

## A $c_0$ -SATURATED BANACH SPACE WITH NO LONG UNCONDITIONAL BASIC SEQUENCES

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ABSTRACT. We present a Banach space  $\mathfrak{X}$  with a Schauder basis of length  $\omega_1$  which is saturated by copies of  $c_0$  and such that for every closed decomposition of a closed subspace  $X = X_0 \oplus X_1$ , either  $X_0$  or  $X_1$  has to be separable. This can be considered as the non-separable counterpart of the notion of hereditarily indecomposable space. Indeed, the subspaces of  $\mathfrak{X}$  have “few operators” in the sense that every bounded operator  $T : X \rightarrow \mathfrak{X}$  from a subspace  $X$  of  $\mathfrak{X}$  into  $\mathfrak{X}$  is the sum of a multiple of the inclusion and a  $\omega_1$ -singular operator, i.e., an operator  $S$  which is not an isomorphism on any non-separable subspace of  $X$ . We also show that while  $\mathfrak{X}$  is not distortable (being  $c_0$ -saturated), it is arbitrarily  $\omega_1$ -distortable in the sense that for every  $\lambda > 1$  there is an equivalent norm  $\|\cdot\|$  on  $\mathfrak{X}$  such that for every non-separable subspace  $X$  of  $\mathfrak{X}$  there exist  $x, y \in S_X$  such that  $\|x\|/\|y\| \geq \lambda$ .

### 1. INTRODUCTION

The solutions of the unconditional basic sequence problem [8] and the distortion problem ([6], [11]) have an intricate connection. They both have profited on one side from the development of the Tsirelson-like constructions of conditional norms and on the other from the development of the infinite dimensional Ramsey theory. These connections are, however, well understood only at the level of separable spaces. This paper, which can be considered as a natural continuation of the paper [1], is an attempt to explain these connections in the non-separable context as well. As byproducts we discover some new non-separable as well as separable phenomena. Our results are all based on a method of constructing Banach spaces with long Schauder bases of length  $\omega_1$ , a method which among other things crucially uses the information about the way the classical Ramsey theorem [13] fails in the uncountable context [15].

Recall that an infinite dimensional Banach space  $X$  is *indecomposable* if for every closed decomposition  $X = X_0 \oplus X_1$  one of the subspaces  $X_0$  or  $X_1$  must be finite dimensional. The space  $X$  is *hereditary indecomposable*, HI in short, if every closed infinite dimensional subspace of  $E$  is indecomposable. The first example of such a space was constructed by W. T. Gowers and B. Maurey [8] as a byproduct of their solution of the unconditional basic sequence problem. Later, S. A. Argyros and A. Tolias [4] produced the first non-separable HI space, as the dual of a separable HI space. The paper [1] considers the unconditional basic sequence problem in

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the context of reflexive Banach spaces that are not necessarily separable. In particular [1] produces a reflexive Banach space  $\mathfrak{X}_{\omega_1}$  with a Schauder basis  $(e_\alpha)_{\alpha < \omega_1}$  of length  $\omega_1$  with no infinite unconditional basic sequence. Being reflexive and non-separable,  $\mathfrak{X}_{\omega_1}$  has to have non-trivial projections (explicitly here, for example  $\mathfrak{X}_{\omega_1} = \overline{\langle e_\alpha \rangle_{\alpha < \omega}} \oplus \overline{\langle e_\alpha \rangle_{\alpha \geq \omega}}$ ). Applying Gowers' dichotomy [7], one concludes that, while  $\mathfrak{X}_{\omega_1}$  is decomposable it is saturated by hereditarily indecomposable subspaces (HI-*saturated* in short); i.e., every closed infinite dimensional subspace of  $\mathfrak{X}_{\omega_1}$  contains an hereditarily indecomposable Banach space.

Motivated by a specific question of Argyros asking if it is possible to have a non-separable Banach space saturated with unconditional sequences but without uncountable ones, in this paper we present an example of a Banach space  $\mathfrak{X}$  with a Schauder basis of length  $\omega_1$  with some extreme discrepancies between properties of the class of separable subspaces of  $\mathfrak{X}$  and the class of non-separable subspaces of  $\mathfrak{X}$ . For example, at the separable level,  $\mathfrak{X}$  is *saturated* by copies of  $c_0$ ; i.e. every closed infinite dimensional subspace of  $\mathfrak{X}$  contains an isomorphic copy of  $c_0$ . So, in particular, every closed infinite dimensional subspace contains an infinite unconditional basic sequence. On the non-separable level,  $\mathfrak{X}$  contains no unconditional basic sequence of length  $\omega_1$ . More precisely, for every closed subspace  $X$  of  $\mathfrak{X}$  and every decomposition  $X = X_0 \oplus X_1$ , one of the spaces  $X_0$  or  $X_1$  must be separable. In fact every bounded operator  $T : X \rightarrow \mathfrak{X}$ , where  $X$  is a closed subspace of  $\mathfrak{X}$ , can be decomposed as

$$T = \lambda i_{X, \mathfrak{X}} + S,$$

where  $i_{X, \mathfrak{X}} : X \rightarrow \mathfrak{X}$  is the inclusion mapping and  $S$  is an  $\omega_1$ -*singular operator*, the non-separable counterpart of the notion of a *strictly singular operator*, which requires that  $S$  is not an isomorphism on any non-separable subspace of  $X$ .

Another discrepancy between the behavior of separable subspaces of  $\mathfrak{X}$  and non-separable ones (as well as a striking distinction between  $\mathfrak{X}$  and  $\mathfrak{X}_{\omega_1}$ ) comes when one considers the distortion constants of its equivalent norms. Recall that the *distortion* of an equivalent norm  $\|\cdot\|$  of a Banach space  $(X, \|\cdot\|)$  is the constant

$$d(X, \|\cdot\|) = \inf_Y \sup \left\{ \frac{\|x\|}{\|y\|} : x, y \in S_{(Y, \|\cdot\|)} \right\},$$

where the infimum is taken over all *infinite dimensional* subspaces  $Y$  of  $X$ . One says that  $(X, \|\cdot\|)$  is *arbitrarily distortable* if

$$\sup_{\|\cdot\|} d(X, \|\cdot\|) = \infty,$$

where the supremum is taken over all equivalent norms  $\|\cdot\|$  of  $(X, \|\cdot\|)$ . The analysis of our space  $\mathfrak{X}$  suggests the following variation of this notion of distortion. Given an equivalent norm  $\|\cdot\|$  of a Banach space  $(X, \|\cdot\|)$ , let

$$d_{\omega_1}(X, \|\cdot\|) = \inf_Y \sup \left\{ \frac{\|x\|}{\|y\|} : x, y \in S_{(Y, \|\cdot\|)} \right\},$$

where the infimum now is taken over all *non-separable* subspaces  $Y$  of  $X$ . We say that  $(X, \|\cdot\|)$  is *arbitrarily  $\omega_1$ -distortable* when

$$\sup_{\|\cdot\|} d_{\omega_1}(X, \|\cdot\|) = \infty,$$

where again the supremum is taken over all equivalent norms  $\|\cdot\|$  of  $(X, \|\cdot\|)$ . It has been shown in [1] that the space  $\mathfrak{X}_{\omega_1}$  is arbitrarily distortable. Note, however,

that our space  $\mathfrak{X}$  is *not* distortable at all, i.e.  $d(\mathfrak{X}, \|\cdot\|) = 1$  for every equivalent norm  $\|\cdot\|$  on  $\mathfrak{X}$ . This is a consequence of the fact that  $\mathfrak{X}$  is  $c_0$ -saturated and the well-known result of R. C. James [9] which states that if a Banach space contains isomorphic copies of  $c_0$ , then it contains almost isometric copies of  $c_0$ . Nevertheless, it turns out that the space  $\mathfrak{X}$  is distortable in the non-separable sense, i.e.,  $\mathfrak{X}$  is arbitrarily  $\omega_1$ -distortable. It follows that, while the arbitrary distortion of a Banach space  $X$  implies its arbitrary  $\omega_1$ -distortion, the converse implication is not true.

2. DEFINITION OF THE SPACE  $\mathfrak{X}$

The construction of the space  $\mathfrak{X}$  relies on the construction of the Banach space  $\mathfrak{X}_{\omega_1}$  from [1], so, in order to be as self-contained as possible, we will recall the main notions taken from there.

We start by giving some hints on the construction. We build a norming set  $K \subseteq c_{00}(\omega_1)$  to define the norm on  $c_{00}(\omega_1)$  by declaring that  $\|x\| = \sup\{\langle f, x \rangle : f \in K\}$ , and then taking the completion of it.  $K$  is based on the construction of the Banach space  $\mathfrak{X}_{\omega_1}$  from [1], which in turn relies on the separable Gowers-Maurey construction [8]. We explicitly mention that this example is a separable Banach space, because its construction uses a coding based on the countable enumeration of the finite sequences of vectors of  $c_{00}(\mathbb{N})$  with rational coordinates. Obviously, this is not going to work in the non-separable context. This obstacle is solved by “coding” in a controlled way vectors from  $c_{00}(\omega_1)$  by vectors of  $c_{00}(\mathbb{N})$ . If one tries to do this with a mapping  $f : \omega_1 \rightarrow \omega$ , one immediately realizes that this will not work since any such  $f$  will be constant in an uncountable set, and this spoils the coding. However the situation is quite different if one starts from a particular 2 dimensional mapping  $f : [\omega_1]^2 \rightarrow \omega$ .

**2.1.  $\varrho$ -function and separated sequences.** We recall the semi-distance distance on  $\omega_1$ . The reader is referred to [16] for full discussion of this notion and construction of various  $\varrho$ -functions.

**Definition 2.1.** By a  $\varrho$ -function we mean a function  $\varrho : [\omega_1]^2 \rightarrow \omega$  with the following properties:

- (0)  $\varrho(\alpha, \alpha) = 0$  for every  $\alpha < \omega_1$ .
- (i)  $\varrho(\alpha, \gamma) \leq \max\{\varrho(\alpha, \beta), \varrho(\beta, \gamma)\}$  for all  $\alpha < \beta < \gamma < \omega_1$ .
- (ii)  $\varrho(\alpha, \beta) \leq \max\{\varrho(\alpha, \gamma), \varrho(\beta, \gamma)\}$  for all  $\alpha < \beta < \gamma < \omega_1$ .
- (iii)  $\{\alpha < \beta : \varrho(\alpha, \beta) \leq n\}$  is finite for all  $\beta < \omega_1$  and  $n \in \mathbb{N}$ .

**Definition 2.2.** Given a finite set  $F \subseteq \omega_1$ , let

$$p_F = \max\{\varrho(\alpha, \beta) : \alpha, \beta \in F\}.$$

For a finite set  $F \subseteq \omega_1$  and an integer  $p \in \mathbb{N}$ , let

$$\overline{F}^p = \{\alpha \leq \max F : \text{there is some } \beta \in F \text{ with } \alpha \leq \beta \text{ and } \varrho(\alpha, \beta) \leq p\}.$$

We list some properties of the operator  $F \mapsto \overline{F}^p$  which are not difficult to prove (for more details, see [1, Remark 2.7 and Proposition 2.8]).

- Proposition 2.3.**
- (a)  $\overline{F}^p$  is a finite set.
  - (b)  $F \mapsto \overline{F}^p$  is an idempotent operator.
  - (c) Suppose that  $p \geq p_F, p_G$ . Then  $\overline{F \cap G}^p = \overline{F}^p \cap \overline{G}^p$ . Therefore if  $F$  and  $G$  are  $p$ -closed, then  $F \cap G$  is also  $p$ -closed and it is an initial part of both  $F$  and  $G$ .

Since  $\varrho$  has some properties reminiscent of those of an ultrametric it can be used to measure “distances” between various subsets of  $\omega_1$ . This motivates the terminology used in the following definition.

**Definition 2.4.** For two subsets  $s$  and  $t$  of  $\omega_1$ , set

$$\varrho(s, t) = \min\{\varrho(\alpha, \beta) : \alpha \in s, \beta \in t\}.$$

Given an integer  $p$  we say that  $s$  and  $t$  are  $p$ -separated if  $\varrho(s, t) \geq p$ . A sequence  $(s_i)_i$  is  $p$ -separated if it is pairwise  $p$ -separated, i.e.  $s_i$  and  $s_j$  are  $p$ -separated for every  $i \neq j$ . The sequence  $(s_i)_i$  is separated if it is  $|\bigcup_i s_i|$ -separated.

Every notion we introduce here for sets of ordinals can be naturally transferred, via their supports, to vectors of  $c_{00}(\omega_1)$ .

The following simple, but useful statement is about the existence of separated subsequences of any sufficiently long sequence of finite sets. This will be used later to show that the norm of the space  $\mathfrak{X}$  we build in the present paper keeps a substantial conditional structure when restricted on an arbitrary non-separable subspace of  $\mathfrak{X}$ . Recall that a sequence  $(s_i)_{i \in I}$  of sets of ordinals indexed in a totally ordered set  $I$  is called a *block sequence* if  $i < j$  in  $I$  implies that  $s_i < s_j$ , i.e.  $\max s_i < \min s_j$ . This notion is extended in the natural way to sequences of vectors of  $c_{00}(\omega_1)$  via their supports.

**Proposition 2.5.** *Let  $(A_i)_{i < n}$  be a block sequence of subsets of  $\omega_1$ , each of them of order-type  $\omega$ . Then for every block sequence  $(s_\alpha)_{\alpha \in \bigcup_{i < n} A_i}$  of finite sets of countable ordinals and every integer  $p$ , there are  $\alpha_i \in A_i$  ( $i < n$ ) such that  $(s_{\alpha_i})_{i < n}$  is  $p$ -separated.*

*Proof.* This is done by induction on  $n$ . Fix all data as in the statement for  $n > 1$ . Then let  $\alpha_{n-1} = \min A_{n-1}$ . Since the set

$$\overline{s_{\alpha_{n-1}}}^p = \{\beta < \omega_1 : \text{there is some } \gamma \in s_{\alpha_{n-1}} \text{ with } \beta \leq \gamma \text{ and } \varrho(\beta, \gamma) \leq p\}$$

is, by property (iii) of  $\varrho$ , finite, one easily obtains infinite subsets  $B_i$  of  $A_i$  ( $i < n-1$ ) such that  $s_\alpha \cap \overline{s_{\alpha_{n-1}}}^p = \emptyset$  for every  $\alpha \in B_i$  and all  $i < n-1$ . By inductive hypothesis, there are  $\alpha_i \in B_i$  ( $i < n-1$ ) such that  $(s_{\alpha_i})_{i < n-1}$  is  $p$ -separated. Obviously  $(s_{\alpha_i})_{i < n}$  is the desired  $p$ -separated sequence.  $\square$

**Corollary 2.6.** *Let  $n$  be an integer and let  $(s_\alpha^i)_{\alpha < \omega_1}$  be a block sequence of finite sets of countable ordinals for every  $i < n$ . Then there are  $\alpha_0 < \dots < \alpha_{n-1}$  such that  $(s_{\alpha_i}^i)_{i < n}$  is a separated block sequence.*

*Proof.* Let  $A$  be an uncountable set such that  $|s_\alpha^i| = p_i$  for every  $i \in A$  and every  $i < n$ . Now for each  $i < n$ , let  $A_i \subseteq A$  be of order type  $\omega$  and such that  $A_i < A_j$  and  $s_\alpha^i < s_\beta^j$  if  $i < j < n$  and  $\alpha \in A_i, \beta \in A_j$  are such that  $\alpha < \beta$ . Then apply the previous proposition to  $(t_\alpha)_{\alpha \in A_0 \cup \dots \cup A_{n-1}}$  and  $\sum_{i < n} p_i$ , where  $t_\alpha = s_\alpha^i$  for the unique  $i < n$  such that  $\alpha \in A_i$ .  $\square$

**2.2. Special sequences.** We fix a  $\varrho$ -function. Let us denote by  $\mathbb{Q}_s(\omega_1)$  the set of finite sequences

$$(\phi_1, w_1, p_1, \phi_2, w_2, p_2, \dots, \phi_d, w_d, p_d)$$

such that

- (S1) for all  $i \leq d$ ,  $\phi_i \in c_{00}(\omega_1)$  has rational coordinates and  $\phi_1 < \phi_2 < \dots < \phi_d$ ,
- (S2)  $(w_i)_{i=1}^d, (p_i)_{i=1}^d \in \mathbb{N}^d$  are strictly increasing, and
- (S3)  $p_i \geq p(\cup_{k=1}^i \text{supp } \phi_k)$  for every  $i \leq d$ .

Let  $\mathbb{Q}_s$  be the set of finite sequences  $(\phi_1, w_1, p_1, \phi_2, w_2, p_2, \dots, \phi_d, w_d, p_d)$  satisfying (S1) and (S2) above and in addition for every  $i \leq d$ ,  $\phi_i \in c_{00}(\mathbb{N})$ . Notice that  $\mathbb{Q}_s$  is a countable set. Fix a one-to-one function  $\sigma : \mathbb{Q}_s \rightarrow \{2j : j \text{ odd}\}$  such that

$$\sigma(\phi_1, w_1, p_1, \phi_2, w_2, p_2, \dots, \phi_d, w_d, p_d) > \max\{p_d^2, \frac{1}{\varepsilon^2}, \max \text{supp } \phi_d\},$$

where  $\varepsilon = \min\{|\phi_k(e_\alpha)| : \alpha \in \text{supp } \phi_k, k = 1, \dots, d\}$ . Given a finite subset  $F$  of  $\omega_1$ , we denote by  $\pi_F : \{1, 2, \dots, \#F\} \rightarrow F$  the natural order preserving map. Given  $\Phi = (\phi_1, w_1, p_1, \phi_2, w_2, p_2, \dots, \phi_d, w_d, p_d) \in \mathbb{Q}_s(\omega_1)$  we set  $G_\Phi = \overline{\cup_{i=1}^d \text{supp } \phi_i}^{p_d}$  and then we consider the family

$$\pi_{G_\Phi}(\Phi) = (\pi_G(\phi_1), w_1, p_1, \pi_G(\phi_2), w_2, p_2, \dots, \pi_G(\phi_d), w_d, p_d) \in \mathbb{Q}_s,$$

where

$$\pi_G(\phi_k)(n) = \begin{cases} \phi_k(\pi_{G_\Phi}(n)) & \text{if } n \in G_\Phi, \\ 0 & \text{otherwise.} \end{cases}$$

Finally,  $\sigma_\rho : \mathbb{Q}_s(\omega_1) \rightarrow \{2j : j \text{ odd}\}$  is defined as  $\sigma_\rho(\Phi) = \sigma(\pi_G(\Phi))$ .

Let  $(m_j)_j$  and  $(n_j)_j$  be the following two sequences of integers defined recursively by

$$m_1 = 2, \text{ and } m_{j+1} = m_j^4, n_1 = 4, \text{ and } n_{j+1} = (4n_j)^{s_j} \text{ where } s_j = \log_2 m_{j+1}^3.$$

Let  $W$  be the minimal subset of  $c_{00}(\omega_1)$  satisfying that:

- (1) It contains  $(e_\gamma^*)_{\gamma < \omega_1}$ , is symmetric (i.e.,  $\phi \in W$  implies  $-\phi \in W$ ) and is closed under the restriction on ordinal intervals of  $\omega_1$ .
- (2) For every  $j \in \mathbb{N}$  and every  $\{\phi_i : i = 1, \dots, n_j\} \subseteq W$  with  $\text{supp } \phi_1 < \dots < \text{supp } \phi_{n_j}$ , the functional  $\phi = (1/m_j) \sum_{i=1}^{n_j} \phi_i \in W$ .
- (3) It is rationally convex.

By minimality of  $W$ , every element  $\phi$  of  $W$  is either of the form  $\pm e_\alpha^*$ , or a rational convex combination, or a sum  $\phi = (1/m_j) \sum_{i=1}^{n_j} \phi_i$  as in (2). In this last case, we say that that  $\phi$  has *type I* and that its weight is  $w(\phi) = m_j$  (note that the weight is not unique).

**Definition 2.7.** A sequence  $\Phi = (\phi_1, \dots, \phi_{n_{2j+1}})$  of vectors of  $c_{00}(\omega_1)$  is said to be a *2j + 1-special sequence* if:

- (1)  $\text{supp } \phi_1 < \text{supp } \phi_2 < \dots < \text{supp } \phi_{n_{2j+1}}$ , each  $\phi_k \in W$  is of type I,  $w(\phi_k) = m_{2j_k}$  and  $w(\phi_1) = m_{2j_1}$  with  $j_1$  even and satisfying  $m_{2j_1} > n_{2j+1}^2$ .
- (2) There exists a strictly increasing sequence  $(p_1^\Phi, \dots, p_{n_{2j+1}-1}^\Phi)$  of natural numbers such that for all  $1 \leq i \leq n_{2j+1} - 1$  one has that  $w(\phi_{i+1}) = m_{\sigma_\rho(\Phi_i)}$ , where  $\Phi_i = (\phi_1, w(\phi_1), p_1^\Phi, \phi_2, w(\phi_2), p_2^\Phi, \dots, \phi_i, w(\phi_i), p_i^\Phi)$ .

As we have mentioned before, the weight of a type I element of  $W$  is not uniquely determined. However in the case of the elements  $\phi_i$  of a  $2j + 1$ -special sequence  $\Phi$ ,  $w(\phi_i)$  will denote the unique weight involved in the definition of the special sequence  $\Phi$ .

**2.3. The norming set  $K$  of  $\mathfrak{X}$ .** The space  $\mathfrak{X}$  is defined as the completion of  $(c_{00}, \|\cdot\|)$  under the norm  $\|\cdot\|$  induced by a set of functionals  $K \subseteq c_{00}(\omega_1)$ . For this we use the notions of special sequence from Definition 2.7 and of separated sequence from Definition 2.4.

**Definition 2.8.** Let  $K$  be the minimal subset of  $W$  satisfying the following conditions:

- (i) It contains  $(e_\gamma^*)_{\gamma < \omega_1}$ , is symmetric (i.e.,  $\phi \in K$  implies  $-\phi \in K$ ) and is closed under the restriction on intervals of  $\omega_1$ .
- (ii) For every *separated* block sequence  $(\phi_i)_{i=1}^d \subseteq K$  with  $d \leq n_{2j}$ , one has that the combination  $(1/m_{2j}) \sum_{i=1}^{n_{2j}} \phi_i \in K$ .
- (iii) For every separated *special* sequence  $(\phi_i)_{i=1}^d \subseteq K$  with  $d \leq n_{2j+1}$ , one has that  $\phi = (1/m_{2j+1}) \sum_{i=1}^{n_{2j+1}} \phi_i$  is in  $K$ .
- (iv) It is rationally convex.

Finally, the norm on  $c_{00}(\omega_1)$  is defined as

$$\|x\| = \sup\{\phi(x) = \sum_{\alpha} \phi(\alpha) \cdot x(\alpha) : \phi \in K\},$$

and  $\mathfrak{X}$  is the completion of  $(c_{00}(\omega_1), \|\cdot\|)$ .

Before we discuss the new notion of separated sequence, let us give a list of direct consequences from the definition of  $\mathfrak{X}$  as well as the role of each of the properties (i)–(iv) of  $K$ .

*Remark 2.9.* (a) It is clear by the definition that the norming set  $K$  presented here is a subset of the one introduced in [1, Subsection 2.2] for the Banach space  $\mathfrak{X}_{\omega_1} = (\mathfrak{X}_{\omega_1}, \|\cdot\|_{\omega_1})$ . So, it follows that for every  $x \in c_{00}(\omega_1)$  one has that  $\|x\| \leq \|x\|_{\omega_1}$ . This fact will be used frequently in this paper.

- (b) The property (i) makes the natural Hamel basis  $(e_\alpha)_{\alpha < \omega_1}$  of  $c_{00}(\omega_1)$  a *transfinite bimonotone Schauder basis* of  $\mathfrak{X}$ ; i.e.  $(e_\alpha)_{\alpha < \omega_1}$  is total, and for every interval  $I \subseteq \omega_1$  the corresponding projection  $P_I : \mathfrak{X} \rightarrow \overline{\langle e_\alpha \rangle_{\alpha \in I}}$  has norm 1. Let us set  $P_\gamma = P_{[0, \gamma]}$  and  $\mathfrak{X}_\gamma = \mathfrak{X}_{[0, \gamma]}$  for every countable ordinal  $\gamma$ . It follows that every closed infinite dimensional subspace contains a further subspace isomorphic to the closed linear span of a block sequence of the basis  $(e_\alpha)_\alpha$  (see [1, Proposition 1.3] for full details). This goes in contrast with the corresponding property of a Banach space with a Schauder basis  $(x_k)_{k \in \mathbb{N}}$  for which it is well-known that every closed infinite dimensional subspace contains almost isometric copies of the closed linear span of a certain block sequence.
- (c) The second property (ii) is responsible of the existence of semi-normalized averages in the span of every uncountable block sequence of  $\mathfrak{X}$ . The third (iii) and fourth (iv) properties make every operator from a closed subspace of  $\mathfrak{X}$  into  $\mathfrak{X}$  a multiple of the identity plus a  $\omega_1$ -singular operator.
- (d) The basis  $(e_\alpha)_{\alpha < \omega_1}$  is shrinking; i.e.  $(e_{\alpha_n})_n$  is shrinking in the usual sense for every increasing sequence  $(\alpha_n)_n$  of countable ordinals (the proof is essentially equal to that of the space  $\mathfrak{X}_{\omega_1}$  provided in [1, Proposition 4.13]). It follows that  $(e_\alpha)_{\alpha < \omega_1}$  is an uncountable weakly-null sequence; i.e. for every  $x^* \in \mathfrak{X}^*$  the numerical sequence  $(x^*(e_\alpha))_{\alpha < \omega_1} \in c_0(\omega_1)$ . This last property readily implies the following.

- (e) Suppose that  $T : \mathfrak{X} \rightarrow \mathfrak{X}$  is a bounded operator. Then for every uncountable subset  $A$  of  $\omega_1$  and every countable ordinal  $\gamma$  one has that  $P_\gamma(T(e_\alpha)) = 0$  for all but countably many  $\alpha \in A$ .

**2.4. Rapidly increasing sequences and deciding pairs.** We now introduce some standard technical tools in this field. Particularly, a type of vector, called  $\ell_1^k$ -averages, and the so-called rapidly increasing sequences (RIS for short). The importance of rapidly increasing sequences  $(x_n)$  is that it is possible to estimate the norm of linear combinations of  $(x_n)$  in terms of norms of linear combinations of the basis  $(e_\alpha)_{\alpha < \omega_1}$ . In this sense RIS behave like a subsequence  $(e_{\alpha_k})_k$  of basis  $(e_\alpha)_{\alpha < \omega_1}$ . The role of  $\ell_1^k$ -averages is that they are useful in creating RIS. We give the precise definitions now.

**Definition 2.10.** Let  $C, \varepsilon > 0$ . A normalized block sequence  $(x_k)_k$  of  $X$  is called a  $(C, \varepsilon)$ -rapidly increasing sequence ( $(C, \varepsilon)$ -RIS in short) iff there is an increasing sequence  $(j_k)_k$  of integers such that for all  $k$ ,

- (i)  $\|x_k\| \leq C$ ,
- (ii)  $|\text{supp } x_k| \leq m_{j_{k+1}} \varepsilon$  and
- (iii) for every functional  $\phi \in K$  of type I with  $w(\phi) < m_{j_k}$ , one has that  $|\phi(x_k)| \leq C/w(\phi)$ .

Let  $C > 0$  and  $k \in \mathbb{N}$ . A normalized vector  $y$  is called a  $C - \ell_1^k$ -average iff there is a finite block sequence  $(x_0, \dots, x_{k-1})$  such that  $y = (x_0 + \dots + x_{k-1})/k$  and  $\|x_i\| \leq C$ , or equivalently if  $x$  is the sum of a block sequence  $(x_i)_{i < k}$  of vectors  $x_i$  of norm at most  $Ck^{-1}$ .

Notice that any  $\omega$ -subsequence  $(e_{\alpha_n})_n$  of the basis is a  $(1, \varepsilon)$ -RIS for every  $\varepsilon$ . Note also that it follows easily from the definition that if  $(x_n)$  is a  $(C, \varepsilon)$ -RIS, then for every  $\varepsilon' > 0$  there is a subsequence  $(x_n)_{n \in A}$  of  $(x_n)$  which is a  $(C, \varepsilon')$ -RIS.

As the norming set  $K$  we defined here is not saturated by “free” combinations of the form  $(1/m_{2j}) \sum_{i < n_{2j}} f_i$ , one cannot expect that there are  $\ell_1^k$ -averages in the span of arbitrary block sequence. Indeed, as Theorem 3.2 shows, this is not the case for most of the block sequences, since clearly if  $(x_n)$  is  $C$ -equivalent to the  $c_0$ -basis, then  $\|(x_0 + \dots + x_k)/k\| \leq C/k$  for every  $k$ . However, the next proposition guarantees their existence in some cases. We extract its proof from [2] and [8].

**Proposition 2.11.** *For every  $k \in \mathbb{N}$  there is  $l = l(k) \in \mathbb{N}$  such that the following holds: Suppose that  $(x_i)_{i < l}$  is a normalized block sequence with the property that there exists a separated block sequence  $(\phi_i)_{i < l}$  in  $K$  which is biorthogonal to  $(x_i)_{i < l}$ . Then  $\langle x_i \rangle_{i < l}$  contains  $2\text{-}\ell_1^k$ -averages.*

*Proof.* Fix  $k$ . Let  $j$  be such that  $k \leq n_{2j-1}$ . Then the integer  $l = k^s$ , where  $s = \log_2(m_{2j}^3)$ , works: Fix  $(x_i)_{i=1}^l$  and  $(\phi_i)_{i=1}^l$  as in the hypothesis. By the choice of  $j$  and  $s$  it follows that  $k^s \leq (n_{2j-1})^s \leq n_{2j}$ . Also, the sequence  $(\phi_i)_{i=1}^l$  is separated, so by property (ii) of the norming set  $K$  the functional  $\phi = m_{2j}^{-1} \sum_{i=1}^l \phi_i$  is in  $K$ , and hence

$$(1) \quad \left\| \sum_{i=1}^l x_i \right\| \geq \phi \left( \sum_{i=1}^l x_i \right) = \frac{k^s}{m_{2j}}.$$

Now suppose that there are no  $2\text{-}\ell_1^k$ -averages in  $\langle x_i \rangle_{i < l}$ . For each  $0 \leq i \leq s$  and each  $1 \leq j \leq k^{s-i}$ , let  $x(i, j) = \sum_{r=(j-1)k^{i+1}}^{jk^i} x_r$ . It readily follows that  $x(0, j) = x_j$

and that for every  $i > 0$  one has that  $x(i, j) = \sum_{r=1}^k x(i-1, (j-1)k+r)$ . Since no  $x(i, j)$  is a  $2\text{-}\ell_1^k$ -average it follows that  $\|x(i, j)\| < \|x(i-1, (j-1)k+r)\|k/2$ , and so

$$(2) \quad \left\| \sum_{i=1}^l x_i \right\| = \|x(s, 1)\| < \frac{k^s}{2^s}.$$

Now (1) and (2) imply that  $m_{2j} > 2^s = m_{2j}^3$ , a contradiction. □

The following is also a well-known fact about  $\ell_1^k$ -averages, connecting them to RIS (see [8, Lemma 4]).

**Proposition 2.12.** *Suppose that  $y$  is a  $C\text{-}\ell_1^k$ -average and suppose that  $E_0 < \dots < E_{l-1}$  are intervals. Then*

$$\sum_{i=0}^{l-1} \|E_i y\| \leq C \left( 1 + \frac{2l}{k} \right).$$

As a consequence, if  $y$  is a  $C\text{-}\ell_1^{n_j}$ -average and  $\phi \in K$  is with  $w(\phi) < m_j$ , then  $|\phi(y)| \leq 3C/2w(\phi)$ . In particular, for  $2\text{-}\ell_1^{n_j}$ -averages we get that  $|\phi(y)| \leq 3/w(\phi)$  if  $w(\phi) < m_j$ . □

*Remark 2.13.* It follows that if  $\mathbf{x} = (x_k)_k$  is a normalized block sequence such that each  $x_k$  is a  $2\text{-}\ell_1^{n_{j_k}}$ -average with  $|\text{supp } x_k| \leq m_{j_{k+1}}$ , then  $\mathbf{x}$  is a  $(3, 1)$ -RIS. It readily follows that if  $(x_k)_k$  is a normalized block sequence having a separated biorthogonal pair, then the linear span of  $(x_k)_k$  contains a  $(3, \varepsilon)$ -RIS for every  $\varepsilon > 0$ .

Let  $<_{\text{antilex}}$  denote the anti-lexicographical ordering on  $\mathbb{N} \times \omega_1$ . Whenever we say that a sequence  $(x_k^\alpha)_{(k,\alpha) \in A}$  indexed on a subset  $A$  of  $\mathbb{N} \times \omega_1$  is a block sequence, we mean that  $x_k^\alpha$  is finitely supported and that  $x_{k_0}^{\alpha_0} < x_{k_1}^{\alpha_1}$  whenever  $(k_0, \alpha_0) <_{\text{antilex}} (k_1, \alpha_1)$ . In the following definitions and lemma, we introduce two genuinely non-separable tools. They are necessary for us because it not true in general that a non-separable subspace of  $\mathfrak{X}$  contains an almost isometric copy of the closed linear span of some uncountable block sequence. Recall that if  $X$  is a separable Banach space with a Schauder basis of length  $\omega$ , the corresponding result is true and very frequently used.

**Definition 2.14.** We call a double-indexed sequence  $\mathbf{x} = (x_k^\alpha)_{(k,\alpha) \in \mathbb{N} \times \omega_1}$  a *long rapidly increasing sequence (LRIS in short)* iff

- (i)  $x_{k_0}^{\alpha_0} < x_{k_1}^{\alpha_1}$  for every  $(k_0, \alpha_0) <_{\text{antilex}} (k_1, \alpha_1)$ , and
- (ii) the cardinality of  $\text{supp } x_k^\alpha$  only depends on  $k$ ,
- (iii) there is a sequence of integers  $(j_k)$  such that  $x_k^\alpha$  is a  $2\text{-}\ell_1^{n_{j_k}}$ -average and  $|\text{supp } x_k^\alpha| < m_{j_{k+1}}$  for every  $(k, \alpha)$ .

*Remark 2.15.* The name was chosen because it follows from the definition and Remark 2.13 that if  $f = (f_0, f_1) : \mathbb{N} \rightarrow \mathbb{N} \times \omega_1$  is such that

$$\text{if } k < l, \text{ then } f_0(k) < f_0(l) \text{ and } f_1(k) \leq f_1(l),$$

then the subsequence  $(x_{f_0(k)}^{f_1(k)})_{k \in \mathbb{N}}$  is a  $(3, 1)$ -RIS.



The following is an extension of the block order  $x < y$  on vectors of  $c_0(\omega_1)$ .

**Definition 2.16.** Given  $\varepsilon > 0$  and two vectors  $x, y \in c_0(\omega_1)$ , with  $x \neq 0$ , we write  $x <_\varepsilon y$  to denote that

$$\|P_{\text{sup supp } x}(y)\| < \varepsilon.$$

By technical reasons, we declare  $0 <_\varepsilon y$  for every  $y$ .

We recall that for a vector  $x \in c_0(\omega_1)$ , one sets  $\text{ran } x = [\min \text{supp } x, \text{sup supp } x]$ .

**Definition 2.17.** Given a bounded operator  $T : X \rightarrow \mathfrak{X}$ , where  $X$  is a closed non-separable subspace of  $\mathfrak{X}$ , we say that the couple  $(\mathbf{x}, \mathbf{y})$  is a *deciding pair* for  $T$  if the following holds:

- (i)  $\mathbf{x} = (x_k^\alpha)_{(k,\alpha) \in \mathbb{N} \times \omega_1} \subseteq X$  and  $\mathbf{y} = (y_k^\alpha)_{(k,\alpha) \in \mathbb{N} \times \omega_1}$  is a LRIS and  $\mathbf{x} \subseteq X$ .
- (ii)  $x_k^\alpha \in X$  and  $\text{ran } y_k^\alpha \subseteq \text{ran } x_k^\alpha$  for every pair  $(k, \alpha)$ .
- (iii)  $\sum_{k \in \mathbb{N}} \|x_k^\alpha - y_k^\alpha\| \leq 1$  for every  $\alpha$ .
- (iv)  $x_{k_0}^{\alpha_0} <_{\varepsilon_{k_1}} x_{k_1}^{\alpha_1}$  and  $T(x_{k_0}^{\alpha_0}) <_{\varepsilon_{k_1}} T(x_{k_1}^{\alpha_1})$  for every  $(k_0, \alpha_0) <_{\text{antilex}} (k_1, \alpha_1)$ .

A *transversal subsequence* of a double-indexed sequence  $(x_k^\alpha)_{(k,\alpha)}$  is a finite subsequence of the form  $(x_{k_i}^{\alpha_i})_i$ , where  $k_i < k_{i+1}$  and  $\alpha_i < \alpha_{i+1}$  for every  $i$ .

In other words, a deciding pair is nothing else but an uncountable ordered sequence of rapidly increasing sequences  $(y_k^\alpha)_{k \in \mathbb{N}}$  which are asymptotically closed to a sequence  $(x_k^\alpha)_{k \in \mathbb{N}}$  in  $X$  for which the corresponding sequence of images  $(Tx_k^\alpha)_{k \in \mathbb{N}}$  is “almost” block ordered.

Before we prove that deciding pairs always exist, we give some explanation of this notion.

**Proposition 2.18.** *Suppose that  $(\mathbf{x}, \mathbf{y})$  is a deciding pair for  $T$ . Then for every  $\varepsilon > 0$  and every integer  $l$  there is a transversal subsequence  $\mathbf{z} = (y_{k_i}^{\alpha_i})_{i < l}$  of  $\mathbf{y}$  such that*

- (i)  $\mathbf{z}$  is a  $(3, \varepsilon)$ -RIS.
- (ii)  $\mathbf{z}$  has a biorthogonal separated block sequence in the norming set  $K$ .
- (iii)  $\sum_{i < l} \|x_{k_i}^{\alpha_i} - y_{k_i}^{\alpha_i}\| \leq \varepsilon$ .

*Proof.* Fix all data. Let  $M \subseteq \mathbb{N}$  be such that for every  $\alpha < \omega_1$  one has that  $(y_k^\alpha)_{k \in M}$  is a  $(3, \varepsilon)$ -RIS and such that  $\sum_{k \in M} \|x_k^\alpha - y_k^\alpha\| \leq \varepsilon$ . Also fix for each pair  $(k, \alpha)$  a functional  $\phi_k^\alpha \in K$  such that  $\phi_k^\alpha(y_k^\alpha) = 1$  and with  $\text{ran } \phi_k^\alpha \subseteq \text{ran } y_k^\alpha$ . Now apply Corollary 2.6 to  $(\text{supp } \phi_k^\alpha)_{\alpha < \omega_1}$  ( $k \in M$ ) to find a transversal subsequence  $(\phi_{k_i}^{\alpha_i})_{i < l}$ . Then  $(y_{k_i}^{\alpha_i})_{i < l}$  is the desired sequence.  $\square$

**Lemma 2.19.** *Every bounded operator  $T : X \rightarrow \mathfrak{X}$  with  $X$  non-separable has a deciding pair.*

*Proof.* First we make the following approximation to the final result.

*Claim 1.* For every integer  $k$  and every  $\varepsilon > 0$  there are two normalized sequences  $(x_\alpha)_{\alpha < \omega_1}$  and  $(y_\alpha)_{\alpha < \omega_1}$  such that

- (a)  $x_\alpha \in X$  ( $\alpha < \omega_1$ ).
- (b)  $\text{ran } y_\alpha \subseteq \text{ran } x_\alpha$  and  $\|x_\alpha - y_\alpha\| \leq \varepsilon$  ( $\alpha < \omega_1$ ).
- (c)  $y_\alpha$  is a  $2 - \ell_1^k$ -average and  $|\text{supp } y_\alpha|$  is independent of  $\alpha < \omega_1$ .
- (d)  $x_\alpha <_\varepsilon x_\beta$  and  $Tx_\alpha <_\varepsilon Ty_\alpha$  for every  $\alpha < \beta$ .

Let us show the desired result from this claim: Find recursively for each  $k \in \mathbb{N}$  two sequences  $(z_\alpha^k)_{\alpha < \omega_1}$  and  $(t_\alpha^k)_{\alpha < \omega_1}$  as the result of the application of the previous claim to the integer  $n_{j_k}$  and  $\varepsilon_k$  and where  $j_k$  is chosen such that  $|\text{supp } t_\alpha^{k-1}| < m_{j_k}$ . Finally, it is not difficult to see that one can extract for every  $k$  a subsequence  $(x_k^\alpha, y_k^\alpha)_{\alpha < \omega_1}$  of  $(z_\alpha^k, t_\alpha^k)_{\alpha < \omega_1}$  with the property that  $x_{k_0}^{\alpha_0} <_{\varepsilon_{k_1}} x_{k_1}^{\alpha_1}$  and  $T(x_{k_0}^{\alpha_0}) <_{\varepsilon_{k_1}} T(x_{k_1}^{\alpha_1})$  for every  $(k_0, \alpha_0) <_{\text{antilex}} (k_1, \alpha_1)$ .

Let us now give a proof of the claim. Fix  $k$  and  $\varepsilon > 0$ , and let  $l$  be any integer given by Proposition 2.11 when applied to  $k$ . Set  $\delta = \varepsilon/2l$ . Now use the fact that the bounded operator  $U = U(T, \gamma) : X \rightarrow X_\gamma \oplus X_\gamma$  defined by  $U(x) = (P_\gamma(x), P_\gamma(T(x)))$  is  $\omega_1$ -singular (because it has separable range) to find two normalized sequences  $\mathbf{z} = (z_\alpha)_{\alpha < \omega_1}$  and  $\mathbf{t} = (t_\alpha)_{\alpha < \omega_1}$ , and a block sequence  $(F_\alpha)_{\alpha < \omega_1}$  of finite sets  $F_\alpha \subseteq \omega_1$  of size  $l$  such that

- (e)  $\mathbf{z} \subseteq X$ , and  $\mathbf{t}$  is a block sequence.
- (f)  $\text{ran } t_\alpha \subseteq \text{ran } z_\alpha$  and  $\|z_\alpha - t_\alpha\| \leq \delta$  for every  $\alpha$ .
- (g)  $z_\alpha <_\delta z_\beta$  and  $T(z_\alpha) <_\delta T(z_\beta)$  for every  $\alpha < \beta$ .
- (h) For every countable ordinal  $\alpha$  there is a separated block sequence  $(f_\xi)_{\xi \in F_\alpha}$  that is biorthogonal to  $(t_\xi)_{\xi \in F_\alpha}$ .

We observe that (h) can be achieved by a simple application of Corollary 2.6. By Proposition 2.11, we can find a  $2 - \ell_1^k$ -average  $y_\alpha \in \langle t_\beta \rangle_{\beta \in F_\alpha}$  for each  $\alpha < \omega_1$ . It is easy to see that if  $x_\alpha$  is an arbitrary normalized vector in  $\langle z_\beta \rangle_{\beta \in F_\alpha}$  such that  $\text{ran } y_\alpha \subseteq \text{ran } x_\alpha$  and  $\|x_\alpha - y_\alpha\| \leq \varepsilon$  (and there is such a vector), then the corresponding sequences  $(x_\alpha)_{\alpha < \omega_1}$  and  $(y_\alpha)_{\alpha < \omega_1}$  fulfill all the properties (a) to (d). □

*Remark 2.20.* It is easy to find for given two bounded operators  $T_0, T_1 : X \rightarrow \mathfrak{X}$  a deciding pair  $(\mathbf{x}, \mathbf{y})$  for, simultaneously,  $T_0$  and  $T_1$ . This can be done simply by replacing  $U(T, \gamma)$  in the above proof of Lemma 2.19 by the mapping  $\bar{U} : X \rightarrow \mathfrak{X}_\gamma^3$  defined by  $x \mapsto \bar{U}(x) = (P_\gamma(x), P_\gamma(T_0(x)), P_\gamma(T_1(x)))$ .

### 3. MAIN PROPERTIES OF THE SPACE $\mathfrak{X}$

**3.1.  $c_0$ -saturation.** We are ready to prove the fact that  $\mathfrak{X}$  is  $c_0$ -saturated. We start with the following more informative result.

**Lemma 3.1.** *Suppose that  $(x_k)$  is a normalized block sequence such that*

$$\lim_{k \rightarrow \infty} \|x_k\|_\infty = 0.$$

*Then  $(x_k)$  has a subsequence which is 5-equivalent to the natural basis of  $c_0$ .*

*Proof.* Let  $(y_k)_k$  be a subsequence of  $(x_k)_k$  with  $\|y_k\|_\infty \leq 1/(2^{k+2}k)$ .

*Claim 2.* There is an infinite set  $M$  of integers such that for every triple  $k_0 < k_1 < k_2$  in  $M$  one has that

$$(3) \quad \max\{\varrho(\alpha, \beta) : \alpha \in \text{supp } y_{k_0}, \beta \in \text{supp } y_{k_1}\} < k_2.$$

*Proof of Claim 2.* We color every triple of integers  $k_0 < k_1 < k_2$  by 1 if (3) holds and 0 otherwise. By the classical Ramsey theorem we can find an infinite set  $M$  all of whose triples are equally colored. If this color is 1, then we are done. Otherwise, suppose that this color is 0, and let us derive a contradiction. Fix two integers  $k_0 < k_1$  in  $M$ . Then for every  $k \in M$  with  $k > k_1$  let  $\alpha_k \in \text{supp } y_{k_0}$  and

$\beta_k \in \text{supp } y_{k_1}$  with  $\varrho(\alpha_k, \beta_k) \geq k$ . Find an infinite  $P \subseteq N$  with  $(\alpha_k, \beta_k) = (\alpha, \beta)$  for every  $k \in P$ . Then  $\varrho(\alpha, \beta) \geq k$  for every  $k \in P$ , a contradiction.  $\square$

Fix such  $M$  from the claim. We show that  $(y_n)_{n \in M}$  is 5-equivalent to the natural basis of  $c_0$ . Observe that since the basis  $(e_\alpha)_\alpha$  is bimonotone one has that  $\|\sum_{n \in M} a_n y_n\| \geq \|(a_n)_{n \in M}\|_\infty$  for every sequence  $(a_n)_{n \in M}$  of scalars. So it remains to show that  $\|\sum_{n \in M} a_n y_n\| \leq 5\|(a_n)_{n \in M}\|_\infty$ . This is done in the next claim.

*Claim 3.* For every  $\phi \in K$  and every sequence  $(a_n)_{n \in M}$  of scalars, one has that

$$(4) \quad |\phi(\sum_n a_n y_n)| \leq 5\|(a_n)_{n \in M}\|_\infty.$$

*Proof of Claim 3.* Fix all data, and set  $y = \sum_n a_n y_n$ . The proof of (4) is done by induction on the complexity of  $\phi$ . If  $\phi = \pm e_\alpha^*$ , the result is trivial. Suppose that  $\phi$  is a rational convex combination  $\phi = \sum_{i < k} c_i f_i$  of elements  $(f_i)_{i < n}$  of  $K$ . Then, by applying the inductive hypothesis to  $f_i$ 's, one has that  $|\phi(y)| \leq \sum_{i < k} c_i |f_i(y)| \leq 5\|(a_n)_{n \in M}\|_\infty$ . Suppose now that  $\phi = (1/m_j) \sum_{i < d} f_i$ , where  $d \leq n_j$  and  $(f_i)_{i < d}$  is a separated block sequence in  $K$ . Let

$$\begin{aligned} a &= \{i < d : \text{supp } y \cap \text{supp } f_i \neq \emptyset\}, \text{ and for } i \in a \text{ let} \\ k(i) &= \min\{k : \text{supp } y_k \cap \text{supp } f_i \neq \emptyset\}, \text{ and} \\ L &= \{k(i) : i \in a\}. \end{aligned} \quad \square$$

- Claim 4.* (a) If  $k < \min L$ , then  $\text{supp } y_k \cap \text{supp } f = \emptyset$ .  
 (b) There is some  $\bar{i} \in a$  such that for every  $k > \max L$ , then  $\text{supp } y_k \cap \text{supp } f = \text{supp } y_k \cap \text{supp } f_{\bar{i}}$ .  
 (c) For every two consecutive  $k_0 < k_1$  in  $L$  there is some  $\bar{i}(k_0, k_1) \in a$  such that for every  $k$  with  $k_0 < k < k_1$  one has that  $\text{supp } y_k \cap \text{supp } f = \text{supp } y_k \cap \text{supp } f_{\bar{i}(k_0, k_1)}$ .

*Proof of Claim 4.* Statement (a) is clear. It is not difficult to show that statements (b) and (c) proceed from the following, also non-difficult, fact: Suppose that  $I$  is an interval of integers, and suppose that there are at least two integers  $i_0$  and  $i_1$  in  $a$  such that  $\text{supp } f_{i_\varepsilon} \cap \bigcup_{k \in I} \text{supp } y_k \neq \emptyset$  for  $\varepsilon = 0, 1$ . Then  $I \cap L \neq \emptyset$ .  $\square$

Now we consider two cases:

*Case 1.* The cardinality of  $L$  is at most two. If  $L = \emptyset$ , then  $f(x) = 0$ . Suppose that  $L = \{\bar{k}\}$ . Then using (a), (b) and the inductive hypothesis one obtains that

$$|f(x)| \leq \frac{1}{m_j} |f_{\bar{i}}(\sum_{k > \bar{k}} a_k y_k)| + |a_{\bar{k}}| \leq 2\|(a_k)_k\|_\infty.$$

Finally suppose that  $L = \{k_0, k_1\}$  with  $k_0 < k_1$ . Then one has that

$$|f(x)| \leq |a_{k_0}| + |a_{k_1}| + \frac{1}{m_j} \left| f_{\bar{i}(k_0, k_1)}(\sum_{k=k_0+1}^{k_1-1} a_k y_k) + f_{\bar{i}}(\sum_{k > k_1} a_k y_k) \right| \leq 5\|(a_k)_k\|_\infty.$$

*Case 2.* The set  $L$  has cardinality at least three. Let  $k_0 < k_1 < k_2$  be the least three elements of  $L$ . Find  $i_0 < i_1$  in  $a$  such that  $k(i_\varepsilon) = k_\varepsilon$  for  $\varepsilon = 0, 1$ , and then  $\alpha_\varepsilon \in \text{supp } f_{i_\varepsilon} \cap \text{supp } y_{k_\varepsilon}$  for  $\varepsilon = 0, 1$ . It follows that

$$\|f\|_{\ell_1} \leq |\text{supp } f| \leq \varrho(\alpha_0, \alpha_1) < k_2.$$

Hence for every  $k \geq k_2$  one has that

$$(5) \quad |f(y_k)| \leq \|f\|_{\ell_1} \|y_k\|_\infty \leq \frac{1}{2^{k+2}}.$$

Using the inequality (5), conditions (a)-(c) above and the inductive hypothesis applied to  $f_{i(i_0, i_1)}$  and  $f_{i_1}$  one obtains that

$$\begin{aligned} |f(x)| \leq & |a_{k_0}| + |a_{k_1}| + \frac{1}{m_j} \left| f_{i(k_0, k_1)} \left( \sum_{k=k_0+1}^{k_1-1} a_k y_k \right) + f_{i(k_1, k_2)} \left( \sum_{k=k_1+1}^{k_2-1} a_k y_k \right) \right| \\ & + |f(\sum_{k \geq k_2} a_k y_k)| \leq 2\|(a_k)_k\|_\infty + \frac{5}{2}\|(a_k)_k\|_\infty + \frac{1}{2}\|(a_k)\|_\infty = 5\|(a_n)\|_\infty, \end{aligned}$$

as desired. □

**Theorem 3.2.**  $\mathfrak{X}$  is  $c_0$ -saturated.

Before we start with the proof we recall the notion of an exact pair.

**Definition 3.3.** A pair  $(x, \phi)$  with  $x \in \mathfrak{X}$  and  $\phi \in K$  is said to be a  $(C, j)$ -exact pair if  $\|x\| \leq C$ ,  $w(\phi) = m_j$ ,  $\phi(x) = 1$ , and for every  $\psi \in K$  of type I and  $w(\psi) = m_i$ ,  $i \neq j$ , one has that

$$(6) \quad |\psi(x)| \leq \begin{cases} \frac{2C}{m_i} & \text{if } i < j, \\ \frac{C}{m_j^2} & \text{if } i > j. \end{cases}$$

*Proof of Theorem 3.2.* Fix a closed infinite dimensional subspace  $X$  of  $\mathfrak{X}$ . We may assume, by Remark 2.9 (b), that indeed  $X$  is the closed linear span of a normalized block sequence  $(x_k)$ . Suppose that  $c_0$  does not embed isomorphically into  $X$ , and let us derive a contradiction. Consider the norm-one operator  $\sum_k a_k x_k \in X \mapsto (a_k)_k \in c_0$ . This is, by hypothesis, strictly singular. So we can find a normalized block subsequence  $(y_k)_k$  of  $(x_k)_k$  with  $\lim_{k \rightarrow \infty} \|y_k\|_\infty = 0$ . Then by Lemma 3.1  $(y_k)_k$  has a subsequence equivalent to the  $c_0$ -basis, a contradiction. □

**3.2. Distortion.** From the previous Theorem 3.2 one immediately obtains the following.

**Corollary 3.4.**  $\mathfrak{X}$  is not distortable.

In contrast to this, we have a strong distortion phenomenon in the level of non-separable subspaces of  $\mathfrak{X}$ :

**Theorem 3.5.**  $\mathfrak{X}$  is arbitrarily  $\omega_1$ -distortable.

*Proof.* We follow the ideas used to show that  $\mathfrak{X}_{\omega_1}$  is arbitrarily distortable (see Corollary 5.36 in [1]). For  $j \in \mathbb{N}$ , and  $x \in \mathfrak{X}_{\omega_1}$ , let

$$\|x\|_j = \sup\{\phi(x) : \phi \in K, w(\phi) = m_{2j}\}.$$

Notice that, obviously, for every  $\phi \in K$  one has that  $(\phi)$  is a dependent sequence, hence  $(1/m_{2j})\phi \in K$ . So, it follows that  $\|\cdot\|_j \leq \|\cdot\| \leq m_{2j}\|\cdot\|_j$ . Fix a closed non-separable subspace  $X$  of  $\mathfrak{X}$ . Let  $(x, y)$  be a deciding pair for the inclusion mapping  $i_{X, \mathfrak{X}} : X \rightarrow \mathfrak{X}$ .

Now fix an integer  $l$  and  $\varepsilon > 0$ . Use Proposition 2.18 to find a transversal subsequence  $\mathbf{z} = (y_{k_i}^{\alpha_i})_{i < n_{2l}}$  of  $\mathbf{y}$  such that

- (a)  $\sum_{i < l} \|x_{k_i}^{\alpha_i} - y_{k_i}^{\alpha_i}\| \leq \varepsilon$ , and
- (b)  $\mathbf{z}$  is a  $(3, n_{2l}^{-2})$ -RIS with a biorthogonal separated block sequence  $(f_i)_{i < n_{2l}}$  in  $K$ . Set

$$z_l = \frac{m_{2l}}{n_{2l}} \sum_{i < l} y_{k_i}^{\alpha_i}, \phi_l = \frac{1}{m_{2l}} \sum_{i < l} f_i.$$

Then  $\phi_l \in K$ , and the pair  $(z_l, \phi_l)$  is a  $(6, 2l)$ -exact pair (see [1, the proof of Proposition 4.12]). So, it follows that if  $l = j$ , then one has

$$1 \leq \|z_j\|_j \leq \|z_j\| \leq 6,$$

while if  $l > j$ , then

$$1 \leq \|z_l\| \leq 6 \text{ and } \|z_l\|_j \leq \frac{12}{m_{2j}}.$$

Hence for every  $l > j$  one has the discrepancy

$$(7) \quad \frac{\|z_j/\|z_j\|\|_j}{\|z_l/\|z_l\|\|_j} \geq \frac{1/6}{12/m_{2j+1}} = \frac{m_{2j+1}}{72}.$$

Since the vectors  $z_k$  and  $z_l$  can be found to be arbitrarily close to  $X$ , it follows that one can obtain a similar inequality to (7) for vectors in  $X$ . Hence  $(\mathfrak{X}, \|\cdot\|)$  is arbitrarily  $\omega_1$ -distortable. □

**3.3. Operators.** It turns out that while the space  $\mathfrak{X}$  is  $c_0$ -saturated it has the following form of indecomposability at the non-separable level.

**Definition 3.6.** A Banach space  $X$  is  $\omega_1$ -indecomposable if for every decomposition  $X = Y \oplus Z$  one has that either  $Y$  or  $Z$  is separable. We say that  $X$  is  $\omega_1$ -hereditarily indecomposable ( $\omega_1$ -HI in short) if every subspace of  $X$  is  $\omega_1$ -indecomposable.

This kind of indecomposability corresponds to the following notion of singularity for operators.

**Definition 3.7.** An operator  $T : X \rightarrow Y$  is  $\omega_1$ -singular if  $T$  is not an isomorphism on any non-separable subspace of  $X$ .

Observe that strictly singular and separable range operators are  $\omega_1$ -singular. While the strictly singular operators and operators with separable ranges form closed ideals of the Banach algebra  $\mathcal{L}(X)$  of all bounded operators from  $X$  into  $X$ , we do not know if, in general, this is also the case for the family  $\mathcal{S}_{\omega_1}(X)$  of  $\omega_1$ -singular operators of  $X$ . Indeed we do not even know if  $\mathcal{S}_{\omega_1}(X)$  is closed under sums. We shall show however that for subspaces  $X$  of our space  $\mathfrak{X}$  one does have the property that  $\mathcal{S}_{\omega_1}(X)$  forms a closed ideal in the algebra  $\mathcal{L}(X)$  (Corollary 3.20).

Using this notion one can have the following sufficient condition for being  $\omega_1$ -HI.

**Proposition 3.8.** Suppose that  $X$  has the property that for every subspace  $Y$  of  $X$  every bounded operator  $T : Y \rightarrow X$  is of the form  $T = \lambda i_{Y,X} + S$ , where  $S$  is  $\omega_1$ -singular and  $\lambda \in \mathbb{R}$ . Then  $X$  is  $\omega_1$ -HI.

*Proof.* Otherwise, fix two non-separable subspaces  $Y$  and  $Z$  of  $X$  with the property that  $d(S_X, S_Y) > 0$ . It follows that the two natural projections  $P_Y : Y \oplus Z \rightarrow Y$

and  $P_Z : Y \oplus Z \rightarrow Z$  are both bounded. Fix  $\lambda \in \mathbb{R}$  such that  $T = i_{Y,X} \circ P_Y = \lambda i_{Y \oplus Z, X} + S$  with  $S$   $\omega_1$ -singular. Since  $T^2 = T$ , we have that

$$(8) \quad (\lambda^2 - \lambda)i_{Y \oplus Z, X} = ((1 - 2\lambda)i_{Y \oplus Z, X} - S) \circ S.$$

Since it is clear that  $U \circ S$  is  $\omega_1$ -singular if  $S$  is  $\omega_1$ -singular, it follows from (8) that  $\lambda^2 = \lambda$ . Without loss of generality, we may assume that  $\lambda = 1$  (if  $\lambda = 0$  we replace  $Y$  by  $Z$  in the preceding argument). Since  $P_Y|_Z = 0$ , we obtain that  $S = -i_{Z,X}$ , a contradiction.  $\square$

*Remark 3.9.* Recall that V. Ferenczi has shown in [5] that if a complex Banach space  $X$  is HI, then every operator from a subspace  $Y$  of  $X$  into  $X$  is a multiple of the inclusion plus a strictly singular operator. We do not know if the analogous result is true for  $\omega_1$ -singular operators or, in other words, if the converse implication of Proposition 3.8 is true in the case of complex Banach spaces.

The main purpose of this subsection is the study of the operator space  $\mathcal{L}(X, \mathfrak{X})$ , where  $X$  is an arbitrary closed infinite dimensional subspace of  $\mathfrak{X}$ . For the next few lemmas we fix a bounded operator  $T : X \rightarrow \mathfrak{X}$  from a closed infinite dimensional subspace  $X$  of  $\mathfrak{X}$  into the space  $\mathfrak{X}$ . We also fix a deciding pair  $(\mathbf{x}, \mathbf{y})$  for  $T$  (see Definition 2.17).

**Lemma 3.10.** *For all but countably many  $\alpha < \omega_1$  one has that*

$$\lim_{k \rightarrow \infty} d(Tx_k^\alpha, \mathbb{R}x_k^\alpha) = 0.$$

In the proof we use the following slight modification of the concept of the  $(0, j)$ -dependent sequence used in [1] to control the operators of the space  $\mathfrak{X}_{\omega_1}$  defined there.

**Definition 3.11.** A sequence  $(x_1, \phi_1, \dots, x_{n_{2j+1}}, \phi_{n_{2j+1}})$  is called an  $(\varepsilon, j)$ -dependent sequence if the following conditions are fulfilled:

- (1)  $\Phi = (\phi_1, \dots, \phi_{n_{2j+1}})$  is a  $2j + 1$ -special sequence and  $|\phi_i x_{i'}| \leq \varepsilon$  for every  $1 \leq i, i' \leq n_{2j+1}$ .
- (2) There are  $\psi_1, \dots, \psi_{n_{2j+1}}$  with the properties that  $w(\psi_i) = w(\phi_i)$ ,  $\#\text{supp } x_i \leq w(\phi_{i+1})/n_{2j+1}^2$  and  $(x_i, \psi_i)$  is a  $(6, 2j_i)$ -exact pair for every  $1 \leq i \leq n_{2j+1}$ .
- (3) If  $H = (h_1, \dots, h_{n_{2j+1}})$  is an arbitrary  $2j + 1$ -special sequence, then

$$(9) \quad \left( \bigcup_{\kappa_{\Phi, H} < i < \lambda_{\Phi, H}} \text{supp } x_i \right) \cap \left( \bigcup_{\kappa_{\Phi, H} < i < \lambda_{\Phi, H}} \text{supp } h_i \right) = \emptyset.$$

The main property of  $(\varepsilon, j)$ -dependent sequences is the following, whose proof is essentially the same as the one of [1, Proposition 5.24].

**Proposition 3.12.** *For every  $(\varepsilon, j)$ -dependent sequence  $(x_1, \phi_1, \dots, x_{n_{2j+1}}, \phi_{n_{2j+1}})$  with  $\varepsilon \leq n_{2j+1}^{-1}$  one has that*

$$\left\| \frac{1}{n_{2j+1}}(x_1 + \dots + x_{n_{2j+1}}) \right\| \leq \frac{1}{m_{2j+1}^2}.$$

*Proof.* Otherwise, using property (iii) of the deciding pair  $(\mathbf{x}, \mathbf{y})$  and going to subsequences if necessary, we may assume that there is  $\varepsilon > 0$  such that

$$\inf_{k \in \mathbb{N}} d(Tx_k^\alpha, \mathbb{R}y_k^\alpha) > \varepsilon$$

for every countable ordinal  $\alpha$ . Now using the Hahn-Banach Theorem and the fact that the norming set  $K$  is closed under rational convex combinations and restrictions on intervals, we can find for every pair  $(k, \alpha) \in \mathbb{N} \times \omega_1$  a functional  $f_k^\alpha \in K$  such that one has that:

- (a)  $(f_k^\alpha)_{(k, \alpha) \in \mathbb{N} \times \omega_1}$  is a block sequence and  $\text{ran } f_k^\alpha \subseteq \text{ran } x_k^\alpha$  for every  $(k, \alpha)$ , and
- (b)  $|f_k^\alpha(y_k^\alpha)| \leq \varepsilon_k$ , while  $f_k^\alpha(T(x_k^\alpha)) \geq \varepsilon$  for every  $(k, \alpha)$ .

Fix an integer  $j$  such that  $\varepsilon m_{2j+1} > 2\|T\|$ . Now use Proposition 2.18 to find a sequence  $(F_i^\alpha)_{(i, \alpha) \in n_{2j+1} \times \omega_1}$  of finite sets of pairs from  $\mathbb{N} \times \omega_1$  such that

- (c)  $|F_i^\alpha| = n_{2j_i}$  for every  $(i, \alpha) \in n_{2j+1} \times \omega_1$ ,
- (d)  $(y_k^\xi)_{(k, \xi) \in F_i^\alpha}$  is a  $(3, (n_{2j_i})^{-2})$ -RIS and  $(f_k^\xi)_{(k, \xi) \in F_i^\alpha}$  is a separated block sequence for every  $(i, \alpha) \in n_{2j+1} \times \omega_1$ ,
- (e)  $|f_k^\xi(T(y_k^\xi))|, \|x_k^\xi - y_k^\xi\| \leq n_{2j_i}^{-3}$  for every  $(k, \xi) \in F_i^\alpha$  and every  $(i, \alpha) \in n_{2j+1} \times \omega_1$ ,
- (f)  $2j_i = \sigma_\varrho(\phi_0^\alpha, m_{2j_0}, p_0, \dots, \phi_{i-1}^\alpha, m_{2j_{i-1}}, p_{i-1})$  for every  $(i, \alpha) \in n_{2j+1} \times \omega_1$ , where  $p_i$  is an integer such that

$$p_i \geq \max\{p_0, \dots, p_{i-1}, n_{2j+1}^2, p_\varrho(\bigcup_{k < i} (\text{supp } \phi_k^\alpha \cup \text{supp } t_k^\alpha)), |\text{supp } t_{i-1}^\alpha| n_{2j+1}^2\},$$

and where

$$\begin{aligned} \phi_i^\alpha &= \frac{1}{m_{2j_i}} \sum_{(k, \xi) \in F_i^\alpha} f_k^\xi, \\ t_i^\alpha &= \frac{n_{2j_i}}{m_{2j_i}} \sum_{(k, \xi) \in F_i^\alpha} y_k^\xi, \end{aligned}$$

and  $\sigma_\varrho$  and  $p_\varrho$  are the coding and the  $\varrho$ -number defined in section 2. Also set

$$z_i^\alpha = \frac{n_{2j_i}}{m_{2j_i}} \sum_{(k, \xi) \in F_i^\alpha} x_k^\xi.$$

Now again using Corollary 2.6 we can find countable ordinals  $\alpha_0 < \dots < \alpha_{n_{2j+1}-1}$  such that  $(\phi_i^{\alpha_i})_{i < n_{2j+1}}$  is a separated block sequence. It follows that the sequence  $((t_i^{\alpha_i}, \phi_i^{\alpha_i}))_{i < n_{2j+1}}$  is a  $(n_{2j+1}^{-2}, j)$ -dependent sequence (see Definition 3.11). It follows that

$$(10) \quad \left\| \frac{1}{n_{2j+1}} \sum_{i < n_{2j+1}} t_i^{\alpha_i} \right\| \leq \frac{1}{m_{2j+1}^2}.$$

Hence, setting  $z = (1/n_{2j+1}) \sum_{i < n_{2j+1}} z_i^{\alpha_i}$ , one has that

$$(11) \quad \|z\| \leq \frac{2}{m_{2j+1}^2}.$$

By the other hand, since  $\phi = (1/m_{2j+1}) \sum_{i < n_{2j+1}} \phi_i^{\alpha_i}$  is in  $K$ , it follows that

$$(12) \quad \|T(z)\| \geq \phi(T(z)) \geq \frac{\varepsilon}{m_{2j+1}}.$$

Putting (11) and (12) together one gets

$$\frac{\varepsilon}{m_{2j+1}} \leq \|T(z)\| \leq \|T\| \|z\| \leq \frac{2\|T\|}{m_{2j+1}^2},$$

and this is contradictory with the choice of  $j$ . □

Now for each countable ordinal  $\alpha$ , let  $\lambda_k^\alpha = \lambda_k^\alpha(T, \mathbf{x}, \mathbf{y}) \in \mathbb{R}$  be such that

$$d(T(x_k^\alpha), \mathbb{R}x_k^\alpha) = \|T(x_k^\alpha) - \lambda_k^\alpha x_k^\alpha\|.$$

**Lemma 3.13.** *For all but countably many  $\alpha < \omega_1$ , the numerical sequence  $(\lambda_k^\alpha)_k$  is convergent.*

*Proof.* Otherwise, using Lemma 3.10, one can find two real numbers  $\delta < \varepsilon$ , an uncountable set  $A \subseteq \omega_1$ , for each  $\alpha \in A$  two infinite disjoint subsets  $L_\alpha$  and  $R_\alpha$  of  $\mathbb{N}$ , and a block sequence  $(f_k^\alpha)_{(k,\alpha) \in (L_\alpha \cup R_\alpha) \times A}$  in  $K$  such that

- (i)  $\text{ran } f_k^\alpha \subseteq \text{ran } y_k^\alpha$ ,  $f_k^\alpha(y_k^\alpha) = 1$  for every  $(k, \alpha) \in (L_\alpha \cup R_\alpha) \times A$ .
- (ii) For every  $\alpha \in A$  one has that  $f_k^\alpha(T(x_k^\alpha)) < \delta$  if  $k \in L_\alpha$ , and  $f_k^\alpha(T(x_k^\alpha)) > \varepsilon$  if  $k \in R_\alpha$ .

Let  $j \in \mathbb{N}$  be such that  $(\varepsilon - \delta)m_{2j+1} \geq 4\|T\|$ . We find, as in the proof of the previous Lemma 3.10 a sequence  $(F_i^\alpha)_{(i,\alpha) \in n_{2j+1} \times \omega_1}$  of finite sets of pairs from  $\mathbb{N} \times \omega_1$  such that (c), (d), and (f) as there holds, and also

- (e') For every  $(k, \xi) \in F_i^\alpha$  one has that  $\|x_k^\xi - y_k^\xi\| \leq n_{2j_i}^{-3}$  for every  $(i, \alpha) \in n_{2j+1} \times \omega_1$ , and  $f_k^\xi(T(x_k^\xi)) \geq \varepsilon$  if  $i$  is odd and  $f_k^\xi(T(x_k^\xi)) \leq \delta$  if  $i$  is even.

We also set  $\phi_i^\alpha$ ,  $u_i^\alpha$  and  $z_i^\alpha$  as there. Let  $\alpha_0 < \dots < \alpha_{n_{2j+1}-1}$  be such that  $(\phi_i^{\alpha_i})_{i < n_{2j+1}}$  is a separated block sequence. It follows that the corresponding sequence  $((i_i^{\alpha_i}, \phi_i^{\alpha_i}))_{i < n_{2j+1}}$  is a  $(1, j)$ -dependent sequence (see Definition 3.3 in [1]), hence

$$(13) \quad \left\| \frac{1}{n_{2j+1}} \sum_{i < n_{2j+1}} (-1)^i u_i^{\alpha_i} \right\| \leq \frac{1}{m_{2j+1}^2},$$

and so by property (e') one has that

$$(14) \quad \|z\| \leq \frac{2}{m_{2j+1}^2},$$

where  $z = (1/n_{2j+1}) \sum_{i < n_{2j+1}} (-1)^i z_i^{\alpha_i}$ . Setting  $\phi = (1/m_{2j+1}) \sum_{i < n_{2j+1}} \phi_i^{\alpha_i}$ , it follows that

$$(15) \quad \|T(z)\| \geq |\phi(Tz)| = \left| \frac{1}{m_{2j+1}n_{2j+1}} \sum_{i < n_{2j+1}} (-1)^i \phi_i^{\alpha_i}(T(z_i^{\alpha_i})) \right| \geq \frac{\varepsilon - \delta}{2m_{2j+1}}.$$

From (14) and (15) one easily gets a contradiction with the choice of  $j$ . □

For every  $\alpha < \omega_1$  let (if it exists)  $\lambda_\alpha = \lambda_\alpha(T, \mathbf{x}, \mathbf{y}) = \lim_{k \rightarrow \infty} \lambda_k^\alpha$ .

**Corollary 3.14.** *There is a real number  $\lambda = \lambda(T, \mathbf{x}, \mathbf{y})$  such that  $\lambda_\alpha = \lambda$  for all but countably many  $\alpha$ .*

*Proof.* Otherwise, there are two reals  $\delta < \varepsilon$  such that both sets  $A_0 = \{\alpha < \omega_1 : \lambda_\alpha < \delta\}$  and  $A_1 = \{\alpha < \omega_1 : \lambda_\alpha > \varepsilon\}$  are uncountable. Find for every  $(k, \alpha) \in \mathbb{N} \times \omega_1$  a countable ordinal  $\beta(k, \alpha)$   $((k, \alpha) \in \mathbb{N} \times \omega_1)$  such that

- (i)  $\beta(k, \alpha) \in A_i$  if  $k$  is equal to  $i \pmod 2$ .
- (ii)  $((x_k^{\beta(k,\alpha)}, y_k^{\beta(k,\alpha)}))_{(k,\alpha) \in \mathbb{N} \times \omega_1}$  is a deciding pair  $(\mathbf{z}, \mathbf{t})$  for  $T$ .

It follows that  $(\lambda_k^\alpha(\mathbf{z}, \mathbf{t}))_k$  is never converging, contradicting Lemma 3.13. □



**Corollary 3.15.** *The scalar  $\lambda(T, \mathbf{x}, \mathbf{y})$  is independent of  $\mathbf{x}$  and  $\mathbf{y}$ . We call it  $\lambda(T)$ .*

*Proof.* Fix two deciding pairs  $(\mathbf{x}, \mathbf{y})$  and  $(\mathbf{z}, \mathbf{u})$  for  $T$ . It is easy to define a third one  $(\mathbf{v}, \mathbf{w})$  for  $T$  such that the sets

$$\begin{aligned} &\{\alpha < \omega_1 : (\exists \beta < \omega_1)(\forall k \in \mathbb{N})(x_k^\alpha, y_k^\alpha) = (z_k^\alpha, u_k^\alpha)\}, \\ &\{\alpha < \omega_1 : (\exists \beta < \omega_1)(\forall k \in \mathbb{N})(v_k^\alpha, w_k^\alpha) = (z_k^\alpha, u_k^\alpha)\} \end{aligned}$$

are both uncountable. It follows that  $\lambda(T, \mathbf{x}, \mathbf{y}) = \lambda(T, \mathbf{v}, \mathbf{w}) = \lambda(T, \mathbf{z}, \mathbf{u})$ . □

**Theorem 3.16.** *The mapping  $\lambda : \mathcal{L}(X, \mathfrak{X}) \rightarrow \mathbb{R}$  which sends  $T$  to  $\lambda(T)$  is a bounded linear functional whose kernel  $\text{Ker}(\lambda)$  is equal to the family of all  $\omega_1$ -singular operators from  $X$  into  $\mathfrak{X}$ .*

*Proof.* It is obvious from the definition of  $\lambda(T)$  that  $|\lambda(T)| \leq \|T\|$ . We now show the linearity of  $\lambda$ . It is easy to see that  $\lambda(\mu T) = \mu\lambda(T)$ . Let us now prove that  $\lambda(T_0 + T_1) = \lambda(T_0) + \lambda(T_1)$ . Let  $(\mathbf{x}, \mathbf{y})$  be a deciding pair for both  $T_0$  and  $T_1$  (see Remark 2.20). Let  $\lambda_k^{\alpha,0}, \lambda_k^{\alpha,1}, \lambda_k^\alpha \in \mathbb{R}$  be such that  $d((T_0 + T_1)(x_k^\alpha), \mathbb{R}x_k^\alpha) = \|(T_0 + T_1)(x_k^\alpha) - \lambda_k^\alpha x_k^\alpha\|$ , and  $d(T_i(x_k^\alpha), \mathbb{R}x_k^\alpha) = \|T(x_k^\alpha) - \lambda_k^{\alpha,i} x_k^\alpha\|$  for  $i = 0, 1$ . It follows from Lemma 3.10, applied to  $T_0, T_1$  and  $T_0 + T_1$ , that

$$(16) \quad \lim_{k \rightarrow \infty} (\lambda_k^{\alpha,0} + \lambda_k^{\alpha,1}) - \lambda_k^\alpha = 0,$$

for all but countably many  $\alpha$ . The desired result now follows from (16) and Lemma 3.13.

Now we prove that  $\text{Ker}(\lambda)$  is the family of the  $\omega_1$ -singular operators. Suppose first that  $\lambda(T) = 0$ . We are going to show that  $T$  is not an isomorphism when restricted to a non-separable subspace of  $X$ . To do this, let  $\varepsilon > 0$ , and let  $Z$  be a non-separable subspace of  $X$ . Let  $(\mathbf{x}, \mathbf{y})$  be any deciding pair for  $T$  with  $\mathbf{x} \subseteq Z$ . Since  $\lambda(T) = 0$ , by the previous Lemma 3.10 we can find  $(k, \alpha) \in \mathbb{N} \times \omega_1$  such that  $\|T(x_k^\alpha)\| < \varepsilon$ , as desired.

Finally, suppose that  $T$  is a  $\omega_1$ -singular operator. Our intention is to provide a deciding pair  $(\mathbf{x}, \mathbf{y})$  such that  $\|T(x_k^\alpha)\| \leq 2^{-k}$  for every  $(k, \alpha)$ . Then  $\lambda_\alpha(T, \mathbf{x}, \mathbf{y}) = 0$  for every  $\alpha$ , and so  $\lambda(T) = 0$ .

*Claim 5.* For every non-separable  $X_0 \hookrightarrow X$ , every  $\varepsilon > 0$  and every  $k$  there are two normalized vectors  $x$  and  $y$  such that  $x \in X_0$ ,  $y$  is a  $2 - \ell_1^k$ -average,  $\|T(x)\| \leq \varepsilon$  and  $\|x - y\| \leq \varepsilon$ .

It is easy to find the desired deciding pair  $(\mathbf{x}, \mathbf{y})$  from a simple use of the previous claim. Now we pass to give a proof of the claim. Fix all data. Let  $l \in \mathbb{N}$  be the result of the application of Proposition 2.11 to our fixed  $k$ , and let  $\delta = \varepsilon/2l$ . Since  $T$  is  $\omega_1$ -singular, one can find two normalized uncountable sequences  $(x_\alpha)_{\alpha < \omega_1}$  and  $(y_\alpha)_{\alpha < \omega_1}$  such that

- (a)  $x_\alpha \in X_0$  for every  $\alpha < \omega_1$ .
- (b)  $\mathbf{y} = (y_\alpha)_{\alpha < \omega_1}$  is a block sequence, and there an uncountable block sequence  $(f_\alpha)_{\alpha < \omega_1}$  in  $K$  biorthogonal to  $\mathbf{y}$ .
- (c)  $\|x_\alpha - y_\alpha\|, \|T(x_\alpha)\| \leq \delta$ .

It follows from Corollary 2.6 when applied to  $(\text{supp } f_\alpha)_{\alpha < \omega_1}$  that there is  $F \subseteq \omega_1$  of size  $l$  such that  $(f_\alpha)_{\alpha \in F}$  is a separated sequence. So, by Proposition 2.11 one

can find a  $2 - \ell_1^k$  in  $\langle y_\alpha \rangle_{\alpha \in F}$  and then, using property (c) above, the counterpart  $x \in \langle x_\alpha \rangle_{\alpha \in F}$  so that  $x$  and  $y$  fulfill the desired conditions.  $\square$

From Theorem 3.16 one easily gets the main conclusion of this section which gives us the description of the spaces of operators  $\mathcal{L}(X, \mathfrak{X})$ , where  $X$  is an arbitrary closed infinite dimensional subspace of  $\mathfrak{X}$ .

**Corollary 3.17.** *Every bounded operator  $T : X \rightarrow \mathfrak{X}$  from a closed subspace of  $\mathfrak{X}$  into  $\mathfrak{X}$  can be expressed as the sum*

$$T = \lambda(T)i_{X, \mathfrak{X}} + S,$$

where  $S$  is an  $\omega_1$ -singular operator.  $\square$

**Corollary 3.18.** *The space  $\mathfrak{X}$  is  $\omega_1$ -hereditarily indecomposable, and therefore it contains no uncountable unconditional basic sequence.*

*Proof.* This follows from Proposition 3.8.  $\square$

*Remark 3.19.* It follows that  $\mathfrak{X}$  is an example of an  $\omega_1$ -hereditarily indecomposable Banach space that is unconditionally saturated. This answers a question communicated to us by S. A. Argyros.

It is well known that the class of strictly singular operators on a Banach space  $X$  is a closed ideal of the Banach algebra  $\mathcal{L}(X)$  of all bounded operators from  $X$  into  $X$ . We do not know if the same is true for the class of  $\omega_1$ -singular operators. However, we have the following.

**Corollary 3.20.** *If  $X$  is a closed infinite dimensional subspace of  $\mathfrak{X}$ , then the family  $\mathcal{S}_{\omega_1}(X)$  of  $\omega_1$ -singular operators in  $\mathcal{L}(X)$  forms a closed ideal in the Banach algebra  $\mathcal{L}(X)$ .*

*Proof.* We show that  $\lambda : \mathcal{L}(X) \rightarrow \mathbb{R}$ , formally defined by  $\lambda \mapsto \lambda(T) = \lambda(i_{X, X} \circ T)$ , is a bounded operator between Banach algebras. From this one easily gets the desired result. Observe that we did almost all the work in Theorem 3.16. It remains to show that  $\lambda(T_1 \circ T_0) = \lambda(T_1)\lambda(T_0)$ . Fix a deciding pair  $(\mathbf{x}, \mathbf{y})$  for both  $T_0$  and  $T_1$ . It follows that for all but countably many  $\alpha < \omega_1$  one has that

$$\begin{aligned} \lim_{k \rightarrow \infty} \|(T_1 \circ T_0)(x_k^\alpha) - \lambda(T_1 \circ T_0)x_k^\alpha\| &= 0, \\ \lim_{k \rightarrow \infty} \|(T_1 \circ T_0)(x_k^\alpha) - T_1(\lambda(T_0)x_k^\alpha)\| &= 0, \\ \lim_{k \rightarrow \infty} \|T_1(\lambda(T_0)x_k^\alpha) - \lambda(T_1)\lambda(T_0)x_k^\alpha\| &= 0, \end{aligned}$$

and so  $\lambda(T_1 \circ T_0) = \lambda(T_1)\lambda(T_0)$ , as desired.  $\square$

In the case of  $X = \mathfrak{X}$  we obtain the following slightly more informative result.

**Corollary 3.21.** *If  $T : \mathfrak{X} \rightarrow \mathfrak{X}$  is a bounded operator, then  $T(e_\alpha) = \lambda(T)e_\alpha$  for all but countably many  $\alpha$ . It follows that  $T$  is the sum of a multiple of the identity plus an operator with separable range.*

*Proof.* Otherwise we can find an uncountable subset  $A \subseteq \omega_1$  and  $\varepsilon > 0$  such that

$$(17) \quad \|T(e_\alpha) - \lambda(T)e_\alpha\| > \varepsilon$$

for all  $\alpha \in A$  and also, as a consequence of Remark 2.9 (e), such that  $T(e_\alpha) < T(e_\beta)$  for every  $\alpha < \beta$  in  $A$  (where we set  $0 < x$  for every vector  $x$ ). Let  $\theta_A : \omega \times \omega_1 \rightarrow A$  be the unique order-preserving onto mapping, and define  $x_k^\alpha = e_{\theta_A(k,\alpha)}$ . Then  $((x_k^\alpha), (x_k^\alpha))$  is a deciding pair for  $T$ . Observe that this is what it makes so peculiar the situation  $X = \mathfrak{X}$ . So, by Lemma 3.10, for all but countably many  $\alpha$  one has that

$$\lim_{k \rightarrow \infty} \|T(x_k^\alpha) - \lambda(T)x_k^\alpha\| = 0,$$

which is contradictory with (17). □

The following result shows that, among the bounded operators from  $\mathfrak{X}$  into  $\mathfrak{X}$ , there is no distinction between the notions of  $\omega_1$ -singular operators and the notion of operators with separable range.

**Corollary 3.22.** *The following are equivalent for  $T \in \mathcal{L}(\mathfrak{X})$ :*

- (a)  $(T(e_\alpha))_{\alpha < \omega_1}$  is eventually zero.
- (b)  $T$  has separable range.
- (c)  $T$  is  $\omega_1$ -singular.

*Proof.* (c) implies (a): Suppose that  $T$  is  $\omega_1$ -singular. Then by Theorem 3.16 one has that  $\lambda(T) = 0$ , so by Corollary 3.21 it follows that  $T(e_\alpha) = 0$  eventually. □

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