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BESSEL POTENTIALS, HITTING DISTRIBUTIONS AND GREEN FUNCTIONS

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ABSTRACT. The purpose of the paper is to find explicit formulas for basic objects pertaining to the potential theory of the operator $(I-\Delta)^{\alpha/2}$, which is based on Bessel potentials $J_\alpha = (I-\Delta)^{-\alpha/2}$, $0<\alpha<2$. We compute the harmonic measure of the half-space and obtain a concise form for the corresponding Green function of the operator $(I-\Delta)^{\alpha/2}$. As an application we provide sharp estimates for the Green function of the half-space for the relativistic process.

1. INTRODUCTION

In this paper we deal with the potential theory of $(I-\Delta)^{\alpha/2}$, $0<\alpha<2$, where Δ is the Laplace operator on $\mathbb{R}^d$. The (formal) inverse $J_\alpha = (I-\Delta)^{-\alpha/2}$ is called *the Bessel potential operator*, and it has an integral representation with the following (*Bessel*) convolution kernel:

$$\frac{2^{1-(d+\alpha)/2}}{\Gamma(\alpha/2)\pi^{d/2}} \frac{K_{(d-\alpha)/2}(|x|)}{|x|^{(d-\alpha)/2}}, \tag{1.1}$$

where K_ϑ denotes *the modified Bessel function of the third kind* (see Section 2, Preliminaries). The reader interested in properties of Bessel potentials is referred to an exhaustive treatise [A1], [A2]. The significance of Bessel potentials is that *the Sobolev space* $L^p_\alpha(\mathbb{R}^d)$ can be defined in terms of J_α. To be more specific, we define $L^p_\alpha(\mathbb{R}^d)$ as the subspace of $L^p(\mathbb{R}^d)$, consisting of all f which can be written in the form $f = J_\alpha g$, $g \in L^p(\mathbb{R}^d)$. The norm of f is written as $||f||_{p,\alpha}$, and it is defined as equal to the L^p norm of g (see [S], ch. V). The spaces $L^p_\alpha(\mathbb{R}^d)$ play an important rôle in Harmonic Analysis and Partial Differential Equations (see e.g. [S] and [H]).

Closely related to the operator $(I-\Delta)^{\alpha/2}$ is *the fractional Laplacian* $(-\Delta)^{\alpha/2}$. Its inverse $I_\alpha = (-\Delta)^{-\alpha/2}$, called *the Riesz potential operator*, has an integral (*Riesz*) convolution kernel of the form

$$\frac{\Gamma((d-\alpha)/2)}{2^\alpha \pi^{d/2}\Gamma(\alpha/2)} \frac{1}{|x|^{d-\alpha}}, \quad \text{if } \alpha < d,\ 0<\alpha<2. \tag{1.2}$$

If $\alpha \geq d = 1$ we consider instead the so-called *compensated potentials* (see e.g. [BGR]). For $\alpha > d = 1$ they are of the same form as above. When $\alpha = d = 1$

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the compensated (*logarithmic*) potential is $(1/\pi)\ln(1/|x|)$. The reader interested in potential theory based on Riesz kernels is referred to [La].

The potential theories based on kernels J_α and I_α can be analyzed in terms of stochastic processes (for $0 < \alpha < 2$): the Riesz potentials can be expressed by means of *the α-stable rotation invariant Lévy process*, and the Bessel potentials are related to the so-called *relativistic α-stable process*.

The homogeneity of Riesz kernels yields elegant and transparent formulas of the potential theory of $(-\Delta)^{\alpha/2}$, much like in the Newtonian case. In particular, explicit formulas for the Poisson kernel and the Green function for the ball and half-space in $\mathbb{R}^d$ (see e.g. [BGR]) are available.

In contrast to this situation, up to now, there were no explicit formulas known either for harmonic measure or the Green function for the operator $(I-\Delta)^{\alpha/2}$, for sets such as half-spaces or balls. The purpose of the present paper is to fill in this gap by providing explicit formulas for half-spaces.

To formulate our main results denote by $\mathbb{H} = \{(x_1, \ldots, x_d);\, x_d > 0\}$ the upper half-space in $\mathbb{R}^d$ and let $P_{\mathbb{H}}(x,u)$ be *the Poisson kernel* for $\mathbb{H}$, that is, the u-density of *the harmonic measure* of $\mathbb{H}$ for $(I-\Delta)^{\alpha/2}$. Our first main result is the following:

Poisson kernel of $\mathbb{H}$ for $(I-\Delta)^{\alpha/2}$.

$$P_{\mathbb{H}}(x,u) = \frac{2}{\pi}\frac{\sin(\pi\alpha/2)}{(2\pi)^{d/2}}\left(\frac{x_d}{-u_d}\right)^{\alpha/2}\frac{K_{d/2}(|x-u|)}{|x-u|^{d/2}}, \quad u_d < 0 < x_d.$$

The second main result can be stated as follows:

Green function of $\mathbb{H}$ for $(I-\Delta)^{\alpha/2}$.

$$G_{\mathbb{H}}(x,y) = \frac{2^{1-\alpha}|x-y|^{\alpha-d/2}}{(2\pi)^{d/2}\Gamma(\alpha/2)^2}\int_0^{\frac{4x_d y_d}{|x-y|^2}} \frac{t^{\frac{\alpha}{2}-1}}{(t+1)^{d/4}} K_{d/2}(|x-y|(t+1)^{1/2})dt, \quad x,y \in \mathbb{H}.$$

We note that, in spite of the fact that our kernels are not homogeneous, the resulting formulas are transparent and very similar to those for the α-stable case.

Throughout the paper we employ a mixture of probabilistic and analytic methods, even though an analytic approach to the above results is fully feasible. We consider the operator $H_\alpha = I - (I-\Delta)^{\alpha/2}$, which is *the infinitesimal generator* of the relativistic α-stable process. We then identify Bessel kernels as kernels of a 1-*resolvent* for the semigroup associated with the relativistic α-stable process and compute the corresponding 1-Poisson kernel and 1-Green function for half-spaces.

Let us note that the relativistic α-stable process, apart from its usefulness in analyzing $(I-\Delta)^{\alpha/2}$, is an interesting subject of study on its own, mainly because of its applications in relativistic quantum mechanics. To be more specific, let us point out that for $\alpha = 1$ the generator of this process has the form

$$H_1 = I - (I-\Delta)^{1/2}$$

and represents the kinetic energy of a relativistic particle with the unit mass. Generators of this kind were investigated for example by E. Lieb [L] in connection with the problem of stability of relativistic matter. An interested reader will find references on this subject in the papers [C], [Ry].

The organization of the paper is as follows. We first compute the formulas for the harmonic measure and Green function for the one-dimensional case. To this end we apply complex-variable methods and some real-variable manipulations with definite integrals to obtain a satisfactory form of the Green function. The

d-dimensional case is then settled via an application of $(d-1)$-dimensional Fourier transform. For technical reasons, we have to consider the Poisson kernel and Green function not only for the operator $(I-\Delta)^{\alpha/2}$ but also for $(m^{2/\alpha}I-\Delta)^{\alpha/2}$. Let us point out that we do not apply Kelvin's transform (see [La], IV. 5), which was an indispensable tool in the multi-dimensional α-stable case so far. By a limiting procedure we obtain the well-known formulas for the α-stable case (see e.g. [R]). The last section is devoted to various estimates for the Green function of the half-space $\mathbb{H}$, computed for the relativistic α-stable process (that is, for the operator $H_\alpha = I-(I-\Delta)^{\alpha/2}$). To distinguish it from the corresponding object computed for the operator $(I-\Delta)^{\alpha/2}$ we call it a *0-Green function*. Our estimates are sharp for $x, y \in \mathbb{H}$ with $|x-y|<1$.

2. Preliminaries

In this section we collect some preliminary material. For more detailed information regarding the α-stable relativistic process, see [Ry] and [C]. For questions regarding Markov processes, semigroup properties and basic potential theory the reader is referred to [ChZ] and [BG].

2.1. **Basic notation.** By $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$, we denote the set of all *positive integers, integers, real numbers, and complex numbers*, respectively. $\mathbb{R}^d$ denotes the *d-dimensional Euclidean space* and (x,y) denotes the standard *inner product* of $x, y \in \mathbb{R}^d$, $(x,y)=\sum_{i=1}^d x_i y_i$, $x=(x_1,\ldots,x_d), y=(y_1,\ldots,y_d)$, and $|\cdot|$ is the norm induced by $(\cdot,\cdot)$. We denote $\mathbb{R}_+=[0,\infty)$. For any subset A of $\mathbb{R}^d$, we denote its *complement* by $A^c=\mathbb{R}^d\setminus A$, its *closure* by $\bar{A}$, and its *boundary* by ∂A. By $\mathbf{1}_A$ we denote the indicator function of a subset A of $\mathbb{R}^d$.

2.2. **Bessel functions.** Various potential-theoretic objects in the theory of the relativistic process are expressed in terms of modified Bessel functions K_ϑ of the third kind, also called Macdonald functions. For convenience here we collect basic information about these functions.

The modified Bessel function I_ϑ of the first kind is defined by (see e.g. [E], 7.2.2 (12))

$$I_\vartheta(z)=\left(\frac{z}{2}\right)^\vartheta \sum_{k=0}^\infty \left(\frac{z}{2}\right)^{2k}\frac{1}{k!\Gamma(k+\vartheta+1)}, \quad z\in\mathbb{C}\setminus(-\mathbb{R}_+), \tag{2.1}$$

where $\vartheta\in\mathbb{R}$ is not an integer and z^ϑ is the branch that is analytic on $\mathbb{C}\setminus(-\mathbb{R}_+)$ and positive on $\mathbb{R}_+\setminus\{0\}$. The modified Bessel function of the third kind is defined by (see [E], 7.2.2 (13) and (36))

$$K_\vartheta(z) = \frac{\pi}{2\sin(\vartheta\pi)}\left[I_{-\vartheta}(z)-I_\vartheta(z)\right], \quad \vartheta\notin\mathbb{Z}, \tag{2.2}$$

$$K_n(z) = \lim_{\vartheta\to n} K_\vartheta(z) = (-1)^n\frac{1}{2}\left[\frac{\partial I_{-\vartheta}}{\partial\vartheta}-\frac{\partial I_\vartheta}{\partial\vartheta}\right]_{\vartheta=n}, \quad n\in\mathbb{Z}. \tag{2.3}$$

The asymptotic expansion of $K_\vartheta(z)$ is given by (see [E], 7.4.1. (4))

$$K_\vartheta(z)=\left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\left[\sum_{k=0}^{M-1}\frac{\Gamma(1/2+\vartheta+k)}{k!\Gamma(1/2+\vartheta-k)}(2z)^{-k}+O(|z|^{-M})\right], \tag{2.4}$$

where $M=1,2,\ldots$, and $-3\pi/2<\arg z<3\pi/2$.

Observe that the series in (2.1) defines an analytic function of the complex variable z. From (2.1), (2.2) and (2.3) we have that $K_\vartheta(z)$ has a branch cut along the negative real axis in the complex z-plane. Moreover, the following formula for any integer k holds (see [E], 7.11. (45)):

$$K_\vartheta(ze^{ik\pi}) = e^{-ik\pi\vartheta}K_\vartheta(z) - i\pi\frac{\sin(k\pi\vartheta)}{\sin(\pi\vartheta)}I_\vartheta(z), \quad \vartheta \notin \mathbb{Z}. \tag{2.5}$$

When ϑ is an integer n we put

$$\frac{\sin(k\pi\vartheta)}{\sin(\pi\vartheta)} := k(-1)^{n(k+1)} \quad \left(= \lim_{\zeta\notin\mathbb{Z},\zeta\to n}\frac{\sin(k\pi\zeta)}{\sin(\pi\zeta)}\right).$$

We will also use the following integral representations of the function $K_\vartheta(z)$ ([E], 7.11 (23) or [GR], 8.432 (6)):

$$K_\vartheta(z) = 2^{-\vartheta-1}z^\vartheta\int_0^\infty e^{-t}e^{-\frac{z^2}{4t}}t^{-\vartheta-1}\,dt, \tag{2.6}$$

where $\Re(z^2) > 0$, $|\arg z| < \frac{\pi}{2}$. Moreover (see [GR], 8.432 (3)),

$$K_\vartheta(z) = \left(\frac{z}{2}\right)^\vartheta\frac{\Gamma(1/2)}{\Gamma(\vartheta+1/2)}\int_1^\infty\frac{e^{-zt}}{(t^2-1)^{1/2-\vartheta}}\,dt, \tag{2.7}$$

where $\Re(\vartheta+1/2) > 0$, $|\arg z| < \frac{\pi}{2}$. In the sequel we will use the asymptotic behaviour of K_ϑ, $\vartheta > 0$, as a function of the real variable r:

$$K_\vartheta(r) \cong \frac{\Gamma(\vartheta)}{2}\left(\frac{r}{2}\right)^{-\vartheta} \qquad r \to 0^+, \tag{2.8}$$

$$K_\vartheta(r) \cong \frac{\sqrt{\pi}}{\sqrt{2r}}\,e^{-r}, \quad r \to \infty, \tag{2.9}$$

where $g(r) \cong f(r)$ means that the ratio of g and f tends to 1. For $\vartheta < 0$ we have $K_\vartheta(r) = K_{-\vartheta}(r)$, which determines the asymptotic behaviour for negative indices. We also state the following results concerning differentiability properties of the functions $K_\vartheta(r)$ with respect to the real variable $r > 0$ (see [E], 7.21, 7.22):

$$\frac{d}{dr}[r^{-\vartheta}K_\vartheta(r)] = -r^{-\vartheta}K_{\vartheta+1}(r)\,, \quad \frac{d}{dr}[r^{\vartheta}K_\vartheta(r)] = -r^{\vartheta}K_{\vartheta-1}(r)\,.$$

Consequently, for fixed $\vartheta > 0$, we obtain

$$r^{-\vartheta}K_\vartheta(r),\ r^\vartheta K_\vartheta(r) \text{ are decreasing in } r > 0\,. \tag{2.10}$$

2.3. **Relativistic processes.** A stochastic process $Y_t, t \geq 0$, is called *a Lévy* process on $\mathbb{R}^d$ if it has stationary independent increments, it is stochastically continuous, that is, $\lim_{s\to t} P(|Y_s - Y_t| > \epsilon) = 0$ for every $\epsilon > 0$, and it is right-continuous with left-hand limits (see e.g. [Sa]). Observe that we do not assume that $Y_0 = 0$, because our processes start from various points of $\mathbb{R}^d$. As usual, by E^x we denote the expectation with respect to the distribution P^x of the process starting from $x \in \mathbb{R}^d$. Every Lévy process Y_t is Markov with transition probabilities given by $P_t(x, A) = P^x(Y_t \in A) = \mu_t(A-x)$, where μ_t is the one-dimensional distribution of Y_t with respect to P^0. When $P_t(x, A) = \int_A p_t(x, y)\,dy$ then $p_t(x, y) = p_t(x-y, 0)$, and, with a certain abuse of notation, we denote this last object by $p_t(x-y)$ and call $p_t(x)$ *the transition density function* of the process Y_t.

A subordinator $W_t, t \geq 0$, is a one-dimensional Lévy process with increasing sample paths and such that $W_0 = 0$. According to general theory (see again [Sa], Ch. 6), if Y_t is a Lévy process on $\mathbb{R}^d$ and W_t is a subordinator and the processes Y_t

and W_t are independent, then $X_t = Y_{W_t}$ is another Lévy process. We now adapt this procedure to our needs, specifying processes Y_t and W_t.

We begin by recalling the definition of the standard $\alpha/2$-stable subordinator S_t^α with the Laplace transform $E^0 e^{-\lambda S_t^\alpha} = e^{-t\lambda^{\alpha/2}}$. Throughout the entire paper α denotes the stability index of the process and we always assume $0 < \alpha < 2$. The transition density function of S_t^α will be denoted by $\theta_t^\alpha(u)$. Here $u, t > 0$.

For $m > 0$ we define another subordinating process $T_t^{\alpha,m}$ by modifying $\theta_t^\alpha(u)$ as follows:

$$\theta_t^{\alpha,m}(u) = e^{mt}\,\theta_t^\alpha(u)\,e^{-m^{2/\alpha}u}, \quad u > 0\,. \tag{2.11}$$

The Laplace transform of $T_t^{\alpha,m}$ is

$$E^0 e^{-\lambda T_t^{\alpha,m}} = e^{mt}\,e^{-t(\lambda+m^{2/\alpha})^{\alpha/2}}\,.$$

Let B_t be the Brownian motion in $\mathbb{R}^d$ with the characteristic function $E^0 e^{i(\xi, B_t)} = e^{-t|\xi|^2}$. The transition density function of B_t is denoted by g_t and is of the form

$$g_t(u) = \frac{1}{(4\pi t)^{d/2}}\,e^{-|u|^2/4t}\,.$$

Assume that the processes $T_t^{\alpha,m}$ and B_t are stochastically independent. Then the process $X_t^{\alpha,m} = B_{T_t^{\alpha,m}}$ is called *the α-stable relativistic process* (with parameter m). In the sequel we will use the generic notation X_t^m instead of $X_t^{\alpha,m}$. If $m = 1$, we then write T_t^α instead of $T_t^{\alpha,1}$ and X_t instead of X_t^1.

When $m = 0$ we obtain *the α-stable rotation invariant Lévy process* which is denoted by Z_t.

We obtain

$$\begin{aligned} P^x(X_t^m \in A) \quad &= \quad E^x[\mathbf{1}_A(B_{T_t^{\alpha,m}})] \\ = \int_0^\infty \left[\int_A g_u(x-y)\,dy\right]\theta_t^{\alpha,m}(u)\,du \quad &= \quad \int_A \left[\int_0^\infty g_u(x-y)\,\theta_t^{\alpha,m}(u)\,du\right]dy\,. \end{aligned}$$

This provides the formula for the transition density function of the process X_t^m:

$$p_t^m(x) = \int_0^\infty \theta_t^{\alpha,m}(u)\,g_u(x)\,du\,, \tag{2.12}$$

where $p_t^m(x)\,, t > 0$, is a semigroup under convolution. A particular case when $\alpha = 1$ yields *the relativistic Cauchy semigroup* on $\mathbb{R}^d$ with parameter m and is denoted by $\tilde{p}_t^m$. The formula below exhibits the explicit form of this transition density function.

Lemma 2.1 (Relativistic Cauchy semigroup)**.**

$$\tilde{p}_t^m(x) = 2(m/2\pi)^{(d+1)/2}\,te^{mt}\frac{K_{(d+1)/2}(m(|x|^2+t^2)^{1/2})}{(|x|^2+t^2)^{\frac{d+1}{4}}}\,. \tag{2.13}$$

Proof. Observe that $\theta_t^1(u)$, the transition density function of the 1/2-stable subordinator, is of the form

$$\theta_t^1(u) = \frac{t}{\sqrt{4\pi}}u^{-3/2}e^{-t^2/4u}\,,$$

so, taking into account (2.6), we obtain

$$\begin{aligned}\tilde{p}_t^m(x) &= e^{mt}\int_0^\infty \frac{1}{(4\pi u)^{d/2}} e^{-|x|^2/4u} e^{-m^2 u}\frac{t}{\sqrt{4\pi}} u^{-3/2} e^{-t^2/4u}\, du\\ &= \frac{te^{mt}}{(4\pi)^{\frac{d+1}{2}}}\int_0^\infty e^{-m^2u} e^{-(|x|^2+t^2)/4u}\frac{du}{u^{\frac{d+1}{2}+1}}\\ &= 2(m/2\pi)^{(d+1)/2}\, te^{mt}\frac{K_{(d+1)/2}(m(|x|^2+t^2)^{1/2})}{(|x|^2+t^2)^{\frac{d+1}{4}}}. \end{aligned}$$ □

In the next lemma we compute the Fourier transform of the transition density function (2.12).

Lemma 2.2 (Fourier transform of p_t^m)**.** *The Fourier transform of the α-stable relativistic transition density function p_t^m is of the form*

$$\widehat{p_t^m}(z) = e^{mt} e^{-t(|z|^2+m^{2/\alpha})^{\alpha/2}}.$$

Proof.

$$\begin{aligned}\widehat{p_t^m}(z) &= \int_{\mathbb{R}^d} p_t^m(x) e^{i(z,x)}\, dx = \int_{\mathbb{R}^d}\int_0^\infty e^{mt} g_u(x) e^{-m^{2/\alpha}u}\theta_t^\alpha(u)\, du\, e^{i(z,x)}\, dx\\ &= e^{mt}\int_0^\infty e^{-u|z|^2} e^{-m^{2/\alpha}u}\theta_t^\alpha(u)\, du = e^{mt}\int_0^\infty e^{-u(|z|^2+m^{2/\alpha})}\theta_t^\alpha(u)\, du\\ &= e^{mt} e^{-t(|z|^2+m^{2/\alpha})^{\alpha/2}}. \end{aligned}$$ □

Specializing this to the case $\alpha = 1$ we obtain

$$\widehat{\tilde{p}_t^m}(z) = e^{mt} e^{-t(|z|^2+m^2)^{1/2}}. \tag{2.14}$$

From the Fourier transform we obtain the following scaling property:

$$p_t^m(x) = m^{d/\alpha} p_{mt}^1(m^{1/\alpha} x). \tag{2.15}$$

In terms of one-dimensional distributions of the relativistic process (starting from the point 0) (2.15) reads as

$$X_t^m \sim m^{-1/\alpha} X_{mt}^1,$$

where X_t^m denotes the relativistic α-stable process with parameter m and "$\sim$" denotes the equality of distributions. Because of this scaling property, we often restrict our attention to the case when $m = 1$, if not specified otherwise. When $m = 1$ we omit the superscript "1", i.e. we write $p_t(x)$ instead of $p_t^1(x)$.

In what follows we will work within the framework of the so-called λ-*potential theory*, for $\lambda > 0$.

The kernel of the λ-resolvent of the semigroup generated by X_t^m will be denoted by $U_\lambda^m(x)$ and will be called the λ-*potential* of the process X_t^m. We have

$$U_\lambda^m(x) = \int_0^\infty e^{-\lambda t}\, p_t^m(x)\, dt.$$

The function has a particularly simple expression when $\lambda = m$, and we state it for further reference.

Lemma 2.3 (m-potential for the relativistic process with parameter m)**.**

$$U_m^m(x) = \frac{2^{1-(d+\alpha)/2}}{\Gamma(\alpha/2)\pi^{d/2}} \frac{m^{\frac{d-\alpha}{2\alpha}}\, K_{(d-\alpha)/2}(m^{1/\alpha}|x|)}{|x|^{(d-\alpha)/2}}\,. \tag{2.16}$$

Proof. We provide calculations for $m = 1$; the general case follows from (2.15). Observe first that the potential kernel of the $\alpha/2$-stable subordinator is well known (and easy to obtain via the Laplace transform). Namely, we have

$$\int_0^\infty \theta_t^\alpha(u)dt = \frac{u^{\alpha/2-1}}{\Gamma(\alpha/2)}\,.$$

This and (2.6) yield

$$\begin{aligned} U_1(x) &= \int_0^\infty e^{-t} p_t(x)\,dt = \int_0^\infty \int_0^\infty g_u(x) e^{-u} \theta_t^\alpha(u)\,du\,dt \\ &= \int_0^\infty \frac{1}{(4\pi u)^{d/2}} e^{-\frac{|x|^2}{4u}} e^{-u} \left(\int_0^\infty \theta_t^\alpha(u)dt \right) du \\ &= \frac{1}{\Gamma(\alpha/2)(4\pi)^{d/2}} \int_0^\infty e^{-\frac{|x|^2}{4u}} e^{-u} \frac{du}{u^{\frac{d-\alpha}{2}+1}} \\ &= \frac{2^{1-(d+\alpha)/2}}{\Gamma(\alpha/2)\pi^{d/2}} \frac{K_{(d-\alpha)/2}(|x|)}{|x|^{(d-\alpha)/2}}\,. \end{aligned}$$

□

In what follows we denote by U_1 the λ-potential for $\lambda = m = 1$.

We also recall the formula for the density function $\nu^m(x)$ of the Lévy measure of the relativistic α-stable process (see [Ry]).

Lemma 2.4 (Lévy measure of relativistic process with parameter m)**.**

$$\nu^m(x) = \frac{\alpha 2^{\frac{\alpha-d}{2}}}{\pi^{d/2}\Gamma(1-\frac{\alpha}{2})} \frac{m^{\frac{d+\alpha}{2\alpha}}\, K_{\frac{d+\alpha}{2}}(m^{1/\alpha}|x|)}{|x|^{\frac{d+\alpha}{2}}}\,. \tag{2.17}$$

When $m = 1$ we write ν instead of ν^1.

Remark. Observe that the density function of the Lévy measure of the α-stable rotation invariant Lévy process is of the form

$$\nu^\#(x) = \frac{2^\alpha \Gamma(\frac{d+\alpha}{2})}{\pi^{d/2}|\Gamma(-\alpha/2)|} \frac{1}{|x|^{d+\alpha}}\,. \tag{2.18}$$

The formulas (2.18) and (2.17) exemplify a correspondence between the potential theories of $(-\Delta)^{\alpha/2}$ and $(I-\Delta)^{\alpha/2}$, which can be stated as follows: if an object in the potential theory of $(-\Delta)^{\alpha/2}$ is expressed in terms of the kernel $1/|x|^\vartheta$, then the corresponding object in the potential theory of $(I-\Delta)^{\alpha/2}$ has the kernel $K_{\vartheta/2}(|x|)/|x|^{\vartheta/2}$. We observe that this principle holds true for the Lévy measures with $\vartheta = d+\alpha$ and for the potential kernels (1.2) and (1.1) with $\vartheta = d-\alpha$. The objects of the Riesz potential theory are denoted with the supercript "#".

The first exit time from an (open) set $D \subset \mathbb{R}^d$ by the process X_t^m is defined by the formula

$$\tau_D = \inf\{t \geq 0;\, X_t^m \notin D\}\,.$$

To facilitate the discussion we will assume (in what follows) that D is regular, e.g., with smooth boundary. In fact below D will be either a ball or a half-space.

The λ*-harmonic measure* of the set D represents the basic object in the potential theory of X_t^m. In probabilistic terms it is defined as follows:

$$P_D^{\lambda,m}(x,A) = E^x[\tau_D < \infty;\ e^{-\lambda\tau_D}\mathbf{1}_A(X^m_{\tau_D})]. \tag{2.19}$$

Observe that for $x \in D^c$ we obtain $P_D^{\lambda,m}(x,\cdot) = \delta_x(\cdot)$, the point mass at x. Obviously, $P_D^{\lambda,m}(x,\mathbb{R}^d) \le 1$. On the other hand, if $x \in D$, then, by the right-continuity of sample paths, $P_D^{\lambda,m}(x,\cdot)$ is concentrated on D^c. The kernel of the measure $P_D^{\lambda,m}(x,A)$, for $x \in D$ (if it exists), is called the λ-*Poisson kernel* of the set D. When $\lambda = m$ we denote the corresponding λ-harmonic measure by $P_D^m(x,A)$ and its kernel by $P_D^m(x,y)$. The function P_D^m is the primary object of our investigations. When $\lambda = m = 1$ we use the notation $P_D(x,y)$.

In Sections 3 and 4 we prove the existence of $P_{\mathbb{H}}^m(x,y)$, while providing at the same time an explicit formula for it. The existence of a λ-Poisson kernel for general λ can be deduced from papers [IW] and [St], but analogous explicit formulas are not available.

The process $X_t^{m,D}$ *killed* when exiting the set D is described in terms of sample paths up to time τ_D. Its transition probability, $P_t^{m,D}$, is defined by $P_t^{m,D}(x,A) = P^x(t<\tau_D; X_t^m \in A)$, $t>0$. Correspondingly, $p_t^{m,D}$, the transition density function of $X_t^{m,D}$, can be expressed as

$$p_t^{m,D}(x,y) = p_t^m(x-y) - E^x[t \ge \tau_D;\ p^m_{t-\tau_D}(X^m_{\tau_D} - y)]\,, \quad x,y\in\mathbb{R}^d\,.$$

Obviously, we obtain $p_t^{m,D}(x,y) \le p_t^m(x,y)$, $x,y \in \mathbb{R}^d$ and $p_t^{m,D}(x,y) = 0$ if x or y does not belong to D.

$P_t^{m,D}$ is a strongly contractive semigroup (under composition), and it shares many properties of the semigroup P_t^m. In particular, it is strongly Feller and its transition density function is symmetric: $p_t^{m,D}(x,y) = p_t^{m,D}(y,x)$. When $m=1$, we write, as before, p_t^D instead of $p_t^{1,D}$.

The λ-potential kernel of the process $X_t^{m,D}$ is called the λ-*Green function* and is denoted by $G_D^{\lambda,m}$. Thus, we have

$$G_D^{\lambda,m}(x,y) = \int_0^\infty e^{-\lambda t}\, p_t^{m,D}(x,y)\,dt\,.$$

The "first passage time relation" (see [BGR]) provides another important formula for the λ-Green function of the set D, which is expressed in terms of the λ-harmonic measure:

$$G_D^{\lambda,m}(x,y) = U_\lambda^m(x-y) - \int_{\mathbb{R}^d} U_\lambda^m(z-y)\, P_D^{\lambda,m}(x,dz)\,, \quad x\ne y\,, \quad x,y\in\mathbb{R}^d\,. \tag{2.20}$$

Observe that if $x \ne y$ and x or y belongs to D^c, then we obtain $G_D^{\lambda,m}(x,y) = 0$. This yields the following:

$$\int_{D^c} U_\lambda^m(z-y)\, P_D^{\lambda,m}(x,dz) = U_\lambda^m(x-y)\,, \quad x\in D\,, \quad y\in D^c\,. \tag{2.21}$$

We are mainly interested when $\lambda = m$, and we then write G_D^m. If $\lambda = m = 1$ we write G_D. The formula (2.21) is a particular case of "balayage" or "sweeping out" (see [La], V.5) and, together with the following uniqueness lemma, is crucial for obtaining the explicit form of $P_{\mathbb{H}}^m(x,y)$ as well as $G_{\mathbb{H}}^m(x,y)$.

Lemma 2.5 (Uniqueness)**.** *Suppose that μ is a finite signed measure concentrated on a closed set $F \subseteq \mathbb{R}^d$ with the (finite energy) property*

$$\int_F \int_F \frac{K_{(d-\alpha)/2}(m^{1/\alpha}|z-y|)}{|z-y|^{(d-\alpha)/2}} |\mu|(dz)\,|\mu|(dy) < \infty . \tag{2.22}$$

If for every $z \in F$ we have

$$\int_F \frac{K_{(d-\alpha)/2}(m^{1/\alpha}|z-y|)}{|z-y|^{(d-\alpha)/2}}\,\mu(dy) = 0 , \tag{2.23}$$

then $\mu = 0$.

Proof. The proof of the above lemma is standard (see e.g. [BGR]); we include it for convenience of the reader. We provide the details for $m = 1$ only; the general case follows by scaling. Observe that condition (2.22) enables us to integrate equation (2.23) over the set F and apply Fubini's theorem in the calculations below:

$$\begin{aligned} 0 &= \frac{2^{1-(d+\alpha)/2}}{\Gamma(\alpha/2)\pi^{d/2}} \int_F \int_F \frac{K_{\frac{d-\alpha}{2}}(|z-y|)}{|z-y|^{\frac{d-\alpha}{2}}}\,\mu(dz)\,\mu(dy) \\ &= \int_F \int_F \int_0^\infty e^{-t}\,p_t(z-y)\,dt\,\mu(dz)\,\mu(dy) \\ &= (2\pi)^{-d} \int_F \int_F \int_0^\infty \int_{\mathbb{R}^d} e^{-t}\,e^{-i(\xi,z-y)}\,e^t e^{-t(|\xi|^2+1)^{\alpha/2}}\,d\xi\,dt\,\mu(dz)\,\mu(dy) \\ &= (2\pi)^{-d} \int_0^\infty \int_{\mathbb{R}^d} e^{-t(|\xi|^2+1)^{\alpha/2}} \left| \int_F e^{-i(\xi,z)}\mu(dz) \right|^2 d\xi\,dt . \end{aligned}$$

This shows that the Fourier transform of the measure μ is zero, so the measure itself vanishes, which concludes the proof. ☐

We now state some scaling properties for $P_D^m(x,y)$, $G_D^m(x,y)$. The proof employs the scaling property (2.15) and consists of elementary but tedious calculation; hence it is omitted.

Lemma 2.6 (Scaling property)**.** *We have*

$$\begin{aligned} P_D^m(x,u) &= m^{d/\alpha} P_{m^{1/\alpha}D}(m^{1/\alpha}x, m^{1/\alpha}u), \quad x \in D, u \in D^c , \\ G_D^m(x,y) &= m^{(d-\alpha)/\alpha} G_{m^{1/\alpha}D}(m^{1/\alpha}x, m^{1/\alpha}y), \quad x \in D, y \in D . \end{aligned}$$

In particular, if $D = \mathbb{H}$ we obtain

$$\begin{aligned} P_{\mathbb{H}}^m(x,u) &= m^{d/\alpha} P_{\mathbb{H}}(m^{1/\alpha}x, m^{1/\alpha}u), \quad x \in \mathbb{H}, u \in \mathbb{H}^c , \\ G_{\mathbb{H}}^m(x,y) &= m^{(d-\alpha)/\alpha} G_{\mathbb{H}}(m^{1/\alpha}x, m^{1/\alpha}y), \quad x \in \mathbb{H}, y \in \mathbb{H} . \end{aligned}$$

3. One-dimensional case

This section is basic for the whole paper. Here we establish the formulas for the Poisson kernel and Green function of $(0,\infty)$ for the operator $(m^{2/\alpha}I - \frac{d^2}{dx^2})^{\alpha/2}$. The presentation is divided into three parts. The first one relies on complex integration.

3.1. **Complex-variable method.** The following lemma is crucial for further purposes.

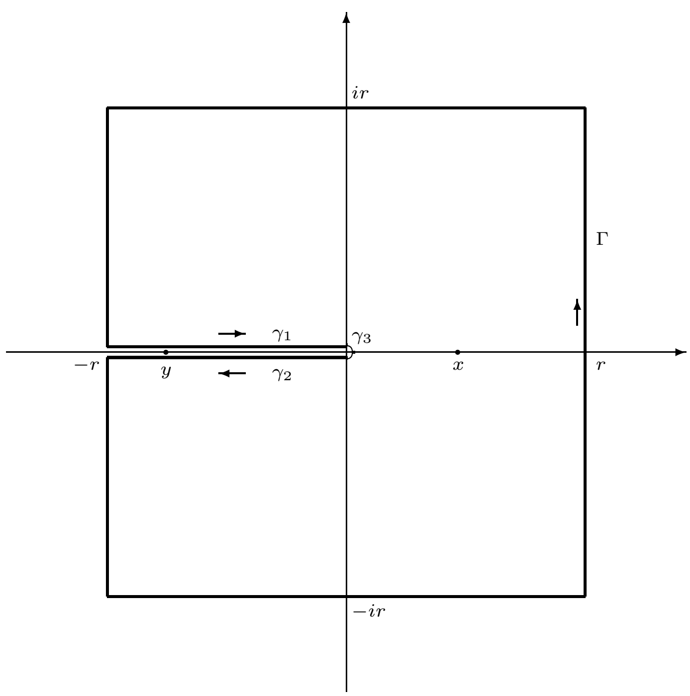
Lemma 3.1. *For $x > 0 \geq y$ we have*

$$\frac{\sin(\frac{\pi\alpha}{2})}{\pi} \int_{-\infty}^{0} \left(\frac{x}{-u}\right)^{\frac{\alpha}{2}} \frac{e^{-|x-u|}}{|x-u|} \frac{K_{\frac{1-\alpha}{2}}(|u-y|)}{|u-y|^{\frac{1-\alpha}{2}}}\, du = \frac{K_{\frac{1-\alpha}{2}}(|x-y|)}{|x-y|^{\frac{1-\alpha}{2}}}\,. \tag{3.1}$$

Proof. Let $x > 0 > y$ and consider the following function of the complex variable z:

$$f(z) = \frac{1}{z^{\frac{\alpha}{2}}} \frac{e^{z-x}}{z-x} \frac{K_{\frac{1-\alpha}{2}}(z-y)}{(z-y)^{\frac{1-\alpha}{2}}}\,.$$

Due to the properties of the function K_ϑ given in the Preliminaries, it is easy to see that f is a holomorphic function in $\mathbb{C}\backslash(-\infty,0]\backslash\{x\}$ with a branch cut along the negative real axis and has a simple pole at $z = x$. We are going to integrate the above function over the contour described below. We make the branch cut along the axis $(-\infty, 0]$ and make the contour of integration wrap around this line (see the picture below). We choose the intervals γ_1 and γ_2 in such a way that their points are at the distance $0 < \epsilon < 1$ from the negative axis.



By the Cauchy theorem, we get

$$\frac{1}{2\pi i} \int_{\Gamma} f(z)\, dz + \frac{1}{2\pi i} \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} \right) f(z)\, dz = \mathrm{Res}_{z=x} f(z)\,. \tag{3.2}$$

The asymptotic expansion for the Macdonald function (2.4), which is valid for $-\frac{3}{2}\pi < \arg z < \frac{3}{2}\pi$, gives

$$\begin{aligned} f(z) &= \frac{1}{z^{\frac{\alpha}{2}}} \frac{e^{z-x}}{z-x} \frac{e^{y-z}}{(z-y)^{1-\frac{\alpha}{2}}} R_0(z-y) \\ &= e^{y-x} \frac{1}{z^{\frac{\alpha}{2}}(z-x)(z-y)^{1-\frac{\alpha}{2}}} R_0(z-y)\,, \end{aligned} \tag{3.3}$$

where $R_0(z) = O(1)$ and z is large enough. Using (3.3) it is easy to show that the expression $|f(z)||z|^2$ is bounded for large z, and this implies that the integral over Γ of $f(z)$ vanishes when $r \to \infty$.

The function $f(z)$ behaves like $|z|^{-\alpha/2}$ near the origin (notice that $y < 0$), and consequently the integral over γ_3 vanishes when $\epsilon \searrow 0$.

To calculate the limits of the integrals over γ_1 and γ_2 we examine the behaviour of the function f near the branch cut. For every $u < y$, putting $k = 1$ and $k = -1$ in (2.5), we get

$$\begin{aligned} \lim_{\epsilon\to 0+} K_{\frac{1-\alpha}{2}}(u-y+i\epsilon) &= e^{\frac{i\pi(\alpha-1)}{2}} K_{\frac{1-\alpha}{2}}(|u-y|) - i\pi I_{\frac{1-\alpha}{2}}(|u-y|)\,, \\ \lim_{\epsilon\to 0+} K_{\frac{1-\alpha}{2}}(u-y-i\epsilon) &= e^{\frac{i\pi(1-\alpha)}{2}} K_{\frac{1-\alpha}{2}}(|u-y|) + i\pi I_{\frac{1-\alpha}{2}}(|u-y|)\,. \end{aligned}$$

We also have for $u < y$

$$\begin{aligned} \lim_{\epsilon\to 0+} (u-y+i\epsilon)^{-\frac{1-\alpha}{2}} &= e^{\frac{i\pi(\alpha-1)}{2}} |u-y|^{-\frac{1-\alpha}{2}}\,, \\ \lim_{\epsilon\to 0+} (u-y-i\epsilon)^{-\frac{1-\alpha}{2}} &= e^{\frac{i\pi(1-\alpha)}{2}} |u-y|^{-\frac{1-\alpha}{2}}\,. \end{aligned}$$

Next, for $u < 0$ we have

$$\begin{aligned} \lim_{\epsilon\to 0+} (u+i\epsilon)^{-\frac{\alpha}{2}} &= e^{-\frac{i\pi\alpha}{2}} |u|^{-\frac{\alpha}{2}}\,, \\ \lim_{\epsilon\to 0+} (u-i\epsilon)^{-\frac{\alpha}{2}} &= e^{\frac{i\pi\alpha}{2}} |u|^{-\frac{\alpha}{2}}\,. \end{aligned}$$

Using all the relations given above we obtain that for $u < y$

$$\begin{aligned} f_1^+(u) = \lim_{\epsilon\to 0+} f(u+i\epsilon) &= -\frac{1}{|u|^{\frac{\alpha}{2}}} \frac{e^{u-x}}{u-x} \left[e^{\frac{i\pi\alpha}{2}} \frac{K_{\frac{1-\alpha}{2}}(|u-y|)}{|u-y|^{\frac{1-\alpha}{2}}} + \pi \frac{I_{\frac{1-\alpha}{2}}(|u-y|)}{|u-y|^{\frac{1-\alpha}{2}}} \right]\,, \\ f_1^-(u) = \lim_{\epsilon\to 0+} f(u-i\epsilon) &= -\frac{1}{|u|^{\frac{\alpha}{2}}} \frac{e^{u-x}}{u-x} \left[e^{-\frac{i\pi\alpha}{2}} \frac{K_{\frac{1-\alpha}{2}}(|u-y|)}{|u-y|^{\frac{1-\alpha}{2}}} + \pi \frac{I_{\frac{1-\alpha}{2}}(|u-y|)}{|u-y|^{\frac{1-\alpha}{2}}} \right]\,. \end{aligned}$$

Similarly for $y < u < 0$ we get

$$\begin{aligned} f_2^+(u) = \lim_{\epsilon\to 0+} f(u+i\epsilon) &= \frac{1}{|u|^{\frac{\alpha}{2}}} \frac{e^{u-x}}{u-x} e^{-\frac{i\pi\alpha}{2}} \frac{K_{\frac{1-\alpha}{2}}(|u-y|)}{|u-y|^{\frac{1-\alpha}{2}}}\,, \\ f_2^-(u) = \lim_{\epsilon\to 0+} f(u-i\epsilon) &= \frac{1}{|u|^{\frac{\alpha}{2}}} \frac{e^{u-x}}{u-x} e^{\frac{i\pi\alpha}{2}} \frac{K_{\frac{1-\alpha}{2}}(|u-y|)}{|u-y|^{\frac{1-\alpha}{2}}}\,. \end{aligned}$$

Observe that the family of functions $g_\epsilon(u) := f(u+i\epsilon) - f(u-i\epsilon),\ u < 0$, indexed by positive $\epsilon < 1$, has the following properties:

$$\begin{aligned} |g_\epsilon(u)| &= O(|u|^{-2}), && u \to \infty, \\ |g_\epsilon(u)| &= O(|u|^{-\alpha/2}), && u \to 0, \end{aligned}$$

uniformly with respect to ϵ.

For $u \to y$, using (2.1) and (2.2), we get that

$$\begin{aligned} |g_\epsilon(u)| &= O(|u-y|^{\alpha-1}), \qquad && \alpha<1, \\ |g_\epsilon(u)| &= O(1), && \alpha>1, \end{aligned}$$

uniformly with respect to ϵ.

Moreover, $|g_\epsilon(u)| = O(-\log(|u-y|))$ uniformly with respect to ϵ, when $\alpha = 1$ (see [E], 7.2.6 (39)).

Combining the above and using the dominated convergence theorem we find that

$$\begin{aligned} \left(\int_{\gamma_1} + \int_{\gamma_2}\right) f(z)\,dz \quad &\longrightarrow \quad \int_{-\infty}^{y} (f_1^+(u) - f_1^-(u))\,du + \int_y^0 (f_2^+(u) - f_2^-(u))\,du \\ &= \quad (2\pi i)\frac{\sin(\frac{\pi\alpha}{2})}{\pi} \int_{-\infty}^{0} \frac{1}{(-u)^{\frac{\alpha}{2}}} \frac{e^{-|x-u|}}{|x-u|} \frac{K_{\frac{1-\alpha}{2}}(|u-y|)}{|u-y|^{\frac{1-\alpha}{2}}}\,du\,, \end{aligned}$$

as $r \to \infty$ and $\epsilon \to 0$. We also have

$$\mathrm{Res}_{z=x} f(z) \quad = \quad \frac{1}{x^{\frac{\alpha}{2}}} \frac{K_{\frac{1-\alpha}{2}}(|x-y|)}{|x-y|^{\frac{1-\alpha}{2}}}\,.$$

Using (3.2) and the relations given above we obtain the desired formula. To end the proof we observe that both sides of (3.1) are right-continuous at $y = 0$, as functions of y, $y \le 0 < x$, so the formula (3.1) also holds for $y = 0 < x$. □

3.2. **Poisson kernel of** $(0,\infty)$**.** For $x > 0$ we denote

$$Q^m_{(0,\infty)}(x,u) = \begin{cases} \frac{\sin(\pi\alpha/2)}{\pi} \left(\frac{x}{-u}\right)^{\alpha/2} \frac{e^{-m^{1/\alpha}(x-u)}}{x-u}\,, & u<0\,, \\ 0, & u \ge 0\,. \end{cases}$$

For $m = 1$ we write $Q_{(0,\infty)}(x,u)$ instead of $Q^1_{(0,\infty)}(x,u)$.

Theorem 3.2 (Poisson kernel)**.** *$Q^m_{(0,\infty)}(x,u)$ is the Poisson kernel of $(0,\infty)$ for the operator $(m^{2/\alpha}I - d^2/dx^2)^{\alpha/2}$.*

Let us observe that in probabilistic terms we thus identify the density function of the m-harmonic measure $P^m_{(0,\infty)}(x,\cdot)$ (see (2.19)) for the α-stable relativistic process X^m_t; that is $E^x[e^{-m\tau_{(0,\infty)}}; X^m_{\tau_{(0,\infty)}} \in \cdot]$.

Proof. By the scaling property we may assume that $m = 1$. To prove the theorem it is enough to check that condition (2.22) of Lemma 2.5 is satisfied for the measure $Q_{(0,\infty)}(x,u)\,du$ concentrated on the set $F = (-\infty, 0]$, where $x > 0$ is fixed. Observe that the above measure is finite:

$$\begin{aligned} \int_{-\infty}^0 Q_{(0,\infty)}(x,u)\,du \quad &= \quad \int_{-\infty}^0 \left(\frac{x}{-u}\right)^{\alpha/2} \frac{e^{-(x-u)}}{x-u}\,du \\ = e^{-x}x^{\alpha/2}\int_0^\infty w^{-\alpha/2}\frac{e^{-w}}{x+w}\,dw \quad &\le \quad e^{-x}x^{\alpha/2-1}\int_0^\infty e^{-w}\,w^{(1-\alpha/2)-1}\,dw \\ &= \quad e^{-x}x^{\alpha/2-1}\,\Gamma(1-\alpha/2)\,. \end{aligned}$$

This property, Lemma 3.1, and the first property in (2.10) yield for fixed $x > 0$:

$$\begin{aligned} &\int_{-\infty}^{0}\int_{-\infty}^{0}\frac{K_{(1-\alpha)/2}(|z-y|)}{|z-y|^{(1-\alpha)/2}}Q_{(0,\infty)}(x,z)\,dz\,Q_{(0,\infty)}(x,y)\,dy \\ = &\int_{-\infty}^{0}\frac{K_{(1-\alpha)/2}(|x-z|)}{|x-z|^{(1-\alpha)/2}}Q_{(0,\infty)}(x,z)\,dz \\ \le &\frac{K_{(1-\alpha)/2}(|x|)}{|x|^{(1-\alpha)/2}}\int_{-\infty}^{0}Q_{(0,\infty)}(x,z)\,dz < \infty\,. \end{aligned}$$

Observe that the measure $P^1_{(0,\infty)}(x,\cdot)$ is finite. From (2.21) and the last part of the calculations above, carried out for $P^1_{(0,\infty)}$ instead of $Q_{(0,\infty)}$, it follows that $P^1_{(0,\infty)}$ satisfies condition (2.22) as well. Thus, the measure $\mu_x(dz) = P^1_{(0,\infty)}(x,dz) - Q_{(0,\infty)}(x,z)\,dz$ satisfies all the conditions of Lemma 2.4, for $x \in (0,\infty)$. Thus, $\mu_x(dz) = 0$, so $Q_{(0,\infty)}(x,z)$ is the desired Poisson kernel. The proof is complete. □

3.3. **Green function of** $(0,\infty)$.

Theorem 3.3 (Green function). *The Green function of* $(0,\infty)$ *for the operator* $(m^{2/\alpha}I - d^2/dx^2)^{\alpha/2}$ *is of the form*

$$G^m_{(0,\infty)}(x,y) = \frac{|x-y|^{\alpha-1}}{2^\alpha\Gamma(\alpha/2)^2}\int_0^{\frac{4xy}{(x-y)^2}} e^{-m^{1/\alpha}|x-y|(t+1)^{1/2}}t^{\alpha/2-1}(t+1)^{-1/2}\,dt\,,$$

where $x, y > 0$.

Proof. By the scaling property we may assume that $m = 1$. Due to the symmetry of the Green function, it is enough to determine $G_{(0,\infty)}(x,y)$ for $0 < x < y$. We compute the compensator of the 1-Green function for the one-dimensional α-stable relativistic process, that is, the integral in the formula (2.20). We have

$$\begin{aligned} H(x,y) &= E^x[e^{-\tau_{(0,\infty)}}U_1(X_{\tau_{(0,\infty)}} - y)] \\ &= C(\alpha,1)\frac{\sin(\pi\alpha/2)}{\pi}\int_{-\infty}^{0}\left(\frac{x}{-u}\right)^{\alpha/2}\frac{e^{-(x-u)}}{x-u}\,\frac{K_{\frac{1-\alpha}{2}}(y-u)}{(y-u)^{\frac{1-\alpha}{2}}}\,du\,, \end{aligned} \tag{3.4}$$

where $C(\alpha,1) = \frac{2^{(1-\alpha)/2}}{\Gamma(\alpha/2)\pi^{1/2}}$. Substituting $(-u)^{1-\alpha/2} = v$ and taking into account the two following well-known identities (see (2.7) for the second one)

$$\begin{aligned} \frac{1}{x+v^{2/(2-\alpha)}} &= \int_0^\infty e^{-w(x+v^{2/(2-\alpha)})}\,dw\,, \\ U_1(z) &= C(\alpha,1)\frac{K_{\frac{1-\alpha}{2}}(z)}{z^{\frac{1-\alpha}{2}}} \\ &= \frac{1}{\Gamma(\alpha/2)\Gamma(1-\alpha/2)}\int_1^\infty \frac{e^{-zt}}{(t^2-1)^{\alpha/2}}\,dt, \quad z > 0\,, \end{aligned}$$

we obtain that the right-hand side of (3.4) is of the form

$$\begin{aligned} &C(x)\int_0^\infty\Big\{\int_0^\infty e^{-wx}e^{-wv^{2/(2-\alpha)}}\,dw\Big\}e^{-v^{2/(2-\alpha)}}\int_1^\infty\frac{e^{-(y+v^{2/(2-\alpha)})t}}{(t^2-1)^{\alpha/2}}\,dt\,dv \\ = &\; C(x)\int_1^\infty\int_0^\infty e^{-wx}\int_0^\infty e^{-(w+t+1)v^{2/(2-\alpha)}}\,dv\,\frac{e^{-yt}}{(t^2-1)^{\alpha/2}}\,dw\,dt, \end{aligned}$$

where $C(x) = \frac{2\sin(\pi\alpha/2)x^{\alpha/2}e^{-x}}{(2-\alpha)\pi\Gamma(\alpha/2)\Gamma(1-\alpha/2)}$. The interior integral can be expressed as

$$\begin{aligned}\int_0^\infty e^{-(w+t+1)v^{2/(2-\alpha)}}\,dv &= (w+t+1)^{(\alpha-2)/2}\int_0^\infty e^{-u^{2/(2-\alpha)}}\,du\\ &= \frac{2-\alpha}{2}\Gamma(1-\alpha/2)(w+t+1)^{(\alpha-2)/2}.\end{aligned}$$

Consequently, we get

$$\begin{aligned}H(x,y) &= \frac{\sin(\pi\alpha/2)x^{\alpha/2}e^{-x}}{\pi\Gamma(\alpha/2)}\int_1^\infty\int_0^\infty e^{-wx}(w+t+1)^{(\alpha-2)/2}\,dw\frac{e^{-yt}}{(t^2-1)^{\alpha/2}}\,dt\\ &= \frac{\sin(\pi\alpha/2)}{\pi\Gamma(\alpha/2)}\int_1^\infty\left[x^{\alpha/2}\int_0^\infty e^{-x(w+t+1)}(w+t+1)^{(\alpha-2)/2}\,dw\right]\frac{e^{-(y-x)t}}{(t^2-1)^{\alpha/2}}\,dt.\end{aligned}$$

Observe that the expression in brackets can be computed as follows:

$$\begin{aligned}x^{\alpha/2}\int_0^\infty e^{-x(w+t+1)}(w+t+1)^{(\alpha-2)/2}\,dw &= \int_{x(t+1)}^\infty e^{-s}s^{\alpha/2-1}\,ds\\ &= \Gamma(\alpha/2) - \int_0^{x(t+1)} e^{-s}s^{\alpha/2-1}\,ds\\ &= \Gamma(\alpha/2) - \frac{2}{\alpha}x^{\alpha/2}(t+1)^{\alpha/2}\int_0^1 e^{-w^{2/\alpha}x(t+1)}\,dw.\end{aligned}$$

Thus we get

$$\begin{aligned}H(x,y) &= \frac{\sin(\pi\alpha/2)}{\pi}\int_1^\infty\frac{e^{-(y-x)t}}{(t^2-1)^{\alpha/2}}\,dt\\ &\quad -\frac{2\sin(\pi\alpha/2)x^{\alpha/2}}{\alpha\pi\Gamma(\alpha/2)}\int_1^\infty\int_0^1 e^{-w^{2/\alpha}x(t+1)}dw\frac{e^{-(y-x)t}}{(t-1)^{\alpha/2}}\,dt\\ &= U_1(x-y) - \frac{2\sin(\pi\alpha/2)x^{\alpha/2}}{\alpha\pi\Gamma(\alpha/2)}\int_1^\infty\int_0^1 e^{-w^{2/\alpha}x(t+1)}\,dw\frac{e^{-(y-x)t}}{(t-1)^{\alpha/2}}\,dt.\end{aligned}$$

Since

$$G_{(0,\infty)}(x,y) = U_1(x-y) - H(x,y),$$

we have

$$\begin{aligned}G_{(0,\infty)}(x,y) &= \frac{2x^{\alpha/2}}{\alpha\Gamma(\alpha/2)^2\Gamma(1-\alpha/2)}\int_1^\infty\int_0^1 e^{-xw^{2/\alpha}(t+1)}\,dw\frac{e^{-(y-x)t}}{(t-1)^{\alpha/2}}\,dt\\ &= \frac{2x^{\alpha/2}}{\alpha\Gamma(\alpha/2)^2\Gamma(1-\alpha/2)}\int_0^1 e^{-xw^{2/\alpha}}\int_1^\infty e^{-xtw^{2/\alpha}}\frac{e^{-(y-x)t}}{(t-1)^{\alpha/2}}\,dt\,dw\\ &= \frac{2x^{\alpha/2}e^{x-y}}{\alpha\Gamma(\alpha/2)^2\Gamma(1-\alpha/2)}\int_0^1 e^{-2xw^{2/\alpha}}\int_0^\infty e^{-u(xw^{2/\alpha}+y-x)}\frac{du}{u^{\alpha/2}}\,dw\\ &= \frac{2x^{\alpha/2}e^{x-y}}{\alpha\Gamma(\alpha/2)^2}\int_0^1 e^{-2xw^{2/\alpha}}(xw^{2/\alpha}+y-x)^{\alpha/2-1}\,dw\\ &= \frac{e^{x-y}}{\Gamma(\alpha/2)^2}\int_0^x e^{-2v}v^{\alpha/2-1}(v+y-x)^{\alpha/2-1}\,dv.\end{aligned}$$

Substituting $4v(v+y-x)=u$ in the last integral we finally obtain

$$G_{(0,\infty)}(x,y) = \frac{1}{2^\alpha \Gamma(\alpha/2)^2}\int_0^{4xy} e^{-(u+(y-x)^2)^{1/2}} u^{\alpha/2-1}(u+(y-x)^2)^{-1/2}\,du$$
$$= \frac{(y-x)^{\alpha-1}}{2^\alpha \Gamma(\alpha/2)^2}\int_0^{\frac{4xy}{(y-x)^2}} e^{-(y-x)(t+1)^{1/2}} t^{\alpha/2-1}(t+1)^{-1/2}\,dt. \qquad \square$$

4. Multi-dimensional case

In this section we rely on the computation of the $(d-1)$-dimensional Fourier transform of the d-dimensional Poisson kernel as well as the corresponding d-dimensional Green function. We show that this Fourier transform can be expressed in terms of the corresponding one-dimensional object, which we can explicitly invert.

4.1. **Notation.**

In the proofs below one-dimensional quantities play an important rôle. Hence, to distinguish one-dimensional and d-dimensional objects, we introduce the following notation:

- $q_t^m(x)$ denotes the one-dimensional α-stable relativistic density with parameter m,
- $V_\lambda^m(x)$ denotes the corresponding one-dimensional λ-potential kernel.

Recall that by $Q^m_{(0,\infty)}(x,u)$ we denote the one-dimensional m-Poisson kernel of $(0,\infty)$.

For $x\in\mathbb{R}^d$ we write $x=(\mathbf{x},x_d)\in\mathbb{R}^d$, where $\mathbf{x}\in\mathbb{R}^{d-1}$. We begin with computation of the $(d-1)$-dimensional Fourier transform of $U_\lambda((\mathbf{x},x_d),(\mathbf{y},y_d)) = U_\lambda((\mathbf{x}-\mathbf{y}),(x_d-y_d))$, that is, the one obtained by integrating with respect to the variable $\mathbf{y}\in R^{d-1}$. We denote it by $U_\lambda\widehat{(x,y_d,\cdot)}(\mathbf{z})$. We employ this notation throughout the entire section.

4.2. **Poisson kernel of $\mathbb{H}$.**

We have

Lemma 4.1.

$$U_\lambda\widehat{(x,y_d,\cdot)}(\mathbf{z}) = e^{i(\mathbf{z},\mathbf{x})}V_{\tilde\lambda}^{\kappa^\alpha}(x_d-y_d),$$

where $\kappa=(|\mathbf{z}|^2+1)^{1/2}$ and $\tilde\lambda=\kappa^\alpha+\lambda-1$. Specifying to the case $\lambda=1$:

$$U_1\widehat{(x,y_d,\cdot)}(\mathbf{z}) = e^{i(\mathbf{z},\mathbf{x})}V_{\kappa^\alpha}^{\kappa^\alpha}(x_d-y_d)$$
$$= \frac{2^{\frac{1-\alpha}{2}}}{\sqrt{\pi}\Gamma(\alpha/2)} e^{i(\mathbf{z},\mathbf{x})}\,\kappa^{\frac{1-\alpha}{2}}\frac{K_{\frac{1-\alpha}{2}}(\kappa|x_d-y_d|)}{|x_d-y_d|^{\frac{1-\alpha}{2}}}.$$

Proof. We begin with computation of the $(d-1)$-dimensional Fourier transform of the transition density function $g_u(x-y)$ of the normal distribution:

$$g_u\widehat{(x,y_d,\cdot)}(\mathbf{z}) = \int_{\mathbb{R}^{d-1}} g_u(x-y)e^{i(\mathbf{z},\mathbf{y})}d\mathbf{y} = \frac{e^{i(\mathbf{z},\mathbf{x})}}{(4\pi u)^{1/2}}e^{-|\mathbf{z}|^2u}e^{-\frac{(x_d-y_d)^2}{4u}}.$$

In the next step we use this to find the $(d-1)$-dimensional Fourier transform of $p_t(x-y)$:

$$\begin{aligned}
p_t\widehat{(x, y_d, \cdot)}(\mathbf{z}) &= \int_{\mathbb{R}^{d-1}} p_t(x-y) e^{i(\mathbf{z},\mathbf{y})} d\mathbf{y} \\
&= e^t \int_0^\infty g_u\widehat{(x, y_d, \cdot)}(\mathbf{z}) e^{-u}\, \theta_t^\alpha(u)\, du \\
&= e^{i(\mathbf{z},\mathbf{x})} e^t \int_0^\infty \frac{1}{(4\pi u)^{1/2}} e^{-(|\mathbf{z}|^2+1)u} e^{-\frac{(x_d-y_d)^2}{4u}} \theta_t^\alpha(u)\, du \\
&= e^{i(\mathbf{z},\mathbf{x})} e^{t(1-(|\mathbf{z}|^2+1)^{\alpha/2})} \int_0^\infty \frac{1}{(4\pi u)^{1/2}} e^{t(|\mathbf{z}|^2+1)^{\alpha/2}} e^{-(|\mathbf{z}|^2+1)u} e^{-\frac{(x_d-y_d)^2}{4u}} \theta_t^\alpha(u)\, du \\
&= e^{i(\mathbf{z},\mathbf{x})} e^{-(\kappa^\alpha-1)t} \int_0^\infty e^{\kappa^\alpha t} e^{-\kappa^2 u} \frac{1}{(4\pi u)^{1/2}} e^{-\frac{(x_d-y_d)^2}{4u}} \theta_t^\alpha(u)\, du\,.
\end{aligned}$$

Note that the the integral expression in the last line is the one-dimensional α-stable relativistic transition density function with parameter κ^α which we denoted by $q_t^{\kappa^\alpha}$. Hence we obtain

$$p_t\widehat{(x, y_d, \cdot)}(\mathbf{z}) = e^{i(\mathbf{z},\mathbf{x})} e^{(1-\kappa^\alpha)t} q_t^{\kappa^\alpha}(x_d - y_d).$$

As a consequence we obtain the $(d-1)$-dimensional Fourier transform of the λ-potential:

$$\begin{aligned}
U_\lambda\widehat{(x, y_d, \cdot)}(\mathbf{z}) &= e^{i(\mathbf{z},\mathbf{x})} \int_0^\infty e^{(1-\kappa^\alpha-\lambda)t} q_t^{\kappa^\alpha}(x_d-y_d)\, dt \\
&= e^{i(\mathbf{z},\mathbf{x})} V^{\kappa^\alpha}_{(\kappa^\alpha+\lambda-1)}(x_d-y_d) = e^{i(\mathbf{z},\mathbf{x})} V^{\kappa^\alpha}_{\tilde{\lambda}}(x_d-y_d)\,,
\end{aligned}$$

where $\tilde{\lambda} = (\kappa^\alpha + \lambda - 1)$ and $V^{\kappa^\alpha}_{\tilde{\lambda}}$ is the $\tilde{\lambda}$ potential for the one-dimensional relativistic α-stable semigroup $q_t^{\kappa^\alpha}$ with parameter κ^α. Now, if we take $\lambda = 1$, then $\tilde{\lambda} = (\kappa^\alpha + \lambda - 1) = (|\mathbf{z}|^2 + 1)^{\alpha/2} = \kappa^\alpha$. By (2.16) we have

$$V^{\kappa^\alpha}_{\tilde{\lambda}}(x_d-y_d) = V^{\kappa^\alpha}_{\kappa^\alpha}(x_d-y_d) = \frac{2^{\frac{1-\alpha}{2}}}{\sqrt{\pi}\Gamma(\alpha/2)} \kappa^{\frac{1-\alpha}{2}} \frac{K_{\frac{1-\alpha}{2}}(\kappa|x_d-y_d|)}{|x_d-y_d|^{\frac{1-\alpha}{2}}}\,. \qquad \square$$

For $x \in \mathbb{H}$ we denote

$$f(x,u) = \begin{cases} 2\frac{\sin(\pi\alpha/2)}{\pi(2\pi)^{d/2}} \left(\frac{x_d}{-u_d}\right)^{\alpha/2} \frac{K_{d/2}(|x-u|)}{|x-u|^{d/2}}, & u \in \overline{\mathbb{H}}^c, \\ 0, & \text{otherwise}\,. \end{cases}$$

Lemma 4.2. *The $(d-1)$-dimensional Fourier transform of $f(x,u)$ with respect to* $\mathbf{u}$ *is given by*

$$f\widehat{(x, u_d, \cdot)}(\mathbf{z}) = e^{i(\mathbf{u},\mathbf{z})} \frac{\sin(\pi\alpha/2)}{\pi} \left(\frac{x_d}{-u_d}\right)^{\alpha/2} \frac{e^{-\kappa(x_d-u_d)}}{x_d-u_d}\,, \quad u_d < 0 < x_d\,.$$

For any $x \in \mathbb{H}$ the function $f(x,u)$ is a density of a finite measure supported on $\mathbb{H}^c$.

Proof. Taking into account (2.14) and (2.13) with $(d-1)$ instead of d and $x_d - y_d$ instead of t, we obtain the proof for the first part of the lemma.

The second part follows from the observation that

$$\int_{\mathbb{H}^c} f(x,u)\,du = \int_{-\infty}^{0} \widehat{f(x,u_d,\cdot)}(\mathbf{0})\,du_d < \infty\,.$$

The finiteness of the last integral was shown in the proof of Theorem 3.2. □

Theorem 4.3 (Poisson kernel)**.** *The Poisson kernel of* $\mathbb{H}$ *for* $(m^{2/\alpha}I-\Delta)^{\alpha/2}$ *exists and is given by*

$$P^m_{\mathbb{H}}(x,u) = 2\frac{\sin(\pi\alpha/2)m^{d/2\alpha}}{\pi(2\pi)^{d/2}}\left(\frac{x_d}{-u_d}\right)^{\alpha/2}\frac{K_{d/2}(m^{1/\alpha}|x-u|)}{|x-u|^{d/2}}\,,\quad u_d<0<x_d\,.$$

Proof. We will consider only the case $m=1$, since the general case follows from the scaling property.

We claim that

$$U_1(x-y) = \int_{\overline{\mathbb{H}}^c} f(x,u)U_1(u-y)\,du,\quad x\in\mathbb{H}, y\in\mathbb{H}^c\,. \tag{4.1}$$

If we show that the measure with the u-density $f(x,u)$ has the finite energy (2.22), then the application of Lemma 2.5 will complete the argument. Define

$$R(x,y) = \int_{u_d<0} f(x,u)U_1(u,y)\,du\,.$$

From Theorem 3.2 (with $m=\kappa^\alpha$) and Lemma 4.2 it follows that

$$\widehat{f(x,y_d,\cdot)}(\mathbf{z}) = e^{i(\mathbf{x},\mathbf{z})}Q^{\kappa^\alpha}_{(0,\infty)}(x_d,u_d)\,.$$

Since $U_1\widehat{(u,y_d,\cdot)}(\mathbf{z}) = e^{i(\mathbf{z},\mathbf{u})}V^{\kappa^\alpha}_{\kappa^\alpha}(u_d-y_d)$ we obtain

$$\begin{aligned}
R\widehat{(x,y_d,\cdot)}(\mathbf{z}) &= \int_{u_d<0} f(x,u)U_1\widehat{(u,y_d,\cdot)}(\mathbf{z})\,du \\
&= \int_{u_d<0} e^{i(\mathbf{z},\mathbf{u})}f(x,u)V^{\kappa^\alpha}_{\kappa^\alpha}(u_d-y_d)\,du \\
&= \int_{-\infty}^{0} \widehat{f(x,u_d,\cdot)}V^{\kappa^\alpha}_{\kappa^\alpha}(u_d-y_d)\,du_d \\
&= e^{i(\mathbf{x},\mathbf{z})}\int_{-\infty}^{0} Q^{\kappa^\alpha}_{(0,\infty)}(x_d,u_d)\,V^{\kappa^\alpha}_{\kappa^\alpha}(u_d-y_d)\,du_d\,. \qquad (4.2)
\end{aligned}$$

Next, Theorem 3.2 applied for $m=\kappa^\alpha$ together with the "sweeping out" formula (2.21) yield

$$V^{\kappa^\alpha}_{\kappa^\alpha}(x_d-y_d) = \int_{-\infty}^{0} Q^{\kappa^\alpha}_{(0,\infty)}(x_d,u_d)\,V^{\kappa^\alpha}_{\kappa^\alpha}(u_d-y_d)\,du_d\,,\quad y_d\le 0<x_d\,.$$

Hence, by the uniqueness of Fourier transforms, we verify (4.1). It remains only to show the finitness of energy. By (4.1),

$$\begin{aligned}
\int_{\overline{\mathbb{H}}^c}\int_{\overline{\mathbb{H}}^c} U_1(u-y)f(x,u)f(x,y)\,du\,dy &= \int_{\overline{\mathbb{H}}^c} U_1(x-y)f(x,y)\,dy \\
&\le \sup_{y\in\overline{\mathbb{H}}^c} U_1(x-y)\int_{\overline{\mathbb{H}}^c} f(x,y)\,dy < \infty\,,
\end{aligned}$$

since $\sup_{y\in\overline{\mathbb{H}}^c} U_1(x-y)$ is finite, as $x_d>0$. □

4.3. **Green function of $\mathbb{H}$.**

Theorem 4.4 (Green function)**.** *The Green function of $\mathbb{H}$ for $(m^{2/\alpha}I-\Delta)^{\alpha/2}$ is given by the formula*
(4.3)
$$G^m_{\mathbb{H}}(x,y)=\frac{2^{1-\alpha}m^{d/2\alpha}|x-y|^{\alpha-d/2}}{(2\pi)^{d/2}\Gamma(\alpha/2)^2}\int_0^{\frac{4x_dy_d}{|x-y|^2}}\frac{t^{\frac{\alpha}{2}-1}}{(t+1)^{d/4}}K_{d/2}(m^{1/\alpha}|x-y|(t+1)^{1/2})\,dt,$$
where $x,y\in\mathbb{H}$.

Proof. We will consider only the case $m=1$, since the general case follows from the scaling property. Also it is enough to consider $x=(\mathbf{0},x_d)$. We will find the $(d-1)$-dimensional Fourier transform of the Green function. If we write
$$G_{\mathbb{H}}(x,y)=U_1(x-y)-\int_{\overline{\mathbb{H}}^c}P_{\mathbb{H}}(x,u)U_1(u-y)du=U_1(x-y)-R(x,y),$$
then by (4.2) and by Theorem 3.3
$$\begin{aligned}
&G_{\mathbb{H}}\widehat{(x,y_d},\cdot)(\mathbf{z})=U_1\widehat{(x,y_d},\cdot)(\mathbf{z})-R\widehat{(x,y_d},\cdot)(\mathbf{z})\\
&= V^{\kappa^\alpha}_{\kappa^\alpha}(x_d-y_d)-\int_{-\infty}^0 Q^{\kappa^\alpha}_{(0,\infty)}(x_d,u_d)V^{\kappa^\alpha}_{\kappa^\alpha}(u_d-y_d)\,du_d\\
&= G^{\kappa^\alpha}_{(0,\infty)}(x_d,y_d)\\
&= \frac{1}{2^\alpha\Gamma(\alpha/2)^2}\int_0^{4x_dy_d}\frac{s^{\frac{\alpha}{2}-1}}{(s+(x_d-y_d)^2)^{1/2}}e^{-(s+(x_d-y_d)^2)^{1/2}(|\mathbf{z}|^2+1)^{1/2}}\,ds\,.
\end{aligned}$$

Taking into account (2.14) and (2.13), with $(d-1)$ instead of d, and with $(s+(x_d-y_d)^2)^{1/2}$ instead of t, we obtain for $d>1$
$$\begin{aligned}
G_{\mathbb{H}}(x,y)&=\frac{2^{1-\alpha}}{(2\pi)^{d/2}\Gamma(\alpha/2)^2}\int_0^{4x_dy_d}s^{\frac{\alpha}{2}-1}\frac{K_{d/2}((|\mathbf{y}|^2+(x_d-y_d)^2+s)^{1/2})}{(|\mathbf{y}|^2+(x_d-y_d)^2+s)^{d/4}}\,ds\\
&=\frac{2^{1-\alpha}}{(2\pi)^{d/2}\Gamma(\alpha/2)^2}\int_0^{4x_dy_d}s^{\frac{\alpha}{2}-1}\frac{K_{d/2}((|x-y|^2+s)^{1/2})}{(|x-y|^2+s)^{d/4}}\,ds\\
&=\frac{2^{1-\alpha}|x-y|^{\alpha-d/2}}{(2\pi)^{d/2}\Gamma(\alpha/2)^2}\int_0^{\frac{4x_dy_d}{|x-y|^2}}\frac{t^{\frac{\alpha}{2}-1}}{(t+1)^{d/4}}K_{d/2}(|x-y|(t+1)^{1/2})\,dt\,.
\end{aligned}$$
□

We now compute $E^ze^{-m\tau_{\mathbb{H}}}$, where $\tau_{\mathbb{H}}$ is the first exit time from $\mathbb{H}$ of the relativistic α-stable process with parameter $m>0$. Its importance, among other things, is due to the fact that it is harmonic on $\mathbb{H}$ for the operator $(m^{2/\alpha}I-\Delta)^{\alpha/2}$. Indeed, according to the theory of Schrödinger operators (see [ChZ], 4.5), the function $\phi(x)=E^x[\exp(\int_0^{\tau_{\mathbb{H}}}q(X^m_t)\,dt)]$ is harmonic on $\mathbb{H}$ with respect to the Schrödinger operator $mI-(m^{2/\alpha}I-\Delta)^{\alpha/2}+qI$ based on the generator of our relativistic process (with parameter m) with the potential q. Taking $q=-m$ we obtain that $E^x[\exp(-m\int_0^{\tau_{\mathbb{H}}}\mathbf{1}\,dt)]=E^xe^{-m\tau_{\mathbb{H}}}$ is harmonic on $\mathbb{H}$ for $(m^{2/\alpha}I-\Delta)^{\alpha/2}$.

Corollary 4.5.
$$E^ze^{-m\tau_{\mathbb{H}}}=\frac{1}{\Gamma(\alpha/2)}\int_{m^{1/\alpha}z_d}^\infty t^{\alpha/2-1}e^{-t}\,dt,\quad z\in\mathbb{H}\,.$$

Proof. The proof consists of computing the mass of the m-Poisson kernel and will be carried out for $m=1$, since the general case follows from the scaling property. It is obvious that we may assume that $d=1$.

Substituting $(-u) = v^{\frac{2}{2-\alpha}}$ and taking into account the following identity:

$$\frac{1}{x + v^{\frac{2}{2-\alpha}}} = \int_0^\infty e^{-w(x+v^{\frac{2}{2-\alpha}})}\, dw\,,$$

we obtain, after changing order of integration

$$\begin{aligned} E^z e^{-\tau_{\mathbb{H}}} &= \frac{\sin(\alpha\pi/2)}{\pi} \int_{-\infty}^0 \left(\frac{z}{-u}\right)^{\alpha/2} \frac{e^{-(z-u)}}{z-u}\, du \\ &= \frac{2\sin(\pi\alpha/2)}{(2-\alpha)\pi} z^{\alpha/2} e^{-z} \int_0^\infty \{\int_0^\infty e^{-wz} e^{-wv^{\frac{2}{2-\alpha}}}\, dw\} e^{-v^{\frac{2}{2-\alpha}}}\, dv \\ &= \frac{2z^{\alpha/2}e^{-z}}{(2-\alpha)\Gamma(\alpha/2)\Gamma(1-\alpha/2)} \int_0^\infty e^{-wz} \{\int_0^\infty e^{-v^{\frac{2}{2-\alpha}}(w+1)}\, dv\}\, dw \\ &= \frac{1}{\Gamma(\alpha/2)} \int_0^\infty [z(w+1)]^{\alpha/2-1} e^{-(w+1)z}\, z\, dw \\ &= \frac{1}{\Gamma(\alpha/2)} \int_z^\infty t^{\alpha/2-1} e^{-t}\, dt. \end{aligned}$$

□

4.4. **Riesz potential theory.** We provide an alternative proof for the formulas of the Poisson kernel and the Green function of $\mathbb{H}$ for the α-stable rotation invariant process Z_t. As far as we know, it is the first proof in the multi-dimensional case which does not use Kelvin's transform. We define

$$P^{\#}_{\mathbb{H}}(x,u) = \lim_{m\to 0^+} P^m_{\mathbb{H}}(x,u) = \frac{\sin(\pi\alpha/2)\Gamma(d/2)}{\pi^{1+d/2}} \left(\frac{x_d}{-u_d}\right)^{\alpha/2} \frac{1}{|x-u|^d},$$

where $u_d < 0 < x_d$ and

$$G^{\#}_{\mathbb{H}}(x,y) = \lim_{m\to 0^+} G^m_{\mathbb{H}}(x,y) = \frac{\Gamma(d/2)}{\pi^{d/2} 2^\alpha \Gamma(\alpha/2)^2} |x-y|^{\alpha-d} \int_0^{\frac{4x_d y_d}{|x-y|^2}} \frac{t^{\frac{\alpha}{2}-1}}{(t+1)^{d/2}}\, dt,$$

where $x_d,\ y_d > 0$. Note that the above limits are obtained using (2.8). Also observe that $P^m_{\mathbb{H}}(x,u)$, $G^m_{\mathbb{H}}(x,y)$ and $U^m_m(x,y)$ are decreasing functions of the parameter m. This easily follows from the fact that $r^\vartheta K_\vartheta(r)$ is a decreasing function of $r > 0$; see (2.10). This shows in particular that

$$G^m_{\mathbb{H}}(x,y) \le G^{\#}_{\mathbb{H}}(x,y) \tag{4.4}$$

and

$$P^m_{\mathbb{H}}(x,y) \le P^{\#}_{\mathbb{H}}(x,y).$$

Corollary 4.6. *$P^{\#}_{\mathbb{H}}$, $G^{\#}_{\mathbb{H}}$ are the 0-Poisson kernel of $\mathbb{H}$ or 0-Green function of $\mathbb{H}$, respectively, for the d-dimensional α-stable rotation invariant process.*

Remark. Comparing the formulas for $P^{\#}_{\mathbb{H}}$, $P_{\mathbb{H}}$ and for $G^{\#}_{\mathbb{H}}$, $G_{\mathbb{H}}$, we observe that the correspondence between the potential theories of $(-\Delta)^{\alpha/2}$ and $(I-\Delta)^{\alpha/2}$ we mentioned in the Remark below Lemma 2.4 extends onto the Poisson kernel and the Green function of $\mathbb{H}$. More specifically, the expression $\frac{1}{|x-u|^d}$ $\left(\frac{1}{(|x-u|(t+1)^{1/2})^d}\right)$ in the formula for $P^{\#}_{\mathbb{H}}$ $(G^{\#}_{\mathbb{H}})$, respectively, is replaced by $\frac{K_{d/2}(|x-u|)}{|x-u|^{d/2}}$ $\left(\frac{K_{d/2}(|x-u|(t+1)^{1/2})}{(|x-u|(t+1)^{1/2})^{d/2}}\right)$ in the one for $P_{\mathbb{H}}$ $(G_{\mathbb{H}})$, respectively. No such correspondence is known so far for the corresponding Poisson kernels or the Green functions for balls.

Proof. We provide the arguments only in the case $\alpha < d$. The case $d = 1 \leq \alpha$ can be handled in a similar way but requires the compensated potential kernels. We omit the details.

To deal with the Poisson kernel we write the sweeping out formula (2.21) for the m-Poisson kernel:

$$U_m^m(x,y) = \int_{\mathbb{H}^c} P_{\mathbb{H}}^m(x,u)U_m^m(u,y)du, \quad x \in \mathbb{H},\ y \in \overline{\mathbb{H}}^c. \tag{4.5}$$

Next, note that for $\alpha < d$

$$\lim_{m \to 0^+} U_m^m(u,y) = C^{\#}(\alpha,d)|u-y|^{\alpha-d},$$

where $C^{\#}(\alpha,d) = 2^{-\alpha}\pi^{-d/2}\Gamma((d-\alpha)/2)/\Gamma(\alpha/2)$ and $C^{\#}(\alpha,d)|u-y|^{\alpha-d}$ is the Riesz kernel. Observe that

$$P_{\mathbb{H}}^m(x,u)U_m^m(u,y) \nearrow C^{\#}(\alpha,d)P_{\mathbb{H}}^{\#}(x,u)|u-y|^{\alpha-d}, \text{ as } m \searrow 0.$$

Then, after taking the limit in (4.5) and using the monotone convergence theorem, we obtain

$$|x-y|^{\alpha-d} = \int_{\mathbb{H}^c} P_{\mathbb{H}}^{\#}(x,u)|u-y|^{\alpha-d}du, \quad x \in \mathbb{H},\ y \in \overline{\mathbb{H}}^c,$$

which gives the formula (4.5) rewritten in terms of Riesz kernels and $P_{\mathbb{H}}^{\#}$.

Applying the uniqueness theorem for Riesz kernels (see [BGR]) we obtain that $P_{\mathbb{H}}^{\#}(x,y)$ is the Poisson kernel of $\mathbb{H}$ for the rotation invariant d-dimensional α-stable process.

To deal with the Green function we use the identity (2.20), written for the m-Poisson kernel:

$$G_{\mathbb{H}}^m(x,y) = U_m^m(x,y) - \int_{\mathbb{H}^c} U_m^m(u,y)P_{\mathbb{H}}^m(x,u)du, \quad x,y \in \mathbb{H}.$$

After taking the limit and using the monotone convergence theorem we obtain

$$G_{\mathbb{H}}^{\#}(x,y) = C^{\#}(\alpha,d)|x-y|^{\alpha-d} - \int_{\mathbb{H}^c} C^{\#}(\alpha,d)|u-y|^{\alpha-d}P_{\mathbb{H}}^{\#}(x,u)du, \quad x,y \in \mathbb{H},$$

which shows that $G_{\mathbb{H}}^{\#}(x,y)$ is the Green function of $\mathbb{H}$ for the rotation invariant d-dimensional α-stable process. □

5. Appendix

Throughout this section by $c, C, C_1, \ldots$ we denote positive constants. The notation $p(u) \approx q(u)$, $u \in A$, means that the ratio $p(u)/q(u)$, $u \in A$, is bounded from below and above by positive constants. For any $y = (\mathbf{y}, y_d) \in \mathbb{R}^d$ we denote $y^* = (\mathbf{y}, -y_d)$.

In this section we apply results obtained so far in the paper to obtain estimates of the Green function of $\mathbb{H}$ for the relativistic α-stable process X_t^m, under the assumption that $m = 1$. Equivalently, it is the Green function of $\mathbb{H}$ for the operator $I - (I-\Delta)^{\alpha/2}$. To distinguish it from the previously considered Green function $G_{\mathbb{H}}$ for the operator $(I-\Delta)^{\alpha/2}$, we call it *the 0-Green function* and denote it as $G_{\mathbb{H}}^0$. From the point of view of the potential theory of X_t^m the Green function $G_{\mathbb{H}}^0$ is of prime importance. In particular, questions of boundary behaviour of harmonic functions or of potential theory of the Schrödinger operator based on $I - (I-\Delta)^{\alpha/2}$ require more detailed information about the Green function (cf. [ChZ]). For $m = 0$

we obtain the α-stable rotation invariant Lévy process Z_t whose potential theory is considerably more advanced. The process X_t^m is another important example of a Lévy process whose potential theory is still in its early stages.

In principle we can use the resolvent equation to represent $G_{\mathbb{H}}^0$ by a series involving powers of $G_{\mathbb{H}}$. That is, we have the following identity:

$$G_{\mathbb{H}}^0(x,y) - G_{\mathbb{H}}(x,y) = \int_{\mathbb{H}} G_{\mathbb{H}}(x,u)\, G_{\mathbb{H}}^0(u,y)\, du,\ \ x,y \in \mathbb{H}\,.$$

The iteration of the above identity yields

$$G_{\mathbb{H}}^0(x,y) = \sum_{n=1}^{\infty} G_{\mathbb{H}}^{(n)}(x,y),\ \ x,y \in \mathbb{H}\,, \tag{5.1}$$

where $G_{\mathbb{H}}^{(n)}$ is the n-th iteration of the kernel $G_{\mathbb{H}}$. The validity of the representation (5.1) requires further justification, which is omitted here. Besides, the consecutive terms of the series (5.1) are not decaying sufficiently fast, as $n \to \infty$, in order to obtain an appropriate estimate for $G_{\mathbb{H}}^0(x,y)$. The present, alternative approach, exploits the following formula for the Green function:

$$G_{\mathbb{H}}^0(x,y) = \int_0^{\infty} p_t^{\mathbb{H}}(x,y)\, dt$$

and relies on estimation of the transition density function $p_t^{\mathbb{H}}(x,y)$.

Comparison results. One of our motivations to explore bounds for $G_{\mathbb{H}}^0$ stems from comparison results for the relativistic 0-Green function and its stable counterpart. We recall that the potential theoretic objects corresponding to $(-\Delta)^{\alpha/2}$, that is, in probabilistic terms, to the potential theory of the α-stable rotation invariant process Z_t, are denoted with the superscript "#". In several papers ([Ry], [CS2], [GRy1], [KL]) it was shown, under various assumptions on an open bounded set D, that there is a constant C depending on D such that

$$C^{-1} G_D^{\#}(x,y) \le G_D^0(x,y) \le C G_D^{\#}(x,y),\ \ x,y \in D, \tag{5.2}$$

where $G_D^{\#}$ is the 0-Green function for Z_t. One of the important questions which may be raised is: to what extent does the above comparison hold if an unbounded set is considered? We show that for $D = \mathbb{H}$ such comparison holds if $d \ge 3$, and we restrict x, y to $\mathbb{H}$ with $|x-y| \le 1$. On the other hand (5.2) does not hold for $d \le 2$ even if $|x-y| \le 1$. In the forthcoming paper [GRy2] the optimal estimates for $G_{\mathbb{H}}^0$ are provided, and they extend the results obtained in this section.

We start with several lemmas leading to estimates of the semigroup $p_t^{\mathbb{H}}(x,y)$.

5.1. **Estimates of transition densities $p_t^{\mathbb{H}}(x,y)$.** We begin with the following estimate [Ry]:

Lemma 5.1. *There exists a constant $c = c(\alpha, d)$ such that*

$$\max_{x \in \mathbb{R}^d} p_t(x) \le c(t^{-d/2} + t^{-d/\alpha})\,.$$

The next lemma will be very useful in the sequel.

Lemma 5.2. *There is C such that*

$$p_t(x-y) - p_t(x-y^*) \le p_t^{\mathbb{H}}(x,y) \le C(t^{-d/2} + t^{-d/\alpha})\, P^x(\tau_{\mathbb{H}} \ge t/3)\, P^y(\tau_{\mathbb{H}} \ge t/3)$$

for $x, y \in \mathbb{H}$.

Proof. We start with the upper bound. Since $p_t^{\mathbb{H}}(x,y)$ is a transition density function, the semigroup property and the inequality $p_t^{\mathbb{H}}(x,y) \le \max_{z\in\mathbb{R}^d} p_t(z)$ yield

$$\begin{aligned} p_{2t}^{\mathbb{H}}(x,y) &= \int_{\mathbb{H}} p_t^{\mathbb{H}}(x,z)\, p_t^{\mathbb{H}}(z,y)\, dz \le \max_{z\in\mathbb{R}^d} p_t(z) \int_{\mathbb{H}} p_t^{\mathbb{H}}(x,z)\, dz \\ &= \max_{z\in\mathbb{R}^d} p_t(z)\, P^x(\tau_{\mathbb{H}} \ge t). \end{aligned}$$

We repeat this argument to obtain

$$\begin{aligned} p_{3t}^{\mathbb{H}}(x,y) &= \int_{\mathbb{H}} p_{2t}^{\mathbb{H}}(x,z)\, p_t^{\mathbb{H}}(z,y)\, dz \le \max_{z\in\mathbb{R}^d} p_t(z)\, P^x(\tau_{\mathbb{H}} \ge t) \int_{\mathbb{H}} p_t^{\mathbb{H}}(z,y)\, dz \\ &= \max_{z\in\mathbb{R}^d} p_t(z)\, P^x(\tau_{\mathbb{H}} \ge t)\, P^y(\tau_{\mathbb{H}} \ge t)\,. \end{aligned}$$

By Lemma 5.1, $\max_{z\in\mathbb{R}^d} p_t(z) \le C(t^{-d/2} + t^{-d/\alpha})$; hence we obtain the upper bound.

To get the lower bound we use the subordination of the process to the Brownian motion: $X_t = B_{T_t^\alpha}$. Let $[y, y+\epsilon) = \bigotimes_{k=1}^d [y_k, y_k+\epsilon) \subseteq \mathbb{H}, \epsilon > 0$. Then

$$\begin{aligned} P^x(X_t \in [y,y+\epsilon), t < \tau_{\mathbb{H}}) &= P^x(B_{T_t^\alpha} \in [y,y+\epsilon), B_{T_s^\alpha} \in \mathbb{H}, 0 \le s < t) \\ &\ge P^x(B_{T_t^\alpha} \in [y,y+\epsilon), B_s \in \mathbb{H}, 0 \le s < T_t^\alpha). \end{aligned}$$

As a consequence of *the reflection principle* we have that the transition densities for the the Brownian motion killed on exiting $\mathbb{H}$ are equal to $g_t(x-y) - g_t(x-y^*)$ (see e.g. [ChZ]). Using the independence of T^α and B we obtain

$$\begin{aligned} &P^x(B_{T_t^\alpha} \in [y,y+\epsilon), B_s \in \mathbb{H}, 0 \le s < T_t^\alpha | T^\alpha) \\ &\qquad = \int_{[y,y+\epsilon)} \left(g_{T_t^\alpha}(x-v) - g_{T_t^\alpha}(x-v^*) \right) dv. \end{aligned}$$

Taking the expectation, passing $\epsilon \searrow 0$ and using the fact that $E^0 g_{T_t^\alpha}(z) = p_t(z)$, we obtain the lower bound. □

The next lemma is taken from [G], but for the reader's convenience we provide its proof.

Lemma 5.3. *For $t \ge 2$, $x_d > 0$ we have*

$$P^x(\tau_{\mathbb{H}} \ge t) \le C \frac{x_d + \ln t}{t^{1/2}}\,. \tag{5.3}$$

Proof. Let $Y_t = X_t^{(d)}$, where $X_t = (X_t^{(1)}, \dots, X_t^{(d)})$ and let ν^Y be the density of the Lévy measure of Y_t. By the symmetry of the process Y_t we obtain

$$P^x(\tau_{\mathbb{H}} \ge t) = P^x(\inf_{s\le t} Y_s \ge 0) = P^0(\inf_{s\le t}(-Y_s + x_d) \ge 0) = P^0(\sup_{s\le t} Y_s \le x_d).$$

Using a version of the Lévy inequality ([B], Ch.7, 37.9) we have for any $\varepsilon, y > 0$ that

$$2P^0(Y_t \ge y + 2\varepsilon) - 2\sum_{k=1}^n P^0(Y_{\frac{tk}{n}} - Y_{\frac{t(k-1)}{n}} \ge \varepsilon) \le P^0(\sup_{k\le n} Y_{\frac{tk}{n}} \ge y)\,, \quad n \in \mathbb{N}\,.$$

Note that $\sum_{k=1}^{n} P^0(Y_{\frac{tk}{n}} - Y_{\frac{t(k-1)}{n}} \geq \varepsilon) = nP^0(Y_{\frac{t}{n}} \geq \varepsilon) \to t \int_\varepsilon^\infty \nu^Y(x)\,dx$; hence, by the symmetry again

$$\begin{aligned} P^0(|Y_t| \geq y + 2\varepsilon) - 2t \int_\varepsilon^\infty \nu^Y(x)\,dx &= 2P^0(Y_t \geq y + 2\varepsilon) - 2t \int_\varepsilon^\infty \nu^Y(x)\,dx \\ &\leq P^0(\sup_{s\leq t} Y_s \geq y). \end{aligned}$$

This implies that

$$P^x(\tau_{\mathbb{H}} \geq t) = P^0(\sup_{s\leq t} Y_s \leq x_d) \leq P^0(|Y_t| \geq x_d + 2\varepsilon) + 2t \int_\varepsilon^\infty \nu^Y(x)\,dx\,. \tag{5.4}$$

From (2.17) and (2.9) we obtain

$$\int_\varepsilon^\infty \nu^Y(x)\,dx \leq Ce^{-\varepsilon}\varepsilon^{-\alpha/2},\ \varepsilon \geq 1.$$

Because of Lemma 5.1 we have the fact that the density of $Y(t)$ is bounded by $Ct^{-1/2}, t \geq 2$. Hence taking $\varepsilon = \frac{3}{2}\ln t$, we obtain by (5.4)

$$P^x(\tau_{\mathbb{H}} \geq t) \leq C\,(x_d + \ln t)\,t^{-1/2}.$$ □

In order to improve the above estimate for x close to the boundary, we use the following result proved recently in [GRy1].

Lemma 5.4. *Assume that $d = 1$. Let $D = (0,1)$ and $x \in D$. Then for $y > 1$*

$$P_D^0(x,y) \approx \frac{(x(1-x))^{\alpha/2}}{(y-1)^{\alpha/2}(y-x)}\ e^{-y}.$$

We also have

$$E^x[X_{\tau_D} > 1; X_{\tau_D}] \approx P^x(X_{\tau_D} > 1) \approx x^{\alpha/2}$$

and

$$E^x\tau_D \approx (x(1-x))^{\alpha/2}.$$

We note that in [GRy1] it was shown that the 0-Green function of D is comparable with the 0-Green function of the corresponding stable process (see (5.2)). Then we can use the result of Ikeda-Watanabe [IW] which describes the 0-Poisson kernel in terms of the 0-Green function and the Lévy measure:

$$P_D^0(x,z) = \int_D G_D^0(x,y)\,\nu(y-z)\,dy, \quad x \in D, z \in \overline{D}^c.$$

The conclusion of the lemma follows from arguments used in [Ry] (see Theorem 6).

We further need the following strengthening of Lemma 5.3.

Lemma 5.5. *For $0 < x_d < 1/2$ we have*

$$P^x(\tau_{\mathbb{H}} \geq t) \leq Cx_d^{\alpha/2}\ \ln t/t^{1/2}, \quad t \geq 4. \tag{5.5}$$

Proof. It is enough to prove the claim for $d = 1$. Let $D = (0,1)$ and assume that $0 < x < 1/2$.

By the strong Markov property and then by Lemma 5.3 we obtain for $t \geq 4$:

$$\begin{aligned} P^x(\tau_{\mathbb{H}} \geq t) &= P^x(\tau_D \geq t/2, \tau_{\mathbb{H}} \geq t) + P^x(\tau_D < t/2, \tau_{\mathbb{H}} \geq t) \\ &\leq P^x(\tau_D \geq t/2) + P^x(\tau_{\mathbb{H}} - \tau_D \geq t/2) \\ &= P^x(\tau_D \geq t/2) + E^x[\tau_D < \tau_{\mathbb{H}}\,; P^{X_{\tau_D}}(\tau_{\mathbb{H}} \geq t/2)] \\ &\leq P^x(\tau_D \geq t/2) + CE^x[\tau_D < \tau_{\mathbb{H}}; X_{\tau_D} + \ln t]/t^{1/2} \\ &\leq 2E^x\tau_D/t + CE^x[X_{\tau_D} > 1\,; X_{\tau_D}]/t^{1/2} + C\ln t\, P^x(X_{\tau_D} > 1)/t^{1/2} \\ &\leq Cx^{\alpha/2}\ln t/t^{1/2}. \end{aligned}$$

The last inequality follows from Lemma 5.4. The proof is complete. □

5.2. **Bounds for 0-Green function.** We start with a lower estimate of $G^0_{\mathbb{H}}(x,y)$ by the 0-Green function of the Brownian motion for $\mathbb{H}$ which we denote by $G^g_{\mathbb{H}}(x,y)$. As a consequence of the reflection principle we have

$$G^g_{\mathbb{H}}(x,y) = \int_0^\infty (g_u(x-y) - g_u(x-y^*))\,du, \quad x,y \in \mathbb{H},$$

which yields the following formulas (see e.g. [ChZ]) depending on the dimension d:

$$G^g_{\mathbb{H}}(x,y) = \begin{cases} x \wedge y\,, & d=1, \\ \frac{1}{2\pi}\ln\frac{|x^*-y|}{|x-y|}\,, & d=2, \\ \frac{\Gamma((d-2)/2)}{4\pi^{d/2}}\left(\frac{1}{|x-y|^{d-2}} - \frac{1}{|x^*-y|^{d-2}}\right), & d\geq 3. \end{cases} \tag{5.6}$$

Lemma 5.6. *For any $x, y \in \mathbb{H}$*

$$G^0_{\mathbb{H}}(x,y) \geq G^g_{\mathbb{H}}(x,y)\,.$$

Proof. Let $V(x,y) = \int_0^\infty (p_t(x-y) - p_t(x-y^*))\,dt$. By Lemma 5.2 it is enough to prove that $V(x,y) \geq G^g_{\mathbb{H}}(x,y)$. For this purpose, let us write the 0-potential of the subordinator T^α_t (2.11):

$$G(u) = e^{-u}\int_0^\infty e^t\,\theta^\alpha_t(u)\,dt\,.$$

Using (2.12) we have

$$\begin{aligned} V(x,y) &= \int_0^\infty (p_t(x-y) - p_t(x-y^*))\,dt \\ &= \int_0^\infty e^t \int_0^\infty (g_u(x-y) - g_u(x-y^*))\,e^{-u}\,\theta^\alpha_t(u)\,du\,dt \\ &= \int_0^\infty (g_u(x-y) - g_u(x-y^*))e^{-u}\int_0^\infty e^t\,\theta^\alpha_t(u)\,dt\,du \\ &= \int_0^\infty (g_u(x-y) - g_u(x-y^*))\,G(u)\,du\,. \end{aligned}$$

It was proved in [RSV] that $G(u)$ is a completely monotone (hence decreasing) function and $\inf_{u>0} G(u) = \lim_{u\to\infty} G(u) = C$. We find the constant $C = \lim_{u\to\infty} G(u)$

by taking into account the asymptotics of the Laplace transform of $G(u)$ at the origin:

$$\begin{aligned}\int_0^\infty e^{-\lambda u}\, G(u)\, du &= \int_0^\infty e^t \int_0^\infty e^{-u(1+\lambda)}\, \theta_t^\alpha(u)\, du\, dt \\ &= \int_0^\infty e^t\, e^{-(1+\lambda)^{\alpha/2} t}\, dt = \frac{1}{(1+\lambda)^{\alpha/2}-1} \sim \frac{2}{\lambda\alpha}.\end{aligned}$$

Applying the Karamata Tauberian Theorem (see [Bi], 1.7) we obtain that $C = 2/\alpha > 1$. Since $g_u(x-y) - g_u(x-y^*) \geq 0$ we finally obtain

$$\begin{aligned}V(x,y) &= \int_0^\infty (g_u(x-y) - g_u(x-y^*))\, G(u)\, du \\ &\geq \int_0^\infty (g_u(x-y) - g_u(x-y^*))\, du = G^g_{\mathbb{H}}(x,y). \qquad \square\end{aligned}$$

Lemma 5.7. *Assume that* $|x-y| \leq 1$*. Then there is* $C = C(\alpha, d)$ *such that*

$$C\,(x_d\, y_d \wedge 1)^{\alpha/2} \leq G_{\mathbb{H}}(x,y) \leq G^{\#}_{\mathbb{H}}(x,y).$$

Moreover for $d \geq 2$ *there is a constant* $C_1 = C_1(\alpha, d)$ *such that*

$$G^{\#}_{\mathbb{H}}(x,y) \leq C_1 G_{\mathbb{H}}(x,y).$$

Proof. The inequality

$$G_{\mathbb{H}}(x,y) \leq G^{\#}_{\mathbb{H}}(x,y), \quad x, y \in \mathbb{H},$$

is obviously identical with (4.4) written for $m = 1$.

Since $\frac{4x_d y_d}{|x-y|^2} \geq x_d y_d \wedge 1$, applying the formulas (4.3) and (2.8) we obtain

$$\begin{aligned}G_{\mathbb{H}}(x,y) &= C|x-y|^{\alpha-d/2} \int_0^{\frac{4x_d y_d}{|x-y|^2}} \frac{t^{\frac{\alpha}{2}-1}}{(t+1)^{d/4}} K_{d/2}(|x-y|(t+1)^{1/2})\, dt \\ &\geq C|x-y|^{\alpha-d/2} \int_0^{x_d y_d \wedge 1} \frac{t^{\frac{\alpha}{2}-1}}{(t+1)^{d/4}} K_{d/2}(|x-y|(t+1)^{1/2})\, dt \\ &\geq C|x-y|^{\alpha-d} \int_0^{x_d y_d \wedge 1} \frac{t^{\frac{\alpha}{2}-1}}{(t+1)^{d/2}}\, dt \geq C(x_d y_d \wedge 1)^{\alpha/2}.\end{aligned}$$

Assume that $d \geq 2$. If $\frac{x_d y_d}{|x-y|^2} \leq 1$ and $|x-y| \leq 1$, then using the formulas (4.3) and (2.8) we obtain $G_{\mathbb{H}}(x,y) \geq C G^{\#}_{\mathbb{H}}(x,y)$. If $\frac{x_d y_d}{|x-y|^2} \geq 1$ and $|x-y| \leq 1$ we apply (2.8) to arrive at

$$\begin{aligned}G_{\mathbb{H}}(x,y) &= C|x-y|^{\alpha-d/2} \int_0^{\frac{4x_d y_d}{|x-y|^2}} \frac{t^{\alpha 2-1}}{(t+1)^{d/4}} K_{d/2}(|x-y|(t+1)^{1/2})\, dt \\ &\geq C|x-y|^{\alpha-d/2} \int_0^1 \frac{t^{\frac{\alpha}{2}-1}}{(t+1)^{d/4}} K_{d/2}(|x-y|(t+1)^{1/2})\, dt \\ &\geq C|x-y|^{\alpha-d} \int_0^1 \frac{t^{\frac{\alpha}{2}-1}}{(t+1)^{d/2}}\, dt \\ &\geq C_1|x-y|^{\alpha-d} \int_0^{\frac{4x_d y_d}{|x-y|^2}} \frac{t^{\frac{\alpha}{2}-1}}{(t+1)^{d/2}}\, dt \\ &= C_2 G^{\#}_{\mathbb{H}}(x,y).\end{aligned}$$

Note that the only place where we used the assumption $d \geq 2$ is the passage from the third to the fourth line in the above string of inequalities. $\square$

The next result provides a general bound for the Green function. Before proving it we introduce some notation. For $d \in \mathbb{N}$ we define a function ψ_d in the following way:

$$\psi_d(v) = \begin{cases} v^{\alpha/2}, & 0 < v < 1, \quad d \in \mathbb{N}, \\ v^{1/2}, & v \geq 1, \quad d = 1, \\ \ln^{1/2}(1+v), & v \geq 1, \quad d = 2, \\ 1, & v \geq 1, \quad d \geq 3. \end{cases}$$

Theorem 5.8. *There is a constant C such that*

$$\max\{G_{\mathbb{H}}(x,y), G^g_{\mathbb{H}}(x,y)\} \leq G^0_{\mathbb{H}}(x,y) \leq C[\psi_d(x_d)\psi_d(y_d) + G_{\mathbb{H}}(x,y)], \quad x, y \in \mathbb{H}.$$

Proof. The lower bound is an obvious consequence of Lemma 5.6. The proof of the upper bound will rely on the estimates of $P^x(\tau_{\mathbb{H}} \geq t)$ and the application of Lemma 5.2.

We proceed to estimate the Green function from above. We split the integration,

$$G^0_{\mathbb{H}}(x,y) = \int_0^\infty p_t^{\mathbb{H}}(x,y)\,dt = \int_0^6 p_t^{\mathbb{H}}(x,y)\,dt + \int_6^\infty p_t^{\mathbb{H}}(x,y)\,dt.$$

The first integral is estimated as follows:

$$\int_0^6 p_t^{\mathbb{H}}(x,y)\,dt \leq e^6 \int_0^6 e^{-t}\, p_t^{\mathbb{H}}(x,y)\,dt \leq e^6\, G_{\mathbb{H}}(x,y);$$

in the second one we apply Lemma 5.2 to obtain

$$\int_6^\infty p_t^{\mathbb{H}}(x,y)dt \leq C \int_2^\infty P^x(\tau_{\mathbb{H}} \geq t)\, P^y(\tau_{\mathbb{H}} \geq t)\, \frac{dt}{t^{d/2}} = CR(x,y).$$

For $d \geq 3$ we have

$$R(x,y) \leq CP^x(\tau_{\mathbb{H}} \geq 2)\, P^y(\tau_{\mathbb{H}} \geq 2) \int_2^\infty \frac{dt}{t^{d/2}} \leq C(x_d \wedge 1)^{\alpha/2}(y_d \wedge 1)^{\alpha/2},$$

because of (5.5), which completes the proof in this case.

To deal with $d = 1$ and 2 we observe that by the Schwarz inequality

$$R^2(x,y) \leq R(x,x)\, R(y,y),$$

so we need to estimate $R(x,x)$. For the case $x_d \leq 2$, using (5.3) and (5.5), we have

$$R(x,x) = \int_2^\infty \left(P^x(\tau_{\mathbb{H}} \geq t)\right)^2 \frac{dt}{t^{d/2}} \leq Cx_d^\alpha \int_2^\infty \frac{(\ln t)^2}{t} \frac{dt}{t^{d/2}}.$$

If $x_d > 2$, then applying (5.3) we estimate

$$\begin{aligned} R(x,x) &= \int_2^\infty \left(P^x(\tau_{\mathbb{H}} \geq t)\right)^2 \frac{dt}{t^{d/2}} \\ &= \int_2^{x_d^2} \left(P^x(\tau_{\mathbb{H}} \geq t)\right)^2 \frac{dt}{t^{d/2}} + \int_{x_d^2}^\infty \left(P^x(\tau_{\mathbb{H}} \geq t)\right)^2 \frac{dt}{t^{d/2}} \\ &\leq C \int_2^{x_d^2} \frac{dt}{t^{d/2}} + C \int_{x_d^2}^\infty \left(\frac{x_d + \ln t}{t^{1/2}}\right)^2 \frac{dt}{t^{d/2}}. \end{aligned}$$

Thus for $d=1$ we have $R(x,x)\le Cx$, while for $d=2$ we arrive at $R(x,x)\le C\ln x_2$. Taking into account all cases we get

$$\int_6^\infty p_t^{\mathbb{H}}(x,y)\,dt \le C\psi_d(x_d)\psi_d(y_d).$$

The proof of the theorem is complete. □

Our final result provides optimal bounds for $G^0_{\mathbb{H}}(x,y)$ when the points x,y are close to each other.

Theorem 5.9. *For $d=1$ and $|x-y|<1$ we have that*

$$G^0_{\mathbb{H}}(x,y)\approx G_{\mathbb{H}}(x,y)+x\wedge y\,. \tag{5.7}$$

For $d=2$ and $|x-y|<1$ we have that

$$G^0_{\mathbb{H}}(x,y)\approx G_{\mathbb{H}}(x,y)+\ln(1\vee(x_2\wedge y_2))\,.$$

For $d\ge 3$ and $|x-y|<1$ we have that

$$G^0_{\mathbb{H}}(x,y)\approx G_{\mathbb{H}}(x,y).$$

Proof. Step 1. We first show that if $|x-y|<1$, then

$$G_{\mathbb{H}}(x,y)\approx G^0_{\mathbb{H}}(x,y)\quad \text{if}\quad x_d\wedge y_d\le 2\quad\text{or}\quad d\ge 3.$$

Recall that by Lemma 5.7 we have

$$(x_d\,y_d\wedge 1)^{\alpha/2}\le CG_{\mathbb{H}}(x,y). \tag{5.8}$$

Let $x_d\wedge y_d\le 2$. The condition $|x-y|<1$ implies that $x_d\vee y_d\le 3$, hence

$$x_d\,y_d\le 9(1\wedge x_d\,y_d).$$

Then due to Theorem 5.8 and (5.8) we obtain

$$G_{\mathbb{H}}(x,y)\le G^0_{\mathbb{H}}(x,y)\le C_1(G_{\mathbb{H}}(x,y)+(x_d\,y_d)^{\alpha/2})\le C_2G_{\mathbb{H}}(x,y)\,. \tag{5.9}$$

We now consider the case when $d\ge 3$. It is elementary that

$$(x_d\wedge 1)(y_d\wedge 1)\le x_d\,y_d\wedge 1.$$

Then the inequality (5.9) can be rewritten in the following way regardless of the assumption on $x_d\wedge y_d$:

$$\begin{aligned} G_{\mathbb{H}}(x,y) &\le G^0_{\mathbb{H}}(x,y)\le C(G_{\mathbb{H}}(x,y)+((x_d\wedge 1)(y_d\wedge 1))^{\alpha/2})\\ &\le C(G_{\mathbb{H}}(x,y)+(x_d\,y_d\wedge 1)^{\alpha/2})\\ &\le C_1G_{\mathbb{H}}(x,y), \end{aligned}$$

where in the last line we again applied (5.8). This completes the proof of Step 1 and proves the theorem for $d\ge 3$.

Step 2. In this step we complete the proof of the case $d=1$.

The lower bound follows from the estimate proved in Lemma 5.6 and (5.6): $G^0_{(0,\infty)}(x,y)\ge G^g_{(0,\infty)}(x,y)=x\wedge y$. In the case $x\wedge y\le 2$ we also get the upper bound since $G^0_{(0,\infty)}(x,y)\approx G_{(0,\infty)}(x,y))$ by Step 1.

If $x\wedge y\ge 2$ we obtain by Theorem 5.8,

$$G^0_{(0,\infty)}(x,y)\le C_2(G_{(0,\infty)}(x,y)+(xy)^{1/2}).$$

Since $|x-y|<1$ and $x\wedge y\ge 2$ we have $(xy)^{1/2}\le 2(x\wedge y)$, which completes the proof of (5.7).

Step 3. Now we deal with the case $d = 2$. We claim that

$$G^g_{\mathbb{H}}(x,y) \geq C \ln(1 \vee (x_2 \wedge y_2)).$$

It is enough to show it for $x_2 \wedge y_2 \geq 1$. If $|x-y| \leq 1$, then using (5.6) we arrive at

$$\begin{aligned} G^g_{\mathbb{H}}(x,y) &= \frac{1}{2\pi} \ln \frac{|x^*-y|}{|x-y|} \geq \frac{1}{2\pi} \ln |x^*-y| \\ &\geq \frac{1}{2\pi} \ln (x_2+y_2) \geq \frac{1}{2\pi} \ln(1 \vee (x_2 \wedge y_2)). \end{aligned}$$

As a consequence of the above inequality and Lemma 5.6 we obtain

$$\frac{1}{2\pi} \ln(1 \vee (x_2 \wedge y_2)) \leq G^g_{\mathbb{H}}(x,y)) \leq G^0_{\mathbb{H}}(x,y),$$

which proves the lower bound.

If $x_2 \wedge y_2 \leq 2$, then by Step 1, $G_{\mathbb{H}}(x,y) \approx G^0_{\mathbb{H}}(x,y)$, which yields the upper bound in this case.

It remains to consider the case $x_2 \wedge y_2 \geq 2$. Note that $1+x_2 \vee y_2 \leq (1 \vee (x_2 \wedge y_2))^2$, so

$$\ln^{1/2}(1+y_2) \ln^{1/2}(1+x_2) \leq \ln(1+x_2 \vee y_2) = 2\ln(1 \vee (x_2 \wedge y_2)).$$

Thus, by Theorem 5.8,

$$G^0_{\mathbb{H}}(x,y) \leq C(G_{\mathbb{H}}(x,y) + \ln^{1/2}(1+y_2)\ln^{1/2}(1+x_2)) \leq C(G_{\mathbb{H}}(x,y) + \ln(1 \vee (x_2 \wedge y_2))),$$

which completes the proof of Step 3 and of the theorem. □

We conclude with a partial answer to the question posed at the beginning of this section.

Corollary 5.10. *For $d \leq 2$ we obtain*

$$\sup_{|x-y|\leq 1} \frac{G^0_{\mathbb{H}}(x,y)}{G^{\#}_{\mathbb{H}}(x,y)} = \infty.$$

For $d \geq 3$

$$G^0_{\mathbb{H}}(x,y) \approx G^{\#}_{\mathbb{H}}(x,y), \ |x-y| \leq 1.$$

Proof. For $d \geq 3$ the proof easily follows from Lemma 5.7 and the preceding theorem.

For $d \leq 2$ from the formula (4.4) we easily obtain the following upper bounds for $G^{\#}_{\mathbb{H}}(x,y)$:

$$G^{\#}_{\mathbb{H}}(x,y) \leq \begin{cases} C\frac{(x \wedge y)^{\alpha/2}}{|x-y|^{1-\alpha/2}}, & x,y \in \mathbb{H}, \quad d=1, \\ C\frac{1}{|x-y|^{2-\alpha}}, & x,y \in \mathbb{H}, \quad d=2. \end{cases}$$

The application of the above inequalities together with Theorem 5.9 completes the proof. □

Remark. It would be very interesting to obtain optimal estimates for the 0-Green function for balls. For balls of moderate radii one may show that (5.2) holds with a universal constant, so optimal estimates are easily derived from the well-known estimates in the stable case. However for large balls the comparison (5.2) is not satisfactory, as the constant grows to ∞ with the radius of the ball.

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