pair and with X $\mathbb{Q}$-factorial with Δ effective. Let $h : X \to Y$ be a Fano fibration obtained by contracting an extremal ray. Y is automatically $\mathbb{Q}$-factorial by ([KM98], 3.36). Let $\Delta = \sum_i a_i\Delta_i$, where Δ_i are irreducible. Let $n = dim(X)$ and $r = dim(Y)$.

Lemma 5.1. *Let D be any prime Weil divisor on X. Then either $h(D) = Y$ or $h(D)$ is a divisor.*

Proof. Assume that $\dim(h(D)) = k \leq r - 2$. After cutting Y and then cutting X by general very ample hypersurfaces, we may assume $h^*(H_Y)^k \cdot H_X^{n-k-2}$ is a surface whose image in Y is still 2 dimensional (because $r-k \geq 2$) and $D \cdot h^*(H_Y)^k \cdot H_X^{n-k-2}$ is a curve which is contracted to a point on Y; then we have $D^2 \cdot h^*(H_Y)^k \cdot H_X^{n-k-2} < 0$. On the other hand, let E be a curve on the fibre over a general point. Since E and D do not intersect $E \cdot D = 0$, $D \cdot h^*(H_Y)^k \cdot H_X^{n-k-2}$ and E cannot be numerically proportional. But this contradicts the fact that the relative Picard number of h is one. □

Lemma 5.2. *Let D be a prime Weil divisor on X which does not dominate Y. Then* $\mathrm{Supp}(D) = \mathrm{Supp}(h^{-1}h(D))$. *In particular if Δ_i and Δ_j do not dominate Y, then $h(\Delta_i)$ and $h(\Delta_j)$ are distinct divisors on Y unless $i = j$.*

Proof. Assume there exists an irreducible component D' of $h^{-1}h(D)$ different from D. Then $h(D') \subset h(D)$. But by the previous Lemma 5.1, both are irreducible divisors in Y. Hence $h(D') = h(D)$. Cut Y and X by general very ample hypersurfaces so that $h^*(H_Y)^{r-1} \cdot H_X^{n-r-1}$ is a surface whose image is a curve. $D' \cdot h^*(H_Y)^{r-1} \cdot H_X^{n-r-1}$ and $D \cdot h^*(H_Y)^{r-1} \cdot H_X^{n-r-1}$ are curves on $h^*(H_Y)^{r-1} \cdot H_X^{n-r-1}$, which are mapped to the same point on Y. Then by Zariski's lemma ([BPV84], III.8.2) they cannot be numerical proportional, which is a contradiction since h has relative Picard number one. □

The hypothesis of Theorem 1.2 says that X is smooth and the divisor Δ whose support is X_0 is *snc* (each irreducible component of Δ appearing with coefficient one). However, we know (X, Δ) is *dlt* (cf. [KM98]), which is a property preserved under the log minimal model program. Now we pinpoint the property we need for the *dlt* condition as follows,

5.3 (Hypothesis)**.** Let $\pi : (X, \Delta) \to S$ be a projective log canonical pair where Δ is a vertical divisor. Assume X is $\mathbb{Q}$-factorial. Let $\{\Delta_i\}_{1 \leq i \leq m}$ be the set of all irreducible components of Δ and assume each Δ appears with coefficient one in Δ. Assume that the only log canonical centers of (X, Δ) are given by Δ_I, where $I \subset \{1, ..., m\}$ and $\Delta_I = \bigcap_{i \in I} \Delta_i$.

Remark 5.4. Section 5.3 in particular implies that for each $i \in \{1, ..., m\}$, (X, Δ_i) is *plt*. Hence each irreducible component Δ_i is *klt* by inversion of adjunction. We will show (see Lemma 4.1) that even after performing divisorial contractions and flips, the resulting divisor has *klt* components. The following proposition proves the same for Fano fibrations.

Proposition 5.5. *Let (X, Δ) satisfy the hypothesis of Section* 5.3*. Let $X \to Y$ be a Fano fibration obtained by contracting an extremal ray. Let D_i be the image of Δ_i in Y and let $D = \sum_{i=1}^m D_i$. Then (Y, D) also satisfies Section* 5.3*.*

Remark 5.6. To prove Proposition 5.5 we need Kawamata's canonical bundle formula as proved in [Kol05b]. A similar method has been used before in [Am04] to study singularities of Fano contractions.

Definition 5.7 (Standard normal crossing assumptions; see [Kol05b])**.** We say that a projective morphism $f : X \to Y$ together with $\mathbb{Q}$-divisors R, B satisfy the standard normal crossing assumptions if the following hold:

(1) X,Y are smooth,
(2) $R+f^*B$ and B are *snc* divisors,
(3) f is smooth over $Y\setminus B$, and
(4) R is a relative *snc* divisor over $Y\setminus B$.

Theorem 5.8 (Kawamata's canonical bundle formula; see [Kol05b])**.** *Let $f:X\to Y$ and R,B satisfy the standard normal crossing assumptions. Let F be the generic fibre of f. Assume that $K_X+R\sim_{\mathbb{Q}} f^*H_R$ for some $\mathbb{Q}$-divisor H_R on Y. Let $R=R_h+R_v$ be the horizontal and vertical parts of R and assume that when we write $R_h=R_{h(\geq 0)}-R_{h(\leq 0)}$ as the difference of its positive and negative parts, we have $h^0(F,\lfloor R_{h(\leq 0)}|_F\rfloor)=1$. Then we can write*

$$K_{X/Y}+R\sim_{\mathbb{Q}} f^*(J(X/Y,R)+B_R),$$

where

(1) *the moduli part $J(X/Y,R)$ is nef and depends only on $(F,R_h|_F)$ and Y,*
(2) *the boundary part B_R depends only on $f:X\to Y$ and R_v. More precisely, B_R is the unique smallest $\mathbb{Q}$-divisor supported on B such that*

$$red(f^*B)\geq R_v+f^*(B-B_R).$$

Moreover,

(3) *the pair (Y,B_R) is lc iff the pair (X,R) is lc,*
(4) *if $\lfloor R_h\rfloor\leq 0$, then the pair (Y,B_R) is klt iff the pair (X,R) is klt,*
(5) *B_i appears in B_R with nonnegative coefficient if $f_*\mathcal{O}_X(\lfloor -R_v\rfloor)\subset\mathcal{O}_{B_i,Y}$.*

Remark 5.9. In the above formula, $J(X/Y,R)$ is only defined as a $\mathbb{Q}$-divisor class. However, B_R is a $\mathbb{Q}$-divisor.

Proof of Proposition 5.5. Since h is a contraction of a $-(K_X+\Delta)$-negative ray, $-(K_X+\Delta)$ is h-ample. Thus there exist ample $\mathbb{Q}$-divisors H_X (resp. H_Y) on X (resp. Y) in general position such that

$$K_X+\Delta+H_X\sim_{\mathbb{Q}} h^*(H_Y).$$

Choose log resolutions $g:\overline{X}\to(X,\Delta)$ and $d:\overline{Y}\to Y$ such that we have the following commutative diagram:

$$\begin{array}{ccc}\overline{X} & \xrightarrow{g} & X\\ \downarrow{\scriptstyle\overline{h}} & & \downarrow{\scriptstyle h}\\ \overline{Y} & \xrightarrow{d} & Y\end{array}$$

There exists a Cartier divisor R on $\overline{X}$ such that

$$K_{\overline{X}/X}+R=g^*(\Delta+H_X).$$

After further blowup we may assume $\overline{h}:\overline{X}\to\overline{Y}$, the divisor R, and a reduced divisor B on $\overline{Y}$ containing $\mathrm{Supp}(d^{-1}(D))$ satisfy the standard normal crossing assumptions (see Definition 5.7).

We can choose the above $H_X = \sum d_i H_i$, where $0 < d_i \ll 1$ and H_i are divisors which are in general position. We now apply Theorem 5.8 to $\overline{h} : \overline{X} \to \overline{Y}, R$ and B. Since the negative part of R consists of exceptional divisors, $h^0(F, \lfloor R_{h(\leq 0)} \rfloor) = 1$. Thus by Theorem 5.8 we have

$$K_{\overline{X}} + R \sim_{\mathbb{Q}} \overline{h}^*(K_{\overline{Y}} + J(\overline{X}/\overline{Y}, R) + B_R).$$

Lemma 5.10. *For every i the coefficient in B_R of the birational transform of D_i in $\overline{Y}$ is 1. Furthermore, every log canonical center W of $(\overline{Y}, B_R)$ satisfies the fact that $d(W) = \bigcap_J D_j$ for some $J \subset \{1, ..., m\}$.*

Proof. The first part of the lemma is because of Theorem 5.8(2) and the fact that R is a subboundary (i.e. the coefficients are less than or equal to 1, not necessarily nonnegative) which contains a component dominating D_i for every i, whose coefficient in R is 1.

The reason for the second part is similar to the one just given. Let D be an exceptional divisor of $\overline{Y}$ whose center in $\overline{Y}$ is a log canonical center W of $(\overline{Y}, D)$. To show $d(W)$ is of the form $\bigcap_J D_J$ in Y, we may blow up $\overline{Y}$ and $\overline{X}$ further and assume $D = W$ and the coefficient of D in B_R is 1. But this implies that there is a component E of R_v dominating B such that the coefficient of E in R_v is one. The image of E in X is a log canonical center of (X, Δ) and hence is of the form $\bigcap_{i \in J} \Delta_i$ for some $J \in \{1, ..., m\}$. Thus $d(W) = \bigcap_{i \in J} D_J$. □

Now we only need to prove that (Y, D) is log canonical. For this we first claim that

Lemma 5.11. *$d_* B_R$ is an effective divisor.*

Proof. By Theorem 5.8(2) it follows that if an irreducible component B_1 appears in B_R with negative coefficient, then every component of $\overline{h}^* B_1$ appears in R_v with negative coefficient. But since all negative components of R have to be g exceptional, it follows that B_1 is also d-exceptional. This proves the lemma. □

To check (Y, D) is log canonical, it is enough to check that for every divisor E on $\overline{Y}$, $a(E, Y, D) \geq -1$. Since $J(\overline{X}/\overline{Y})$ is nef, for every $\epsilon > 0$ we can choose an effective divisor $J_\epsilon \sim_{\mathbb{Q}} J(\overline{X}/\overline{Y}) + d^* H_Y$ such that for every divisor E on Y, $a(E, \overline{Y}, J_\epsilon + B_R) > -1 - \epsilon$. Because $K_{\overline{Y}} + J_\epsilon + B_R \sim_{\mathbb{Q}} d^*(A)$ for some $\mathbb{Q}$-divisor A on Y, then we have

$$K_{\overline{Y}} + J_\epsilon + B_R \sim_{\mathbb{Q}} d^*(K_Y + d_*(J_\epsilon + B_R)).$$

But $D \leq d_*(J_\epsilon + B_R)$. Hence for every divisor E on Y we have

$$a(E, Y, D) \geq -1 - \epsilon.$$

Taking the limit as $\epsilon \to 0$ we see that (Y, D) is log canonical. □

Proof of Theorem 1.10. The idea of the proof is similar to that of Theorem 1.2. Let us assume that the relative dimension of $X \to C$ is 3. For the case of smaller dimension, the argument is similar but easier. Let Δ be the sum of all irreducible components of X_0, each with coefficient one. Then after running the relative minimal model program for (X, Δ) we arrive at one of the following two cases:

Case (1)

$$\begin{tikzcd}
(X,\Delta) \arrow[r, dashed, "\phi"] \arrow[d, "\pi"'] & (X',\Delta') \arrow[d, "\pi'"] \\
C \arrow[r, equal] & C
\end{tikzcd}$$

where π' has relative Picard number one.

Case (2)

$$\begin{tikzcd}
(X,\Delta) \arrow[r, dashed, "\phi"] \arrow[d, "\pi"'] & (X',\Delta') \arrow[r, "h"] \arrow[d, "\pi'"] & Y \arrow[dl, "f"] \\
C \arrow[r, equal] & C &
\end{tikzcd}$$

where

(i) the relative Picard number of h is one,
(ii) X and X' are birational, and $h : X' \to Y$ is a Fano contraction of relative dimension two, and
(iii) Y/C is of relative dimension one with generic fibre isomorphic to $\mathbb{P}^1_{k(C)}$.

Case (3)

$$\begin{tikzcd}
(X,\Delta) \arrow[r, dashed, "\phi"] \arrow[d, "\pi"'] & (X',\Delta') \arrow[r, "h"] \arrow[d, "\pi'"] & Y \arrow[dl, "f"] \\
C \arrow[r, equal] & C &
\end{tikzcd}$$

where

(i) the relative Picard number of h is one,
(ii) X and X' are birational, and $h : X' \to Y$ is a Fano contraction of relative dimension one, and
(iii) Y/C is of relative dimension two.

Case (1): Every irreducible component of Δ' contributes to the relative Picard number of π'. Since the relative Picard number is one, Δ' must be irreducible. It is also geometrically connected since fibres of π' are geometrically connected. By Lemma 4.1 Δ' is *klt*, in particular normal. A normal geometrically connected variety over k is geometrically irreducible. Hence Δ' is geometrically irreducible. Since Δ' is a degeneration of a proper rationally chain connected scheme, it is rationally chain connected. But Δ' is *klt*; hence by [HM07] it is in fact rationally connected. This proves that Δ' is geometrically irreducible as well as rationally connected. By Lemma 4.1(i), $\Delta' = \phi_*\Delta_i$ for some irreducible component Δ_i of Δ. Hence Δ_i is also rationally connected and geometrically irreducible.

Case (2): Let $\{\Delta'_i\}_{1\le i\le s}$ be the irreducible components of Δ' and let $\Delta'_i = \phi_*\Delta_i$. Let $D = \sum_{i=1}^s D_i$. By Proposition 5.5 $(Y, D) \to C$ satisfies the hypothesis of Section 5.3.

We first claim that there exists an irreducible component D_{j_0} of D which is rationally connected and in particular geometrically irreducible. The generic fibre of Y/C is isomorphic to $\mathbb{P}^1$ and hence $(K_Y + D)$ is not pseudo-effective. Thus after running the minimal model program for $(Y, D)/C$ we arrive at the following situation:

$$\begin{tikzcd} (Y,D) \arrow[r, dashed, "\phi"] \arrow[d, "\pi_Y"'] & (Y',D') \arrow[d, "\pi'_Y"] \\ C \arrow[r, equal] & C \end{tikzcd}$$

where π'_Y has relative Picard number one. The claim now follows by an argument similar to that in Case (1).

Now look at Δ'_{j_0}. It is geometrically irreducible by Lemma 5.2. Let $\Delta'' \to \Delta_{j_0}$ be a resolution of singularities. Then by [HM07], (1.2), every fibre of $h' : \Delta'' \to D_{j_0}$ is rationally chain connected. In particular the generic fibre is rationally connected since it is also smooth. This together with [GHS03] and the fact that D_{j_0} is rationally connected proves that Δ'' and hence Δ'_{j_0} is rationally connected. Since Δ_{j_0} and Δ'_{j_0} are birational, Δ_{j_0} is also rationally connected.

Case (3): The proof is similar to Case (2). We need to use induction to prove that there exists a rationally connected subvariety in Y_0. Hence it suffices to prove that when we run the minimal model program for (Y, D), it terminates with a Fano contraction. In other words, it suffices to show the non-pseudo-effectivity of $K_Y + D$. Let X_η (resp. Y_η) denote the generic fibre of X/C (resp. Y/C). Now $\phi_\eta : X_\eta \to Y_\eta$ is a Fano contraction of an extremal ray of K_{X_η}, and X_η is a terminal three-fold. By applying ([MP06], 1.2.7) we conclude that Y_η contains only Du Val singularity of type A. In particular, since Y_η is rationally connected, it is a rational surface with at worse canonical singularities. Let $d : \overline{Y}_\eta \to Y_\eta$ be the minimal resolution of Y_η. Since $\overline{Y}_\eta$ is a smooth rationally connected variety, $K_{\overline{Y}_\eta}$ is not pseudo-effective. But $d^*K_{Y_\eta} = K_{\overline{Y}_\eta}$. Thus K_{Y_η} cannot be pseudo-effective. Since $K_Y + D|_{Y_\eta} = K_{Y_\eta}$, we conclude that $K_Y + D$ is not pseudo-effective. $\square$

Remark 5.12. One can also try to answer Questions 1.1 or 1.8 in higher dimensions by the same method as above. If the argument of the above type works for higher dimensions, one would not have to deal with the case of degeneration of Fano varieties separately. The missing property here is the non-pseudo-effectivity of $K_Y + D$ where (Y, D) is a result of a Fano contraction. This non-pseudo-effectivity is by no means automatic except in the case when relative dimension of Y/C is one. In fact, in dimension ≥ 2, Kollár has constructed a family of rational varieties with only quotient singularities, but whose canonical divisor is $\mathbb{Q}$-ample (cf. [Kol06]). As a stronger evidence to show the limitation of our approach towards Question 1.8, Prokhorov even constructs an example of $h : X \to Y$, a Fano contraction, where X is a rationally connected variety of terminal singularity but Y has a $\mathbb{Q}$-effective canonical bundle.

Example 5.13 (Prokhorov)**.** Let S be the surface $x^n + y^n + z^n + w^n = 0$ in $\mathbb{P}^3$. The group $G = \mathbb{Z}/n$ acts on $\mathbb{P}^3$ and S with weights $(0, 0, 1, 1)$. Let $Y := S/G$, and $\pi : S \to Y$ is the quotient morphism. Since π is étale outside finite points, $\pi^*(K_Y) = K_S$. So when $n \geq 4$, K_Y is $\mathbb{Q}$-effective. We claim Y is rational. To see this, note that the projection map $p : S \to \mathbb{P}^1$ by sending (x, y, z, w) to (x, y) factors through Y. The generic fibre of the map p is birational to the affine curve $(z^n + w^n + \lambda = 0) \subset \mathbb{A}^2$, so the generic fibre of the map from Y to $\mathbb{P}^1$ is birational to the curve $(z^n + w^n + \lambda = 0)/G$, which is also rational. Thus Y is a rational surface.

Now consider the action of $G(=\mathbb{Z}/n)$ on $\mathbb{P}^{n-1}$ with the weights $(0, 1, ..., n-1)$. We define $X := (\mathbb{P}^{n-1} \times S)/G$, where G acts diagonally. For a fixed point x of a $g \in G$ on $(\mathbb{P}^{n-1} \times S)/G$, write the eigenvalue of g on T_x as $e(r_1), ..., e(r_{n+1})$, where $e(x) := e^{2\pi i x}$ and $0 \leq r_i < 1$. Then to prove X contains at worse terminal singularities, it suffices to prove that for any nontrivial element g and any x stablized by g, the *age*, defined as follows, satisfies

$$\mathrm{age}_x(g) := r_1 + \cdots + r_{n+1} > 1$$

(cf. [Re], Theorem 4.6). When $n \geq 4$, we can see that the above condition holds in our case.

We define the map $h : X \to Y$ by descending the second projection of $p_2 : \mathbb{P}^{n+1} \times S \to S$, which is equivariant. It is easy to see that h is a Mori contraction of an extremal ray. Since any terminal variety is an output of a minimal model program which starts with a smooth variety (simply reverse the desingularization process), we in fact have an example where we start with a smooth rationally connected variety, but running the minimal model program terminates with a contraction to a variety of ample canonical bundle.

Finally we give an example of a simple normal crossing variety of dimension 3 which is rationally chain connected but none of its components are rationally connected.

Example 5.14. Let C be any smooth curve in $\mathbb{P}^2$ of genus $g \geq 2$. Let $i : C \hookrightarrow \mathbb{P}^2$ be the closed embedding of C in $\mathbb{P}^2$. Let

$$\begin{tikzcd} & \mathbb{P}^2 \times_k C = X_1 \\ C \times_k C \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & \\ & \mathbb{P}^2 \times_k C = X_2 \end{tikzcd}$$

be two maps defined by $i_1 = (i, id_C)$ and $i_2 = (id_C, i)$. Glue X_1 and X_2 along the maps i_1, i_2. The resulting three dimensional variety $X = X_1 \cup X_2$ is a simple normal crossing as well as rationally chain connected. But neither X_1 nor X_2 is rationally connected.

6. Rationally connected subvarieties of the exceptional locus

In this section we prove Theorem 1.3. The proof of the following lemma is left to the reader.

Lemma 6.1. *Let X/k be a normal quasi-projective variety. Let $\pi : X' \to X$ be a proper birational morphism. Then after further blowing up X' if necessary we can find an effective exceptional divisor A on X' such that $-A$ is π-ample.*

Proof of Theorem 1.3. For getting hold of a geometrically irreducible subvariety of $\pi^{-1}(p)$ we follow the general idea of the proof of main theorem in [Kol05a]. First by a base extending to $k(p)$ we may assume p is a k-point. By replacing X by a suitable open neighbourhood of p we may assume X is affine. We claim that there exists an effective $\mathbb{Q}$-divisor $\Delta_1 \sim_{\mathbb{Q}} 0$ passing through p such that $(X, \Delta + \Delta_1)$ is *klt* outside p and log canonical at p. Since X is affine, taking Δ_1 to be a suitable rational multiple of a sufficiently general section of $\mathcal{O}_X$ passing through p with high multiplicity does the job.

By further blowing up X' we may assume

(i) $\pi : X' \to X$ is a log resolution of $(X, \Delta + \Delta_1)$,
(ii) $\pi^{-1}(p)$ is a divisor, and
(iii) there is an effective exceptional divisor A on X' such that $-A$ is π-ample.

Let $\{E_i\}_{1\leq i\leq r}$ be the exceptional divisors lying over p and let $\{F_j\}_{1\leq j\leq s}$ be the remaining exceptional divisors of π.

Henceforth we will use the notation "$\sim_{\mathbb{Q}}$" to denote $\mathbb{Q}$-numerical equivalence of $\mathbb{Q}$-divisors relative to π. (Since X is affine, numerical equivalence relative to π is nothing but usual numerical equivalence.) In other words, for two $\mathbb{Q}$-Cartier divisors A and B, we write $A \sim_{\mathbb{Q}} B$ iff

$$A \cdot \eta = B \cdot \eta,$$

for any $\eta \in N_1(X'/X)$. Then we have

$$K_{X'} \sim_{\mathbb{Q}} \sum_{i=1}^{r} a_i E_i + \sum_{j=1}^{s} b_j F_j - \Delta',$$

where

(i) $a_i \geq -1 \ \forall\, i$ with equality holding for at least one i,
(ii) $b_j > -1 \ \forall j$, and
(iii) Δ' is the birational transform of $\Delta + \Delta_1$ hence $\lfloor \Delta' \rfloor = 0$.

Let

$$A = \sum_{i=1}^{r} c_i E_i + \sum_{j=1}^{s} d_j F_j.$$

By replacing A by $\frac{1}{n}A$ for n large enough we may assume that the coefficients c_i and d_j are sufficiently small as compared to one. For small rational numbers $\epsilon > 0$ and $\{\delta_i\}_{1\leq i\leq r}$ consider the following divisor:

$$K_{X'} - \pi^*(K_X + \Delta + (1-\epsilon)\Delta_1) - (A + \sum_i \delta_i E_i) \sim_{\mathbb{Q}} K_{X'} - (A + \sum_i \delta_i E_i) - \Delta'(\epsilon)$$
$$\sim_{\mathbb{Q}} \sum_{i=1}^{r} a_i(\epsilon, \delta) E_i + \sum_{j=1}^{s} b_j(\epsilon) F_j - \Delta'(\epsilon),$$

where $\Delta'(\epsilon)$ is the birational transform of $\Delta + (1-\epsilon)\Delta_1$. Notice that $b_j(\epsilon) > -1$ for all $\epsilon > 0$ and all j. Choose ϵ and $\delta_i' s$ such that

(i) $A + \sum_i \delta_i E_i = A'$ is effective and such that $-A'$ is π-ample (this is true for all δ_i small enough because ampleness is an open condition), and
(ii) $a_i(\epsilon, \delta) \geq -1$ with equality holding for exactly one i (say $i = 1$).

Thus we can write

$$K_{X'} - A' \sim_{\mathbb{Q}} -E_1 + E - M,$$

where

$$E = \sum_{i\geq 2} \lceil a_i(\epsilon, \delta) \rceil E_i + \sum_j \lceil b_j(\epsilon) \rceil F_j$$

is an effective exceptional divisor and M is an effective *snc* divisor (having no common components with E_1) and such that $\lfloor M \rfloor = 0$.

Thus by the relative Kawamata-Viehweg vanishing theorem we get

$$R^1\pi_*\mathcal{O}_{X'}(-E_1+E)=0.$$

This gives us a surjection

$$\mathcal{O}_X \cong \pi_*\mathcal{O}_{X'}(E) \to \pi_*\mathcal{O}_{E_1} \to 0$$

which proves E_1 is geometrically irreducible.

We claim that E_1 is also rationally connected. Since $-A'$ is π-ample and since X is affine, it is actually ample. Choose a general divisor W such that $-A' \sim_{\mathbb{Q}} W$ and such that $\lfloor W \rfloor = 0$. Since W is general, it has simple normal crossings with all exceptional divisors and also with M. Let $M' = M + W$. Then M' supports an ample divisor, and we have

$$K_{X'} + E_1 + M' \sim_{\mathbb{Q}} E.$$

But E is an effective divisor having no common components with E_1. Thus $K_{E_1} + M'_{|E_1}$ is effective and as in proof of Theorem 3.1 rational connectivity of E_1 will follow by ([HM07], 4.1) once we show $\kappa(K_{E_1} + M'_{|E_1}) = 0$. As in the proof of Lemma 3.4 this follows from Theorem 3.3 and using the fact that E is effective exceptional. □

Remark 6.2. Theorem 1.3 can fail for varieties having log canonical singularities. See ([Kol05a], 9) for an example.

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Department of Mathematics, Princeton University, Princeton, New Jersey 08544
E-mail address: amit@math.princeton.edu

Current address: School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Mumbai 400005 India

Department of Mathematics, Princeton University, Princeton, New Jersey 08544
E-mail address: chenyang@math.princeton.edu