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A CONSTRUCTION OF NUMERICAL CAMPEDELLI SURFACES WITH TORSION $\mathbb{Z}/6$

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Abstract. We produce a family of numerical Campedelli surfaces with $\mathbb{Z}/6$ torsion by constructing the canonical ring of the étale 6 to 1 cover using serial unprojection. In Section 2 we develop the necessary algebraic machinery. Section 3 contains the numerical Campedelli surface construction, while Section 4 contains remarks and open questions.

1. Introduction

A numerical Campedelli surface is a smooth minimal surface of general type over the complex numbers with $K^2 = 2$ and $q = p_g = 0$. It is known that the algebraic fundamental group π_1^{alg} of such a surface is finite, of order at most 9 (cf. [BPHV] Chap. VII.10). Two recent papers about numerical Campedelli surfaces are [MP] and [LP]; the first classifies the case where the algebraic fundamental group has order exactly 9, while the second gives simply connected examples.

In the present work we give a construction of numerical Campedelli surfaces with algebraic fundamental group equal to $\mathbb{Z}/6$. To our knowledge, there were no such examples previously known, and it settles the existence question for numerical Campedelli surfaces with algebraic fundamental group of order 6, since by [Na] there are no numerical Campedelli surfaces with algebraic fundamental group equal to the symmetric group of order 6.

Our approach is to construct, using serial unprojection of type Kustin–Miller, the canonical ring of the étale 6 to 1 cover together with a suitable basepoint free action of $\mathbb{Z}/6$. The cover is a regular canonical surface with $p_g = 5$, and $K^2 = 12$, canonically embedded in $\mathbb{P}(1^5, 2^3)$.

The number of moduli of our construction is at most 4; cf. Remark 3.17 below. Since the expected number of moduli of a numerical Campedelli is 6 (cf. [BPHV], p. 295), it follows that there exist numerical Campedelli surfaces with torsion $\mathbb{Z}/6$ that cannot be obtained by our construction.

In Section 2 we define, for $n \geq 2$, what we call the generic $\binom{n}{2}$ Pfaffians ideal (Definition 2.2) and prove that it is Gorenstein of codimension equal to $n+1$ (Theorem 2.3). A special case of the construction is due to Frantzen ([Fr], Section 2.4).

In Section 3 we apply the results of the previous sections in the case of $n = 4$ to our specific geometric situation. The main results are Theorems 3.12 and 3.16, where we settle the existence of a nonsingular regular surface with $p_g = 5$ and

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$K^2 = 12$ endowed with a $\mathbb{Z}/6$ basepoint free action. Finally, Section 4 contains some remarks and open questions.

The way in which we have arrived at the family constructed in this article is strongly influenced by the general theory in [R1]. More precisely, one assumes that a hypothetical étale 6 to 1 cover of a numerical Campedelli surface is a quadratic section of an anticanonically embedded Fano 3-fold V, as in many similar constructions of surfaces of general type. Then, standard numerical Hilbert series calculations (cf. [R1], Section 3) lead to the expectation that the anticanonical model of V is a codimension 5 projectively Gorenstein subscheme of $\mathbb{P}(1^5, 2^4)$ with a certain Hilbert series. Combining this with the knowledge of how the Hilbert series changes during unprojection (or, alternatively, and more easily, reading the result directly from [Br]) one concludes that V can be realized as the result of a series of 4 unprojections of Kustin–Miller type, starting from a degree 4 hypersurface in $\mathbb{P}(1^5)$. Hence, starting from a degree 4 hypersurface in $\mathbb{P}(1^5)$ and unprojecting an arrangement of 4 codimension 1 loci, one could obtain a 3-fold V in $\mathbb{P}(1^5, 2^4)$ with the right Hilbert series. Then, taking a suitable member of $|-2K_V|$ we would obtain the étale 6 to 1 cover of a numerical Campedelli.

However, to set up a free $\mathbb{Z}/6$ action, motivated by empirical evidence showing that unprojection is best calculated in a general framework, we were driven to the unprojection of a general set of 4 linear subspaces of dimension 5 in a degree 4 6-fold hypersurface in $\mathbb{P}(1^8)$. Our main motivation was that we could then assume that these loci were defined by $x_1 = x_2 = 0$, $x_3 = x_4 = 0$, etc. After the unprojection of these subspaces we obtained a 6-fold in $\mathbb{P}(1^8, 2^4)$, and then taking 3 linear sections and 1 quadratic section we constructed a family of surfaces with $p_g = 5, q = 0, K^2 = 12$. Afterwards, calculations with characters and G-Hilbert series helped us to discover a suitable subfamily endowed with a good $\mathbb{Z}/6$ action.

We believe that a similar approach could be useful to other situations as well; compare Remark 4.3 below.

Bearing in mind the expectation that under mild conditions unprojections commute (cf. Remark 2.6 below) and the previously done calculations of [P2] and [Fr], Section 2.4, we obtained the generic $\binom{n}{2}$ Pfaffians ideal format for $n = 4$. We then discovered that the arguments for the Gorensteinness of the format for general n were a rather straightforward generalisation of those needed in the special case.

2. The generic $\binom{n}{2}$ Pfaffians ideal

Notation 2.1. Let us make the following notation:

(1) Let $n \geq 2$. Let $A_0 = \mathbb{Z}[x_1, \dots, x_n, z_1, \dots, z_n, r_{d_1 \cdots d_n}]$ be the polynomial ring over the integers in n variables $x_1, \dots, x_n$, n variables $z_1, \dots, z_n$ and 2^n variables $r_{d_1 \cdots d_n}$, indexed by $(d_1, \dots, d_n) \in \{0,1\}^n$.

(2) Define the polynomial algebra extensions $A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n$ by setting inductively $A_i = A_{i-1}[y_i]$ for $i = 1, \dots, n$.

(3) Make these rings graded by setting the degree of x_i, z_i and $r_{d_1 \cdots d_n}$ equal to 1, for all $i = 1, \dots, n$ and $(d_1, \dots, d_n) \in \{0,1\}^n$ and by setting the degree of y_i equal to $n-1$, for all $i = 1, \dots, n$.

(4) Consider the degree $n+1$ homogeneous polynomial defined by

$$Q = \sum r_{d_1 \cdots d_n} a_{1,d_1} \cdots a_{n,d_n} \in A_0,$$

where the summation is for $(d_1, \ldots, d_n) \in \{0,1\}^n$, and, by definition, a_{i,d_i} is equal to x_i if $d_i = 0$ and equal to z_i if $d_i = 1$.

(5) For each $1 \le i < j \le n$, let

$$Q_{ij}^{xx} = \frac{\partial^2 Q}{\partial x_i \partial x_j}, \quad Q_{ij}^{xz} = \frac{\partial^2 Q}{\partial x_i \partial z_j}, \quad Q_{ij}^{zx} = \frac{\partial^2 Q}{\partial z_i \partial x_j} \quad \text{and} \quad Q_{ij}^{zz} = \frac{\partial^2 Q}{\partial z_i \partial z_j}.$$

Then each of the Q_{ij}^{ab}, where $a, b \in \{x, y\}$, is homogeneous of degree $n-1$, and clearly,

$$Q = x_i x_j Q_{ij}^{xx} + x_i z_j Q_{ij}^{xz} + z_i x_j Q_{ij}^{zx} + z_i z_j Q_{ij}^{zz}.$$

(6) For each $1 \le i < j \le n$ let

$$M_{ij} = \begin{pmatrix} 0 & x_i & z_i & -x_j & -z_j \\ & 0 & y_j & Q_{ij}^{zz} & -Q_{ij}^{zx} \\ & & 0 & -Q_{ij}^{xz} & Q_{ij}^{xx} \\ & \text{-sym} & & 0 & -y_i \\ & & & & 0 \end{pmatrix}$$

be a skew–symmetric 5×5 matrix with entries in A_n. The 5 submaximal Pfaffians[1] of this matrix are:

$$\begin{gathered} y_i y_j - Q_{ij}^{xz} Q_{ij}^{zx} + Q_{ij}^{xx} Q_{ij}^{zz}, \\ x_i y_i + (x_j Q_{ij}^{zx} + z_j Q_{ij}^{zz}), \qquad z_i y_i - (x_j Q_{ij}^{xx} + z_j Q_{ij}^{xz}), \\ x_j y_j + (x_i Q_{ij}^{xz} + z_i Q_{ij}^{zz}), \qquad z_j y_j - (x_i Q_{ij}^{xx} + z_i Q_{ij}^{zx}), \end{gathered} \tag{2.1}$$

which are all homogeneous elements of A_n. Fixing $1 \le i \le n$ and varying j we see that several Pfaffians involve the monomial $x_i y_i$. Notice, however, that $x_j Q_{ij}^{zx} + z_j Q_{ij}^{zz} = \frac{\partial Q}{\partial z_i}$, which does not depend on j. Hence, in the set of Pfaffians of all possible M_{ij}, there is only 1 polynomial in which the monomial $x_i y_i$ occurs. A similar observation applies to the Pfaffians in which the monomial $z_i y_i$ occurs.

(7) For each $0 \le p \le n$ we define an homogeneous ideal $I_p \subset A_p$ by:
 (a) $I_0 = (Q) \subset A_0$;
 (b) $I_1 = \left(x_1 y_1 + \frac{\partial Q}{\partial z_1}, z_1 y_1 - \frac{\partial Q}{\partial x_1} \right) \subset A_1$, the ideal of A_1 generated by the two Pfaffians of M_{12} which involve $x_1 y_1$ and $z_1 y_1$;
 (c) for $2 \le p \le n$, $I_p \subset A_p$ is the ideal of A_p generated by the union of all the submaximal Pfaffians of all matrices M_{ij} for $1 \le i < j \le p$.

(8) In the set of Pfaffians of all possible M_{ab}, $1 \le a < b \le n$, denote by
 (a) l_i the polynomial in which $x_i y_i$ occurs with coefficient 1;
 (b) m_i the polynomial in which $z_i y_i$ occurs with coefficient 1;
 (c) e_{ij} the polynomial in which $y_i y_j$ occurs with coefficient 1

for all $1 \le i \le n$ in (a) and (b), and $1 \le i < j \le n$ in (c). In particular,

$$I_p = (\{e_{ij}, l_t, m_t : 1 \le i < j \le p \text{ and } 1 \le t \le p\}) \subset A_p,$$

and in addition $I_1 = (l_1, m_1) \subset A_1$. Let us stress that we can take the expressions in (2.1) for any given j to write the polynomials l_i, m_i and e_{ij}.

(9) For $0 \le p \le n-1$, define the homogeneous ideals $J_p \subset A_p$ as follows:
 (a) $J_0 = (x_1, z_1) \subset A_0$;
 (b) $J_p = (x_{p+1}, z_{p+1}, y_1, \ldots, y_p) \subset A_p$, for $p \ge 1$.

[1]For a discussion about the Pfaffians of a skew–symmetric matrix see, for example, [BH], Section 3.4.

(10) Finally, notice that $I_p \subset J_p \subset A_p$. Set $R_p = A_p/I_p$ and consider J_p as an ideal of R_p.

Definition 2.2. We call the ideal I_n of A_n the generic $\binom{n}{2}$ Pfaffians ideal.

The main aim of this section is to prove by induction on $p = 0, 1, \dots, n$ that $R_p = A_p/I_p$ is a Gorenstein graded ring whose dimension is equal to $\dim A_0 - 1$ (hence I_p has codimension $p+1$ in A_p). Our strategy is to establish inductively that R_p is the result of serial unprojection of type Kustin–Miller ([PR], Definition 1.2). Our main algebraic result is the following theorem, which we will prove in Subsection 2.1.

Theorem 2.3. *Let all the notation be as above.*

(a) *For $p = 1, \dots, n$, R_p is the unprojection ring of type Kustin–Miller of the pair $J_{p-1} \subset R_{p-1}$.*
(b) *For $p = 0, 1, \dots, n$, R_p is a normal Gorenstein graded integral domain of dimension equal to* $\dim R_0$ *(which is equal to $2n + 2^n$ since* $\dim \mathbb{Z} = 1$*).*
(c) *There are natural inclusions*

$$R_0 \subset R_1 \subset \cdots \subset R_p \subset K(R_0),$$

where $K(R_0)$ is the field of fractions of R_0, all except the last induced by the chain of inclusions $A_0 \subset A_1 \subset \cdots \subset A_p$.
(d) *For $p = 0, \dots, n$ there exists a Zariski closed subset $F_p \subset$* $\operatorname{Spec} R_p$*, with the codimension of F_p in* $\operatorname{Spec} R_p$ *at least 2, such that the open subscheme* $\operatorname{Spec} R_p \setminus F_p$ *is naturally isomorphic with an open subscheme of* $\operatorname{Spec} R_0$.
(e) *For $p = 1, \dots, n$, x_p, z_p is a regular sequence of R_p.*
(f) *For $p = 0, 1, \dots, n$ and $1 \leq i < j \leq n$, x_i, x_j is a regular sequence of R_p.*

Remark 2.4. The most important conclusions of Theorem 2.3 are (a) and (b). However, for the purposes of the inductive step we need all six statements.

Remark 2.5. For $1 \leq t \leq n$ the inclusion $R_{t-1} \subset R_t$ of (c) of Theorem 2.3 is given by $R_t = R_{t-1}[s_t]$, where $s_t \in K(R_0)$ is the rational function given by

$$s_t = \frac{x_t y_t - l_t}{x_t} = \frac{z_t y_t - m_t}{z_t}$$

and

$$s_t = \frac{y_i y_t - e_{it}}{y_i}$$

for $1 \leq i \leq t-1$.

Remark 2.6. Fix $2 \leq p \leq n$. Inside $\operatorname{Spec} R_0$ we have the p codimension 1 subschemes $V(x_i, z_i)$ for $1 \leq i \leq p$. We can interpret Theorem 2.3 and Corollary 2.13 below as saying that the order we perform the unprojections of the subschemes is irrelevant. An interesting open question is to find general conditions that will guarantee this kind of commutativity of unprojections.

2.1. **Proof of Theorem 2.3.** We begin the proof of Theorem 2.3. The following proposition will be used in the proof of Proposition 2.8.

Proposition 2.7. *Denote by $\mathbb{Z}[X]$ the polynomial ring $\mathbb{Z}[x_1, ..., x_n]$, and assume $I \subset \mathbb{Z}[X]$ is a proper ideal generated by monic monomials. For any field k we have*

$$\dim \left(\frac{\mathbb{Z}[X]}{I} \otimes_{\mathbb{Z}} k \right) = \dim \left(\frac{\mathbb{Z}[X]}{I} \right) - 1.$$

Proof. Let k be a field. Denote, for $d \geq 1$, by $\widetilde{S_d}$ the set of monic monomials of $\mathbb{Z}[X]$ of degree d which are not in I. Since I is generated by monic monomials, it follows that $\widetilde{S_d}$ is a k-basis of the degree d homogeneous component of $\mathbb{Z}[X]/I \otimes_{\mathbb{Z}} k$. As a consequence, the Hilbert polynomial of $\mathbb{Z}[X]/I \otimes_{\mathbb{Z}} k$ is independent of the choice of the field k. Therefore, by [Ei], Corollary 13.7 the dimension of $\mathbb{Z}[X]/I \otimes_{\mathbb{Z}} k$ is also independent of the choice of the field k.

Being a free $\mathbb{Z}$-module, the polynomial ring $\mathbb{Z}[X]$ is a faithfully flat $\mathbb{Z}$-module. Moreover, the height of a homogeneous ideal of a graded Cohen–Macaulay ring is equal to its codimension; cf. [BH], Corollary 2.1.4. The result now follows from [BV], Proposition 3.14 (for $J = 0$), taking into account that $\dim \mathbb{Z}[X] = \dim k[X] + 1$. $\square$

Proposition 2.8. *Fix $1 \leq p \leq n$. Assume R_p is Cohen–Macaulay with $\dim R_p = \dim R_0$. Then x_p, z_p is a regular sequence for R_p.*

Proof. Denote by $T \subset A_p$ the ideal of A_p generated by all $r_{d_1 \cdots d_n}$ for $(d_1, \ldots, d_n) \in \{0,1\}^n$ with $(d_1, \ldots, d_n) \neq (0, 0, \ldots, 0)$ and $(d_1, \ldots, d_n) \neq (1, 1, \ldots, 1)$.

Using the assumptions about the dimension and the Cohen–Macaulayness of R_p, to prove the proposition it is enough to show that

$$\dim R_p/(x_p, z_p) \leq \dim R_0 - 2 = \dim A_0 - 3,$$

and for that it is enough to show that

$$\dim A_p/(I_p + (x_p, z_p, y_p) + T) \leq (\dim A_0 - 3) - (2^n - 2) - 1 = 2n - 1.$$

We denote by $\eta(l_i)$ the polynomial obtained from l_i by setting the variables x_p, z_p, y_p and $r_{d_1 \cdots d_n}$ for $(d_1, \ldots, d_n) \in \{0,1\}^n$ with $(d_1, \ldots, d_n) \neq (0, 0, \ldots, 0)$ and $(d_1, \ldots, d_n) \neq (1, 1, \ldots, 1)$ equal to 0, and similarly for $\eta(m_i)$ and $\eta(e_{ij})$.

For $1 \leq i < p$ we have

$$\eta(l_i) = x_i y_i, \qquad \eta(m_i) = z_i y_i,$$

for $1 \leq i < j < p$ we have

$$\eta(e_{ij}) = y_i y_j,$$

and for $1 \leq i < p$ we have

$$\begin{aligned}
\eta(e_{ip}) &= r_{00\cdots 0} r_{11\cdots 1} \left[\prod_{t=1}^{i-1} x_t z_t\right] \left[\prod_{t=i+1}^{p-1} x_t z_t\right] \left[\prod_{t=p+1}^{n} x_t z_t\right], \\
\eta(l_p) &= r_{11\cdots 1} \left[\prod_{t=1}^{p-1} z_t\right] \left[\prod_{t=p+1}^{n} z_t\right], \\
\eta(m_p) &= -r_{00\cdots 0} \left[\prod_{t=1}^{p-1} x_t\right] \left[\prod_{t=p+1}^{n} x_t\right].
\end{aligned}$$

For the proof of the first equality, substitute $x_p = z_p = 0$ to

$$l_i = x_i y_i + x_p \frac{\partial^2 Q'}{\partial z_i \partial x_p} + z_p \frac{\partial^2 Q'}{\partial z_i \partial z_p},$$

where $Q' = r_{00\cdots 0}\ x_1 \cdots x_n + r_{11\cdots 1}\ z_1 \cdots z_n$. The proof of the other equalities is similar.

To calculate the codimension of the ideal $(I_p + (x_p, z_p, y_p) + T)$ of A_p using Proposition 2.7, we can make a base change from $\mathbb{Z}$ to an algebraically closed field k and argue geometrically by studying zero loci. Since

$$I_p = (l_1, \dots, l_p, m_1, \dots, m_p, e_{ij}) \subset A_p$$

(with indices $1 \le i < j \le p$), using the vanishing of $\eta(e_{ij}) = y_i y_j$, for $1 \le i < j < p$, we get two cases.

Case 1. All y_i are 0, for $1 \le i \le p-1$. Then, the vanishing of $\eta(l_p)$ and $\eta(m_p)$ imply that two more variables vanish, so we get the desired codimension.

Case 2. There exists unique nonzero y_a, with $1 \le a \le p-1$. Using the vanishing of $\eta(l_a)$ and $\eta(m_a)$ we get the vanishing of both x_a and z_a, and using the vanishing of $\eta(e_{ap})$ we get that at least 1 more variable should vanish, so we again reach the desired codimension which finishes the proof of the proposition. □

We now start the induction for the proof of Theorem 2.3.

Lemma 2.9. *Theorem* 2.3 *is true for* $p = 0$.

Proof. The ring A_0 is a Gorenstein normal integral domain, since it is a finitely generated polynomial $\mathbb{Z}$-algebra. Since $Q \in A_0$ is an irreducible polynomial, it follows that $R_0 = A_0/(Q)$ is a Gorenstein integral domain. Therefore, using the fact that m homogeneous elements (of positive degree) of a graded Cohen–Macaulay ring form a regular sequence if and only if the ideal they generate has codimension m (cf. [BH], Theorem 2.1.2), to prove that, for fixed $1 \le i < j \le n$, x_i, x_j is a regular sequence of R_0, it is enough to show that $\dim R_0/(x_i, x_j) = \dim R_0 - 2$. This is true, since $R_0/(x_i, x_j)$ is isomorphic to the quotient of a polynomial ring over the integers in two less variables than A_0 by the ideal generated by a nonzero nonconstant homogeneous polynomial.

We will prove the normality of R_0 by applying [BV], Lemma 16.24, which says that a Noetherian ring R is normal if there exists $y \in R$ which is not a zero–divisor, such that $R/(y)$ is reduced and $R[y^{-1}]$ is normal.

Fix $1 \le i \le n$. The ring $R_0/(x_i)$ is a reduced ring, since the polynomial obtained from Q by substituting 0 for x_i has no multiple factors.

Denote, for $1 \le i \le n$, by $T_i \subset R_0$ the multiplicatively closed subset

$$T_i = \{1, x_i, x_i^2, x_i^3, \dots\} \subset R_0.$$

For notational convenience we also set $T_0 = \{1\} \subset R_0$. We will prove by induction on $t = 0, 1, \dots, n$ that the localization ring

$$B_t = T_{n-t}^{-1} T_{n-t-1}^{-1} \cdots T_1^{-1} T_0^{-1} R_0$$

is a normal integral domain. Since a localization of an integral domain is an integral domain, we only need to prove the normality of B_t.

Assume first that $t = 0$. By the form of Q, B_0 is isomorphic to a localization of the polynomial subring

$$\mathbb{Z}[x_1, \dots, x_n, z_1, \dots, z_n, r_{d_1 \cdots d_n}] \subset A_0,$$

where $(d_1, d_2, \dots, d_n) \in \{0,1\}^n$ and $(d_1, \dots, d_n) \ne (0, 0, \dots, 0)$. Since the localization of a normal ring is again normal, we get that B_0 is a normal domain.

Assume now that for some t with $0 \le t \le n-1$ we have that B_t is normal. By [BV], Lemma 16.24, to prove that the domain B_{t+1} is normal it is enough to prove that $B_{t+1}/(x_{n-t})$ is reduced. Since localization commutes with taking quotients and the localization of a reduced ring is again reduced, we have that $B_{t+1}/(x_{n-t})$ is reduced as a localization of the already proven reduced $R_0/(x_{n-t})$. This finishes the induction, and hence the case $p=0$ of Theorem 2.3 follows. □

Lemma 2.10. *Theorem* 2.3 *is true for* $p=1$.

Proof. Using [P2], Section 4, we get that R_1 is the unprojection of type Kustin–Miller of the pair $J_0 \subset R_0$; hence by the definitions of unprojection ([PR], Section 1) we have that R_0 is contained in a natural way in R_1 and R_1 has the same dimension as R_0. Moreover, by [PR], Theorem 1.5, R_1 is Gorenstein, and by [PR], Remark 1.5, R_1 is a domain contained in a natural way in the field of fractions $K(R_0)$ of R_0. In particular, the ring R_1 is Cohen–Macaulay and hence satisfies Serre's conditions S_i for all $i \geq 0$ (cf. [BH], p. 63).

Proposition 2.8 says that x_1, z_1 is a regular sequence of R_1. Hence by setting

$$F_1 = V(x_1, z_1) \subset \operatorname{Spec} R_1$$

we get that F_1 has codimension 2 in $\operatorname{Spec} R_1$, and by the construction of unprojection $\operatorname{Spec} R_1 \setminus F_1$ is isomorphic in a natural way (induced by the inclusion $R_0 \subset R_1$) to the open subscheme $\operatorname{Spec} R_0 \setminus V(x_1, z_1)$ of $\operatorname{Spec} R_0$. Using Serre's normality criterion ([BH], Theorem 2.2.22), we get that the integral domain R_1 is normal.

We now prove that if $1 \leq i < j \leq n$, then x_i, x_j is a regular sequence of R_1. Assume this is not true. Using the fact that we proved that R_1 is Gorenstein and that m homogeneous elements (of positive degree) of a graded Cohen–Macaulay ring form a regular sequence if and only if the ideal they generate has codimension m (cf. [BH], Theorem 2.1.2), we get that $V(x_i, x_j) \subset \operatorname{Spec} R_1$ has codimension at most 1 in $\operatorname{Spec} R_1$. Using the natural isomorphism of $\operatorname{Spec} R_1 \setminus F_1$ with the open subscheme $\operatorname{Spec} R_0 \setminus V(x_1, z_1)$ and the fact that we proved that $F_1 \subset \operatorname{Spec} R_1$ has codimension 2 in $\operatorname{Spec} R_1$, it follows that x_i, x_j is not a regular sequence for R_0, contradicting Lemma 2.9. This finishes the proof of Theorem 2.3 for $p=1$. □

We now do the inductive step in the proof of Theorem 2.3. We fix q with $1 \leq q \leq n-1$. We assume that Theorem 2.3 is true for all values p with $0 \leq p \leq q$, and we will show that it is also true for the case $p = q+1$.

For the rest of the proof, given $0 \leq t \leq n-1$, we will denote by $L_t \subset R_0$ the ideal $L_t = (x_{t+1}, z_{t+1}) \subset R_0$, and by $i_t \colon J_t \to R_t$ and $i_{1,t} \colon L_t \to R_0$ the natural inclusion homomorphisms.

Lemma 2.11. *There exists a unique homomorphism of Abelian groups*

$$\phi_q \colon \operatorname{Hom}_{R_0}(L_q, R_0) \to \operatorname{Hom}_{R_q}(J_q, R_q)$$

such that

$$\phi_q(f)(x_{q+1}) = f(x_{q+1})$$

for all $f \in \operatorname{Hom}_{R_0}(L_q, R_0)$.

Proof. Recall that if $L \subset R$ is an ideal of a commutative ring R and $w \in L$ a nonzero divisor of R, then $\operatorname{Hom}_R(L, R)$ is isomorphic to the ideal $\{a \in R \colon aL \subset (w)\}$ by the map $f \mapsto f(w)$ (cf. [PR], Remark 1.3.3). In particular, f is uniquely specified by the value $f(w)$.

Accordingly, since by the inductive hypothesis both R_0 and R_q are integral domains with $R_0 \subset R_q$, it is enough to show that

$$\{a \in R_0 \colon aL_q \subset R_0 x_{q+1}\} \subset \{b \in R_q \colon bJ_q \subset R_q x_{q+1}\}.$$

Suppose $a \in R_0$ and $aL_q \subset R_0 x_{q+1}$. In particular, $az_{q+1} \in R_0 x_{q+1}$. Obviously $a \in R_q$, so we need to show that $ay_i \subset R_q x_{q+1}$ for all $1 \leq i \leq q$.

Fix $1 \leq i \leq q$. Using the equation l_i, which is 0 at R_q, we get

$$x_i a y_i = -(a x_{q+1} Q^{zx}_{i,q+1} + a z_{q+1} Q^{zz}_{i,q+1}) \in R_q x_{q+1}. \tag{2.2}$$

By the inductive hypothesis, x_i, x_{q+1} is a regular sequence for R_q. As a consequence, (2.2) implies that $ay_i \in R_q x_{q+1}$. Hence we get the existence of the map ϕ_q. R_q is a domain which implies that x_{q+1} is a regular element of R_q. The uniqueness follows by the fact that an element of $\operatorname{Hom}_{R_q}(J_q, R_q)$ is uniquely specified by its value at x_{q+1}. □

Notice that clearly $\phi_q(i_{1,q}) = i_q$.

Proposition 2.12. *Assume $f \in \operatorname{Hom}_{R_q}(J_q, R_q)$. There exists $b \in R_q$ such that the homomorphism $f - bi_q$ maps x_{q+1} and z_{q+1} inside $R_0 \subset R_q$.*

Proof. We prove by induction that for every $t = 0, \ldots, q$ there exists $b_t \in R_q$ such that $f - b_t i_q$ maps the elements x_{q+1}, z_{q+1} and y_j, for $1 \leq j \leq q - t$, inside $R_{q-t} \subset R_q$.

The result is trivially true when $t = 0$. Assume $0 \leq t \leq q-1$ and that there exists $b_t \in R_q$ such that $f - b_t i_q$ maps the elements x_{q+1}, z_{q+1} and y_j, for $1 \leq j \leq q - t$, inside R_{q-t}. Since, by construction, $R_{q-t} = R_{q-(t+1)}[y_{q-t}]$ (as algebras), there exist $a \in R_{q-(t+1)}$ and $c \in R_q$ with

$$(f - b_t i_q)(y_{q-t}) = a + c y_{q-t}.$$

Set $g = f - (b_t + c) i_q$. We claim that g maps the elements x_{q+1}, z_{q+1} and y_j, for $1 \leq j \leq q-(t+1)$, inside $R_{q-(t+1)}$. Indeed, if u is in the ideal of $R_{q-(t+1)}$ generated by x_{q+1}, z_{q+1} and y_j, for $1 \leq j \leq q - (t+1)$, we have

$$y_{q-t} g(u) = u g(y_{q-t}) \in R_{q-(t+1)}.$$

Since by the inductive hypothesis of Theorem 2.3 we have normality of $R_{q-(t+1)}$ and the fact that R_{q-t} is an unprojection of $R_{q-(t+1)}$, using [PR], Remark 1.3.4 (cf. [P1], Lemma 2.1.7) we get that $g(u) \in R_{q-(t+1)}$, which finishes the proof of Proposition 2.12. □

Corollary 2.13. *Fix $s \in \operatorname{Hom}_{R_0}(L_q, R_0)$ such that s together with $i_{1,q}$ generate the R_0-module $\operatorname{Hom}_{R_0}(L_q, R_0)$. Then $\phi_q(s)$ together with i_q generate the R_q-module $\operatorname{Hom}_{R_q}(J_q, R_q)$.*

Proof. Assume $f \in \operatorname{Hom}_{R_q}(J_q, R_q)$. Using Proposition 2.12, there exists $b \in R_q$ such that, if we set $g = f - bi_q$, we have $g(x_{q+1}) \in R_0$ and $g(z_{q+1}) \in R_0$. Therefore, there exists $g_1 \in \operatorname{Hom}_{R_0}(L_q, R_0)$, with $g_1(x_{q+1}) = g(x_{q+1})$ and $g_1(z_{q+1}) = g(z_{q+1})$. By the assumptions there exist $c_1, c_2 \in R_0$ with $g_1 = c_1 s + c_2 i_{1,q}$. As a consequence,

$$f(x_{q+1}) = c_1 \big[\phi_q(s)(x_{q+1})\big] + (b + c_1) i_q(x_{q+1}).$$

Since by the inductive hypothesis R_q is a domain, x_{q+1} is a regular element of R_q. Arguing as in the beginning of the proof of Lemma 2.11 we get

$$f = c_1\phi_q(s) + (b + c_1)i_q,$$

and the result follows. □

Proposition 2.14. *The ring R_{q+1} is isomorphic to the unprojection ring of the pair $J_q \subset R_q$.*

Proof. To simplify the notation of the proof we set, for $a, b \in \{x, z\}$, $Q^{ab} = Q^{ab}_{i,q+1}$. Using [P2], Section 4, $\operatorname{Hom}_{R_0}(L_q, R_0)$ is generated as an R_0-module by the inclusion map $i_{1,q}$ together with the homomorphism $t\colon L_q \to R_0$ such that

$$t(x_{q+1}) = -(x_iQ^{xz} + z_iQ^{zz}), \qquad t(z_{q+1}) = x_iQ^{xx} + z_iQ^{zx}$$

for all $1 \le i \le q$. Notice that these equations correspond exactly to l_{q+1} and m_{q+1}. Using Corollary 2.13, i_q together with $\phi_q(t)$ generate the R_q-module $\operatorname{Hom}_{R_q}(J_q, R_q)$, so $\phi_q(t)$ can be used to define the unprojection ring.

Fix $1 \le i \le q$. We have inside R_q

$$x_{q+1}\big[\phi_q(t)(y_i)\big] = y_i\big[\phi_q(t)(x_{q+1})\big] = -y_i(x_iQ^{xz} + z_iQ^{zz}).$$

Using the relations $l_i = 0$ and $m_i = 0$ which hold in R_q (since $1 \le i \le q$), we get

$$\begin{aligned}
&-y_i(x_iQ^{xz} + z_iQ^{zz}) = -Q^{xz}(x_iy_i) - Q^{zz}(z_iy_i)\\
&= Q^{xz}(x_{q+1}Q^{zx} + z_{q+1}Q^{zz}) - Q^{zz}(x_{q+1}Q^{xx} + z_{q+1}Q^{xz})\\
&= x_{q+1}(Q^{xz}Q^{zx} - Q^{xx}Q^{zz}).
\end{aligned}$$

Hence, since R_q is a domain,

$$\phi_q(t)(y_i) - (Q^{xz}Q^{zx} - Q^{xx}Q^{zz}) = 0,$$

which corresponds exactly to e_{iq}. As a consequence, Proposition 2.14 follows. □

Proposition 2.15. *The ring R_{q+1} is a Gorenstein integral domain, of dimension equal to $\dim R_0$, containing R_q in a natural way and contained in the field of fractions $K(R_0)$.*

Proof. Using Proposition 2.14, we get by the definitions of unprojection ([PR], Section 1) that R_q is contained in a natural way in R_{q+1} and that R_{q+1} has the same dimension as R_q; hence by the inductive hypothesis it has the same dimension as R_0. Moreover, by [PR], Theorem 1.5 and the inductive hypotheses for R_q we get that R_{q+1} is Gorenstein, and by [PR], Remark 1.5 that it is also a domain contained in a natural way in the field of fractions $K(R_q)$ of R_q. Since by the inductive hypothesis $K(R_q) = K(R_0)$, Proposition 2.15 follows. □

Proposition 2.16. *The sequence x_{q+1}, z_{q+1} is a regular sequence for R_{q+1}.*

Proof. By Proposition 2.15, R_{q+1} is a Gorenstein integral domain, of dimension equal to $\dim R_0$. As a consequence, the result follows by using Proposition 2.8. □

Proposition 2.17. *There exists a Zariski closed subset $F_{q+1} \subset \operatorname{Spec} R_{q+1}$, with the codimension of F_{q+1} in $\operatorname{Spec} R_{q+1}$ at least 2, such that the open subscheme $\operatorname{Spec} R_{q+1} \setminus F_{q+1}$ is naturally isomorphic with an open subscheme of $\operatorname{Spec} R_0$.*

Proof. By the construction of unprojection,

$$\operatorname{Spec} R_{q+1} \setminus V(x_{q+1}, z_{q+1}, y_1, \ldots, y_q)$$

is naturally isomorphic to an open subset of $\operatorname{Spec} R_q$. Using the inductive hypothesis and Proposition 2.16 the result follows. □

Proposition 2.18. *The ring R_{q+1} is a normal domain.*

Proof. By Proposition 2.15, R_{q+1} is a Gorenstein integral domain. The result follows by combining the normality of R_0 (Lemma 2.10), Proposition 2.17 and Serre's normality criterion ([BH], Theorem 2.2.22). □

Proposition 2.19. *If $1 \le i < j \le n$, then x_i, x_j is a regular sequence of R_{q+1}.*

Proof. If this was not true, using the fact that R_{q+1} is Gorenstein (Proposition 2.15) we would have that $V(x_i, x_j) \subset \operatorname{Spec} R_{q+1}$ would have codimension at most 1 in $\operatorname{Spec} R_{q+1}$. Using Proposition 2.17, we get that x_i, x_j is not a regular sequence for R_0, contradicting Lemma 2.9. □

We have now finished the proof of the inductive step, hence the proof of Theorem 2.3.

2.2. **Generic perfection of R_p.** We fix $n \ge 2$ and $0 \le p \le n$. We will prove that the A_p-module R_p is a generically perfect A_p-module. Recall ([BV], Section 3.A) that this means that R_p is a perfect A_p-module and also faithfully flat as a $\mathbb{Z}$-module.

A useful consequence of the generic perfection of R_p is that whenever we substitute the variables of the ideal I_p with elements of an arbitrary Noetherian ring, we get, under mild conditions, good induced properties of the resulting ideal (cf. [BV], Section 3 for precise statements). We will use the generic perfection of R_p in Corollary 2.22, Remark 2.23 and Proposition 3.5.

Remark 2.20. Recall ([BV], Section 16.B) that if A is a Noetherian ring and M a finitely generated A-module, the grade of M is defined to be the maximal length of an A-regular sequence contained in the annihilator ideal $\operatorname{Ann} M$ of M. If in addition A is a graded Cohen–Macaulay and M is a graded module, we have that the grade of M is equal to $\dim A - \dim A/\operatorname{Ann} M$ (cf. [BH], Corollary 2.1.4). As a consequence, using Theorem 2.3, R_p has the grade as an A_p-module equal to $p+1$.

Proposition 2.21. *The A_p-module R_p is generically perfect of the grade $p+1$.*

Proof. Using [BV], Proposition 3.2 it is enough to prove that R_p is a perfect A_p-module, and for every prime integer q the $A_p \otimes_{\mathbb{Z}} \mathbb{Z}/q$-module $R_p \otimes_{\mathbb{Z}} \mathbb{Z}/q$ is perfect.

Using the remark just before [BV], Proposition 16.20, the perfection of R_p as an A_p-module follows from the Gorensteinness of R_p (Theorem 2.3).

Fix an integer prime q. It is clear that all the arguments we used to prove Theorem 2.3 also work if we replace $\mathbb{Z}$ by $\mathbb{Z}/q$. As a consequence, we can argue as in the case of R_p to get that the $A_p \otimes_{\mathbb{Z}} \mathbb{Z}/q$-module $R_p \otimes_{\mathbb{Z}} \mathbb{Z}/q$ is perfect, which finishes the proof of the proposition. □

Corollary 2.22. *Let k be an arbitrary field. The $A_p \otimes k$-module $R_p \otimes_{\mathbb{Z}} k$ is perfect, of the grade equal to $p+1$. Moreover the k-algebra $R_p \otimes_{\mathbb{Z}} k$ is Gorenstein.*

Proof. It follows immediately by combining the Gorensteinness of R_p (Theorem 2.3) and the generic perfection of R_p (Proposition 2.21) with [BV], Theorems 3.3 and 3.6. □

Remark 2.23. Using the construction of unprojection in [KM] which is based on resolution complexes, together with the fact that J_p has Koszul complex as its minimal resolution over A_p (since it is generated by a regular sequence), we can inductively build a free graded resolution of R_p over A_p. Using [BV], Theorem 3.3 this will give us a free graded resolution of $R_p \otimes_{\mathbb{Z}} k$ over $A_p \otimes_{\mathbb{Z}} k$, where k is an arbitrary field. We will use this remark in Proposition 3.5.

More precisely, assume S is a polynomial ring over the integers or over a field, and $I \subset J \subset S$ are two homogeneous ideals of S such that both quotient rings S/I and S/J are Gorenstein and $\dim S/I = \dim S/J + 1$. We define $k_1, k_2 \in \mathbb{Z}$ such that $\omega_{S/I} = S/I(k_1)$ and $\omega_{S/J} = S/J(k_2)$ (cf. [BH], p. 140) and assume that $k_1 > k_2$. Moreover, let

$$0 \to A_g \to A_{g-1} \to \cdots \to A_1 \to A_0 \to S/J \to 0$$

and

$$0 \to B_{g-1} \to \cdots \to B_1 \to B_0 \to S/I \to 0$$

be the minimal free graded resolutions of S/J and S/I respectively as S-modules. Denote by $\widetilde{I} \subset S[v]$ the ideal of the unprojection of the pair $J/I \subset S/I$, where v is a new variable of degree $k_1 - k_2$. Then, whenever $g \geq 3$ (the case $g = 2$ is straightforward), a free graded resolution of the quotient ring $S[v]/\widetilde{I}$ as an $S[v]$-module is equal to

$$0 \to F_g \to F_{g-1} \to \cdots \to F_1 \to F_0 \to S[v]/\widetilde{I} \to 0,$$

where

$$\begin{gathered} F_0 = B_0', \qquad F_1 = B_1' \oplus A_1'(k_2 - k_1), \\ F_i = B_i' \oplus A_i'(k_2 - k_1) \oplus B_{i-1}'(k_2 - k_1) \qquad \text{for } 2 \leq i \leq g-2, \\ F_{g-1} = A_{g-1}'(k_2 - k_1) \oplus B_{g-2}'(k_2 - k_1), \qquad F_g = B_{g-1}'(k_2 - k_1); \end{gathered}$$

cf. [KM], p. 307, Equation (3). In the above expression, for an S-module M we denoted by M' the $S[v]$-module $M \otimes_S S[v]$.

For the following we fix an arbitrary field k.

Lemma 2.24. *The length $n + 2^n - 1$ sequence $z_1, \dots, z_n, r_{d_1 \cdots d_n}$ with indices $(d_1, \dots, d_n) \in \{0,1\}^n$ and $(d_1, \dots, d_n) \neq (0, 0, \dots, 0)$ is regular for $R_p \otimes_{\mathbb{Z}} k$ with respect to any ordering of it.*

Proof. Denote by $T \subset R_p \otimes_{\mathbb{Z}} k$ the ideal of $R_p \otimes_{\mathbb{Z}} k$ generated by the sequence. Since by Corollary 2.22 $R_p \otimes_{\mathbb{Z}} k$ is Gorenstein, hence Cohen–Macaulay, it is enough to prove that

$$\dim (R_p \otimes_{\mathbb{Z}} k)/T = \dim (R_p \otimes_{\mathbb{Z}} k) - (n + 2^n - 1).$$

Denote by T_1 the monomial ideal of $k[x_1, \dots, x_n, y_1, \dots, y_p, r_{00\cdots 0}]$ generated by $x_i y_i$ for $1 \leq i \leq p$, $y_i y_j$ for $1 \leq i < j \leq p$, and w_t for $1 \leq t \leq p$, with

$$w_t = r_{00\cdots 0} \left[\prod_{i=1}^{t-1} x_i \right] \left[\prod_{i=t+1}^{n} x_i \right].$$

Arguing as in the proof of Proposition 2.8 we get that

$$(R_p \otimes_{\mathbb{Z}} k)/T \cong k[x_1, \dots, x_n, y_1, \dots, y_p, r_{00\cdots 0}]/T_1$$

and, moreover, that the right hand side ring has the right dimension. $\square$

Proposition 2.25. *Denote by W_1 the k-vector subspace of $A_p \otimes_{\mathbb{Z}} k$ spanned by all $r_{d_1 \cdots d_n}$ with $(d_1, \dots, d_n) \in \{0,1\}^n$ and $(d_1, \dots, d_n) \neq (0, 0, \dots 0)$. Assume $1 \leq t \leq 2^n - 1$ and that $s_1, \dots s_t$ are elements of W_1 which are k-linearly independent. Then $s_1, \dots s_t$ is a regular sequence for $R_p \otimes_{\mathbb{Z}} k$. As a consequence, Corollary 2.22 implies that the $(A_p \otimes_{\mathbb{Z}} k)/(s_1, \dots, s_t)$-module $(R_p \otimes_{\mathbb{Z}} k)/(s_1, \dots, s_t)$ is perfect, of grade equal to $p+1$, and the k-algebra $(R_p \otimes_{\mathbb{Z}} k)/(s_1, \dots, s_t)$ is Gorenstein.*

Proof. Since by Corollary 2.22 $R_p \otimes_{\mathbb{Z}} k$ is Gorenstein, hence Cohen–Macaulay, it is enough to prove that the dimension drops by t when we divide $R_p \otimes_{\mathbb{Z}} k$ by the ideal generated by $s_1, \dots, s_t$. This follows from Lemma 2.24 after completing s_i to a basis of W_1 and dividing by the ideal generated by the basis together with $z_1, \dots, z_n$. $\square$

3. The numerical Campedelli surface construction

In this section we work over the field $k = \mathbb{C}$ of complex numbers. We will use the algebra developed in Section 2 in order to prove the existence of numerical Campedelli surfaces with torsion group equal to $\mathbb{Z}/6$.

We define the polynomial ring

$$A_4^s = k[x_1, \dots, x_4, z_1 \dots, z_4, y_1, \dots, y_4]$$

(s for specific), and we assign degree 1 to each variable x_i and z_i, for $1 \leq i \leq 4$, and degree 2 to each variable y_i, for $1 \leq i \leq 4$.

Remark 3.1. This is a different choice of degrees for the variables from the one made in Section 2. In Section 2, it was useful, for technical reasons, to give the variables $r_{d_1 \dots d_n}$ positive degrees. However, for the Campedelli cover construction they have to be constants.

Let G be the cyclic group of order 6 and denote by g a generator of G. We define a linear action of G on A_4^s by

$$\begin{aligned} (gx_1, gx_2, gx_3) &= (-x_2, -x_3, -x_1), & gx_4 &= -x_4, \\ (gz_1, gz_2, gz_3) &= (z_2, z_3, z_1), & gz_4 &= z_4, \\ (gy_1, gy_2, gy_3) &= (-y_2, -y_3, -y_1), & gy_4 &= -y_4. \end{aligned}$$

Consider the 16-dimensional k-vector subspace of A_4^s spanned by the monomials $a_1 a_2 a_3 a_4$, where $a_i \in \{x_i, z_i\}$. It is easy to see that it is G-invariant and that its k-vector subspace consisting of the G-invariant polynomials has $F_1, \dots, F_4$ as a k-basis, where

$$\begin{aligned} &F_1 = x_1 x_2 x_3 x_4, \quad F_2 = (x_1 x_2 z_3 + x_1 z_2 x_3 + z_1 x_2 x_3) z_4, \\ &F_3 = (x_1 z_2 z_3 + z_1 x_2 z_3 + z_1 z_2 x_3) x_4, \quad F_4 = z_1 z_2 z_3 z_4. \end{aligned}$$

Fix $(r_1, \dots, r_4) \in k^4$ nonzero. We set

$$Q^s = Q^s(r_t) = \sum_{i=1}^{4} r_i F_i \in A_4^s. \tag{3.1}$$

The polynomial Q^s is homogeneous of degree 4. Similar to item (5) of Notation 2.1, for each $1 \le i < j \le n$, let

$$Q_{ij}^{s,xx} = \frac{\partial^2 Q^s}{\partial x_i \partial x_j}, \quad Q_{ij}^{s,xz} = \frac{\partial^2 Q^s}{\partial x_i \partial z_j}, \quad Q_{ij}^{s,zx} = \frac{\partial^2 Q^s}{\partial z_i \partial x_j} \quad \text{and} \quad Q_{ij}^{s,zz} = \frac{\partial^2 Q^s}{\partial z_i \partial z_j}.$$

We clearly have

$$Q^s = x_i x_j Q_{ij}^{s,xx} + x_i z_j Q_{ij}^{s,xz} + z_i x_j Q_{ij}^{s,zx} + z_i z_j Q_{ij}^{s,zz}.$$

Consider, for $1 \le i < j \le 4$, the 5×5 skew–symmetric matrix

$$M_{ij}^s = \begin{pmatrix} 0 & x_i & z_i & -x_j & -z_j \\ & 0 & y_j & Q_{ij}^{s,zz} & -Q_{ij}^{s,zx} \\ & & 0 & -Q_{ij}^{s,xz} & Q_{ij}^{s,xx} \\ & \text{-sym} & & 0 & -y_i \\ & & & & 0 \end{pmatrix}$$

with entries in A_4^s. We denote by I_4^s the ideal of A_4^s generated by all the submaximal Pfaffians of M_{ij}^s for all values $1 \le i < j \le 4$.

The analogue of item (6) of Notation 2.1 is true, and we denote, for $1 \le i < j \le 4$, by e_{ij}^s the Pfaffian of M_{ij}^s involving $y_i y_j$ with coefficient 1, by l_i^s the Pfaffian of M_{1i}^s (or of M_{12}^s if $i = 1$) involving $x_i y_i$ with coefficient 1, and by m_i^s the Pfaffian of M_{1i}^s (or of M_{12}^s if $i = 1$) involving $z_i y_i$ with coefficient 1. See (3.9) below for the explicit formulas of l_i^s, m_i^s and e_{ij}^s.

It is also clear that

$$I_4^s = (e_{ij}^s, l_t^s, m_t^s) \subset A_4^s \tag{3.2}$$

with indices $1 \le i < j \le 4$ and $1 \le t \le 4$.

We denote $R_4^s = A_4^s / I_4^s$ as the quotient ring (which, of course, also depends on the choice of parameter values (r_t)).

Proposition 3.2. *a) For any* $(r_1, \ldots, r_4) \in k^4$, $\dim R_4^s \ge 7$.

b) Whenever $\dim R_4^s = 7$, *R_4^s is a Gorenstein ring and a perfect A_4^s-module.*

c) There exist parameter values (r_t) such that $\dim R_4^s = 7$.

d) For general parameter values $(r_1, \ldots, r_4) \in k^4$ (in the sense of being outside a proper Zariski closed subset of k^4) the ring R_4^s is Gorenstein with $\dim R_4^s = 7$.

Proof. Since the polynomial Q^s defined in (3.1) can be obtained as a specialization of the polynomial Q defined in (4) of Notation 2.1, it is clear that there exist $s_1, \ldots, s_t \in W_1$, k-linearly independent (where W_1 was defined in Proposition 2.25), such that for $(r_1, \ldots, r_4) \in k^4$, $\operatorname{Spec} R_4^s$ is isomorphic to the fiber over the point with coordinates $r_{d_1 \ldots d_4}$ corresponding to $(r_1, \ldots, r_4)$ of the natural map

$$\operatorname{Spec}(R_4 \otimes_{\mathbb{Z}} k)/(s_1, \ldots, s_t) \to \operatorname{Spec} k[r_{d_1 \ldots d_4}]/(s_1, \ldots, s_t)$$

induced by the inclusion

$$k[r_{d_1 \ldots d_4}]/(s_1, \ldots, s_t) \to (R_4 \otimes_{\mathbb{Z}} k)/(s_1, \ldots, s_t).$$

By [L], Theorem 4.3.12, for a morphism $f \colon X \to Y$ between locally Noetherian schemes and a point $x \in X$ we have

$$\dim \mathcal{O}_{X_y, x} \ge \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,y},$$

where $y = f(x)$ and X_y is the fiber of f over y. Combining it with Proposition 2.25 we get part a).

Part b) follows similarly by combining Proposition 2.25 and [BV], Theorem 3.5.

For part c) we make a specific choice of parameters $r_1 = 1$ and $r_i = 0$ for $2 \leq i \leq 4$. We will prove that $\dim R_4^s = 7$. For that, it is enough to prove that $\dim R_4^s/(z_1, z_2, z_3, z_4) = 3$. Arguing as in the proof of Proposition 2.8, it is easy to see (compare also (3.9)) that

$$R_4^s/(z_1, z_2, z_3, z_4) \cong k[x_1, \ldots, x_4, y_1, \ldots, y_4]/T,$$

where T is the monomial ideal of $k[x_1, \ldots, x_4, y_1, \ldots, y_4]$ generated by the elements $x_i y_i$, for $1 \leq i \leq 4$, together with $y_i y_j$, for $1 \leq i < j \leq 4$, and together with w_t, for $1 \leq t \leq 4$, where

$$w_t = \left[\prod_{i=1}^{t-1} x_i\right]\left[\prod_{i=t+1}^{4} x_i\right]$$

and that we indeed have the right dimension.

Using semicontinuity of the fiber dimension (cf. [Ei], Corollary 14.9) and parts a), b), and c), we have that part d) follows, which completes the proof of Proposition 3.2. □

Remark 3.3. A different way of arguing for the proof of Proposition 3.2 is to suitably modify the arguments used in the proof of Theorem 2.3. One should be able to obtain in this manner the more precise result that R_4^s is Gorenstein with $\dim R_4^s = 7$ whenever there exists i, $1 \leq i \leq 4$, with r_i nonzero. We will not use that in the following.

Denote by $\zeta \in k$ a fixed primitive 6th root of unity. We consider the following homogeneous elements $n_{ij} \in A_4^s$ of degree 1:

$$\begin{aligned} n_{01} &= z_1 + z_2 + z_3, \quad n_{02} = z_4, \quad n_{11} = x_1 + \zeta^2 x_2 + \zeta^4 x_3, \\ n_{21} &= z_1 + \zeta^4 z_2 + \zeta^2 z_3, \quad n_{31} = x_1 + x_2 + x_3, \quad n_{32} = x_4, \\ n_{41} &= z_1 + \zeta^2 z_2 + \zeta^4 z_3, \quad n_{51} = x_1 + \zeta^4 x_2 + \zeta^2 x_3. \end{aligned}$$

By construction, each n_{ij} is an eigenvector for the action of $g \in G$ with eigenvalue equal to ζ^i; that is,

$$g n_{ij} = \zeta^i n_{ij}.$$

We fix 4 more complex numbers $(r_5, \ldots, r_8) \in k^4$, and we define 4 homogeneous elements $h_i = h_i(r_5, \ldots, r_8) \in A_4^s$, for $1 \leq i \leq 4$, by

$$\begin{aligned} h_1 &= n_{01}, \quad h_2 = n_{02}, \quad h_3 = n_{32} + r_5 n_{31}, \\ h_4 &= y_4 + r_6(y_1 + y_2 + y_3) + r_7 n_{11} n_{21} + r_8 n_{41} n_{51}. \end{aligned} \tag{3.3}$$

We have that, for $1 \leq i \leq 4$, the element h_i is an eigenvector for g with eigenvalue equal to 1 for h_1 and h_2 and eigenvalue equal to -1 for h_3 and h_4.

We denote by $T^s \subset A_4^s$ the homogeneous ideal

$$T^s = T^s(r_t) = I_4^s + (h_1, \ldots, h_4) \subset A_4^s.$$

Moreover, we denote by A the polynomial subring

$$A = k[x_1, x_2, x_3, z_1, z_2, y_1, y_2, y_3] \subset A_4^s$$

with the weighting of the variables induced by that of A_4^s. For fixed general parameter values $(r_1, \ldots, r_8) \in k^8$, the composition

$$A \to A_4^s \to A_4^s/T^s$$

is surjective (where the first map is the natural inclusion and the second is the natural projection), so we get an induced isomorphism

$$\frac{A}{L} \cong \frac{A_4^s}{T^s}, \tag{3.4}$$

where $L \subset A$ is the kernel of the composition.

Proposition 3.4. *a) For any choice of parameter values $(r_1, \dots, r_8) \in k^8$ we have $\dim A_4^s/T^s \geq 3$, and whenever $\dim A_4^s/T^s = 3$ we have that A_4^s/T^s is a Gorenstein ring, perfect as A_4^s-module.*

b) There exist parameter values (r_t) such that $\dim A_4^s/T^s = 3$.

c) For general parameter values $(r_1, \dots, r_8) \in k^8$ (in the sense of being outside a proper Zariski closed subset of k^8), $\dim A_4^s/T^s = 3$ and A_4^s/T^s is a Gorenstein ring.

Proof. Part a) follows immediately from Proposition 3.2 by noticing that A_4^s/T^s is isomorphic to $R_4^s/(h_1, \dots, h_4)$.

For part b) we fix the parameter values $r_1 = r_4 = 1$ and $r_j = 0$, for $2 \leq j \leq 8$ with $j \neq 4$. By (3.4) $A_4^s/T^s \cong A/L$, where L is the ideal of A generated by

$$\{x_1y_1, z_1y_1, x_2y_2, z_2y_2, x_3y_3, z_1y_3 + z_2y_3, z_1^2z_2 + z_1z_2^2, x_1x_2x_3, y_1y_2,\\ y_1y_3, x_2x_3z_1z_2 + x_2x_3z_2^2, y_2y_3, x_1x_3z_1^2 + x_1x_3z_1z_2, x_1x_2z_1z_2\}.$$

It is easily checked that each minimal associated prime of L has codimension 5 in A; hence

$$\dim A/L = 3.$$

Part c) is an immediate consequence of parts a) and b), arguing as in the proof of Proposition 3.2. $\square$

Proposition 3.5. *For general parameter values $(r_1, \dots, r_8) \in k^8$ the minimal graded resolution of A/L as an A-module is equal to*

$$\begin{aligned} 0 &\to A(-12) \to A(-9)^8 \oplus A(-8)^6 \to A(-8)^3 \oplus A(-7)^{24} \oplus A(-6)^8 \\ &\to A(-6)^8 \oplus A(-5)^{24} \oplus A(-4)^3 \to A(-4)^6 \oplus A(-3)^8 \to A. \end{aligned} \tag{3.5}$$

Moreover, the dualising module of A/L is equal to $(A/L)(1)$ and the Hilbert series of A/L as a graded A-module is equal to

$$\frac{t^4 + 2t^3 + 6t^2 + 2t + 1}{(1-t)^3} \in \mathbb{Q}(t).$$

Proof. Using Remark 2.23, we can easily calculate inductively the minimal graded resolution of the generically perfect (Proposition 2.21) module R_4 over A_4. Equation (3.5) follows by combining Proposition 3.4, [BV], Theorem 3.5, and the easily observed fact that the minimal graded resolution of R_4 over A_4 remains homogeneous and minimal. The other conclusions of Proposition 3.5 follow easily from (3.5). $\square$

Definition 3.6. For general $(r_1, \dots, r_8) \in k^8$ we denote by S the scheme

$$S = S(r_t) = \operatorname{Proj} A_4^s/T^s \subset \mathbb{P}(1^8, 2^4).$$

Our main aim is to prove that S is an irreducible nonsingular surface with invariants $p_g = 5, q = 0, K^2 = 12$ and a trivial algebraic fundamental group, which is an étale 6 to 1 cover of a numerical Campedelli surface.

Remark 3.7. By (3.4), S has an embedding as a nondegenerate subscheme

$$S \subset \mathbb{P}(1^5, 2^3). \tag{3.6}$$

Proposition 3.8. *a) The homogeneous coordinate ring of the embedding $S \subset \mathbb{P}(1^8, 2^4)$ is isomorphic to A_4^s/T^s.*

b) The scheme S is a projective purely 2-dimensional scheme over k. Moreover, S is connected and $H^1(S, \mathcal{O}_S(t)) = 0$ for all $t \in \mathbb{Z}$.

c) The dualising sheaf ω_S is isomorphic to $\mathcal{O}_S(1)$ as an $\mathcal{O}_S$-module.

Proof. The graded ring A_4^s/T^s is Gorenstein (Proposition 3.4), hence saturated. As a consequence, part a) follows.

Using part a) the homogeneous coordinate ring of the embedding $S \subset \mathbb{P}(1^8, 2^4)$ is Gorenstein, hence Cohen–Macaulay. It is then well known (cf. [Do], [Ei], Ch. 18) that the conclusions of part b) follow. It is also well-known that part c) follows immediately from Proposition 3.5. □

In the following we will also need the affine cone $S^c \subset \mathbb{A}^{12}$ over $S \subset \mathbb{P}(1^8, 2^4)$, so we set

$$S^c = V(T^s) \subset \mathbb{A}^{12}.$$

We denote by S^c_{cl} the set of closed points of S^c, and by S_{cl} the set of closed points of S. Since $k = \mathbb{C}$ is algebraically closed, we can identify S^c_{cl} with the set of points

$$P = (a_1, \dots, a_4;\ \ b_1, \dots, b_4;\ \ c_1, \dots, c_4) \in k^{12} \tag{3.7}$$

such that $f(P) = 0$ for every $f \in T^s$.

By definition, S_{cl} is the quotient of $S^c_{cl} \setminus \{0\}$ under the group action

$$k^* \times (S^c_{cl} \setminus \{0\}) \to (S^c_{cl} \setminus \{0\})$$

with

$$hP = (ha_1, \dots, ha_4;\ \ hb_1, \dots, hb_4;\ \ h^2c_1, \dots, h^2c_4)$$

for $h \in k^*$ and $P \in S^c_{cl}$ as in (3.7).

Since by Proposition 3.14 below the ideal $T^s \subset R_4^s$ is G-invariant, there is an induced G action $G \times R_4^s/T^s \to R_4^s/T^s$, which induces in a natural way two group actions: $G \times S^c_{cl} \to S^c_{cl}$ and $G \times S_{cl} \to S_{cl}$.

Explicitly, for $P \in S^c_{cl}$ as in (3.7) we have

$$gP = (-a_3, -a_1, -a_2, -a_4;\ \ b_3, b_1, b_2, b_4;\ \ -c_3, -c_2, -c_1, -c_4). \tag{3.8}$$

Lemma 3.9. *For general values of parameters $(r_1, \dots, r_8) \in k^8$ there is no nonzero point $P \in S^c_{cl}$ (notation for P as in (3.7)) such that $a_i = b_i = 0$ for all $1 \le i \le 4$.*

Proof. Indeed, if all $a_i = b_i = 0$ we have by looking at e^s_{ij}, for $1 \le i < j \le 4$ (cf. (3.9)), that at least three of the four c_t are 0, and then by looking at the polynomial h_4 we get that the remaining c_t is also 0, a contradiction to $P \neq 0$. □

Proposition 3.10. *Consider $S \subset \mathbb{P}(1^5, 2^3)$ as in (3.6). Denote by $S_1^c \subset \mathbb{A}^8$ the affine cone over S. The scheme S_1^c is smooth outside the vertex of the cone.*

Proof. Unfortunately, we were only able to prove Proposition 3.10 with the help of the computer algebra program Singular [GPS01]. We took a similar approach as in [R2], p. 18 and worked over the finite field of $\mathbb{Z}/103$ after putting values for parameters $r_7 = r_8 = 0$, in order to have everything defined over $\mathbb{Z}$. $\square$

Remark 3.11. Using the birational character of unprojection it is not hard to specify inductively (for general values of the parameters (r_t)) the singularities of the affine cone over the 6-fold $V(I_t^s)$ for $t = 0, \dots, 4$, where I_t^s are the precise analogues of the ideals I_t defined in Section 2. With a little more effort, one can also specify inductively the singularities of the cone over the 3-fold $V(I_t^s + (h_1, h_2, h_3))$. Since $h_2 = z_4$ vanishes, the trick here is to start from the codimension 2 ideal (l_4^s, m_4^s) and then inductively unproject $V(x_1, z_1, y_4), V(x_2, z_2, y_4, y_1)$ and finally $V(x_3, z_1 + z_2, y_4, y_1, y_2)$. What, unfortunately, we were not able to do was to find a way to deduce the nonsingularity (outside the vertex of the affine cone) of the surface from the singularity calculations of the 3-fold.

Theorem 3.12. *Fix general values of parameters $(r_1, \dots, r_8) \in k^8$. $S = S(r_t)$ is an irreducible minimal nonsingular surface of general type with $p_g = 5, q = 0, K^2 = 12$ and canonical ring isomorphic to A_4^s/T^s.*

Proof. By combining Lemma 3.9 and Proposition 3.10 we get that the scheme S is smooth. Since S is also connected (Proposition 3.8), it follows that S is an irreducible nonsingular surface.

By Proposition 3.8 the dualising sheaf ω_S is isomorphic to $\mathcal{O}_S(1)$. Using Lemma 3.9 $\mathcal{O}_S(1)$ is globally generated. As a consequence,

$$\mathcal{O}_S(1)^{\otimes n} \cong \mathcal{O}_S(n)$$

for all $n \geq 1$, hence A_4^s/T^s is isomorphic to the canonical ring of S. Therefore ω_S is ample, which implies that S is minimal.

Since the irregularity q of S is 0 (because by Proposition 3.8 $h^1(S, \mathcal{O}_S) = 0$), the properties $p_g = 5, K^2 = 12$ follow by comparing the Hilbert series calculation of Proposition 3.5 with [R1], Example 3.5. $\square$

Remark 3.13. We will prove below that S has trivial algebraic fundamental group (see the proof of Theorem 3.16).

Our next aim is to prove that S is an étale 6 to 1 cover of a numerical Campedelli surface.

Proposition 3.14. *Assume $g_1 \in G$ and $u \in T^s$; then $g_1 u \in T^s$.*

Proof. Since $G = \langle g \rangle$, it is enough to check that $gu \in T^s$, where u is one of the generators of I_4^s appearing in (3.2).

It is easy to check (compare (3.9)) that for $i \in \{1, 2, 3\}$ we have

$$g l_i^s = l_t^s, \qquad g m_i^s = -m_t^s, \qquad g e_{i4}^s = e_{t4}^s,$$

where $t \in \{1, 2, 3\}$ is uniquely specified by $t \equiv i + 1 \mod 3$, and also that

$$g l_4^s = l_4^s, \quad g m_4^s = -m_4^s, \quad g e_{12}^s = e_{23}^s, \quad g e_{13}^s = e_{12}^s, \quad g e_{23}^s = e_{13}^s.$$

A more conceptual proof can be given by arguing that due to the G-invariance of Q^s, the action of g interchanges (up to sign) the set of $Q_{ij}^{s,ab}$ (for $a, b \in \{x, z\}$), and we use that to argue that the action of g interchanges (up to sign differences of whole columns or rows) the set of the matrices M_{ij}^s. $\square$

The proof of the following proposition will be given in Subsection 3.1.

Proposition 3.15. *Fix general values of the parameters $(r_1, \ldots, r_8) \in k^8$. If $g_1 \in G$ is not the identity element and $u \in S_{cl}$, we have $g_1 u \neq u$. In other words, the action of G on S_{cl} is basepoint free.*

The following is our main result about the existence of numerical Campedelli surfaces with algebraic fundamental group equal to $\mathbb{Z}/6$.

Theorem 3.16. *For general $(r_1, \ldots, r_8) \in k^8$, the action of G on S is basepoint free. As a consequence, the quotient surface S/G is a smooth irreducible minimal complex surface of general type with $p_g = q = 0$ and $K^2 = 2$ (i.e., a numerical Campedelli surface). Moreover S/G has both algebraic fundamental group and torsion group isomorphic to $\mathbb{Z}/6$.*

Proof. Fix general $(r_1, \ldots, r_8) \in k^8$. By Proposition 3.15, the action of G on S is basepoint free. Hence using Theorem 3.12, S/G is a smooth irreducible surface. Denote by $\pi\colon S \to S/G$ the natural projection map. Since π is étale $\pi^*(\omega_{S/G}) \cong \omega_S$ (cf. [MP], p. 3), and since by the proof of Theorem 3.12 ω_S is ample, we have that $\omega_{S/G}$ is ample (cf. [Ha], Exerc. III.5.7). Hence S/G is a minimal surface of general type.

The invariants of S/G follow from those of S calculated in Theorem 3.12. Indeed, π surjective and $q(S) = 0$ imply $q(S/G) = 0$, and π étale 6 to 1 implies $K_S^2 = 6K_{S/G}^2$ and $\chi(S) = 6\chi(S/G)$.

To prove that the algebraic fundamental group of S/G is equal to G, it is enough to show that $\pi_1^{\mathrm{alg}} S = 0$. Assume it is not. Then the group $\pi_1^{\mathrm{alg}}(S/G)$ has $6|\pi_1^{\mathrm{alg}} S| \geq 12$ elements, which contradicts the fact that a Campedelli surface has algebraic fundamental group consisting of at most 9 elements (cf. [BPHV], Chap. VII.10).

Since the torsion group of S/G is the largest abelian quotient of $\pi_1^{\mathrm{alg}}(S/G)$ (cf. [MP], p. 16), we get that S/G has a torsion group isomorphic to $\mathbb{Z}/6$, which finishes the proof of Theorem 3.16. $\square$

Remark 3.17. It can be shown that the $(k^*)^5$ action A_4^s defined by

$$g x_i = g_1 x_i, \quad g z_i = g_2 z_i, \quad g y_i = (g_1 g_2 g_3) y_i$$

for $1 \leq i \leq 3$, and

$$g x_4 = g_4 x_4, \quad g z_4 = g_5 z_4, \quad g y_4 = \frac{g_1^2 g_2^2 g_3}{g_4 g_5} y_4,$$

where $g = (g_1, \ldots, g_5) \in (k^*)^5$, respects our construction, and the induced action on the space of parameters $r_1, \ldots, r_8$ has 1-dimensional kernel. As a consequence, the number of moduli of our family is at most 4. As already observed in the Introduction, it follows that there exist numerical Campedelli surfaces with torsion $\mathbb{Z}/6$ that cannot be obtained by our construction.

3.1. **The proof of Proposition 3.15.** For the proof of Proposition 3.15 we will need the following formulas:

$$
\begin{aligned}
l_1^s &= x_2x_3z_4r_2 + x_3x_4z_2r_3 + x_2x_4z_3r_3 + z_2z_3z_4r_4 + x_1y_1,\\
l_2^s &= x_1x_3z_4r_2 + x_3x_4z_1r_3 + x_1x_4z_3r_3 + z_1z_3z_4r_4 + x_2y_2,\\
l_3^s &= x_1x_2z_4r_2 + x_2x_4z_1r_3 + x_1x_4z_2r_3 + z_1z_2z_4r_4 + x_3y_3,\\
l_4^s &= x_2x_3z_1r_2 + x_1x_3z_2r_2 + x_1x_2z_3r_2 + z_1z_2z_3r_4 + x_4y_4,\\
m_1^s &= -x_2x_3x_4r_1 - x_3z_2z_4r_2 - x_2z_3z_4r_2 - x_4z_2z_3r_3 + z_1y_1,\\
m_2^s &= -x_1x_3x_4r_1 - x_3z_1z_4r_2 - x_1z_3z_4r_2 - x_4z_1z_3r_3 + z_2y_2,\\
m_3^s &= -x_1x_2x_4r_1 - x_2z_1z_4r_2 - x_1z_2z_4r_2 - x_4z_1z_2r_3 + z_3y_3,\\
m_4^s &= -x_1x_2x_3r_1 - x_3z_1z_2r_3 - x_2z_1z_3r_3 - x_1z_2z_3r_3 + z_4y_4,\\
e_{12}^s &= -x_3^2z_4^2r_2^2 + x_3^2x_4^2r_1r_3 - x_3x_4z_3z_4r_2r_3 - x_4^2z_3^2r_3^2\\
&\quad +x_3x_4z_3z_4r_1r_4 + z_3^2z_4^2r_2r_4 + y_1y_2,\\
e_{13}^s &= -x_2^2z_4^2r_2^2 + x_2^2x_4^2r_1r_3 - x_2x_4z_2z_4r_2r_3 - x_4^2z_2^2r_3^2\\
&\quad +x_2x_4z_2z_4r_1r_4 + z_2^2z_4^2r_2r_4 + y_1y_3,\\
e_{14}^s &= x_2^2x_3^2r_1r_2 - x_3^2z_2^2r_2r_3 - x_2x_3z_2z_3r_2r_3 - x_2^2z_3^2r_2r_3\\
&\quad +x_2x_3z_2z_3r_1r_4 + z_2^2z_3^2r_3r_4 + y_1y_4,\\
e_{23}^s &= -x_1^2z_4^2r_2^2 + x_1^2x_4^2r_1r_3 - x_1x_4z_1z_4r_2r_3 - x_4^2z_1^2r_3^2\\
&\quad +x_1x_4z_1z_4r_1r_4 + z_1^2z_4^2r_2r_4 + y_2y_3,\\
e_{24}^s &= x_1^2x_3^2r_1r_2 - x_3^2z_1^2r_2r_3 - x_1x_3z_1z_3r_2r_3 - x_1^2z_3^2r_2r_3\\
&\quad +x_1x_3z_1z_3r_1r_4 + z_1^2z_3^2r_3r_4 + y_2y_4,\\
e_{34}^s &= x_1^2x_2^2r_1r_2 - x_2^2z_1^2r_2r_3 - x_1x_2z_1z_2r_2r_3 - x_1^2z_2^2r_2r_3\\
&\quad +x_1x_2z_1z_2r_1r_4 + z_1^2z_2^2r_3r_4 + y_3y_4.
\end{aligned}
\tag{3.9}
$$

By (3.3) we have $h_1 = z_1 + z_2 + z_3$, $h_2 = z_4$ and $h_3 = x_4 + r_5(x_1 + x_2 + x_3)$.

We set $u_i \in k^{12}$, for $1 \leq i \leq 4$, to be the vector with 1 on the coordinate corresponding to x_i and 0 elsewhere, $v_i \in k^{12}$ to be the vector with 1 on the coordinate corresponding to z_i and 0 elsewhere, and $w_i \in k^{12}$ to be the vector with 1 on the coordinate corresponding to y_i and 0 elsewhere.

Set V_1 to be the vector space spanned by u_1, u_2, u_3, V_2 to be the vector space spanned by v_1, v_2, v_3, V_3 to be the vector space spanned by u_4, V_4 to be the vector space spanned by v_4, V_5 to be the vector space spanned by w_1, w_2, w_3 and V_6 to be the vector space spanned by w_4. We have that, for $1 \leq i \leq 6$, the vector space V_i is G-invariant. More precisely, using (3.8) we get

$$gu_i = -u_t, \qquad gv_i = v_t, \qquad gw_i = -w_t,$$

where $t \in \{1, 2, 3\}$ is uniquely specified by $t \equiv i + 1 \mod 3$, and

$$gu_4 = -u_4, \qquad gv_4 = v_4, \qquad gw_4 = -w_4.$$

Since every element of G different from the identity has a power equal to g^2 or to g^3, to prove Proposition 3.15 it is enough to show that if $t \in \{2, 3\}$ and

$$P = \sum_{i=1}^{4} a_iu_i + \sum_{i=1}^{4} b_iv_i + \sum_{i=1}^{4} c_iw_i \in S_{cl}^c \tag{3.10}$$

(with $a_i, b_i, c_i \in k$) are such that there exists $h \in k^*$ with $g^t P = h * P$, then $P = 0$, where by definition

$$h * P = \sum_{i=1}^{4} h a_i u_i + \sum_{i=1}^{4} h b_i v_i + \sum_{i=1}^{4} h^2 c_i w_i.$$

Step 1. We first study the action of g^2. The action of g^2 on the direct sum of V_i can be described by

$$g^2 = (u_1, u_3, u_2)(v_1, v_3, v_2)(u_4)(v_4)(w_1, w_3, w_2)(w_4)$$

in the sense that $g^2 u_1 = u_3$, $g^2 u_3 = u_2$, $g^2 u_2 = u_1$, etc. We assume

$$g^2(P) = h * P \tag{3.11}$$

for some nonzero P as in (3.10), and we will get a contradiction. Since P is nonzero, by Lemma 3.9 there exists i, with $1 \le i \le 4$, such that $a_i \neq 0$ or $b_i \neq 0$. Since the eigenvalues of g^2 acting on any of $V_1, \dots, V_4$ are contained in the set $\{1, \zeta^2, \zeta^4\}$, we get that $h \in \{1, \zeta^2, \zeta^4\}$.

Step 2. We assume that $h = 1$ in (3.11), and we will get a contradiction. By looking at the action of g^2 on V_1 and V_2 we have

$$a_3 = a_2 = a_1, \quad b_3 = b_2 = b_1.$$

Using the equations h_1, h_2, h_3 we additionally get

$$b_1 = b_2 = b_3 = b_4 = 0, \quad a_4 = -3r_5 a_1.$$

Substituting to m_4^s we get $(a_1)^3 r_1 = 0$. Hence $a_1 = 0$ (since r_1 is general), which implies that all $a_j = 0$ and all $b_j = 0$, contradicting Lemma 3.9.

Step 3. We assume that $h = \zeta^2$ in (3.11), and we will get a contradiction. By looking at the action of g^2 on each V_i we have

$$\begin{aligned} &a_3 = \zeta^2 a_1, \quad a_2 = \zeta^4 a_1, \quad b_3 = \zeta^2 b_1, \quad b_2 = \zeta^4 b_1, \\ &a_4 = b_4 = c_4 = 0, \quad c_3 = \zeta^4 c_1, \quad c_2 = \zeta^2 c_1. \end{aligned}$$

Substituting to l_1^s we get $a_1 c_1 = 0$, and substituting to m_1^s we get $b_1 c_1 = 0$. We can assume that $c_1 = 0$; otherwise we get that all $a_j = 0$ and all $b_j = 0$, contradicting Lemma 3.9. As a consequence, $c_i = 0$ for $1 \le i \le 4$.

Substituting to l_4^s we get

$$3r_2(a_1)^2 b_1 + r_4(b_1)^3 = 0,$$

while substituting to m_4^s we get

$$r_1(a_1)^3 + 3r_3 a_1 (b_1)^2 = 0.$$

It is clear that for general values of the r_i the last two equations have no nonzero common solutions for a_1 and b_1, so we get all $a_j = 0$ and all $b_j = 0$, contradicting P nonzero.

Step 4. We assume that $h = \zeta^4$ in (3.11), and we will get a contradiction. By looking at the action of g^4 on each V_i we have

$$\begin{aligned} &a_3 = \zeta^4 a_1, \quad a_2 = \zeta^2 a_1, \quad b_3 = \zeta^4 b_1, \quad b_2 = \zeta^2 b_1, \\ &a_4 = b_4 = c_4 = 0, \quad c_3 = \zeta^2 c_1, \quad c_2 = \zeta^4 c_1. \end{aligned}$$

Substituting to l_1^s we get $a_1c_1 = 0$, and substituting to m_1^s we get $b_1c_1 = 0$. We can assume that $c_1 = 0$; otherwise we get that all $a_j = 0$ and all $b_j = 0$, contradicting Lemma 3.9. As a consequence, $c_i = 0$ for $1 \le i \le 4$.

Substituting to l_4^s we get

$$3r_2(a_1)^2b_1 + r_4(b_1)^3 = 0,$$

while substituting to m_4^s we get

$$3r_1(a_1)^3 + r_2a_1(b_1)^2 = 0.$$

It is clear that for general values of the r_i the last two equations have no nonzero common solutions for a_1 and b_1, so we again get all $a_j = 0$ and all $b_j = 0$, contradicting P nonzero.

Step 5. We now study the action of g^3. The action of g^3 on the direct sum of the V_i can be described by

$$g^3(u_i) = -u_i, \qquad g^3(v_i) = v_i, \qquad g^3(w_i) = -w_i$$

for $1 \le i \le 4$.

We assume

$$g^3(P) = h * P \tag{3.12}$$

for some nonzero P as in (3.10), and we will get a contradiction. Arguing as in Step 1, we get $h \in \{1, -1\}$.

Step 6. We assume that $h = 1$ in (3.12), and we will get a contradiction. Indeed, by the way g^3 acts we have $a_i = c_i = 0$ for $1 \le i \le 4$. Substituting to $e_{14}^s, e_{24}^s, e_{34}^s$ we get respectively

$$b_2b_3 = b_1b_3 = b_1b_2 = 0;$$

hence at least two of the three b_1, b_2, b_3 are 0. Using h_1 and h_2 we get that $b_i = 0$ for $1 \le i \le 4$, contradicting P nonzero.

Step 7. We assume that $h = -1$ in (3.12), and we will get a contradiction. Indeed, by the way g^3 acts we have $b_i = c_i = 0$ for $1 \le i \le 4$. Fix $1 \le i < j \le 4$. Substituting to e_{ij}^s we get that $a_pa_q = 0$, where the indices p, q have the property $\{p, q, i, j\} = \{1, 2, 3, 4\}$. As a consequence at least three of the four a_i are 0. Using h_3 we get that all a_i are 0, contradicting P nonzero. This finishes the proof of Proposition 3.14.

4. Final remarks and questions

Remark 4.1. In [CR], Corti and Reid pose the problem of interpreting the Gorenstein formats arising from unprojection as solutions to universal problems. What can be said about the generic $\binom{n}{2}$ Pfaffians ideal of Definition 2.2?

Remark 4.2. During the proof of Theorem 3.16, we established that the étale 6 to 1 numerical Campedelli covers of our construction have trivial algebraic fundamental group. We expect that they also have trivial topological fundamental group, but we were unable to prove it.

Remark 4.3. We believe that the ideas of the present paper can also be useful for the study of the numerical Campedelli surfaces with torsion groups $\mathbb{Z}/2$ and $\mathbb{Z}/3$.

Consider first the $\mathbb{Z}/3$ torsion case. The numeric invariants suggest that the étale 3 to 1 cover of such a numerical Campedelli surface could be a suitable member of $|-2K_{V_3}|$, where

$$V_3 \subset \mathbb{P}(1^2, 2^7, 3^5)$$

is a (candidate) codimension 10 Fano 3-fold having a basket of ten $1/2(1,1,1)$ singularities, which appears in Brown's online database of the graded ring [Br]. Moreover, [Br] suggests that V_3 can, perhaps, be constructed as a result of a series of symmetric looking type II unprojections (cf. [R1], [P3]).

Similarly, the numeric invariants for the $\mathbb{Z}/2$ torsion case suggests that the étale 2 to 1 cover of such a numerical Campedelli could be a suitable member of $|-2K_{V_2}|$, where

$$V_2 \subset \mathbb{P}(1, 2^6, 3^8)$$

is a (candidate) codimension 11 Fano 3-fold having a basket of twelve $1/2(1,1,1)$ singularities and also appearing in [Br]. Moreover, [Br] suggests that V_2 can, perhaps, be constructed as a result of a series of again symmetric looking type IV unprojections (cf. [R3]).

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