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SELF DELTA-EQUIVALENCE FOR LINKS WHOSE MILNOR'S ISOTOPY INVARIANTS VANISH

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Dedicated to Professor Tetsuo Shibuya on his 60th birthday.

Abstract. For an n-component link, Milnor's isotopy invariants are defined for each multi-index $I = i_1 i_2 ... i_m$ ($i_j \in \{1, ..., n\}$). Here m is called the length. Let $r(I)$ denote the maximum number of times that any index appears in I. It is known that Milnor invariants with $r = 1$, i.e., Milnor invariants for all multi-indices I with $r(I) = 1$, are link-homotopy invariant. N. Habegger and X. S. Lin showed that two string links are link-homotopic if and only if their Milnor invariants with $r = 1$ coincide. This gives us that a link in S^3 is link-homotopic to a trivial link if and only if all Milnor invariants of the link with $r = 1$ vanish. Although Milnor invariants with $r = 2$ are not link-homotopy invariants, T. Fleming and the author showed that Milnor invariants with $r \leq 2$ are self Δ-equivalence invariants. In this paper, we give a self Δ-equivalence classification of the set of n-component links in S^3 whose Milnor invariants with length $\leq 2n - 1$ and $r \leq 2$ vanish. As a corollary, we have that a link is self Δ-equivalent to a trivial link if and only if all Milnor invariants of the link with $r \leq 2$ vanish. This is a geometric characterization for links whose Milnor invariants with $r \leq 2$ vanish. The chief ingredient in our proof is Habiro's clasper theory. We also give an alternate proof of a link-homotopy classification of string links by using clasper theory.

1. Introduction

For an n-component link L, the *Milnor invariant* $\overline{\mu}_L(I)$ is defined for each multi-index $I = i_1 i_2 ... i_m$ ($i_j \in \{1, ..., n\}$) [18, 19]. Here m is called the *length* of $\overline{\mu}_L(I)$ and denoted by $|I|$. Let $r(I)$ denote the maximum number of times that any index appears in I. For example, $r(1123) = 2$, $r(1231223) = 3$. It is known that if $r(I) = 1$, then $\overline{\mu}_L(I)$ is a *link-homotopy* invariant [18], where link-homotopy is an equivalence relation on links generated by self-crossing changes. Similarly, for a string link L, the Milnor invariant $\mu_L(I)$ is defined [9]. While Milnor invariants are not strong enough to give a link-homotopy classification for links, they are complete for string links. In fact, the following is given by N. Habegger and X. S. Lin [9].

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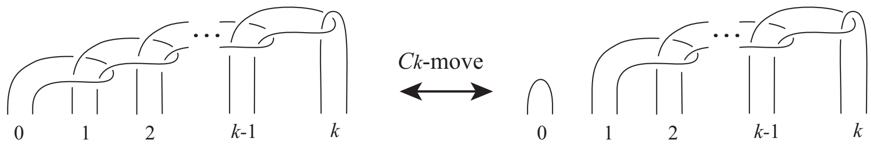

FIGURE 1.1. A C_k-move involves $k+1$ strands of a link

Theorem 1.1 ([9]). *Two n-component string links L and L' are link-homotopic if and only if $\mu_L(I) = \mu_{L'}(I)$ for any I with $r(I) = 1$.*

We will give an alternate proof in section 4 via clasper theory. Actually we will give representatives determined by Milnor link-homotopy invariants for the link-homotopy classes explicitly; see Theorem 4.3. As a corollary, we have that for n-component string links L and L', and for a positive integer k ($k \le n$), $\mu_L(I) = \mu_{L'}(I)$ for any I with $r(I) = 1$ and $|I| \le k$ if and only if L and L' are transformed into each other by combining link-homotopies and C_k-moves; see Corollary 4.5. A *C_k-move* is a local move on (string) links as illustrated in Figure 1.1. (A C_1-move is defined as the crossing change.) These local moves were introduced by Habiro [10]. The C_k-move generates an equivalence relation on (string) links, called the *C_k-equivalence*.

For a string link L, let $\mathrm{cl}(L)$ denote the *closure* of L. It follows from the definitions that $\mu_L(I) = \overline{\mu}_{\mathrm{cl}(L)}(I)$ if $\mu_L(J) = 0$ for any J with $|J| < |I|$. Since the Milnor invariants of trivial (string) links are 0, this and Theorem 1.1 imply the following. The theorem below is also shown by J. Milnor [18].

Theorem 1.2 ([18, 9]). *A link L in S^3 is link-homotopic to a trivial link if and only if $\overline{\mu}_L(I) = 0$ for any I with $r(I) = 1$.*

Although Milnor invariants with $r \ge 2$, i.e., Milnor invariants for all multi-index I with $r(I) \ge 2$, are not necessarily link-homotopy invariants, they are generalized link-homotopy invariants. In fact, Fleming and the author [4] showed that Milnor invariants with $r \le k$ are *self C_k-equivalence* invariants, where the self C_k-equivalence is an equivalence relation on (string) links generated by self C_k-moves, and a *self C_k-move* is a C_k-move where all the strands belong to the same component of a (string) link [26].

The C_k-move can also be defined by using the theory of claspers (see section 2). The (self) C_n-equivalence relation becomes finer as n increases, i.e., the (self) C_m-equivalence implies the (self) C_k-equivalence for $m > k$. We remark that a (self) C_2-move is equivalent to a (self) *Δ-move* defined by [21]. The Δ-move is defined as a local move as illustrated in Figure 1.2. We call the (self) C_2-equivalence the *(self) Δ-equivalence*.

A self Δ-equivalence classification of 2-*component* links was shown by Y. Nakanishi and Y. Ohyama [22]. It is still open for links with at least 3 components. Here we give the following theorem.

Theorem 1.3. *Let L and L' be n-component links. Suppose that $\overline{\mu}_L(I) = \overline{\mu}_{L'}(I) = 0$ for any I with $|I| \le 2n-1$ and $r(I) \le 2$. Then L and L' are self Δ-equivalent if and only if $\overline{\mu}_L(J) = \overline{\mu}_{L'}(J)$ for any J with $|J| = 2n$ and $r(J) = 2$.*

Remark 1.4. (1) The 'only if' part follows directly from the fact that Milnor invariants with $r \leq k$ are self C_k-equivalence invariants [4].

(2) In the last section, we characterize n-component links whose Milnor invariants of length $\leq 2n-1$ and $r \leq 2$ vanish. More precisely, the Milnor invariants of an n-component link with length $\leq 2n-1$ and $r \leq 2$ vanish if and only if, for any integer i in $\{1,...,n\}$, it is self Δ-equivalent to a Brunnian link L_i such that the ith component K of L_i is null-homotopic in $S^3 \setminus (L_i - K)$ (Theorem 6.3). As an example, we will give a 3-component Brunnian link $L = K_1 \cup K_2 \cup K_3$ such that K_1 is not null-homotopic in $S^3 \setminus (L - K_1)$ and K_i is null-homotopic in $S^3 \setminus (L - K_i)$ $(i = 2, 3)$ (Example 6.4). In particular, L is link-homotopic to a trivial link. There is no such link with 2 components, i.e., if a 2-component link is link-homotopic to a trivial link, then it is self Δ-equivalent to a Brunnian link $K_1 \cup K_2$ such that K_i is null-homotopic in $S^3 \setminus K_j$ $(\{i, j\} = \{1, 2\})$.

For 2-component links, Theorem 1.2 can be generalized [22]. Theorem 1.3 gives us the following corollary, which is a generalization of Theorem 1.2 for links with arbitrarily many components. This gives us a geometric characterization for links whose Milnor invariants with $r \leq 2$ vanish.

Corollary 1.5. *A link L is self Δ-equivalent to a trivial link if and only if $\overline{\mu}_L(I) = 0$ for any I with $r(I) \leq 2$.*

Remark 1.6. (1) This corollary gives an affirmative answer for an open question raised in [4].

(2) For string links, Corollary 1.5 does not hold, i.e., there are 2-string links such that their Milnor invariants $\mu(I)$ with $r(I) \leq 2$ vanish and they are not self Δ-equivalent to a trivial string link [5].

(3) Since a C_k-move $(k \geq 3)$ is not an unknotting operation, it is impossible to generalize the corollary above. It is reasonable to consider the following question: If $\overline{\mu}_L(I) = 0$ for any I with $r(I) \leq k$, then is L self C_k-equivalent to a completely split link? Fleming and the author gave a negative answer to the question [4]. In fact, there is a 2-component boundary link L such that L is not self C_3-equivalent to a split link. Note that all Milnor invariants of a boundary link vanish. We will give the definition of a boundary link after Remark 1.8.

By combining Lemma 3.2 ([19, Theorem 7]), Theorem 1.2 and Corollary 1.5, we have the following corollary.

Corollary 1.7. *Let L be an n-component link and let $L(2)$ be a $2n$-component link obtained from L by replacing each component of L with two zero framed parallel copies of it. Then L is self Δ-equivalent to a trivial link if and only if $L(2)$ is link-homotopic to a trivial link.*

Remark 1.8. For an n-component link L, let $L(k)$ be a kn-component link obtained from L by replacing each component of L with k zero framed parallel copies of it.

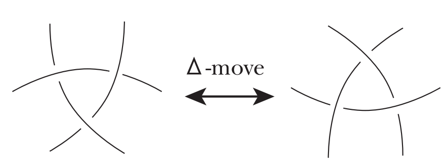

FIGURE 1.2

In the proof of [4, Theorem 2.1], it is shown that if two links L and L' are self C_k-equivalent, then $L(k)$ and $L'(k)$ are link homotopic. So one might expect that if $L(2)$ and $L'(2)$ are link homotopic, then L and L' are self Δ-equivalent. But this is not true. The reason is the following: There are 2-component links L and L' such that they are concordant and are not self Δ-equivalent [23], [24]. The fact that L and L' are concordant implies that $L(2)$ and $L'(2)$ are concordant. Since link-concordance implies link-homotopy [6], [7], $L(2)$ and $L'(2)$ are link-homotopic.

An n-component link $L = K_1 \cup \cdots \cup K_n$ is called a *boundary link* if there is a disjoint union $X = F_1 \cup \cdots \cup F_n$ of orientable surfaces such that $\partial X = L$ and $\partial F_i = K_i$ $(i = 1, 2, ..., n)$. An n-component link L is called a *homology boundary link* if $\pi_1(S^3 \setminus L)$ admits an epimorphism from $\pi_1(S^3 \setminus L)$ to a free group of rank n [28]. Every boundary link is a homology boundary link. T. Shibuya and the author showed that all boundary links are self Δ-equivalent to trivial links [27]. In [25], Shibuya showed that all ribbon links are self Δ-equivalent to trivial links.

Whether homology boundary links are self Δ-equivalent to trivial links and whether slice links are self Δ-equivalent to trivial links have remained as open questions. Since all Milnor invariants of homology boundary links vanish, and since Milnor invariants are concordance invariants [1], we have the following corollary, which gives affirmative answers for the open questions.

Corollary 1.9. *If L is concordant to a homology boundary link, then L is self Δ-equivalent to a trivial link.*

2. Claspers

Let us briefly recall from [11] the basic notions of clasper theory for (string) links. In this paper, we essentially only need the notion of a C_k-tree. For a general definition of claspers, we refer the reader to [11].

Let L be a link in S^3 (resp. a string link in $D^2 \times I$). An embedded disk F in S^3 (resp. $D^2 \times I$) is called a *tree clasper* for L if it satisfies the following (1), (2) and (3):

(1) F is decomposed into disks and bands, called *edges*, each of which connects two distinct disks.

(2) The disks have either 1 or 3 incident edges, called *leaves* or *nodes* respectively.

(3) L intersects F transversely and the intersections are contained in the union of the interior of the leaves.

The *degree* of a tree clasper is the number of leaves *minus* 1. (In [11], a tree clasper and a leaf are called a *strict tree clasper* and a *disk-leaf* respectively.) A degree k tree clasper is called a *C_k-tree* (or a *C_k-clasper*). A C_k-tree is *simple* if each leaf intersects L at one point.

We will make use of the drawing convention for claspers of [11, Fig. 7], except for the following: $\oplus$ on an edge represents a positive half-twist. (This replaces the convention of a circled S used in [11].)

Given a C_k-tree T for a (string) link L, there is a procedure to construct a framed link $\gamma(T)$ in a regular neighborhood of T. *Surgery along T* means surgery along $\gamma(T)$. Since there exists an orientation-preserving homeomorphism, fixing the boundary, from the regular neighborhood $N(T)$ of T to the manifold $N(T)_T$ obtained from $N(T)$ by surgery along T, surgery along the C_k-tree T can be regarded as a local move on L. We say that the resulting link L_T is *obtained from*

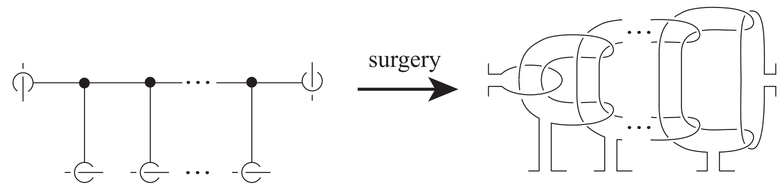

FIGURE 2.1. Surgery along a simple C_k-tree

L *by surgery along* T. In particular, surgery along a simple C_k-tree, illustrated in Figure 2.1, is equivalent to band-summing a copy of the $(k+1)$-component Milnor link (see [18, Fig. 7]) and is equivalent to a C_k-move as defined in the introduction (Figure 1.1). Similarly, for a disjoint union of trees $T_1 \cup \cdots \cup T_m$ for L, we can define $L_{T_1\cup\cdots\cup T_m}$ as a link obtained by surgery along $T_1\cup\cdots\cup T_m$. We often regard $L\cup T_1\cup\cdots\cup T_m$ as $L_{T_1\cup\cdots\cup T_m}$. A C_k-tree T having the shape of the tree clasper in Figure 2.1 is called *linear*, and the left-most and right-most leaves of T in Figure 2.1 are called the *ends* of T. Recall that a C_k-tree is an embedded disk. For a linear C_k-tree T, the ends of T are uniquely determined.

It is known that the C_k-equivalence as defined in section 1 coincides with the equivalence relation on links generated by surgery along C_k-trees and ambient isotopy. Two (string) links L and L' are C_k-equivalent if and only if there is a disjoint union of simple C_k-trees $G_1\cup\cdots\cup G_m$ such that L' is ambient isotopic to $L_{G_1\cup\cdots\cup G_m}$ [11, Theorem 3.17].

Let $L = K_1 \cup \cdots \cup K_n$ be an n-component (string) link. A (simple) C_k-tree T for L is a (simple) *C_k^a-tree* (resp. *C_k^d-tree, C_k^s-tree*) if it satisfies the following:

(1) For each leaf f of T, $f \cap L$ is contained in a single component of L, and

(2) $|\{i \mid T\cap K_i \neq \emptyset\}| = n$ (resp. $= k+1,\ 1$).

Note that n is the number of components of L and that $k+1$ is the number of leaves of T. If T is simple, T always satisfies the condition (1). The superscript a (resp. d and s) stands for trees grasping *all components* (resp. *distinct components* and a *single component*). For an n-component link L, there exist C_k^a-trees for L if and only if $n \leq k+1$, there exist C_k^d-trees for L if and only if $n \geq k+1$, and there exist C_k^s-trees for L for any $k \geq 1$. The *C_k^*-equivalence* $(* = a, d, s)$ is an equivalence relation on (string) links generated by surgery along C_k^*-trees and ambient isotopy. Note that C_k^s-equivalence is the same as self C_k-equivalence. In particular, C_k^s-equivalence coincides with the link-homotopy and self Δ-equivalence for $k=1$ and $k=2$ respectively. For a simple C_k-tree T, the set $\{i \mid T\cap K_i \neq \emptyset\}$ is called the *index* of T, and is denoted by $\mathrm{index}(T)$. Let $r_i(T)$ be the number of intersection points in $T \cap K_i$ $(i = 1, ..., n)$. The $(C_l^s + C_k^*)$-equivalence ($C_k^* = C_k,\ C_k^a$, or C_k^d) is an equivalence relation on (string) links generated by surgery along C_l^s-trees or C_k^*-trees and ambient isotopy. By arguments similar to those in the proof of [11, Theorem 3.17], we have that two (string) links L and L' are $(C_l^s + C_k^*)$-equivalent if and only if there is a disjoint union of simple C_l^s-trees or C_k^*-trees $T_1 \cup \cdots \cup T_m$ such that L' is ambient isotopic to $L_{T_1\cup\cdots\cup T_m}$. We use the notation $L \overset{C_k^*}{\sim} L'$ (resp. $L \overset{C_l^s+C_k^*}{\sim} L'$) for C_k^*-equivalent (resp. $(C_l^s + C_k^*)$-equivalent) links L and L'.

Recall that a string link is a tangle without closed components (see [9] for a precise definition). The set of ambient isotopy classes of n-component string links

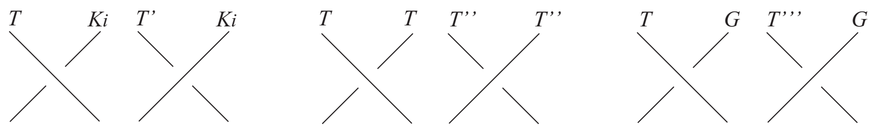

FIGURE 2.2

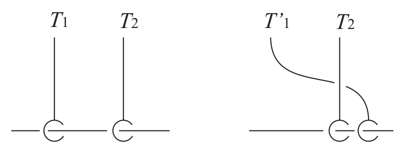

FIGURE 2.3. Sliding a leaf over another leaf

has a monoid structure with composition given by the *stacking product*, denoted by $*$, and with the trivial n-component string link $\mathbf{1}_n$ as the unit element.

In the following, we give several lemmas. The proofs of Lemmas 2.1, 2.2 and 2.4 are essentially given in [11] (see also section 1.4 in [15]), and Lemma 2.5 is essentially shown in [8] (see also [3], [16]). These proofs in [3], [11], [15], [16] rely on Habiro's *zip construction* [11, section 3.3]. Lemmas 2.1, 2.2, 2.4 and 2.5 below follow from straightforward refinements of those proofs, keeping track of the index and r_j of claspers.

Lemma 2.1 (cf. [11, Propositions 4.5, 4.6]). *Let T be a simple C_k-tree for an n-component (string) link L, and let T' (resp. T'', and T''') be obtained from T by changing a crossing of an edge and the ith component K_i of L (resp. an edge of T, and an edge of another simple tree clasper G) (see Figure 2.2). Then*

(1) $L_T \overset{C_{k+1}}{\sim} L_{T'}$, and the C_{k+1}-equivalence is realized by surgery along simple C_{k+1}-trees with index $\mathrm{index}(T) \cup \{i\}$ *and $r_j \geq r_j(T)$ $(j = 1, ..., n)$.*

(2) $L_T \overset{C_{k+1}}{\sim} L_{T''}$, $L_{T\cup G} \overset{C_{k+1}}{\sim} L_{T'''\cup G}$, and the C_{k+1}-equivalence is realized by surgery along simple C_{k+1}-trees with $r_j \geq r_j(T)$ $(j = 1, ..., n)$.

Proof. (1) In the proof of [11, Proposition 4.5], by using zip construction, it is shown that $L_{T'}$ is obtained from L_T by surgery along a simple C_{k+1}-tree H. It is not hard to see that $\mathrm{index}(H) = \mathrm{index}(T) \cup \{i\}$ and $r_j(H) \geq r_j(T)$ $(j = 1, ..., n)$ under the zip construction.

(2) Since T and G are tree claspers, T'' and T''' are obtained from T by combining ambient isotopy and changing crossings of an edge of T and components of L. Lemma 2.1 (1) completes the proof. □

Lemma 2.2 (cf. [11, Propositions 4.4]). *Let T_1 (resp. T_2) be a simple C_k-tree (resp. a simple tree clasper) for an n-component (string) link L, and let T_1' be obtained from T_1 by sliding a leaf of T_1 over a leaf of T_2 (see Figure 2.3). Then $L_{T_1\cup T_2} \overset{C_{k+1}}{\sim} L_{T_1'\cup T_2}$, and the C_{k+1}-equivalence is realized by surgery along simple C_{k+1}-trees with $r_j \geq r_j(T_1)$ $(j = 1, ..., n)$.*

Proof. Note that $L_{T_1\cup T_2} = (L_{T_2})_{T_1}$ and $L_{T_1'\cup T_2} = (L_{T_2})_{T_1'}$. Hence T_1' is obtained from T_1 by changing crossings of edges of T_1 and components of L_{T_2}. By

Lemma 2.1 (1),

$$L_{T_1\cup T_2} = (L_{T_2})_{T_1} \overset{C_{k+1}}{\sim} (L_{T_2})_{T_1'} = L_{T_1'\cup T_2},$$

and the C_{k+1}-equivalence is realized by surgery along simple C_{k+1}-trees with $r_j \geq r_j(T_1)$ $(j = 1, ..., n)$. □

Remark 2.3. In Lemma 2.1 (2), it follows from [11, Proposition 4.6] that if G is a simple C_l-tree, then $L_{T\cup G}$ is C_{k+l+1}-equivalent to $L_{T'''\cup G}$. It is not hard to see that the C_{k+l+1}-equivalence is realized by surgery along a simple C_{k+l+1}-tree with $r_j = r_j(T) + r_j(G)$ $(j = 1, ..., n)$. Similarly, in Lemma 2.2, it follows from [11, Proposition 4.4] that if T_2 is a simple C_l-tree, then $L_{T_1\cup T_2}$ is C_{k+l}-equivalent to $L_{T_1'\cup T_2}$. Moreover, for each $j = 1, ..., n$, the C_{k+l}-equivalence is realized by surgery along a simple C_{k+l}-tree with $r_j = r_j(T_1) + r_j(T_2) - 1$ if the strand of L in Figure 2.3 passing through the leaves of T_1 and T_2 is the jth component of L, or with $r_j = r_j(T_1) + r_j(T_2)$ otherwise. Since we do not need such strong results to show our results in this paper, we only mention them as a remark but not as lemmas.

Lemma 2.4 (cf. [11, Claim on p.36])**.** *Let T be a simple C_k-tree for $\mathbf{1}_n$ and let $\overline{T}$ be a simple C_k-tree obtained from T by adding a half-twist on an edge. Then $(\mathbf{1}_n)_T * (\mathbf{1}_n)_{\overline{T}} \overset{C_{k+1}}{\sim} \mathbf{1}_n$, and the C_{k+1}-equivalence is realized by surgery along simple C_{k+1}-trees with $r_j \geq r_j(T)$ $(j = 1, ..., n)$.*

Proof. In the proof of [11, Claim on p.36], it is seen that there is a simple C_k-tree $\overline{T}'$ for $\mathbf{1}_n$ such that $(\mathbf{1}_n)_{T\cup\overline{T}'}$ is ambient isotopic to $\mathbf{1}_n$ and $\overline{T}'$ is obtained from $\overline{T}$ by changing crossings of edges of $\overline{T}$ and components of $\mathbf{1}_n$. By Lemma 2.1 (1),

$$(\mathbf{1}_n)_T * (\mathbf{1}_n)_{\overline{T}} \overset{C_{k+1}}{\sim} (\mathbf{1}_n)_T * (\mathbf{1}_n)_{\overline{T}'},$$

and by Lemmas 2.1 (2) and 2.2,

$$(\mathbf{1}_n)_T * (\mathbf{1}_n)_{\overline{T}'} \overset{C_{k+1}}{\sim} (\mathbf{1}_n)_{T\cup\overline{T}'} = \mathbf{1}_n.$$

Moreover the C_{k+1}-equivalence above is realized by surgery along simple C_{k+1}-trees with $r_j \geq r_j(\overline{T})(= r_j(T))$ for each $j = 1, ..., n$. □

Lemma 2.5 (cf. [8, Theorem 6.7], [16, Lemma 2.9])**.** *Consider simple C_k-trees T_I, T_H and T_X for $\mathbf{1}_n$ which differ only in a small ball as illustrated in Figure 2.4. Then $(\mathbf{1}_n)_{T_I} \overset{C_{k+1}}{\sim} (\mathbf{1}_n)_{T_H} * (\mathbf{1}_n)_{T_X}$, and the C_{k+1}-equivalence is realized by surgery along simple C_{k+1}-trees with $r_j \geq r_j(T_I)$ $(j = 1, ..., n)$.*

Proof. By arguments similar to those in the proof of [16, Lemma 2.9], we have that there are two simple C_k-trees T_H' and T_X' for $\mathbf{1}_n$ such that $(\mathbf{1}_n)_{T_H'\cup T_X'}$ is ambient isotopic to $(\mathbf{1}_n)_{T_I}$, and T_H' (resp. T_X') is obtained from T_H (resp. T_X) by changing

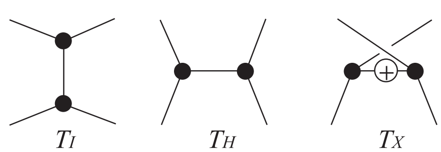

FIGURE 2.4. The IHX relation for C_k-trees

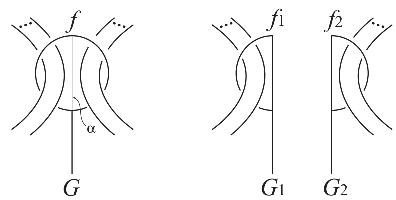

FIGURE 2.5. Splitting a leaf. The bunch of parallel strands passing through f is a bunch of strands of components of $\mathbf{1}_n$

crossings of edges of T_H (resp. T_X) and components of $\mathbf{1}_n$. By Lemmas 2.1 and 2.2,

$$(\mathbf{1}_n)_{T_H} * (\mathbf{1}_n)_{T_X} \overset{C_{k+1}}{\sim} (\mathbf{1}_n)_{T'_H} * (\mathbf{1}_n)_{T'_X} \overset{C_{k+1}}{\sim} (\mathbf{1}_n)_{T'_H \cup T'_X} = (\mathbf{1}_n)_{T_I},$$

and the C_{k+1}-equivalence is realized by surgery along simple C_{k+1}-trees with $r_j \geq r_j(T_I)$ $(j = 1, ..., n)$. □

By combining the proof of [11, Claim on p.26] and Lemmas 2.1 and 2.2 (or [11, Propositions 4.4, 4.5 and 4.6]), we have the following.

Lemma 2.6 (cf. [11, Claim on p.26])**.** *Let G be a C_k-tree for $\mathbf{1}_n$. Let f_1 and f_2 be two disks obtained by splitting a leaf f of G along an arc α as shown in Figure 2.5 (i.e., $f = f_1 \cup f_2$ and $f_1 \cap f_2 = \alpha$). Then, $(\mathbf{1}_n)_G \overset{C_{k+1}}{\sim} (\mathbf{1}_n)_{G_1} * (\mathbf{1}_n)_{G_2}$, where G_i denotes the C_k-tree for $\mathbf{1}_n$ obtained from G by replacing f with f_i $(i = 1, 2)$.*

Proof. In the proof of [11, Claim on p.26], it is shown that there is a simple C_k-tree G'_2 for $\mathbf{1}_n$ such that $(\mathbf{1}_n)_{G_1 \cup G'_2}$ is ambient isotopic to $(\mathbf{1}_n)_G$ and G'_2 is obtained from G_2 by changing crossings of edges of G_2 and components of $\mathbf{1}_n$. By Lemmas 2.1 (1) and 2.2, we have

$$(\mathbf{1}_n)_{G_1} * (\mathbf{1}_n)_{G_2} \overset{C_{k+1}}{\sim} (\mathbf{1}_n)_{G_1} * (\mathbf{1}_n)_{G'_2} \overset{C_{k+1}}{\sim} (\mathbf{1}_n)_{G_1 \cup G'_2} = (\mathbf{1}_n)_G.$$

This completes the proof. □

Remark 2.7. In the proof of Lemma 2.6, we note that the C_{k+1}-equivalence is realized by surgery along a simple C_{k+1}-tree with $r_j \geq r_j(G)$ $(j = 1, ..., n)$.

An n-component (string) link L is *Brunnian* if every proper sublink of L is trivial. In particular, any trivial (string) link is Brunnian. The n-component Brunnian (string) links are characterized by C^a_{n-1}-equivalence as follows.

Proposition 2.8 ([12, 20])**.** *Let L be an n-component (string) link in S^3. Then L is Brunnian if and only if L is obtained from a trivial (string) link by surgery along simple C^a_{n-1}-trees.*

By arguments similar to those in the proof of Theorem 1.2 in [20], we have the following lemma. In [20], the authors proved it with using 'band description' defined by K. Taniyama and the author [29]. Here we give a proof with using clasper.

Lemma 2.9 (cf. [20, Theorem 1.2])**.** *Let L be an n-component Brunnian link in S^3. If L is obtained from a trivial link O by surgery along C^s_1-trees with index $\{i\}$, then L is obtained from O by surgery along simple C^a_n-trees with $r_i = 2$.*

Proof. Set $O = O_1 \cup \cdots \cup O_n$. It is enough to consider the case when $i = 1$. There is a disjoint union F_1 of simple C_1^s-trees with index $\{1\}$ such that $L = O_{F_1}$. Note that $r_1 = 2$ for all C_1^s-trees in F_1. Hence L can be regarded as a union $(O_1)_{F_1} \cup O_2 \cup \cdots \cup O_n$.

Since L is Brunnian, $L \setminus O_2$ is the trivial link. This implies that a split sum of $L \setminus O_2$ and O_2 is trivial. Hence L can be deformed into the trivial link by crossing changes between O_2 and edges of C_1^s-trees of F_1; here $O \cup F_1$ is regarded as $L = O_{F_1}$. By Lemma 2.1 (1), we have that L is obtained from O by surgery along a disjoint union F_2 of simple C_2-trees with index $\{1, 2\}$ and $r_1 = 2$. So we have $L = O_{F_2}(= (O_1 \cup O_2)_{F_2} \cup O_3 \cup \cdots \cup O_n)$.

Since $L \setminus O_3$ is trivial, L can be deformed into the trivial link by crossing changes between O_3 and edges of C_2-trees in F_2. By Lemma 2.1 (1), there is a disjoint union F_3 of simple C_3-trees with index $\{1, 2, 3\}$ and $r_1 = 2$ such that $L = O_{F_3}(= (O_1 \cup O_2 \cup O_3)_{F_3} \cup O_4 \cup \cdots \cup O_n)$.

Repeating this step, we have that there is a disjoint union F_n of simple C_n-trees with index $\{1, ..., n\}$ and $r_1 = 2$ such that $L = O_{F_n}$. This completes the proof. □

By arguments similar to those in the proof of [4, Proposition 3.1], we have

Proposition 2.10 (cf. [4, Proposition 3.1])**.** *If an n-component (string) link L' is obtained from L by surgery along a simple tree clasper T, then for any integers p_i $(0 \leq p_i \leq r_i(T),\ i = 1, ..., n)$ with $k = p_1 + \cdots + p_n \geq 2$, L' is obtained from L by surgery along simple C_{k-1}-trees with $r_i = p_i$ $(i = 1, ..., n)$.*

A simple C_k-tree T for an n-component link is a $C_k^{(l)}$*-tree* if $\max\{r_j(T) \mid j = 1, ..., n\} \geq l$. Two links L and L' are $C_k^{(l)}$*-equivalent* if L is obtained from L' by ambient isotopy and surgery along simple $C_k^{(l)}$-trees. The following proposition is a corollary of Proposition 2.10.

Proposition 2.11 (cf. [4, Proposition 3.1])**.** *If two (string) links are $C_m^{(k+1)}$-equivalent, then they are self C_k-equivalent. Moreover, for some i, if the $C_m^{(k+1)}$-equivalence is realized by surgery along simple $C_m^{(k+1)}$-trees with $r_i \geq k+1$, then the self C_k-equivalence is realized by surgery along simple C_k^s-trees with index $\{i\}$.*

3. Milnor invariants

Milnor defined in [18, 19] a family of invariants of oriented, ordered links in S^3, known as Milnor's $\overline{\mu}$-invariants.

Given an n-component link $L = K_1 \cup \cdots \cup K_n$ in S^3, denote by G the fundamental group of $S^3 \setminus L$, and by G_q the qth subgroup of the lower central series of G. We have a presentation of G/G_q with n generators, given by a meridian m_i of each component K_i. So, for $1 \leq i \leq n$, the longitude l_i of the ith component of L is expressed modulo G_q as a word in the m_i's (abusing notation, we still denote this word by l_i).

The *Magnus expansion* $E(l_i)$ of l_i is the formal power series in non-commuting variables $X_1, ..., X_n$ obtained by substituting $1+X_j$ for m_j and $1-X_j+X_j^2-X_j^3+\cdots$ for m_j^{-1}, $1 \leq j \leq n$.

Let $I = i_1 i_2 ... i_{k-1} j$ $(k \leq q)$ be a multi-index among $\{1, ..., n\}$. Denote by $\mu_L(I)$ the coefficient of $X_{i_1} \cdots X_{i_{k-1}}$ in the Magnus expansion $E(l_j)$. The *Milnor invariant* $\overline{\mu}_L(I)$ is the residue class of $\mu_L(I)$ modulo the greatest common divisor of

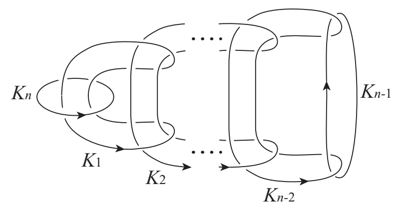

FIGURE 3.1. The Milnor link M_n

all Milnor invariants $\mu_L(J)$ such that J is obtained from I by removing at least one index, and permutating the remaining indices cyclicly. As we mentioned in section 1, $|I| = k$ is called the length of the Milnor invariant $\overline{\mu}_L(I)$. It is known that $\overline{\mu}_L(I)$ are concordance invariants [1].

The indeterminacy comes from the choice of the meridians m_i. Equivalently, it comes from the indeterminacy of representing the link as the closure of a string link [9]. Indeed, $\mu(I)$ is a well-defined invariant for string links.

The following 4 lemmas play an important role in calculating Milnor invariants.

Lemma 3.1 ([18, section 5]). *Let $M_n = K_1 \cup \cdots \cup K_n$ be the n-component Milnor link as illustrated in Figure 3.1. Then the Milnor invariants of length $\leq n-1$ vanish, and*

$$\overline{\mu}_{M_n}(i_1 i_2 ... i_{n-2}\ n-1\ n) = \begin{cases} 1 & \text{if } i_1 i_2 ... i_{n-2} = 12...n-2, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.2 ([19, Theorem 7]). *Let L' be a link obtained from a link L by taking the appropriate number of zero framed parallels of the components of L. Suppose the ith component of L' corresponds to the $h(i)$th component of L. Then*

$$\overline{\mu}_{L'}(i_1 i_2 ... i_m) = \overline{\mu}_L(h(i_1) h(i_2) ... h(i_m)).$$

Lemma 3.3 ([17, Lemma 3.3]). *Let L and L' be n-component string links such that all Milnor invariants of L (resp. L') of length $\leq m$ (resp. $\leq m'$) vanish. Then $\mu_{L*L'}(I) = \mu_L(I) + \mu_{L'}(I)$ for all I of length $\leq m + m'$.*

Lemma 3.4 ([11, Theorem 7.2]). *The Milnor invariants of length $\leq k$ for (string) links are invariants of C_k-equivalence.*

4. LINK-HOMOTOPY OF STRING LINKS

Let $\pi : \{1, ..., k\} \longrightarrow \{1, ..., n\}$ $(k \leq n)$ be an injection such that $\pi(i) < \pi(k-1) < \pi(k)$ $(i \in \{1, ..., k-2\})$, and let $\mathcal{F}_k$ be the set of such injections. For $\pi \in \mathcal{F}_k$, let T_π and $\overline{T}_\pi$ be simple linear C^d_{k-1}-trees as illustrated in Figure 4.1, and set $V_\pi = (\mathbf{1}_n)_{T_\pi}$ and $V_\pi^{-1} = (\mathbf{1}_n)_{\overline{T}_\pi}$. Here, Figure 4.1 gives the images of homeomorphisms from the neighborhoods of T_π and $\overline{T}_\pi$ to the 3-ball. Although V_π and V_π^{-1} are not unique up to ambient isotopy, by Lemmas 2.1 and 2.4, they are unique up to C_k-equivalence. So, for any $\pi \in \mathcal{F}_k$, we may choose V_π and V_π^{-1} uniquely up to C_k-equivalence. In particular, we may choose V_π so that $\mathrm{cl}(V_\pi)$ is the Milnor link $M_\pi = K_{\pi(1)} \cup \cdots \cup K_{\pi(k)}$ as illustrated in Figure 4.2 (cf. Figure 2.1).

For $\pi \in \mathcal{F}_k$, set

$$\mu_\pi(L) = \mu_L(\pi(1)\pi(2)...\pi(k)).$$

By Lemma 3.1, we have the following lemma.

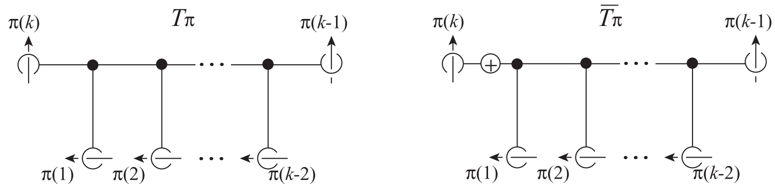

FIGURE 4.1. $\pi(i)$ means the $\pi(i)$th component of $\mathbf{1}_n$ $(i = 1, ..., k)$

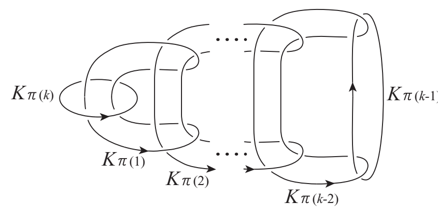

FIGURE 4.2. Milnor link $M_\pi = K_{\pi(1)} \cup \cdots \cup K_{\pi(k)}$

Lemma 4.1. *For any* $\pi, \pi' \in \mathcal{F}_k$,

$$\mu_\pi(V_{\pi'}) = \begin{cases} 1 & \text{if } \pi = \pi', \\ 0 & \text{if } \pi \neq \pi', \end{cases}$$

and the Milnor invariants of $V_{\pi'}$ *of length* $\leq k-1$ *vanish.*

Lemma 4.2. *Let* T *be a simple* C^d_{k-1}*-tree (resp.* C^a_{n-1}*-tree) for an* n*-component string link* L $(k \leq n)$*. Then* L_T *is* C_k*-equivalent (resp.* C^a_n*-equivalent) to* $L * L'$*, where*

$$L' = \prod_{\pi \in \mathcal{F}_k} V_\pi^{\mu_\pi(L_T) - \mu_\pi(L)} \quad \left(\textit{resp.} \ = \prod_{\pi \in \mathcal{F}_n} V_\pi^{\mu_\pi(L_T) - \mu_\pi(L)} \right).$$

Proof. Suppose that T is a simple C^d_{k-1}-tree. By Lemma 2.1, L_T is C_k-equivalent to $L * (\mathbf{1}_n)_{T'}$, where T' is a simple C^d_{k-1}-tree. Set index$(T') = \{i_1, ..., i_k\}$ $(i_j < i_{j+1})$. Consider induction on the length of the path connecting the two leaves grasping the i_{k-1}th and i_kth components of $\mathbf{1}_n$, and apply Lemmas 2.5 and 2.4: we have that $(\mathbf{1}_n)_{T'}$ is C_k-equivalent to a string link which is obtained from $\mathbf{1}_n$ by surgery along simple linear C^d_{k-1}-trees whose ends grasp the i_{k-1}th and i_kth components of $\mathbf{1}_n$. By Lemmas 2.1, 2.2 and 2.4, we have that

$$(\mathbf{1}_n)_{T'} \overset{C_k}{\sim} L'' = \prod_{\pi \in \mathcal{F}_k} V_\pi^{x_\pi}.$$

By Lemmas 3.4, 3.3 and 4.1,

$$\begin{aligned} \mu_{\pi'}(L_T) = \mu_{\pi'}(L * L'') &= \mu_{\pi'}(L) + \mu_{\pi'}(L'') \\ &= \mu_{\pi'}(L) + \sum_{\pi \in \mathcal{F}_k} x_\pi \mu_{\pi'}(V_\pi) = \mu_{\pi'}(L) + x_{\pi'}. \end{aligned}$$

Suppose that T is a simple C^a_{n-1}-tree. Since a C^a_{n-1}-tree is a C^d_{n-1}-tree, we have already shown that

$$L_T \overset{C_n}{\sim} L * \prod_{\pi \in \mathcal{F}_n} V_\pi^{\mu_\pi(L_T) - \mu_\pi(L)}.$$

Lemmas 2.1, 2.2, 2.4 and 2.5 imply that the C_n-equivalence is realized by surgery along C_n^a-trees. □

The following theorem gives representatives, which depend only on Milnor invariants, for the link-homotopy classes.

Theorem 4.3. *Let L be an n-component string link. Then L is link-homotopic to $L_1 * L_2 * \cdots * L_{n-1}$, where*

$$L_i = \prod_{\pi \in \mathcal{F}_{i+1}} V_\pi^{x_\pi},$$

$$x_\pi = \begin{cases} \mu_L(\pi(1)\pi(2)) & \text{if } i = 1, \\ \\ \mu_L(\pi(1)...\pi(i+1)) - \mu_{L_1*\cdots*L_{i-1}}(\pi(1)...\pi(i+1)) & \text{if } i \geq 2. \\ (= \mu_{L_i}(\pi(1)...\pi(i+1))) \end{cases}$$

Remark 4.4. The presentation $L_1 * L_2 * \cdots * L_{n-1}$ of L depends on the choice of order on the elements in $\mathcal{F}_i$ $(i = 2, ..., n)$. If we put $\mathcal{F}_2 \cup \cdots \cup \mathcal{F}_n = \{\pi_1, ..., \pi_q\}$ so that for $i < j$, any element in $\mathcal{F}_i$ appears before the elements in $\mathcal{F}_j$, then by Theorem 4.3 and Lemmas 3.3 and 4.1, L is link-homotopic to

$$V_{\pi_1}^{x_{\pi_1}} * \cdots * V_{\pi_q}^{x_{\pi_q}} \ (x_{\pi_k} = \mu_{\pi_k}(L) - \mu_{\pi_k}(\prod_{i=1}^{k-1} V_{\pi_i}^{x_{\pi_i}})).$$

Note that the representation is unique up to link-homotopy.

Proof. Since the C_1-move is the crossing change, L is C_1-equivalent to the trivial string link $\mathbf{1}_n$. So L is obtained from $\mathbf{1}_n$ by surgery along simple C_1-trees.

Note that a simple C_1-tree is either a simple C_1^s-tree or a simple C_1^d-tree, and that C_1^s-equivalence preserves the value of $\mu(I)$ for any I with $r(I) = 1$. Since L is C_1^s-equivalent to a link which is obtained from $\mathbf{1}_n$ by surgery along C_1^d-trees, by Lemmas 4.2, 2.1, 2.2 and 2.4,

$$L \overset{C_1^s + C_2}{\sim} \prod_{\pi \in \mathcal{F}_2} V_\pi^{\mu_\pi(L)} (= L_1).$$

A C_k-tree $(k \geq 2)$ is either a $C_k^{(2)}$-tree or a C_k^d-tree, and $C_k^{(2)}$-equivalence implies C_1^s-equivalence (Proposition 2.11). It follows that $(C_1^s + C_k)$-equivalence implies $(C_1^s + C_k^d)$-equivalence. So L is obtained from L_1 by surgery along simple C_1^s-trees and C_2^d-trees.

By Lemmas 4.2, 2.1, 2.2 and 2.4,

$$L \overset{C_1^s + C_3}{\sim} L_1 * \prod_{\pi \in \mathcal{F}_3} V_\pi^{\mu_\pi(L) - \mu_\pi(L_1)} (= L_1 * L_2).$$

Therefore L and $L_1 * L_2$ are $(C_1^s + C_3^d)$-equivalent.

Repeating this process, we have that

$$L \overset{C_1^s + C_n}{\sim} L_1 * L_2 * \cdots * L_{n-1}.$$

Since any simple C_n-tree for an n-component string link is a $C_n^{(2)}$-tree, $(C_1^s + C_n)$-equivalence implies C_1^s-equivalence, i.e., link-homotopy. □

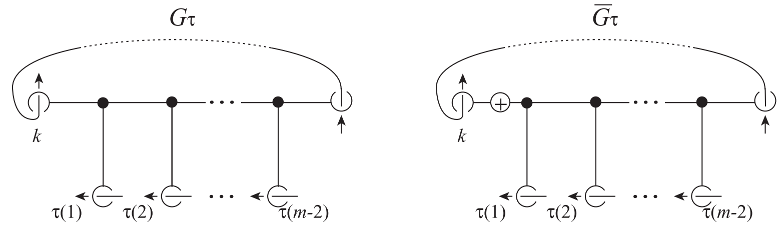

FIGURE 5.1. $\tau(i)$ $(i = 1, ..., m-2)$ and k mean the $\tau(i)$th and kth components of $\mathbf{1}_n$, respectively

By Theorem 4.3, we have the following corollary.

Corollary 4.5. *For a natural number $k \le n$, two n-component string links L and L' are $(C_1^s + C_k)$-equivalent if and only if $\mu_L(I) = \mu_{L'}(I)$ for any I with $r(I) = 1$ and $|I| \le k$.*

Proof. The 'only if' part follows from Lemma 3.4. Now we will prove the 'if' part.

By Theorem 4.3, L and L' are link-homotopic to $L_1 * L_2 * \cdots * L_{n-1}$ and $L'_1 * L'_2 * \cdots * L'_{n-1}$ respectively. Note that both L_i and L'_i are C_i-equivalent to $\mathbf{1}_n$. So L and L' are $(C_1^s + C_k)$-equivalent to $L_1 * L_2 * \cdots * L_{k-1}$ and $L'_1 * L'_2 * \cdots * L'_{k-1}$ respectively. Since $\mu_L(I) = \mu_{L'}(I)$ for any I with $r(I) = 1$ and $|I| \le k$, $L_i = L'_i$ $(i = 1, ..., k-1)$. This completes the proof. □

Theorem 1.1 follows directly from Corollary 4.5.

Proof of Theorem 1.1. It is enough to show the 'if' part. Since a C_n-move for an n-component string link is a $C_n^{(2)}$-move, by Proposition 2.11, C_n-equivalence implies C_1^s-equivalence. Hence $(C_1^s + C_n)$-equivalence implies link-homotopy. By Corollary 4.5, L and L' are link-homotopic. □

Remark 4.6. Let L be an n-component link in S^3. Denote by $\mathcal{L}(L)$ the set of all n-component string links l such that $\mathrm{cl}(l) = L$. Put $\mathcal{F}_2 \cup \cdots \cup \mathcal{F}_n = \{\pi_1, ..., \pi_q\}$ so that any element in $\mathcal{F}_i$ appears before the elements in $\mathcal{F}_j$ $(2 \le i < j \le n)$ and fix this total order on $\mathcal{F}_2 \cup \cdots \cup \mathcal{F}_n$. Then, by Remark 4.4, each l in $\mathcal{L}(L)$ is link-homotopic to $V_{\pi_1}^{x_{\pi_1}} * \cdots * V_{\pi_q}^{x_{\pi_q}}$ $(x_{\pi_k} = \mu_{\pi_k}(l) - \mu_{\pi_k}(\prod_{i=1}^{k-1} V_{\pi_i}^{x_{\pi_i}}))$, which is the unique representation up to link-homotopy. We define a vector v_l as $v_l = (x_{\pi_1}, ..., x_{\pi_q})$, and set $\mathcal{V}_L = \{v_l \mid l \in \mathcal{L}(L)\}$. By the uniqueness of the presentation for l, we have the following: *Two n-component links L and L' in S^3 are link-homotopic if and only if $\mathcal{V}_L \cap \mathcal{V}_{L'} \ne \emptyset$.*

5. Self Δ-equivalence of Brunnian links

Let n and m be integers $(2 \le n < m \le 2n)$. Given $k \in \{1, ..., n\}$, consider a surjection τ from $\{1, ..., m-2\}$ to $\{1, ..., n\} \setminus \{k\}$. Let G_τ and $\overline{G}_\tau$ be the simple linear C_{m-1}^a-trees illustrated in Figure 5.1, and set $V_\tau = (\mathbf{1}_n)_{G_\tau}$ and $V_\tau^{-1} = (\mathbf{1}_n)_{\overline{G}_\tau}$. Here, Figure 5.1 are the images of homeomorphisms from the neighborhoods of G_τ and $\overline{G}_\tau$ to the 3-ball. Although V_τ and V_τ^{-1} are not unique up to ambient isotopy, by Lemmas 2.1 and 2.4, their closures are unique up to C_m^a-equivalence. For each τ, we choose V_τ and V_τ^{-1} and fix them.

Set

$$\mu_\tau(L) = \mu_L(\tau(1)...\tau(m-2)\ k\ k).$$

Recall that $2 \leq n < m \leq 2n$. Let $\mathcal{B}_m(k)$ be the set of all surjections τ from $\{1,...,m-2\}$ to $\{1,...,n\}\setminus\{k\}$ such that $|\tau^{-1}(i)| \leq 2$ $(i=1,...,n)$ and $|\tau^{-1}(j)|=1$ (if $j>k$), and let ρ_m be a surjection from $\{1,...,m-2\}$ to itself defined by $\rho_m(i)=m-1-i$. Note that the definition of $\mathcal{B}_m(k)$ implies that $\mathcal{B}_m(k)=\emptyset$ if $k<m-n$. So we may assume that $k \geq m-n$.

If $m \leq 2n-2$, then for any $\tau \in \mathcal{B}_m(k)$, $\{i \mid \tau(i) \neq \tau(m-1-i)\} \neq \emptyset$. We set

$$\mathcal{P}_m(k) = \left\{ \tau \in \mathcal{B}_m(k) \;\middle|\; \begin{array}{l} \tau(p) < \tau(m-1-p) \\ \text{for } p = \min\{i \mid \tau(i) \neq \tau(m-1-i)\} \end{array} \right\}.$$

See Figure 5.2 (a) and (b): τ_1 is an element in $\mathcal{P}_6(2)$ $(=\mathcal{P}_{2n-2}(m-n))$, and $\tau_1\rho_6$ is an element in $\mathcal{B}_6(2)\setminus\mathcal{P}_6(2)$.

If $m=2n-1$, then $k=n, n-1$ and there is $\tau \in \mathcal{B}_{2n-1}(n)\cup\mathcal{B}_{2n-1}(n-1)$ such that $\tau(i)=\tau(2n-2-i)$ $(i=1,...,n-2)$ and $|\tau^{-1}(\tau(n-1))|=1$. For $k=n,n-1$, set

$$\mathcal{R}_{2n-1}(k) = \left\{ \tau \in \mathcal{B}_{2n-1}(k) \;\middle|\; \begin{array}{l} \tau(i)=\tau(2n-2-i) \ (i=1,...,n-2), \\ |\tau^{-1}(\tau(n-1))|=1 \end{array} \right\},$$

and set

$$\mathcal{P}_{2n-1}(k) = \left\{ \tau \in \mathcal{B}_{2n-1}(k)\setminus\mathcal{R}_{2n-1}(k) \;\middle|\; \begin{array}{l} \tau(p) < \tau(2n-2-p) \\ \text{for } p = \min\{i \mid \tau(i) \neq \tau(2n-2-i)\} \end{array} \right\}.$$

Note that if $\tau \in \mathcal{R}_{2n-1}(n-1)$, then $\tau(n-1)=n$. See Figure 5.2 (c): τ_2 is an element in $\mathcal{R}_7(3)$ $(=\mathcal{R}_{2n-1}(n-1))$.

If $m=2n$, then $k=n$ and there is $\tau \in \mathcal{B}_{2n}(n)$ such that $\tau(i)=\tau(2n-1-i)$ $(i=1,...,n-1)$. Set

$$\mathcal{R}_{2n}(n) = \{\tau \in \mathcal{B}_{2n}(n) \mid \tau(i)=\tau(2n-1-i) \ (i=1,...,n-1)\},$$

and set

$$\mathcal{P}_{2n}(n) = \left\{ \tau \in \mathcal{B}_{2n}(n)\setminus\mathcal{R}_{2n}(n) \;\middle|\; \begin{array}{l} \tau(p) < \tau(2n-1-p) \\ \text{for } p = \min\{i \mid \tau(i) \neq \tau(2n-1-i)\} \end{array} \right\}.$$

See Figure 5.2 (d): τ_3 is an element in $\mathcal{R}_8(4)$ $(=\mathcal{R}_{2n}(n))$.

We note that if $\tau \in \mathcal{R}_m(k)$, then $\tau\rho_m \in \mathcal{R}_m(k)$ (i.e., τ has 'symmetry'); if $\tau \in \mathcal{P}_m(k)$, then $\tau\rho_m \notin \mathcal{P}_m(k)$, and

$$\mathcal{B}_m(k) = \mathcal{P}_m(k) \cup \mathcal{R}_m(k) \cup \{\tau\rho_m \mid \tau \in \mathcal{P}_m(k)\}.$$

For any $\varphi \in \mathcal{B}_m(k)$, V_φ is C_{m-1}-equivalent to $\mathbf{1}_n$. By Lemma 3.4, $\mu_{V_\varphi}(I)=0$ for any I with $|I| \leq m-1$.

By arguments similar to those in the proof of [17, Proposition 5.1], we have the following lemma.

Lemma 5.1. *(1) If $\tau \in \mathcal{P}_m(k)$, then for an n-component string link L,*

$$\mathrm{cl}(L * V_{\tau\rho_m}) \overset{C_m^a}{\sim} \begin{cases} \mathrm{cl}(L * V_\tau) & \text{if } m \text{ is even}, \\ \mathrm{cl}(L * V_\tau^{-1}) & \text{if } m \text{ is odd}. \end{cases}$$

Moreover the C_m^a-equivalence is realized by surgery along simple C_m^a-trees with $r_j \geq r_j(G_\tau)$ $(j=1,...,n)$.
(2) If $\varphi \in \mathcal{R}_{2n-1}(k)$, then for an n-component string link L,

$$\mathrm{cl}(L * V_\varphi) \overset{C_{2n-1}^a}{\sim} \mathrm{cl}(L * V_\varphi^{-1}).$$

Moreover the C_{2n-1}^a-equivalence is realized by surgery along simple C_{2n-1}^a-trees with $r_j \geq r_j(G_\varphi)$ $(j=1,...,n)$.

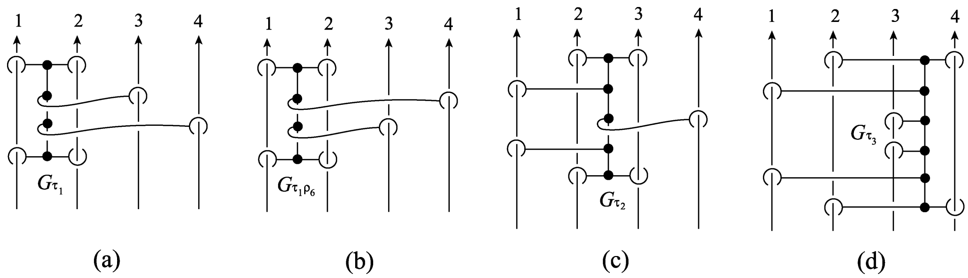

FIGURE 5.2. (a) $\tau_1 \in \mathcal{P}_6(2) = \mathcal{P}_{2n-2}(m-n)$, (b) $\tau_1\rho_6 \in \mathcal{B}_6(2) \setminus \mathcal{P}_6(2)$, (c) $\tau_2 \in \mathcal{R}_7(3) = \mathcal{R}_{2n-1}(n-1)$, and (d) $\tau_3 \in \mathcal{R}_8(4) = \mathcal{R}_{2n}(n)$

Proof. (1) For $V_{\tau\rho_m} = (\mathbf{1}_n)_{G_{\tau\rho_m}}$, we can exchange the relative position of all pairs of leaves of $G_{\tau\rho_m}$ grasping the same component of $\mathbf{1}_n$ by, after taking closure, sliding the upper leaves along the orientation of $\mathrm{cl}(L)$. Lemmas 2.1, 2.2 and 2.4 imply that

$$\mathrm{cl}(L * V_{\tau\rho_m}) \overset{C_m^a}{\sim} \mathrm{cl}(L * (\mathbf{1}_n)_G),$$

where G is obtained from G_τ by adding a positive half twist on each edge adjacent to a leaf (see Figure 5.3). Lemma 2.4 completes the proof.

The same arguments give us a proof of (2). □

Now, for each $\varphi \in \mathcal{R}_m(k) \cup \mathcal{P}_m(k)$, we will calculate some Milnor invariants of the string link V_φ.

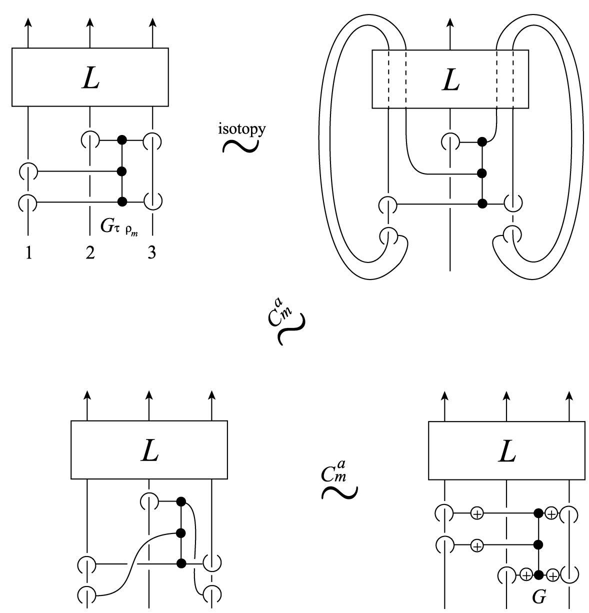

FIGURE 5.3

Lemma 5.2. *For* $\varphi \in \mathcal{P}_m(k)$ $(n+1 \leq m \leq 2n,\ m-n \leq k \leq n)$ *and for* $\tau \in \bigcup_l(\mathcal{R}_m(l) \cup \mathcal{P}_m(l))$,

$$\mu_\tau(V_\varphi) = \begin{cases} 1 & \text{if } \varphi = \tau, \\ 0 & \text{if } \varphi \neq \tau. \end{cases}$$

We recall that $\mu_\tau(V_\varphi) = \mu_{V_\varphi}(\tau(1)...\tau(m-2)ll)$ for $\tau \in \mathcal{R}_m(l) \cup \mathcal{P}_m(l)$.

Proof. We take the following 4 steps to prove this lemma.

Step 1. Make a new link W_φ from V_φ by taking parallels of the components of V_φ so that Lemma 3.2 can be applied.

Consider the C^a_{m-1}-tree G_φ for $\mathbf{1}_n$. Let $\tau \in \mathcal{R}_m(l) \cup \mathcal{P}_m(l)$ $(m-n \leq l \leq n)$. Let $\mathbf{1}_m$ be the m-component trivial string link obtained from $\mathbf{1}_n$ by taking parallel copies of the components of $\mathbf{1}_n$ such that the ith component of $\mathbf{1}_m$ is a parallel copy of either

$$\begin{cases} \text{the } \tau(i)\text{th component of } \mathbf{1}_n \text{ if } i = 1, ..., m-2, \text{ or} \\ \text{the } l\text{th component of } \mathbf{1}_n \text{ if } i = m-1, m, \end{cases}$$

and that $\mathbf{1}_m$ is contained in the tubular neighborhood $N(\mathbf{1}_n)$ of $\mathbf{1}_n$ with $G_\varphi \cap \mathbf{1}_m \subset \mathrm{int}(G_\varphi \cap N(\mathbf{1}_n))$. Since surgery along a C_{m-1}-tree preserves framings, the above correspondance can be naturally extended so that the ith component of $(\mathbf{1}_m)_{G_\varphi}$ is a parallel copy of either

$$\begin{cases} \text{the } \tau(i)\text{th component of } (\mathbf{1}_n)_{G_\varphi} \text{ if } i = 1, ..., m-2, \text{ or} \\ \text{the } l\text{th component of } (\mathbf{1}_n)_{G_\varphi} \text{ if } i = m-1, m. \end{cases}$$

Set $W_\varphi = (\mathbf{1}_m)_{G_\varphi}$. Since G_φ is a C^a_{m-1}-tree, by Lemma 3.4, $\mu_{W_\varphi}(I) = \mu_{V_\varphi}(I) = 0$ for any I with $|I| \leq m-1$. Hence, by Lemma 3.2, we have

$$\mu_{W_\varphi}(12...m) = \overline{\mu}_{\mathrm{cl}(W_\varphi)}(12...m) = \overline{\mu}_{\mathrm{cl}(V_\varphi)}(\tau(1)...\tau(m-2)ll) = \mu_\tau(V_\varphi).$$

Step 2. By applying Lemma 2.6, deform W_φ up to the C_m-equivalence into

$$(\mathbf{1}_m)_{G_{\varphi_1}} * (\mathbf{1}_m)_{G_{\varphi_2}} * \cdots * (\mathbf{1}_m)_{G_{\varphi_s}}$$

so that each G_{φ_j} is a simple linear C_{m-1}-tree.

Set $V_j = (\mathbf{1}_m)_{G_{\varphi_j}}$ $(j = 1, ..., s)$.

Step 3. By Lemma 3.4, we have

$$\mu_{W_\varphi}(12...m) = \mu_{V_1 * \cdots * V_s}(12...m).$$

Since $\mu_{V_j}(I) = 0$ for any I with $|I| \leq m-1$, by Lemma 3.3, we have

$$\mu_{W_\varphi}(12...m) = \mu_{V_1}(12...m) + \cdots + \mu_{V_s}(12...m).$$

Step 4. If G_{φ_j} is a $C^{(2)}_{m-1}$-tree, then by Proposition 2.11, V_j is link-homotopically trivial; hence $\mu_{V_j}(12...m) = 0$. Otherwise, by using Lemma 4.1, calculate each $\mu_{V_j}(12...m)$.

If $(|\varphi^{-1}(1)|, ..., |\varphi^{-1}(n)|) \neq (|\tau^{-1}(1)|, ..., |\tau^{-1}(n)|)$, then each G_{φ_j} is a $C^{(2)}_{m-1}$-tree. This implies that $\mu_{W_\varphi}(12...m) = 0$.

Suppose $(|\varphi^{-1}(1)|, ..., |\varphi^{-1}(n)|) = (|\tau^{-1}(1)|, ..., |\tau^{-1}(n)|)$. Since $|\varphi^{-1}(k)| = 0$ and $|\tau^{-1}(i)| \geq 1$ $(i \neq l)$, we have $k = l$, i.e., $\varphi \in \mathcal{P}_m(l)$.

If $\varphi \neq \tau$, then neither $(\varphi(1), ..., \varphi(m-1))$ nor $(\varphi\rho_m(1), ..., \varphi\rho_m(m-1))$ is equal to $(\tau(1), ..., \tau(m-1))$. By Lemma 4.1, $\mu_{V_j}(12...m) = 0$ for any j $(= 1, 2, ..., s)$.

If $\varphi = \tau$, then by Lemma 4.1 and by the fact that $\varphi\rho_m \neq \varphi$, there is a unique C^d_{m-1}-tree G_{φ_u} in $\{G_{\varphi_1}, ..., G_{\varphi_s}\}$ such that

$$\mu_{V_j}(12...m) = \begin{cases} 1 & \text{if } j = u, \\ 0 & \text{if } j \neq u. \end{cases}$$

This completes the proof. □

Remark 5.3. The calculation method used in the proof of Lemma 5.2 can be applied in a more general context as follows. Let T be a linear, simple C_{m-1}-tree for $\mathbf{1}_n$ whose ends both grasp the kth component, and let $I = i_1...i_{m-2}kk$ be a multi-index. Then, $\mu_{(\mathbf{1}_n)_T}(I)$ can be calculated as follows.

Step 1. Make a new link $W = (\mathbf{1}_m)_T$ from $(\mathbf{1}_n)_T$ by taking parallels of the components of $(\mathbf{1}_n)_T$ so that Lemma 3.2 can be applied.

Step 2. By applying Lemma 2.6, deform W up to the C_m-equivalence into $(\mathbf{1}_m)_{T_1} * (\mathbf{1}_m)_{T_2} * \cdots * (\mathbf{1}_m)_{T_s}$ so that each T_j is a simple linear C_{m-1}-tree.

Step 3. By applying Lemmas 3.4 and 3.3, we have

$$\mu_W(12...m) = \mu_{(\mathbf{1}_m)_{T_1}}(12...m) + \cdots + \mu_{(\mathbf{1}_m)_{T_s}}(12...m).$$

Step 4. If T_j is a $C^{(2)}_{m-1}$-tree, then by Proposition 2.11, $\mu_{(\mathbf{1}_m)_{T_j}}(12...m) = 0$. Otherwise, by using Lemma 4.1, calculate each $\mu_{(\mathbf{1}_m)_{T_j}}(12...m)$.

Lemma 5.4. *(1) For any $\varphi \in \mathcal{R}_{2n-1}(k)$ $(k = n, n-1)$, the Milnor invariants of V_φ of length $\leq 2n-1$ vanish.*
(2) For $\varphi \in \mathcal{R}_{2n-1}(n)$ and $\tau \in \mathcal{R}_{2n}(n) \cup \mathcal{P}_{2n}(n)$, there is a string link U_φ such that $\mathrm{cl}(U_\varphi) \overset{C_{2n-1}}{\sim} \mathrm{cl}(V_\varphi)$ and

$$|\mu_\tau(U_\varphi)| = \begin{cases} 1 & \textit{if } \tau \in \mathcal{R}_{2n}(n) \textit{ and } \varphi(i) = \tau(i) \ (i = 1, ..., n-1), \\ 0 & \textit{otherwise.} \end{cases}$$

(3) For $\tau, \ \varphi \in \mathcal{R}_{2n}(n) \cup \mathcal{P}_{2n}(n)$,

$$\mu_\tau(V_\varphi) = \begin{cases} 1 & \textit{if } \varphi = \tau \in \mathcal{P}_{2n}(n), \\ 2 & \textit{if } \varphi = \tau \in \mathcal{R}_{2n}(n), \\ 0 & \textit{if } \varphi \neq \tau. \end{cases}$$

Remark 5.5. (1) Note that, for any $\varphi \in \mathcal{R}_{2n-1}(n)$, there exists a unique element $\tau \in \mathcal{R}_{2n}(n)$ such that $\varphi(i) = \tau(i)$ $(i = 1, ..., n-1)$, and that the correspondence induces a bijection from $\mathcal{R}_{2n-1}(n)$ to $\mathcal{R}_{2n}(n)$.
(2) For $\varphi \in \mathcal{R}_{2n-1}(n-1)$ and $\tau \in \mathcal{R}_{2n}(n) \cup \mathcal{P}_{2n}(n)$, while the Milnor invariants of V_φ of length $\leq 2n-1$ vanish, $\mu_\tau(V_\varphi)$ is not easily calculated. However, we do not need the calculations to prove Theorem 1.3.

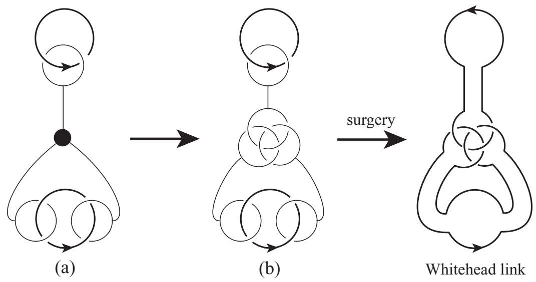

FIGURE 5.4. (a) 2-component trivial link with a simple C_2-clasper. (b) 2-component trivial link with 3 basic claspers

Proof. As illustrated in Figure 5.4, the Whitehead link, which is a link $C(12, 12)$ as defined in [2, subsection 7.11], is obtained from the 2-component trivial link by surgery along a simple C_2-tree. We recall that, for a sequence $i_1...i_k$,

(i) $C(i_1i_2, i_1i_2)$ is a Whitehead link; and

(ii) $C(i_1...i_ki_{k+1}, i_1...i_ki_{k+1}) = K_{i_1} \cup \cdots \cup K_{i_k+1}$ is a $(k+1)$-component link obtained from $C(i_1...i_k, i_1...i_k) = K_{i_1} \cup \cdots \cup K_{i_{k-1}} \cup K'_{i_k}$ by replacing K'_{i_k} by a Bing double $K_{i_k} \cup K_{i_{k+1}}$ of K'_{i_k}.

A 4-component link obtained from the 4-component trivial link by surgery along the 11 *basic claspers* illustrated in Figure 5.5 is ambient isotopic to a link obtained by surgery along the 11 basic claspers of Figure 5.6, and to a link obtained from the trivial link by surgery along a clasper H with *boxes* as illustrated in Figure 5.7 (for the definitions of a basic clasper and a box, see [11]). Apply zip construction [11] twice to the clasper H in Figure 5.7; the first one is for the marking $\{e_1\}$, and the second one is for $\{e_2\}$. Then $\mathrm{Zip}(\mathrm{Zip}(H, \{e_1\}), \{e_2\})$ is C_7-equivalent to a link obtained by surgery along the C_6-tree of Figure 5.8. By Lemmas 2.4 and 5.1 (2), a link illustrated in Figure 5.8 is C_7-equivalent to $\mathrm{cl}((\mathbf{1}_4)_{G_\varphi})$ $(= \mathrm{cl}(V_\varphi))$, where $\varphi \in \mathcal{R}_7(4) = \mathcal{R}_{2n-1}(n)$ is defined by $\varphi(1) = \varphi(5) = 3,\ \varphi(2) = \varphi(4) = 2,\ \varphi(3) = 1$; see Figure 5.9. Since a link illustrated in Figure 5.6 is ambient isotopic to a link $C(1234, 1234)$ (Figure 5.10); the link $C(1234, 1234)$ is C_7-equivalent to $\mathrm{cl}(V_\varphi)$.

Similarly, we can see that, for $\varphi \in \mathcal{R}_{2n-1}(k)$ $(k = n, n-1)$, a link $C(\alpha, \alpha)$ $(\alpha = \varphi(n-1)\varphi(n-2)\cdots\varphi(1)k)$ is C_{2n-1}-equivalent to $\mathrm{cl}(V_\varphi)$. In [2, subsection 7.11], it is shown that $\overline{\mu}_{C(\alpha,\alpha)}(I) = 0$ for any I with $|I| \leq 2n-1$. Hence, by Lemma 3.4, we have the conclusion (1).

Let U_φ be a band sum of $C(\alpha, \alpha)$ and $\mathbf{1}_n$ with $\mathrm{cl}(U_\varphi) = C(\alpha, \alpha)$; for example, see Figure 5.11. Let $\tau \in \mathcal{R}_{2n}(n)$. Then it is not hard to see that $\mathrm{cl}(V_\tau)$ is ambient isotopic to a link $L((\overline{\beta}, \beta))$ $(\beta = \tau(n-1)(\tau(n-2)(\cdots(\tau(1)n)\cdots)),\ \overline{\beta} = ((\cdots(n\tau(1))\cdots)\tau(n-2))\tau(n-1)$ defined in [2, subsection 7.4]. By combining [2, Proposition 6.5, Theorem 7.10 on p.42, and Theorem 7.10 on p.43], we have that if $\alpha = \beta$, i.e., $\varphi(i) = \tau(i)$ $(i = 1, ..., n-1)$, then

$$|2\mu_{U_\varphi}(I)| = |2\overline{\mu}_{C(\alpha,\alpha)}(I)| = |\overline{\mu}_{L((\overline{\beta},\beta))}(I)|$$

for any I with $|I| = 2n$. Then, (2) follows from (3).

For $\tau,\ \varphi \in \mathcal{R}_{2n}(n) \cup \mathcal{P}_{2n}(n)$, by following the 4 steps in Remark 5.3, we have (3). This completes the proof. □

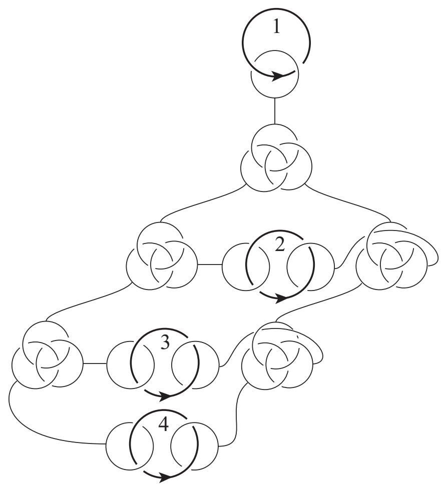

FIGURE 5.5. 4-component trivial link with 11 basic claspers. The numbers 1, 2, 3, and 4 denote the order of the components

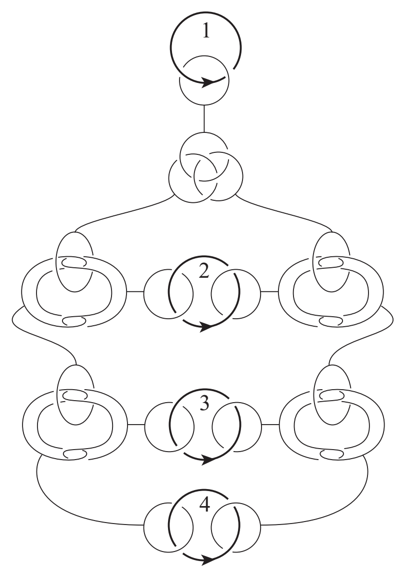

FIGURE 5.6

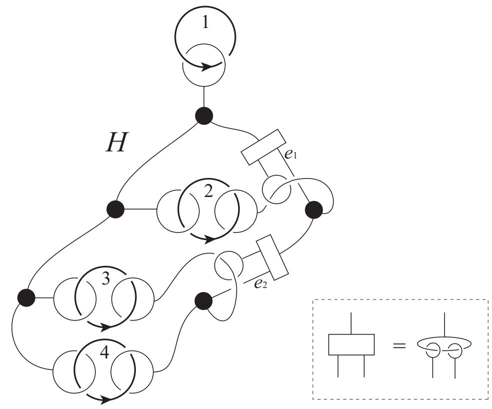

FIGURE 5.7

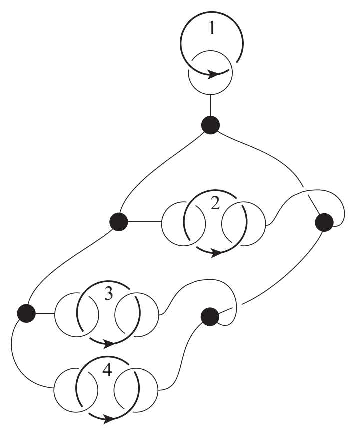

FIGURE 5.8

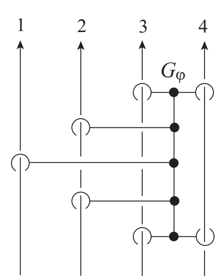

FIGURE 5.9. $\varphi \in \mathcal{R}_7(4) = \mathcal{R}_{2n-1}(n);\ \varphi(1) = \varphi(5) = 3,\ \varphi(2) = \varphi(4) = 2,\ \varphi(3) = 1$

Lemma 5.6. *Let L be an n-component string link and m $(n+1 \leq m \leq 2n)$ an integer. Let T be a simple C^a_{m-1}-tree for L. Suppose that $\mu_L(I) = 0$ for any I with $|I| \leq m-1$ and $r(I) \leq 2$, and that T is not a simple $C^{(3)}_{m-1}$-tree. Set*

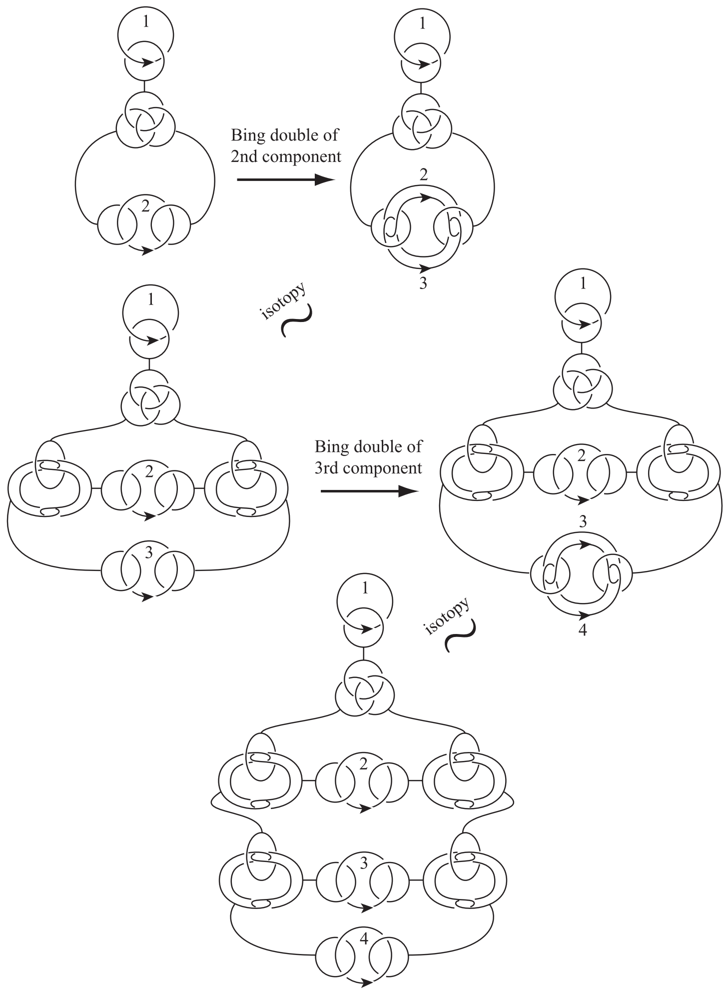

FIGURE 5.10. ‘$\overset{\text{isotopy}}{\sim}$’ means that the links obtained by surgery along claspers are ambient isotopic

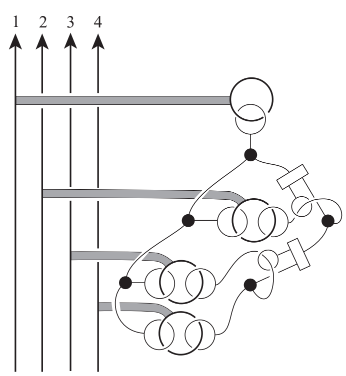

FIGURE 5.11. A band sum of $\mathbf{1}_4$ and $C(1234, 1234)$

$k=\max\{i \mid |T\cap(i\text{th component of }\mathbf{1}_n)|=2\}$. *Then*
(1) $\mathrm{cl}(L_T)$ *is* C_m^a*-equivalent to* $\mathrm{cl}(L*L')$*, where* $L'=$

$$\begin{cases}\displaystyle\prod_{\tau\in\mathcal{P}_m(k)} V_\tau^{\mu_\tau(L_T)-\mu_\tau(L)} & \text{if } n+1\le m\le 2n-2,\\ \displaystyle\prod_{\tau\in\mathcal{P}_{2n-1}(k)} V_\tau^{\mu_\tau(L_T)-\mu_\tau(L)} * \prod_{\substack{\varphi\in\mathcal{R}_{2n-1}(k)\\ (\text{for some } \varepsilon(\varphi)\text{'s in } \{0,1\})}} V_\varphi^{\varepsilon(\varphi)} & \text{if } m=2n-1,\\ \displaystyle\prod_{\tau\in\mathcal{P}_{2n}(n)} V_\tau^{\mu_\tau(L_T)-\mu_\tau(L)} * \prod_{\varphi\in\mathcal{R}_{2n}(n)} V_\varphi^{(\mu_\varphi(L_T)-\mu_\varphi(L))/2} & \text{if } m=2n,\end{cases}$$

and
(2) the C_m^a*-equivalence is realized by surgery along simple* C_m^a*-trees with* $r_k\ge 2$.

Note that if $m\ge n+1$, then a simple C_{m-1}^a-tree is a simple $C_{m-1}^{(2)}$-tree. If T is a simple $C_{m-1}^{(3)}$-tree, then by Proposition 2.11, L_T and L are self Δ-equivalent.

Proof. By Lemma 2.1, L_T is C_m^a-equivalent to $L*(\mathbf{1}_n)_{T'}$, where T' is a simple C_{m-1}^a-tree and not a $C_{m-1}^{(3)}$-tree with

$$\max\{i \mid |T'\cap(i\text{th component of }\mathbf{1}_n)|=2\}=k.$$

By induction on the length of the path connecting the two leaves grasping the kth component, using Lemmas 2.5 and 2.4, we have that $(\mathbf{1}_n)_{T'}$ is C_m^a-equivalent to a string link which is obtained from $\mathbf{1}_n$ by surgery along simple linear C_{m-1}^a-trees whose ends grasp the kth component of $\mathbf{1}_n$. By Lemmas 2.1, 2.2, 2.4, we have that

$$L_T \overset{C_m^a}{\sim} L*\prod_{\tau\in\mathcal{B}_m(k)} V_\tau^{u_\tau}.$$

Moreover, by Lemmas 2.4 and 5.1, the closure $\mathrm{cl}(L*\prod_{\tau\in\mathcal{B}_m(k)} V_\tau^{u_\tau})$ is C_m^a-equivalent to $\mathrm{cl}(L*L')$, where

$$L'=\begin{cases}\displaystyle\prod_{\tau\in\mathcal{P}_m(k)} V_\tau^{y_\tau} & \text{if } n+1\le m\le 2n-2,\\ \displaystyle\prod_{\tau\in\mathcal{P}_m(k)} V_\tau^{y_\tau} * \prod_{\varphi\in\mathcal{R}_m(k)} V_\varphi^{\varepsilon(\varphi)},\ (\varepsilon(\varphi)\in\{0,1\}) & \text{if } m=2n-1,\\ \displaystyle\prod_{\tau\in\mathcal{P}_m(k)} V_\tau^{y_\tau} * \prod_{\varphi\in\mathcal{R}_m(k)} V_\varphi^{z_\varphi} & \text{if } m=2n.\end{cases}$$

Note that the C_m^a-equivalences which are used in the above can be realized by surgery along simple C_m^a-trees with $r_k\ge 2$. By Lemmas 3.3, 5.2, and 5.4, for $\eta\in\mathcal{P}_m(k)\cup\mathcal{R}_m(k)$, we have that

$$\mu_\eta(L*L')=\begin{cases}\mu_\eta(L)+y_\eta & \text{if } \eta\in\mathcal{P}_m(k),\\ \mu_\eta(L)+2z_\eta & \text{if } m=2n \text{ and } \eta\in\mathcal{R}_m(k).\end{cases}$$

Since Milnor invariants of L with length $\le m-1$ and $r\le 2$ vanish, by Lemma 3.4, those of L_T also vanish. By combining this, the fact that $\mathrm{cl}(L_T)$ and $\mathrm{cl}(L*L')$ are C_m^a-equivalent, and Lemma 3.4, we have that

$$\mu_\eta(L_T)=\overline{\mu}_\eta(\mathrm{cl}(L_T))=\overline{\mu}_\eta(\mathrm{cl}(L*L'))=\mu_\eta(L*L').$$

This completes the proof. □

The following is the main result in this section.

Theorem 5.7. *Let L be an n-component Brunnian link. If $\overline{\mu}_L(I)=0$ for any I with $|I|\leq 2n-1$ and $r(I)\leq 2$, then L is self Δ-equivalent to the closure of $L'*L''$, where*

$$L'=\prod_{\varphi\in\mathcal{R}_{2n-1}(n)} U_\varphi^{\varepsilon(\varphi)},\ L''=\prod_{\tau\in\mathcal{R}_{2n}(n)} V_\tau^{(\overline{\mu}_\tau(L)-\overline{\mu}_\tau(L'))/2}*\prod_{\eta\in\mathcal{P}_{2n}(n)} V_\eta^{\overline{\mu}_\eta(L)},$$

and

$$\varepsilon(\varphi)=\begin{cases}1 & \text{if } \overline{\mu}_\tau(L) \text{ is odd for } \tau\in\mathcal{R}_{2n}(n) \text{ with } \tau(i)=\varphi(i)\ (i=1,...,n-1),\\ 0 & \text{if } \overline{\mu}_\tau(L) \text{ is even for } \tau\in\mathcal{R}_{2n}(n) \text{ with } \tau(i)=\varphi(i)\ (i=1,...,n-1).\end{cases}$$

Here U_φ is the string link in Lemma 5.4 *(2).*

Note that, in the theorem above, $L'*L''$ is determined by Milnor invariants of L with length $2n$ and $r=2$.

Proof. By Proposition 2.8, L is obtained from the n-component trivial link O by surgery along simple C_{n-1}^a-trees $T_1,...,T_l$. Hence we have

$$L=\mathrm{cl}((\mathbf{1}_n)_{T_1\cup T_2\cup\cdots\cup T_l}).$$

By Lemmas 4.2, 2.1, 2.2 and 2.4, we have that

$$L\overset{C_n^a}{\sim}\mathrm{cl}(\prod_{\pi\in\mathcal{F}_n} V_\pi^{\overline{\mu}_\pi(L)}).$$

Since $\overline{\mu}_\pi(L)=0$ for any $\pi\in\mathcal{F}_n$, L is C_n^a-equivalent to O; i.e., L is obtained from O by surgery along simple C_n^a-trees. By Lemmas 5.6 (1), 2.1, 2.2 and 2.4, we have that

$$L\overset{C_{n+1}^a}{\sim}\mathrm{cl}(\prod_{1\leq k\leq n}(\prod_{\tau\in\mathcal{P}_{n+1}(k)} V_\tau^{\overline{\mu}_\tau(L)})).$$

Since $\overline{\mu}_\tau(L)=0$ for any $\tau\in\mathcal{P}_{n+1}(k)$ $(k=1,...,n)$, L is C_{n+1}^a-equivalent to O. Note that a simple C_m^a-tree $(m\geq n+1)$ for an n-component link is a simple $C_m^{(2)}$-tree and might be a $C_m^{(3)}$-tree. By Proposition 2.11, surgery along a $C_m^{(3)}$-tree is achieved by surgery alomg C_2^s-trees. By Lemmas 5.6 (1), 2.1, 2.2 and 2.4, we have that

$$L\overset{C_2^s+C_{n+2}^a}{\sim}\mathrm{cl}(\prod_{2\leq k\leq n}(\prod_{\tau\in\mathcal{P}_{n+2}(k)} V_\tau^{\overline{\mu}_\tau(L)})).$$

Since $\overline{\mu}_\tau(L)=0$ for any $\tau\in\mathcal{P}_{n+2}(k)$ $(k=2,...,n)$, by repeating this step, then by similar arguments and Lemma 5.1 (2), we have that

$$L\overset{C_2^s+C_{2n-1}^a}{\sim}\mathrm{cl}(\prod_{\varphi\in\mathcal{R}_{2n-1}(n)} V_\varphi^{\varepsilon(\varphi)}*\prod_{\phi\in\mathcal{R}_{2n-1}(n-1)} V_\phi^{\varepsilon(\phi)})$$

for some $\varepsilon(\varphi)$'s and $\varepsilon(\phi)$'s in $\{0,1\}$.

In the proof of Lemma 5.4 (1), we showed that, for $\phi\in\mathcal{R}_{2n-1}(n-1)$,

$$\mathrm{cl}(V_\phi)\overset{C_{2n-1}}{\sim}C(\alpha,\alpha),$$

where $\alpha=n\phi(n-2)\cdots\phi(1)(n-1)$. Since the Whitehead link $C(12,12)$ is deformed into a trivial link by a single self-crossing change in the first component, $C(\alpha,\alpha)$ is also deformed into a trivial link by a single self-crossing change in the nth component. So $C(\alpha,\alpha)$ is obtained from a trivial link by surgery along a simple C_1^s-tree T

with $r_n(T)=2$. By Lemma 2.9, $C(\alpha,\alpha)$ is obtained from a trivial link by surgery along simple C_n^a-trees with $r_n=2$. Since the Milnor invariants of $C(\alpha,\alpha)$ with length $\leq 2n-1$ vanish, by similar arguments as above and Lemmas 5.6 (for $k=n$, $m\leq 2n-1$), 2.1, 2.2, 2.4 and 5.1 (2), we have that

$$C(\alpha,\alpha) \overset{C_2^s+C_{2n-1}^a}{\sim} \mathrm{cl}(\prod_{\varphi\in\mathcal{R}_{2n-1}(n)} V_\varphi^{\varepsilon'(\varphi)})$$

for some $\varepsilon'(\varphi)$'s in $\{0,1\}$.

Since

$$\prod_{\varphi\in\mathcal{R}_{2n-1}(n)} V_\varphi^{\varepsilon(\varphi)} * \prod_{\phi\in\mathcal{R}_{2n-1}(n-1)} V_\phi^{\varepsilon(\phi)}$$

is obtained from $\mathbf{1}_n$ by surgery along simple C_{2n-2}^a-trees, by Lemmas 2.1, 2.2, and 5.1 (2), L is $(C_2^s+C_{2n-1})$-equivalent to the closure of a product of some V_φ's $(\varphi\in\mathcal{R}_{2n-1}(n))$. By Lemmas 2.4, 5.1 (2) and the proof of Lemma 5.4 (2), L is $(C_2^s+C_{2n-1})$-equivalent to the closure of

$$L'=\prod_{\varphi\in\mathcal{R}_{2n-1}(n)} U_\varphi^{\varepsilon''(\varphi)},$$

where U_φ is the string link in Lemma 5.4 (2), and $\varepsilon''(\varphi)$'s are integers in $\{0,1\}$.

Note that a simple C_{2n-1}-tree for an n-component link is either a C_{2n-1}^a-tree or a $C_{2n-1}^{(3)}$-tree; hence by Proposition 2.11, C_{2n-1}-equivalence implies $(C_2^s+C_{2n-1}^a)$-equivalence. So L is self Δ-equivalent to a link obtained from $\mathrm{cl}(L')$ by surgery along simple C_{2n-1}^a-trees. By Lemmas 5.6 (1) (for $m=2n$), 2.1, 2.2 and 2.4, L is $(C_2^s+C_{2n}^a)$-equivalent to $\mathrm{cl}(L'*L'')$, where

$$L''=\prod_{\tau\in\mathcal{R}_{2n}(n)} V_\tau^{(\mu_\tau(L)-\mu_\tau(L'))/2} * \prod_{\eta\in\mathcal{P}_{2n}(n)} V_\eta^{\mu_\eta(L)}.$$

Since a C_{2n}-tree is a $C_{2n}^{(3)}$-tree, by Proposition 2.11, L is self Δ-equivalent to the closure of $L'*L''$. By Lemmas 3.3, 3.4 and 5.4, we have that for any $\tau\in\mathcal{R}_{2n-1}(n)$,

$$\overline{\mu}_\tau(L)\equiv\varepsilon''(\varphi) \pmod 2,$$

where $\varphi(i)=\tau(i)$ $(i=1,...,n-1)$. This completes the proof. □

By Theorem 5.7, we have the following two corollaries. Corollaries 5.8 and 5.9 are special cases of Theorem 1.3 and Corollary 1.5, respectively. Since these corollaries are needed to show Theorem 1.3, we give the statements.

Corollary 5.8. *Let L and L' be n-component Brunnian links. Suppose that $\overline{\mu}_L(I)=\overline{\mu}_{L'}(I)=0$ for any I with $|I|\leq 2n-1$ and $r(I)\leq 2$. Then L and L' are self Δ-equivalent if and only if $\overline{\mu}_L(J)=\overline{\mu}_{L'}(J)$ for any J with $|J|=2n$ and $r(J)=2$.*

Corollary 5.9. *A Brunnian link L is self Δ-equivalent to a trivial link if and only if $\overline{\mu}_L(I)=0$ for any I with $r(I)\leq 2$.*

6. Links whose Milnor invariants vanish

Before proving Theorem 1.3, we need some preparations.

Let $L=K_1\cup\cdots\cup K_n$ be an n-component link and b a band attaching a single component K_i in a way which is compatible with the orientations, i.e., $b\cap L=K_i\cap b\subset\partial b$ consists of two arcs whose orientations from K_i are opposite to those from ∂b. Then the $(n+1)$-component link $L'=(L\cup\partial b)\setminus \mathrm{int}(b\cap K_i)$ is called a link obtained from L by *fission* (along a band b), and conversely L is called a link obtained from L' by *fusion* [14]. For example, a *ribbon link* is a link obtained from a trivial link by a finite sequence of fusions.

Let $L' = K_{11} \cup \cdots \cup K_{1l_1} \cup \cdots \cup K_{n1} \cup \cdots \cup K_{nl_n}$ be a link obtained from an n-component link $K_1 \cup \cdots \cup K_n$ by a finite sequence of fissions, where $K_{i1} \cup \cdots \cup K_{il_i}$ is obtained from K_i $(i = 1, ..., n)$. We assign a color $c(K_{ij})$ to K_{ij} as $c(K_{ij}) = i$. In this section, for a C_k-tree T, we call T a *C_k^s-tree* (resp. *C_k^d-tree*) if $|\{c(K_{ij}) \mid T \cap K_{ij} \neq \emptyset\}| = 1$ (resp. $= k+1$). A *C_k^s-move* (resp. *C_k^d-move*) is a local move defined by surgery along a simple C_k^s-tree (resp. C_k^d-tree).

Lemma 6.1. *If an n-component link is deformed into a trivial link by a finite sequence of fissions, C_2^s-moves and C_{n-1}^d-moves, then it is self Δ-equivalent to a Brunnian link.*

Proof. Note that L is obtained from a trivial link by a finite sequence of fusions, C_2^s-moves and C_{n-1}^d-moves. By arguments similar to those in the proof of [11, Proposition 3.22], we may assume that the bands of fusion, C_2^s-trees and C_{n-1}^d-trees are mutually disjoint. So there exist an n-component ribbon link L_0 and a disjoint union $F \cup F'$ of simple C_2^s-trees and C_{n-1}^d-trees such that $L = L_{0F\cup F'}$, where F (resp. F') is a disjoint union of C_2^s-trees (resp. C_{n-1}^d-trees). Since ribbon links are self Δ-equivalent to a trivial link [25], L_0 is self Δ-equivalent to the n-component trivial link O. Hence

$$L \overset{C_2^s + C_{n-1}^d}{\sim} O.$$

This implies that L is self Δ-equivalent to a link obtained from O by surgery along simple C_{n-1}^d-trees. Since a C_{n-1}^d-tree is a C_{n-1}^a-tree, by Proposition 2.8, we have the conclusion. □

Theorem 6.2. *Let L be an n-component link such that $\overline{\mu}_L(I) = 0$ for any I with $|I| \leq 2n-2$ and $r(I) \leq 2$. Then L is self Δ-equivalent to a Brunnian link.*

Proof. Set $L = K_1 \cup \cdots \cup K_n$. By Lemma 6.1, it is enough to show that L is deformed into a trivial link by a finite sequence of fissions, C_2^s-moves and C_{n-1}^d-moves.

Since any knot is Δ-equivalent to an unknot [21], we may assume that every component of L is trivial.

Suppose that any k-component sublink of L is Brunnian $(2 \leq k \leq n-1)$. Since surgery along a simple C_{k-1}^d-tree with index $\{i_1, ..., i_k\}$ does not change the link type of a sublink $K_{j_1} \cup \cdots \cup K_{j_k}$ of L for $\{j_1, ..., j_k\} \neq \{i_1, ..., i_k\}$, by Proposition 2.8, L is C_{k-1}^d-equivalent to a link L' whose k-component sublinks are all trivial. Let L_0 be an n-component string link with $\mathrm{cl}(L_0) = L'$, and set $\mathbf{1}_n = \gamma_1 \cup \cdots \cup \gamma_n$. Let $\{S_1, ..., S_m\}$ $(m = \binom{n}{k})$ be the set of subsets of $\{1, ..., n\}$ with $|S_i| = k$ $(i = 1, ..., m)$. By Lemmas 2.1 and 2.2,

$$L \overset{C_k}{\sim} \mathrm{cl}(L_0 * L_1 * \cdots * L_m),$$

where, for each $i = 1, ..., m$, $L_i = U_i \cup L_{i1}$ is a split union of $U_i = \mathbf{1}_n \setminus \bigcup_{j \in S_i} \gamma_j$ and L_{i1} with $L_{i1} = (\bigcup_{j \in S_i} \gamma_j)_{T_i}$ for some union T_i of simple C_{k-1}^d-trees with index S_i, and the C_k-equivalence is realized by surgery along simple C_k-trees with $|\mathrm{index}| \geq k$. This implies that L is C_k^d-equivalent to a link obtained from $\mathrm{cl}(L_0 * L_1 * \cdots * L_m)$ by surgery along simple C_k-trees with $|\mathrm{index}| = k$.

By Lemmas 2.1 and 2.2,

$$L \overset{C_k^d + C_{k+1}}{\sim} \mathrm{cl}(L_0 * L_1^1 * \cdots * L_m^1),$$

where, for each $i = 1, ..., m$, $L_i^1 = U_i \cup L_{i1}^1$ is a split union of U_i and L_{i1}^1 with $L_{i1}^1 = (\bigcup_{j \in S_i} \gamma_j)_{T_i^1}$ for some union T_i^1 of simple C_{k-1}^d-trees and C_k-trees with index S_i, and the C_{k+1}-equivalence is realized by surgery along simple C_{k+1}-trees with $|\mathrm{index}| \geq k$. By Proposition 2.10, surgery along a simple C_{k+1}-tree with $|\mathrm{index}| \geq k+1$ is realized by C_k^d-equivalence. Therefore, L is C_k^d-equivalent to a link obtained from $\mathrm{cl}(L_0 * L_1^1 * \cdots * L_m^1)$ by surgery along simple C_{k+1}-trees with $|\mathrm{index}| = k$.

By Lemmas 2.1 and 2.2,

$$L \overset{C_k^d + C_{k+2}}{\sim} \mathrm{cl}(L_0 * L_1^2 * \cdots * L_m^2),$$

where, for each $i = 1, ..., m$, $L_i^2 = U_i \cup L_{i1}^2$ is a split union of U_i and L_{i1}^2 with $L_{i1}^2 = (\bigcup_{j \in S_i} \gamma_j)_{T_i^2}$ for some union T_i^2 of simple C_{k-1}^d-trees, C_k-trees, and C_{k+1}-trees with index S_i, and the C_{k+2}-equivalence is realized by surgery along simple C_{k+2}-trees with $|\mathrm{index}| \geq k$.

By repeating this procedure, we have that

$$L \overset{C_k^d + C_{2k}}{\sim} \mathrm{cl}(L_0 * L_1^k * \cdots * L_m^k),$$

where, for each $i = 1, ..., m$, $L_i^k = U_i \cup L_{i1}^k$ is a split union of U_i and L_{i1}^k with $L_{i1}^k = (\bigcup_{j \in S_i} \gamma_j)_{T_i^k}$ for some union T_i^k of simple C_{k-1}^d-trees, C_k-trees,..., and C_{2k-1}-trees with index S_i. Note that a simple C_{2k}-tree is either a C_{2k}-tree with index $\geq k+1$ or a $C_{2k}^{(3)}$-tree. By Propositions 2.10 and 2.11,

$$L \overset{C_k^d + C_2^s}{\sim} \mathrm{cl}(L_0 * L_1^k * \cdots * L_m^k).$$

So L is self Δ-equivalent to a link obtained from $\mathrm{cl}(L_0 * L_1^k * \cdots * L_m^k)$ by surgery along simple C_k^d-trees. By Lemmas 2.1 and 2.2,

$$L \overset{C_2^s + C_{k+1}}{\sim} \mathrm{cl}(M_1 * L_0 * L_1^k * \cdots * L_m^k),$$

where M_1 is a string link obtained from $\mathbf{1}_n$ by surgery along simple C_k^d-trees, and the C_{k+1}-equivalence is realized by surgery along simple C_{k+1}-trees with $|\mathrm{index}| \geq k+1$.

By Lemmas 2.1 and 2.2,

$$L \overset{C_2^s + C_{k+2}}{\sim} \mathrm{cl}(M_2 * L_0 * L_1^k * \cdots * L_m^k),$$

where M_2 is a string link obtained from $\mathbf{1}_n$ by surgery along simple C_k^d-trees and C_{k+1}-trees with $|\mathrm{index}| \geq k+1$, and the C_{k+2}-equivalence is realized by surgery along simple C_{k+2}-trees with $|\mathrm{index}| \geq k+1$.

By repeating this step, we have that

$$L \overset{C_2^s + C_{2n}}{\sim} \mathrm{cl}(M_{2n-k} * L_0 * L_1^k * \cdots * L_m^k),$$

where M_{2n-k} is a string link obtained from $\mathbf{1}_n$ by surgery along simple C_k^d-trees, C_{k+1}-trees,..., and C_{2n-1}-trees with $|\mathrm{index}| \geq k+1$. Since a simple C_{2n}-tree is a $C_{2n}^{(3)}$-tree, by Lemma 2.11,

$$L \overset{C_2^s}{\sim} \mathrm{cl}(M_{2n-k} * L_0 * L_1^k * \cdots * L_m^k).$$

Since $|S_i| = k$, the unions of the jth components ($j \in S_i$) of $\mathrm{cl}(M_{2n-k} * L_0 * L_1^k * \cdots * L_m^k)$ and of $\mathrm{cl}(L_0 * L_i^k)$ are equivalent. Let L_{0i} be the union of the jth components ($j \in S_i$) of L_0. Then $\mathrm{cl}(L_{0i} * L_{i1}^k)$ is deformed into a split sum of $\mathrm{cl}(L_{0i})$ and $\mathrm{cl}(L_{i1}^k)$ by a finite sequence of fissions. Since the k-component sublinks of $\mathrm{cl}(L_0)(= L')$ are all trivial, $\mathrm{cl}(L_{0i})$ is the trivial link. Therefore $\mathrm{cl}(L_{0i} * L_{i1}^k)$ is concordant to $\mathrm{cl}(L_{i1}^k)$. This implies that $\mu_{L_{i1}^k}(J) = \overline{\mu}_L(J) = 0$ for any multi-index J in S_i with $|J| \leq 2k$ $(\leq 2n-2)$ and $r(J) \leq 2$. By Corollary 5.9, $\mathrm{cl}(L_{i1}^k)$ is self Δ-equivalent to a trivial link. Since $\mathrm{cl}(M_{2n-k} * L_0 * L_1^k * \cdots * L_m^k)$ is deformed into a split sum of $\mathrm{cl}(M_{2n-k} * L_0)$ and $\mathrm{cl}(L_{11}^k)$,...,$\mathrm{cl}(L_{m1}^k)$ by a finite sequence of fissions, L is deformed into a split sum of $\mathrm{cl}(M_{2n-k} * L_0)$ and a trivial link by a finite sequence of fissions and C_2^s-moves. Note that any $(k+1)$-component sublink of $\mathrm{cl}(M_{2n-k} * L_0)$ is Brunnian.

By the induction, we have that L is deformed into a split sum of an n-component Brunnian link B and a trivial link by a finite sequence of fissions and self Δ-moves. By Proposition 2.8, B is C_{n-1}^d-equivalent to a trivial link. This completes the proof. □

By combining Corollary 5.8 and Theorem 6.2, we can prove Theorem 1.3.

Proof of Theorem 1.3. Let L be an n-component link with $\overline{\mu}_L(I) = 0$ for any I with $|I| \leq 2n-1$ and $r(I) \leq 2$. By Theorem 6.2, L is self Δ-equivalent to a Brunnian link B. Since $\overline{\mu}_B(I) = \overline{\mu}_L(I) = 0$ for any I with $|I| \leq 2n-1$ and $r(I) \leq 2$, by Corollary 5.8, B is determined by Milnor invariants with length $2n$ and $r = 2$. This completes the proof. □

The following theorem characterizes n-component links whose Milnor invariants of length $\leq 2n-1$ and $r \leq 2$ vanish.

Theorem 6.3. *For an n-component link L, $\overline{\mu}_L(I) = 0$ for any I with $|I| \leq 2n-1$ and $r(I) \leq 2$ if and only if, for each $i \in \{1, ..., n\}$, there is a Brunnian link L_i such that L_i is self Δ-equivalent to L and the ith component K of L_i is null-homotopic in $S^3 \setminus (L_i - K)$.*

Proof. For the 'only if' part, it is enough to consider the case when $i = n$. By Theorem 6.2, L is self Δ-equivalent to a Brunnian link. In the proof of Theorem 5.7, we see that a Brunnian link, whose Milnor invariants vanish for any I with $|I| \leq 2n-1$ and $r(I)$, is $(C_2^s + C_{2n-1})$-equivalent to the closure of a product of some V_τ's ($\tau \in \mathcal{R}_{2n-1}(n)$). Hence, by Proposition 2.11, Lemmas 5.4 (1) and 5.6, L is self Δ-equivalent to L_n of a product of some V_φ's ($\varphi \in \mathcal{R}_{2n-1}(n) \cup \mathcal{R}_{2n}(n) \cup \mathcal{P}_{2n}(n)$). Note that, for $\varphi \in \mathcal{R}_{2n-1}(n)$ (resp. $\varphi \in \mathcal{R}_{2n}(n) \cup \mathcal{P}_{2n}(n)$) V_φ is $C_{2n-1}^{(2)}$-equivalent (resp. $C_{2n}^{(2)}$-equivalent) to $\mathbf{1}_n$ and the $C_{2n-1}^{(2)}$-equivalence (resp. $C_{2n}^{(2)}$-equivalence) is realized by surgery along simple $C_{2n-1}^{(2)}$-trees (resp. $C_{2n}^{(2)}$-trees) with $r_n = 2$. By Proposition 2.11, L_n is self C_1-equivalent to the trivial link and the self C_1-equivalence is realized by surgery along simple C_1^s-trees with $r_n = 2$. Hence the nth component K of L_n is null-homotopic in $S^3 \setminus (L_n - K)$.

Now we will show the 'if' part. Let I be a multi-index with $|I| \leq 2n-1$ and $r(I) \leq 2$. Since L is self Δ-equivalent to a Brunnian link, if I does not contain an integer in $\{1, ..., n\}$, then $\overline{\mu}_L(I) = 0$. So we may suppose that I contains any integer in $\{1, ..., n\}$. The condition $|I| \leq 2n-1$ implies that there is an integer i such that i appears in I once. Let L_i be a Brunnian link such that L_i is self Δ-equivalent to

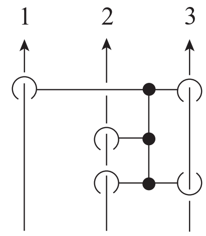

FIGURE 6.1

L and the ith component K of L_i is null-homotopic in $S^3 \setminus (L_i - K)$. This implies that $\overline{\mu}_{L_i}(Ji) = 0$ for any index J in $\{1, ..., n\} \setminus \{i\}$. Since $\overline{\mu}$ has 'cyclic symmetry' ([19, Theorem 8]), $\overline{\mu}_{L_i}(I) = 0$. This completes the proof. □

Example 6.4. Let V be the string link illustrated in Figure 6.1 and let L be the closure of V. By Proposition 2.11, for each i $(i = 2, 3)$, V is self C_1-equivalent to $\mathbf{1}_3$ and the self C_1-equivalence is realized by surgery along C_1^s-trees with index $\{i\}$. Hence the ith component K_i of L is null-homotopic in $S^3 \setminus (L - K_i)$ $(i = 2, 3)$. Suppose that the 1st component K_1 is null-homotopic in $S^3 \setminus (L - K_1)$. Then, by Theorem 6.3, $\overline{\mu}_L(I) = 0$ for any I with $|I| \leq 5$ and $r(I) \leq 2$. By Lemma 5.2, $\mu_V(12233) = 1$. Hence $\overline{\mu}_L(12233) = 1$. This is a contradiction.

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