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SPECTRAL ANALYSIS OF A CLASS OF NONLOCAL ELLIPTIC OPERATORS RELATED TO BROWNIAN MOTION WITH RANDOM JUMPS

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ABSTRACT. Let $D \subset R^d$ be a bounded domain and let $\mathcal{P}(D)$ denote the space of probability measures on D. Consider a Brownian motion in D which is killed at the boundary and which, while alive, jumps instantaneously at an exponentially distributed random time with intensity $\gamma > 0$ to a new point, according to a distribution $\mu \in \mathcal{P}(D)$. From this new point it repeats the above behavior independently of what has transpired previously. The generator of this process is an extension of the operator $-L_{\gamma,\mu}$, defined by

$$L_{\gamma,\mu}u \equiv -\frac{1}{2}\Delta u + \gamma V_\mu(u),$$

with the Dirichlet boundary condition, where V_μ is a nonlocal "μ-centering" potential defined by

$$V_\mu(u) = u - \int_D u \, d\mu.$$

The operator $L_{\gamma,\mu}$ is symmetric only in the case that μ is normalized Lebesgue measure; thus, only in that case can it be realized as a selfadjoint operator. The corresponding semigroup is compact, and thus the spectrum of $L_{\gamma,\mu}$ consists exclusively of eigenvalues. As is well known, the principal eigenvalue gives the exponential rate of decay in t of the probability of not exiting the domain by time t. We study the behavior of the eigenvalues, our main focus being on the behavior of the principal eigenvalue for the regimes $\gamma \gg 1$ and $\gamma \ll 1$. We also consider conditions on μ that guarantee that the principal eigenvalue is monotone increasing or decreasing in γ.

1. Introduction and statement of results

Let $D \subset R^d$ be a bounded domain with $C^{2,\alpha}$-boundary and let $\mathcal{P}(D)$ denote the space of probability measures on D. Fix a measure $\mu \in \mathcal{P}(D)$, and consider a Brownian motion in D which is killed at the boundary and which, while alive, jumps instantaneously at an exponentially distributed random time with intensity $\gamma > 0$ to a new point, according to the distribution μ. From this new point it repeats the above behavior independently of what has transpired previously. Denote this process by $X(t)$, and let τ_D denote its lifetime. Denote probabilities and expectations for the Markov process $X(t)$ starting from $x \in D$ by $P_x^{\gamma,\mu}$ and $E_x^{\gamma,\mu}$.

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Define the contraction semigroup

$$T_t^{\gamma,\mu} f(x) = E_x^{\gamma,\mu}(f(X(t)); \tau_D > t),\ f \in C_0(\bar{D}),$$

where $C_0(\bar{D})$ is the space of continuous functions on $\bar{D}$ vanishing on ∂D. We will show that the infinitesimal generator of this semigroup is an extension of the operator $-L_{\gamma,\mu}$, defined on $C^2(\bar{D}) \cap \{u : u, L_{\gamma,\mu} u \in C_0(\bar{D})\}$ by

$$L_{\gamma,\mu} u \equiv -\frac{1}{2}\Delta u + \gamma V_\mu(u),$$

with the Dirichlet boundary condition, where V_μ is a nonlocal "μ-centering" potential defined by

$$V_\mu(u) = u - \int_D u\ d\mu.$$

We will show that the operator $T_t^{\gamma,\mu}$ is compact; thus, the resolvent operator for $T_t^{\gamma,\mu}$ is compact, and consequently the spectrum $\sigma(L_{\gamma,\mu})$ of $L_{\gamma,\mu}$ consists exclusively of eigenvalues. By the Krein-Rutman theorem, one deduces that $L_{\gamma,\mu}$ possesses a principal eigenvalue, $\lambda_0(\gamma,\mu)$; that is, $\lambda_0(\gamma,\mu)$ is real and simple and satisfies $\lambda_0(\gamma,\mu) = \inf\{\mathrm{Re}(\lambda) : \lambda \in \sigma(L_{\gamma,\mu})\}$ [14]. It is known that $\lambda \in \sigma(L_{\gamma,\mu})$ if and only if $\exp(-\lambda t) \in \sigma(T_t^{\gamma,\mu})$ [12]. Thus, since $||T_t^{\gamma,\mu}|| < 1$, it follows that $\lambda_0(\gamma,\mu) > 0$. We have

$$\sup_{f \in C_0(\bar{D}), ||f|| \le 1} ||T_t^{\gamma,\mu} f|| = \sup_{x \in D} P_x^{\gamma,\mu}(\tau_D > t);$$

thus, a standard result [15] allows us to conclude that

$$\lim_{t\to\infty} \frac{1}{t} \log \sup_{x\in D} P_x^{\gamma,\mu}(\tau_D > t) = -\lambda_0(\gamma,\mu).$$

It is well known that this is equivalent to

$$\lim_{t\to\infty} \frac{1}{t} \log P_x^{\gamma,\mu}(\tau_D > t) = -\lambda_0(\gamma,\mu), \quad x \in D. \tag{1.1}$$

The main focus in this paper is on the behavior of the principal eigenvalue for the regimes $\gamma \gg 1$ and $\gamma \ll 1$. We also consider conditions on μ that guarantee that the principal eigenvalue is monotone increasing or decreasing in γ.

The Brownian motion with random jumps analyzed here is a paradigm for a phenomenon that occurs in various settings and which is best illustrated perhaps in terms of computer games or the game "chutes and ladders". The object of the game is to reach the boundary of D in as little time as possible (or alternatively, to avoid reaching the boundary for as much time as possible). The game is played in rounds; however, time is always accumulating. Various obstacles (modelled by the exponential clock with intensity γ) lead to the end of a round, and each new round begins afresh from a new position, which may be deterministic or random (modelled by the measure μ). Then $\lambda_0(\gamma,\mu)$ is a measure of the probability of long-term failure (or success, depending on the rules).

A number of recent papers have treated Brownian motion with random jumps from the *boundary*, rather than from within the domain. Such a process is ergodic and possesses a unique invariant measure. The principal eigenvalue of the generator of the process is 0, the rest of the spectrum is negative, and the spectral gap, which is the supremum of the real part of the nonzero spectrum, gives the exponential rate of convergence to equilibrium. See [1], [2], [8], [9], [11].

In the past decade or so, a number of papers have treated spectral properties of elliptic operators with a nonlocal reaction term of the form

$$(\nabla \cdot a\nabla u)(x) + b(x)u + c(x)\int_D d(y)u(y)dy,$$

with the Dirichlet boundary condition, where a is positive definite and b, c, and d are functions. These papers study the location and multiplicities of the eigenvalues and the existence of a principal eigenvalue (this last point is automatic in our situation). See, for example, [4], [5], [3] and the references therein for results and applications. See also Remark 3 after Theorem 3 and Remark 1 after Theorem 4.

We now turn to the results, considering first the regime $\gamma \gg 1$. Before stating the theorem, we note that probabilistic intuition suggests the general direction of the result. Since $\gamma \gg 1$, the Brownian motion doesn't get very far before it jumps and gets redistributed according to μ. In particular then, if $\text{supp}(\mu) \subset D$, it will be very difficult for the Brownian motion to exit D, and in light of (1.1) one expects that $\lim_{\gamma\to\infty} \lambda_0(\gamma,\mu) = 0$. More generally, one expects that the leading asymptotic behavior for large γ will depend only on the behavior of μ arbitrarily close to the boundary.

For $\epsilon > 0$, let $D^\epsilon = \{x \in D : \text{dist}(x, \partial D) < \epsilon\}$. We will prove the following result.

Theorem 1. *Let $D \subset R^d$, $d \geq 1$, be a bounded domain and let $\mu \in \mathcal{P}(D)$.*

i. Assume that for some $\epsilon > 0$, the restriction of μ to D^ϵ possesses a density which belongs to $C(\bar{D}^\epsilon)$: $\mu(dx)|_{D^\epsilon} \equiv \mu(x)dx$. Then

$$\lim_{\gamma\to\infty} \gamma^{-\frac{1}{2}}\lambda_0(\gamma,\mu) = \frac{1}{\sqrt{2}}\int_{\partial D} \mu d\sigma.$$

ii. Assume that for some $\epsilon > 0$, the restriction of μ to D^ϵ possesses a density which belongs to $C_b^2(D^\epsilon) \cap C^1(\bar{D}^\epsilon)$ and vanishes on ∂D: $\mu(dx)|_{D^\epsilon} \equiv \mu(x)dx$. Then

$$\lim_{\gamma\to\infty} \lambda_0(\gamma,\mu) = \frac{1}{2}\int_{\partial D} (\nabla\mu \cdot n) d\sigma,$$

where n is the inward unit normal vector on ∂D.

iii. Assume that for some $\epsilon > 0$, the restriction of μ to D^ϵ possesses a density which belongs to $C_b^2(\bar{D}^\epsilon)$ and is such that μ and $\nabla\mu$ vanish on ∂D: $\mu(dx)|_{D^\epsilon} \equiv \mu(x)dx$. Then

$$\lim_{\gamma\to\infty} \gamma^{\frac{1}{2}}\lambda_0(\gamma,\mu) = \frac{1}{2\sqrt{2}}\int_{\partial D} \Delta\mu d\sigma.$$

(One has $\Delta\mu \geq 0$ on ∂D since μ and $\nabla\mu$ vanish on ∂D and μ is nonnegative in D.)

iv. Assume that $\mu \in \mathcal{P}(D)$ is compactly supported. Then

$$\lim_{\gamma\to\infty} \lambda_0(\gamma,\mu) = 0.$$

In fact, letting $l = \text{dist}(\text{supp}(\mu), \partial D)$ and $a = \sup\{|x - z| : z \in \partial D, x \in \text{supp}(\mu)\}$, there exists a constant $c_{l,d}$ such that

(1.2)

$$\frac{\gamma}{1 + (\frac{a^2\pi^2\gamma}{2})^{\frac{1}{4}}}\exp(-\sqrt{2}a\gamma^{\frac{1}{2}}) < \lambda_0(\gamma,\mu) < (2d+1)\gamma\exp(-\frac{l\gamma^{\frac{1}{2}}}{2\sqrt{2}d}), \textit{for } \gamma \geq c_{l,d}.$$

(In parts (i)-(iii), σ denotes Lebesgue measure on ∂D if $d \geq 2$, while it is the counting measure on the endpoints of D if $d = 1$.)

Remark. We expect that if μ and all its partial derivatives up to order k vanish on ∂D, and the derivatives of order $k+1$ do not all vanish identically on ∂D, then $\lambda_0(\gamma, \mu)$ will decay on the order of $\gamma^{-\frac{k}{2}}$. Similarly, if the density is allowed to blow up at the boundary, then the order $\gamma^{\frac{1}{2}}$ in part (i) will increase. By Proposition 1 in Section 2, the order can never be greater than γ.

Example. When $\mu = l_D$, the normalized Lebesgue measure on D, Theorem 1 gives

$$\lim_{\gamma\to\infty} \gamma^{-\frac{1}{2}} \lambda_0(\gamma, l_D) = \frac{1}{\sqrt{2}} \frac{|\partial D|}{|D|}.$$

We now turn to the regime $\gamma \ll 1$. Of course, $\lambda_0(0, \mu) = \lambda_0^D$, where λ_0^D is the principal eigenvalue of $-\frac{1}{2}\Delta$ in D with the Dirichlet boundary condition. (Henceforth, this operator (with the negative sign) will be referred to as the Dirichlet Laplacian.) We wish to determine when $\lambda_0(\gamma, \mu) > \lambda_0^D$ and when $\lambda_0(\gamma, \mu) < \lambda_0^D$, for small γ. In the former (latter) case, random jumps at low intensity cause the probability of the event $\{\tau_D > t\}$ to decay more (less) rapidly than it would for standard Brownian motion without random jumps. Let ϕ_0 denote the principal eigenfunction, normalized by $\phi_0 > 0$ and $\int_D \phi_0^2 dx = 1$, corresponding to the principal eigenvalue λ_0^D for the Dirichlet Laplacian. Let

$$F_0 \equiv \int_D \phi_0 dx \quad \text{and} \quad G_0(\mu) \equiv \int_D \phi_0 d\mu. \tag{1.3}$$

Let V_0 denote the solution to the equation

$$\begin{aligned} &\frac{1}{2}\Delta V + \lambda_0^D V = -1 + F_0\phi_0 \quad \text{in } D; \\ &V = 0 \text{ on } \partial D; \\ &\int_D V\phi_0 dx = 0. \end{aligned} \tag{1.4}$$

(Since $(\frac{1}{2}\Delta + \lambda_0^D)\phi_0 = 0$ and $\int_D(-1 + F_0\phi_0)\phi_0 dx = 0$, the Fredholm alternative guarantees the existence of a unique solution to (1.4).)

Theorem 2. *i.*

$$\frac{d\lambda_0}{d\gamma}(0^+, \mu) = 1 - F_0 G_0(\mu) = 1 - \int_D \phi_0 dx \int_D \phi_0 d\mu.$$

Thus, if $F_0G_0(\mu) < 1$ ($F_0G_0(\mu) > 1$), then $\lambda_0(\gamma, \mu) > \lambda_0^D$ ($\lambda_0(\gamma, \mu) < \lambda_0^D$), for $\gamma \ll 1$.

ii. If $F_0G_0(\mu) = 1$ and $\int_D V_0 d\mu < 0$ ($\int_D V_0 d\mu > 0$), then $\lambda_0(\gamma, \mu) > \lambda_0^D$ ($\lambda_0(\gamma, \mu) < \lambda_0^D$), for $\gamma \ll 1$, where V_0 is the solution to (1.4).

Example 1. Consider the case that $\mu = l_D$, the normalized Lebesgue measure on D. One has $G_0(l_D) = \frac{F_0}{|D|}$. The Cauchy-Schwarz inequality gives $F_0G_0(l_D) = \frac{1}{|D|}(\int_D \phi_0 dx)^2 < \int_D \phi_0^2 dx = 1$, and thus by Theorem 2, $\lambda_0(\gamma, l_D) > \lambda_0^D$, for $\gamma \ll 1$.

Example 2. Consider the case $D = (0, 1)$ with $\mu = \delta_{x_0}$, for some $x_0 \in (0, 1)$. In this case, we have $\phi_0(x) = \sqrt{2}\sin \pi x$, so $F_0 = \frac{2\sqrt{2}}{\pi}$ and $G_0 = \sqrt{2}\sin \pi x_0$. Let $x_c \equiv \frac{1}{\pi} \arcsin \frac{\pi}{4} \approx .288$. Then $F_0G_0(\delta_{x_0}) < 1$ if and only if $x_0 \in (0, x_c) \cup (1 - x_c, 1)$.

Consider now the borderline case, $\mu = \delta_{x_c}$. A long and tedious calculation reveals that the solution V_0 to (1.4) is given by $V_0(x) = \frac{6}{\pi^3}\sin \pi x + \frac{2}{\pi^2}\cos \pi x - \frac{2}{\pi^2} - \frac{4}{\pi^2} x \cos \pi x$. One checks that $\int_0^1 V_0(x) d\mu = V_0(x_c) > 0$. Thus, we conclude from Theorem 2 that for $\gamma \ll 1$, $\lambda_0(\gamma, \delta_{x_0})$ is greater than λ_0^D if $x_0 \in (0, x_c) \cup (1 - x_c, 1)$ and is less than λ_0^D if $x_0 \in [x_c, 1 - x_c]$.

It follows from Theorems 1 and 2 that $\lambda_0(\gamma, \mu)$ is frequently not monotone in γ; one can easily construct examples where it increases and then decreases or vice versa. Note that if there exists a point $x_0 \in D$ such that $\mathcal{P}_{x_0}(\tau_D > t) \geq \mathcal{P}_x(\tau_D > t)$, for all $x \in D$ and all $t > 0$, where $\mathcal{P}_x(\tau_D > t)$ denotes the probability that a standard Brownian motion starting from $x \in D$ has not exited D by time t, then by the definition of the Brownian motion with random jumps along with probabilistic considerations and (1.1), it follows that $\lambda_0(\gamma, \delta_{x_0})$ is decreasing in γ, and furthermore, that $\lambda_0(\gamma, \delta_{x_0}) \leq \lambda_0(\gamma, \mu)$, for all $\mu \in \mathcal{P}(D)$ and all $\gamma > 0$. In particular, this occurs if $D = (0, 1)$ and $x_0 = \frac{1}{2}$.

In the case that $\mu = l_D$, the normalized Lebesgue measure on D, $L_{\gamma,\mu}$ is symmetric and can be realized as a selfadjoint operator. We can express the corresponding quadratic form as

$$\int_D u L_{\gamma,\mu} u dx = \frac{1}{2} \int_D |\nabla u|^2 dx + \gamma |D| \mathrm{Var}_{l_D}(u),$$

where $\mathrm{Var}_{l_D}(u) = \int_D u^2 dl_D - (\int_D u dl_D)^2 \geq 0$ is the variance of u with respect to the probability measure l_D, and $u \in C^2(D) \cap C_0(\bar{D})$. Thus, $\lambda_0(\gamma, \mu)$ is given by the variational formula

$$\lambda_0(\gamma, \mu) = \inf_{0 \neq u \in C^2(D) \cap C_0(\bar{D})} \frac{\frac{1}{2} \int_D |\nabla u|^2 dx + \gamma |D| \mathrm{Var}_{l_D}(u)}{\int_D u^2 dx}.$$

From this it follows that $\lambda_0(\gamma, \mu)$ is strictly monotone increasing in γ. Because of the selfadjointness, it also follows that all of the eigenvalues of $L_{\gamma,\mu}$ are real.

In fact, we can single out two classes of measures μ, each defined by a spectral-theoretic condition, for one of which $\lambda_0(\gamma, \mu)$ is monotone increasing and for the other of which it is monotone decreasing, and for both of which all of the eigenvalues are real, even though $L_{\gamma,\mu}$ is not selfadjoint when $\mu \neq l_D$.

We will need some additional notation to state the result. We have already introduced λ_0^D and ϕ_0. Let $\{\lambda_n^D\}_{n=1}^\infty$ denote all the nonprincipal eigenvalues of the Dirichlet Laplacian, labelled in increasing order, and let $\{\phi_n\}_{n=1}^\infty$ denote the corresponding eigenfunctions, normalized by $\int_D \phi_n^2 dx = 1$. Let

$$F_n \equiv \int_D \phi_n dx \quad \text{and} \quad G_n(\mu) \equiv \int_D \phi_n d\mu. \tag{1.5}$$

When $d \geq 3$, we will sometimes need to assume that the domain D satisfies the following condition.

Assumption 1. $\sum_{n=0}^\infty \frac{F_n}{\lambda_n^D} \phi_n$ *converges uniformly and absolutely on* D.

Remark. When $d = 1, 2$, Assumption 1 always holds [2, 7]. One has $\sum_{n=0}^\infty \frac{1}{(\lambda_n^D)^2} < \infty$ if and only if $d \leq 3$ [2]. Thus, since $\{F_n\} \in l_2$, Assumption 1 holds when $d = 3$ if the ϕ_n are uniformly bounded.

Let $\{\Lambda_n^D\}_{n=0}^\infty$ denote the collection of distinct values among the eigenvalues $\{\lambda_n^D\}_{n=0}^\infty$ of the Dirichlet Laplacian, labelled in increasing order. Let $P_{\Lambda_n^D}$ denote the orthogonal projection onto the eigenspace corresponding to the eigenvalue Λ_n^D. Note that

$$\int_D P_{\Lambda_0^D} 1 d\mu = F_0 G_0(\mu) > 0$$

and that

$$\int_D P_{\Lambda_n^D} 1 d\mu = \sum_{m:\lambda_m^D=\Lambda_n^D} F_m G_m(\mu).$$

Theorem 3. *Let $\mu \in \mathcal{P}(D)$. For part (i), assume that μ possesses an L^2-density. For parts (ii) and (iii), if $d \geq 3$ assume either that μ possesses an L^2-density or that the domain D satisfies Assumption* 1.

i-a. If $\int_D P_{\Lambda_n^D} 1 d\mu = \sum_{m:\lambda_m^D=\Lambda_n^D} F_m G_m(\mu) = 0$, for all $n \geq 1$, then $\lambda_0(\gamma,\mu) = \lambda_0^D$, for all $\gamma > 0$.

i-b. If $\int_D P_{\Lambda_n^D} 1 d\mu = \sum_{m:\lambda_m^D=\Lambda_n^D} F_m G_m(\mu) \geq 0$, for all $n \geq 1$, and is nonzero for at least one value of $n \geq 1$, then $\lambda_0(\gamma,\mu)$ is strictly increasing in γ.

i-c. If $\int_D P_{\Lambda_n^D} 1 d\mu = \sum_{m:\lambda_m^D=\Lambda_n^D} F_m G_m(\mu) \leq 0$, for all $n \geq 1$, and is nonzero for at least one value of $n \geq 1$, then $\lambda_0(\gamma,\mu)$ is strictly decreasing in γ.

ii. If $\int_D P_{\Lambda_n^D} 1 d\mu = \sum_{m:\lambda_m^D=\Lambda_n^D} F_m G_m(\mu)$ is nonnegative for all $n \geq 1$ or nonpositive for all $n \geq 1$, then all of the eigenvalues of $L_{\gamma,\mu}$ are real.

iii. Assume that $\int_D P_{\Lambda_n^D} 1 d\mu = F_n G_n(\mu) > 0$, for all $n \geq 1$, and assume that all the eigenvalues of the Dirichlet Laplacian are distinct. Thus, $\lambda_n^D = \Lambda_n^D$ and $\int_D P_{\Lambda_n^D} 1 d\mu = F_n G_n(\mu)$. Then

$$\gamma + \lambda_{n-1}^D < \lambda_n(\gamma,\mu) < \gamma + \lambda_n^D, \ n \geq 1.$$

Remark 1. Recall that the function 1 is represented in L^2 by $\sum_{n=0}^\infty F_n \phi_n$. The assumption that μ has an L^2-density is used in the proof of part (i) in order to guarantee that $\sum_{n=0}^\infty F_n G_n(\mu) = 1$. (In fact, one can check that the proof of part (i-a) goes through as long as $\sum_{n=0}^\infty F_n G_n(\mu) \leq 1$ and the proof of part (i-b) goes through as long as $\sum_{n=0}^\infty F_n G_n(\mu) \geq 1$.) If $\sum_{n=0}^\infty F_n \phi_n$ converges boundedly pointwise on the support of μ, then the bounded convergence theorem gives $\sum_{n=0}^\infty F_n G_n(\mu) = 1$, and thus part (i) holds in such a case even if μ does not have an L^2-density. In the one-dimensional case, one can show that $\sum_{n=0}^\infty F_n \phi_n$ converges boundedly pointwise away from the endpoints. Thus, when $d = 1$, part (i) can be applied to any compactly supported μ. Of course, such a μ cannot satisfy the condition of part (i-a) or (i-b) since if it did, then $\lambda_0(\gamma,\mu)$ would be nondecreasing in γ, contradicting Theorem 1.

Remark 2. The conditions for monotonicity in part (i) are of course only sufficient. When $D = (0,1)$ and $\mu = \delta_{\frac{1}{2}}$, then as noted earlier, $\lambda_0(\gamma,\delta_{\frac{1}{2}})$ is decreasing in γ; however $F_{2n} G_{2n}(\delta_{\frac{1}{2}}) = \frac{4}{(2n+1)\pi} \sin \frac{2n+1}{2}\pi$, which alternates sign.

Remark 3. In the one-dimensional setting, part (ii) of the theorem was proved in [4].

Remark 4. When one (or both) of the assumptions appearing in the statement of part (iii) is violated, one can still use the method of proof there to give estimates on the eigenvalues; however each particular case must be treated separately.

The proofs of Theorems 1 and 2 utilize a characterization of the principal eigenvalue which appears in Proposition 1 in Section 2. This characterization does not allow for a proof of Theorem 3. Instead, we make use of the following characterization of all of the eigenvalues of $L_{\gamma,\mu}$, which is of some interest in its own right.

Theorem 4. *Let $\mu \in \mathcal{P}(D)$ and $\gamma > 0$. If $d \geq 3$, assume either that μ possesses an L^2-density or that the domain D satisfies Assumption 1. Let d_n denote the dimension of the eigenspace corresponding to the n-th distinct eigenvalue Λ_n^D of the Dirichlet Laplacian. Let*

$$E_{\gamma,\mu}(\lambda) \equiv \sum_{n=0}^{\infty} \frac{\gamma F_n G_n(\mu)}{\gamma + \lambda_n^D - \lambda} = \sum_{n=0}^{\infty} \frac{\gamma \int_D P_{\Lambda_n^D} 1 d\mu}{\gamma + \Lambda_n^D - \lambda}. \tag{1.6}$$

The set of eigenvalues of $L_{\gamma,\mu}$ and their multiplicities are given as follows.

i. The set $\{\lambda : E_{\gamma,\mu}(\lambda) = 1\} \backslash \{\gamma + \Lambda_n^D\}_{n=1}^{\infty}$ consists of simple eigenvalues.

ii. For each $n = 1, 2, \ldots$, the following rule determines whether $\gamma + \Lambda_n^D$ is an eigenvalue, and if so, specifies its multiplicity.

If $d_n = 1$ and neither $F_m = 0$ nor $G_m(\mu) = 0$, for the unique m satisfying $\lambda_m^D = \Lambda_n^D$, then $\gamma + \Lambda_n^D$ is not an eigenvalue. Otherwise, $\gamma + \Lambda_n^D$ is an eigenvalue and its multiplicity is specified as follows:

if $G_m(\mu) \neq 0$ for some m such that $\lambda_m^D = \Lambda_n^D$ and $F_m \neq 0$ for some m such that $\lambda_m^D = \Lambda_n^D$, then the multiplicity is $d_n - 1$;

if $G_m(\mu) = 0$ for all m such that $\lambda_m^D = \Lambda_n^D$ and $F_m \neq 0$ for some m such that $\lambda_m^D = \Lambda_n^D$, or if $G_m(\mu) \neq 0$ for some m such that $\lambda_m^D = \Lambda_n^D$ and $F_m = 0$ for all m such that $\lambda_m^D = \Lambda_n^D$, then the multiplicity is d_n;

if $G_m(\mu) = 0$ for all m such that $\lambda_m^D = \Lambda_n^D$ and $F_m = 0$ for all m such that $\lambda_m^D = \Lambda_n^D$, then the multiplicity is d_n if $E_{\gamma,\mu}(\gamma + \Lambda_n^D) \neq 0$ and is d_{n+1} if $E_{\gamma,\mu}(\gamma + \Lambda_n^D) = 0$.

Remark 1. Partial results along the lines of Theorem 4 can be found in [5] and [3].

Remark 2. For $n \geq 0$, if $\int_D P_{\Lambda_n^D} 1 d\mu$ and $\int_D P_{\Lambda_{n+1}^D} 1 d\mu$ have opposite signs, then there is an eigenvalue between $\gamma + \Lambda_n^D$ and $\gamma + \Lambda_{n+1}^D$, since in this case $E_{\gamma,\mu}(\lambda)$, with $\Lambda_n^D < \lambda < \Lambda_{n+1}^D$, approaches $+\infty$ as λ approaches one endpoint and approaches $-\infty$ as λ approaches the other endpoint.

The rest of the paper is organized as follows. In Section 2 we prove a couple of preliminary results concerning the principle eigenvalue $\lambda_0(\gamma, \mu)$, which will be used to prove Theorems 1 and 2, the proofs of which are given in Sections 3 and 4, respectively. We prove Theorem 4 in Section 5 and Theorem 3, whose proof depends on Theorem 4, in Section 6. In an appendix we show that $-L_{\gamma,\mu}$, suitably extended, is the infinitesimal generator of $T_t^{\gamma,\mu}$, and that $T_t^{\gamma,\mu}$ is compact.

All the results in the paper go through with only cosmetic changes when Brownian motion and the Laplacian are replaced by a general reversible diffusion and its generator $A \equiv \frac{1}{2}\exp(-2Q)\nabla \cdot \exp(2Q) a \nabla = \frac{1}{2}\nabla \cdot a\nabla + \nabla Q \nabla$.

2. Preliminary results for the principal eigenvalue

Let $Y(t)$ denote the standard Brownian motion in D without jumps, which is killed at the boundary, let τ_D denote its lifetime and denote the corresponding probabilities and expectations by $\mathcal{P}.$ and $\mathcal{E}.$.

For $c \in (-\infty, \lambda_0^D)$, let $w_c > 0$ denote the solution to the equation

$$\begin{aligned} &\frac{1}{2}\Delta w + cw = -1 \text{ in } D;\\ &w = 0 \text{ on } \partial D. \end{aligned} \tag{2.1}$$

Applying Ito's formula with the stopping time τ_D gives

$$\mathcal{E}_x \exp(c\tau_D) w_c(Y(\tau_D)) = w_c(x) + \mathcal{E}_x \int_0^{\tau_D} (\frac{1}{2}\Delta + c) w_c(Y(t)) \exp(ct) dt. \tag{2.2}$$

From (2.1) and (2.2) we obtain

$$w_c(x) = \begin{cases} \frac{1}{c}(\mathcal{E}_x \exp(c\tau_D) - 1), \text{ if } c \neq 0;\\ \mathcal{E}_x \tau_D, \text{ if } c = 0. \end{cases} \tag{2.3}$$

(The condition $c < \lambda_0^D$ is necessary and sufficient for the existence of a positive solution to (2.1) and for the finiteness of $\mathcal{E}_x \exp(c\tau_D)$ [14, Chapter 3].) Similarly, the solution u_c to the equation

$$\begin{aligned} &\frac{1}{2}\Delta u + cu = 0 \text{ in } D,\\ &u = 1 \text{ on } \partial D, \end{aligned} \tag{2.4}$$

is given by

$$u_c(x) = \mathcal{E}_x \exp(c\tau_D). \tag{2.5}$$

We begin with a characterization of the principal eigenvalue $\lambda_0(\gamma, \mu)$ of $L_{\gamma,\mu}$.

Proposition 1. *One has $\lambda_0(\gamma, \mu) < \gamma + \lambda_0^D$, for all $\gamma > 0$. More specifically, for $\lambda < \gamma + \lambda_0^D$, consider the equation*

(2.6)

$$\lambda = \begin{cases} \gamma \mathcal{E}_\mu \exp((\lambda - \gamma)\tau_D), \ \gamma \neq \lambda\\ (\mathcal{E}_\mu \tau_D)^{-1}, \ \gamma = \lambda \end{cases} \quad \left(\text{or equivalently}, \lambda = \begin{cases} \gamma \int_D u_{\lambda-\gamma} d\mu, \ \gamma \neq \lambda\\ (\mathcal{E}_\mu \tau_D)^{-1}, \ \gamma = \lambda \end{cases} \right).$$

i. If $\gamma > (\mathcal{E}_\mu \tau_D)^{-1}$, then $\lambda_0(\gamma, \mu) \in (0, \gamma)$ and is equal to the smallest root $\lambda \in (0, \gamma)$ of (2.6).

ii. If $\gamma < (\mathcal{E}_\mu \tau_D)^{-1}$, then $\lambda_0(\gamma, \mu) \in (\gamma, \gamma + \lambda_0^D)$ and is equal to the smallest root $\lambda \in (\gamma, \gamma + \lambda_0^D)$ of (2.6).

iii. If $\gamma = (\mathcal{E}_\mu \tau_D)^{-1}$, then $\lambda_0(\gamma, \mu) = \gamma = (\mathcal{E}_\mu \tau_D)^{-1}$.

Proof. Let w denote the principal eigenfunction corresponding to $\lambda_0(\gamma, \mu)$, normalized by $\int_D w d\mu = 1$. Then

$$\begin{aligned} &\frac{1}{2}\Delta w - \gamma w + \gamma = -\lambda_0(\gamma, \mu) w \text{ in } D;\\ &w = 0 \ \text{ on } \partial D. \end{aligned}$$

Thus, $w = \gamma w_{\lambda_0(\gamma,\mu)-\gamma}$. From the normalization condition above, it then follows that $\lambda = \lambda_0(\gamma, \mu)$ is a solution to the equation

$$\gamma \int_D w_{\lambda-\gamma} d\mu = 1. \tag{2.7}$$

Conversely, if λ solves (2.7), then it is an eigenvalue. Consequently, $\lambda_0(\gamma, \mu)$ is the smallest solution λ to (2.7). By (2.3) and (2.5), it follows that (2.7) is equivalent to (2.6).

Fix $\gamma > 0$ and let $q(\lambda) = \lambda - \gamma\mathcal{E}_\mu \exp((\lambda-\gamma)\tau_D)$. Then $q(0) < 0$ and $q(\gamma) = 0$. If $\gamma > (\mathcal{E}_\mu\tau_D)^{-1}$, then $q'(\gamma) < 0$, and we conclude that the smallest root λ of (2.6) occurs in $(0,\gamma)$. This proves part (i). If $\gamma \le (\mathcal{E}_\mu\tau_D)^{-1}$, then $q'(\lambda) > 0$, for $\lambda \in (0,\gamma)$, and thus (2.6) has no root $\lambda \in (0,\gamma)$. If $\gamma = (\mathcal{E}_\mu\tau_D)^{-1}$, we then conclude that the smallest root of (2.6) occurs at $\lambda = \gamma$. If $\gamma < (\mathcal{E}_\mu\tau_D)^{-1}$, then $q'(\gamma) > 0$. Since $q(\gamma + \lambda_0^D) = -\infty$, we conclude that the smallest root of (2.6) occurs in $(\gamma, \gamma + \lambda_0^D)$. □

The following lemma will be used repeatedly.

Lemma 1.

$$\lim_{\gamma\to\infty} \frac{\lambda_0(\gamma,\mu)}{\gamma} = 0.$$

Proof. By Proposition 1, it is enough to show that as $\gamma \to \infty$, the quotient $\frac{\lambda_0(\gamma,\mu)}{\gamma}$ has no accumulation points in $(0,1]$. We first show that there are no accumulation points in $(0,1)$. If $p \in (0,1)$ were an accumulation point, then substituting $\lambda_0(\gamma,\mu)$ for λ in (2.6), dividing both sides by γ and letting $\gamma \to \infty$ along an appropriate sequence would give the contradiction $p = 0$.

We now show that 1 is not an accumulation point. Assume to the contrary that $\lim_{n\to\infty} \frac{\lambda_0(\gamma_n.\mu)}{\gamma_n} = 1$, where $\lim_{n\to\infty}\gamma_n = \infty$. By Proposition 1, we have $\lambda_0(\gamma_n,\mu) = \gamma_n - c_n$, where $c_n > 0$ for sufficiently large n. Without loss of generality we may assume that $\lim_{n\to\infty} c_n \equiv c_\infty \in [0,\infty]$ exists. First consider the case that $c_\infty \ne 0$. Then substituting $\lambda_0(\gamma_n,\mu)$, γ_n and $-c_n$ respectively for λ, γ and $\lambda - \gamma$ in (2.6), dividing both sides by γ_n and letting $n \to \infty$ gives the contradiction $1 = \mathcal{E}_\mu \exp(-c_\infty\tau_D)$. Now consider the case $c_\infty = 0$. Rewrite (2.6) as $\lambda - \gamma = \gamma(\mathcal{E}_\mu \exp((\lambda-\gamma)\tau_D) - 1)$. Substituting as before, we obtain $-c_n = \gamma_n(\mathcal{E}_\mu \exp(-c_n\tau_D) - 1)$. Dividing both sides by $-c_n$ and letting $n \to \infty$, we obtain the contradiction $1 = \infty$. □

3. Proof of Theorem 1

It will be convenient to prove the results in an order different from that in which they were stated.

Part iv. Clearly $\mathcal{E}_\mu\tau_D$ can be bounded from below by a positive constant depending only on $l \equiv \text{dist}(\text{supp}(\mu), \partial D)$ and d. By Proposition 1, $\lambda_0(\gamma,\mu)$ is the smallest root $\lambda \in (0,\gamma)$ of (2.6) when $\gamma > (\mathcal{E}_\mu\tau_D)^{-1}$. As functions of λ, both the left hand side and the right hand side of (2.6) are increasing. Furthermore, the left hand side is smaller than the right hand side when $\lambda = 0$, and it is larger than the right hand side when $\lambda = \frac{\gamma}{2}$, if $\mathcal{E}_\mu \exp(-\frac{\gamma}{2}\tau_D) < \frac{1}{2}$. This last inequality holds when γ is larger than a constant depending only on l and d. Thus there exists a constant $c_{l,d}$ such that

(3.1)
$$\gamma\int_D \mathcal{E}_x \exp(-\gamma\tau_D)d\mu(x) < \lambda_0(\gamma,\mu) < \gamma\int_D \mathcal{E}_x \exp(-\frac{1}{2}\gamma\tau_D)d\mu(x), \text{ for } \gamma \ge c_{l,d}.$$

One has the following inequality [14, Chapter 2]:

$$\mathcal{P}_x(\tau_D \le t) \le 2d\exp(-\frac{\text{dist}(x,\partial D)^2}{2dt}). \tag{3.2}$$

Letting $t=\sqrt{\frac{2}{\gamma}}\,l$ in (3.2), we have the estimate

$$\begin{aligned}&\mathcal{E}_x \exp(-\frac{1}{2}\gamma\tau_D) \le \mathcal{P}_x(\tau_D \le t) + \exp(-\frac{1}{2}\gamma t)\\ &< 2d\exp(-\frac{l\gamma^{\frac{1}{2}}}{2\sqrt{2}d}) + \exp(-\frac{l\gamma^{\frac{1}{2}}}{\sqrt{2}}),\ \text{if } \operatorname{dist}(x,\partial D)\ge l.\end{aligned} \tag{3.3}$$

Thus, the upper bound in (1.2) follows from (3.3) and (3.1).

By the reflection principle for one-dimensional Brownian motion [10], it follows that in any dimension,

$$\mathcal{P}_x(\tau_D \le t) \ge 2\int_a^\infty \frac{1}{(2\pi t)^{\frac{1}{2}}}\exp(-\frac{|y|^2}{2t})dy,\ \text{where } a=\sup\{|x-z|: z\in\partial D\}. \tag{3.4}$$

One has the following inequality [13, Lemma 3.6]:

$$\begin{aligned}&\int_a^\infty \frac{1}{(2\pi t)^{\frac{1}{2}}}\exp(-\frac{|y|^2}{2t})dy = \int_{at^{-\frac{1}{2}}}^\infty \frac{1}{(2\pi)^{\frac{1}{2}}}\exp(-\frac{|y|^2}{2})dy\\ &\ge \frac{1}{(2\pi)^{\frac{1}{2}}}\frac{1}{at^{-\frac{1}{2}}+(\frac{2}{\pi})^{\frac{1}{2}}}\exp(-\frac{a^2}{2t}).\end{aligned} \tag{3.5}$$

Letting $t=\frac{a}{(2\gamma)^{\frac{1}{2}}}$ in (3.5), and using (3.4), we have the estimate

$$\begin{aligned}&\mathcal{E}_x\exp(-\gamma\tau_D) \ge \exp(-\gamma t)\mathcal{P}_x(\tau_D\le t) \ge \frac{1}{1+(\frac{a^2\pi^2\gamma}{2})^{\frac{1}{4}}}\exp(-\sqrt{2}a\gamma^{\frac{1}{2}}),\\ &\text{if } \sup\{|x-z|: z\in\partial D\}\le a.\end{aligned} \tag{3.6}$$

The lower bound in (1.2) follows from (3.1) and (3.6).

Part ii. By assumption, we can represent μ in the form $\mu=\mu_{\rm reg}dx+\mu_{\rm cs}$, where $\mu_{reg}\in C_b^2(D)\cap C^1(\bar D)$ is a sub-probability density on D which coincides with the density μ in the statement of the theorem in a neighborhood of ∂D, and where $\mu_{\rm cs}$ is a compactly supported sub-probability measure on D. By Proposition 1, we have

$$\begin{aligned}\lambda_0(\gamma,\mu) &= \gamma\mathcal{E}_\mu\exp((\lambda_0(\gamma,\mu)-\gamma)\tau_D)\\ &= \gamma\int_D \mathcal{E}_x\exp((\lambda_0(\gamma,\mu)-\gamma)\tau_D)\mu_{\rm reg}(x)dx\\ &\quad + \gamma\int_D \mathcal{E}_x\exp((\lambda_0(\gamma,\mu)-\gamma)\tau_D)d\mu_{\rm cs}(x).\end{aligned} \tag{3.7}$$

The proof of part (iv) showed that

$$\lim_{\gamma\to\infty}\gamma\int_D \mathcal{E}_x\exp((\lambda_0(\gamma,\mu)-\gamma)\tau_D)d\mu_{\rm cs}(x)=0. \tag{3.8}$$

Using the fact that close to the boundary $\mu_{\rm reg}$ coincides with the density μ in the statement of the theorem along with the assumption that μ vanishes on the boundary, we obtain from (2.5), (2.4) and integration by parts that

(3.9)

$$\begin{aligned}
&\gamma\int_D \mathcal{E}_x \exp((\lambda_0(\gamma,\mu)-\gamma)\tau_D)\mu_{\text{reg}}(x)dx = \gamma\int_D u_{\lambda_0(\gamma,\mu)-\gamma}\mu_{\text{reg}}dx\\
&= \frac{\gamma}{2(\gamma-\lambda_0(\gamma,\mu))}\int_D \mu_{\text{reg}}\Delta u_{\lambda_0(\gamma,\mu)-\gamma}dx\\
&= \frac{\gamma}{2(\gamma-\lambda_0(\gamma,\mu))}\int_{\partial D}\nabla\mu_{\text{reg}}\cdot n d\sigma + \frac{\gamma}{2(\gamma-\lambda_0(\gamma,\mu))}\int_D u_{\lambda_0(\gamma,\mu)-\gamma}\Delta\mu_{\text{reg}}dx,
\end{aligned}$$

where n denotes the inward unit normal on ∂D. By assumption, $\Delta\mu_{\text{reg}}$ is bounded in D. By Lemma 1 and (2.5) it follows that $u_{\lambda_0(\gamma,\mu)-\gamma}(x) = \mathcal{E}_x \exp((\lambda_0(\gamma,\mu)-\gamma)\tau_D) \le \mathcal{E}_x \exp(-\frac{1}{2}\gamma\tau_D)$, for sufficiently large γ. Thus, the bounded convergence theorem and Lemma 1 give

$$\lim_{\gamma\to\infty}\frac{\gamma}{2(\gamma-\lambda_0(\gamma,\mu))}\int_D u_{\lambda_0(\gamma,\mu)-\gamma}\Delta\mu_{\text{reg}}dx = 0. \tag{3.10}$$

From (3.7)–(3.10), Lemma 1 and the fact that $\mu_{\text{reg}} = \mu$ in a neighborhood of ∂D, one concludes that

$$\lim_{\gamma\to\infty}\lambda_0(\gamma,\mu) = \frac{1}{2}\int_{\partial D}\nabla\mu\cdot n d\sigma.$$

Part iii. From (3.7)–(3.9) and the assumption that μ and $\nabla\mu$ vanish on ∂D, we obtain

$$\lambda_0(\gamma,\mu) = \frac{\gamma}{2(\gamma-\lambda_0(\gamma,\mu))}\int_D u_{\lambda_0(\gamma,\mu)-\gamma}\Delta\mu_{\text{reg}}dx. \tag{3.11}$$

By assumption, $\Delta\mu_{\text{reg}}$ is continuous on $\bar{D}$ and coincides with $\Delta\mu$ in a neighborhood of ∂D. Thus, similar to (3.19) in the proof of part (i) below, we have

$$\lim_{\gamma\to\infty}\gamma^{\frac{1}{2}}\int_D u_{\lambda_0(\gamma,\mu)-\gamma}\Delta\mu_{\text{reg}}dx = \frac{1}{\sqrt{2}}\int_{\partial D}\Delta\mu d\sigma. \tag{3.12}$$

From (3.11), (3.12) and Lemma 1, we conclude that

$$\lim_{\gamma\to\infty}\gamma^{\frac{1}{2}}\lambda_0(\gamma,\mu) = \frac{1}{2\sqrt{2}}\int_{\partial D}\Delta\mu d\sigma.$$

Part i. As the proofs of the other parts have shown, we may ignore any compactly supported part of μ. Thus, by assumption, we may assume that μ possesses a continuous density, denoted by μ. From (2.6) we have

$$\lambda_0(\gamma,\mu) = \gamma\int_D u_{\lambda_0(\gamma,\mu)-\gamma}\mu dx. \tag{3.13}$$

Let $\hat{\mu}$ be the harmonic function in D which coincides with μ on ∂D. Using (2.4), integrating by parts and noting that

$$\int_{\partial D} u_{\lambda_0(\gamma,\mu)-\gamma}\nabla\hat{\mu}\cdot N d\sigma = \int_{\partial D}\nabla\hat{\mu}\cdot N d\sigma = \int_D \Delta\hat{u}dx = 0,$$

where N denotes the outward unit normal on ∂D, we have

$$\begin{aligned}\int_D u_{\lambda_0(\gamma,\mu)-\gamma}\hat{\mu}dx &= \frac{1}{2(\gamma-\lambda_0(\gamma,\mu))}\int_D \hat{\mu}\Delta u_{\lambda_0(\gamma,\mu)-\gamma}dx\\ &= \frac{1}{2(\gamma-\lambda_0(\gamma,\mu))}\int_{\partial D}\hat{\mu}\nabla u_{\lambda_0(\gamma,\mu)-\gamma}\cdot Nd\sigma\\ &= \frac{1}{2(\gamma-\lambda_0(\gamma,\mu))}\int_{\partial D}\mu\nabla u_{\lambda_0(\gamma,\mu)-\gamma}\cdot Nd\sigma.\end{aligned} \tag{3.14}$$

We will show below that

$$\lim_{\gamma\to\infty}\gamma^{-\frac{1}{2}}(\nabla u_{-\gamma}\cdot N)(x)=\sqrt{2},\ \text{uniformly over } x\in\partial D. \tag{3.15}$$

From (3.14), (3.15) and Lemma 1, we have

$$\lim_{\gamma\to\infty}\gamma^{\frac{1}{2}}\int_D u_{\lambda_0(\gamma,\mu)-\gamma}\hat{\mu}dx=\frac{1}{\sqrt{2}}\int_{\partial D}\mu d\sigma. \tag{3.16}$$

Now consider $\gamma^{\frac{1}{2}}\int_D u_{\lambda_0(\gamma,\mu)-\gamma}(\mu-\hat{\mu})dx$. The proof of part (iv) showed that $\lim_{\gamma\to\infty}\gamma^{\frac{1}{2}}\int_U u_{\lambda_0(\gamma,\mu)-\gamma}dx=0$, for any set U satisfying $\bar{U}\subset D$. Thus, for any $\epsilon>0$,

$$\begin{aligned}&\limsup_{\gamma\to\infty}|\gamma^{\frac{1}{2}}\int_D u_{\lambda_0(\gamma,\mu)-\gamma}(\mu-\hat{\mu})dx|\\ &\le \sup_{x\in D_\epsilon}|\mu(x)-\hat{\mu}(x)|\limsup_{\gamma\to\infty}\gamma^{\frac{1}{2}}\int_D u_{\lambda_0(\gamma,\mu)-\gamma}dx,\end{aligned} \tag{3.17}$$

where D^ϵ is as in the statement of the theorem. In the case that $\mu=1$ on ∂D, one has $\hat{\mu}\equiv 1$ and (3.16) gives $\lim_{\gamma\to\infty}\gamma^{\frac{1}{2}}\int_D u_{\lambda_0(\gamma,\mu)-\gamma}dx=\frac{1}{\sqrt{2}}|\partial D|$. Substituting this in (3.17), using the fact that $\epsilon>0$ is arbitrary and that μ and $\hat{\mu}$ are continuous and coincide on ∂D, we obtain

$$\lim_{\gamma\to\infty}\gamma^{\frac{1}{2}}\int_D u_{\lambda_0(\gamma,\mu)-\gamma}(\mu-\hat{\mu})dx=0. \tag{3.18}$$

Now (3.16) and (3.18) give

$$\lim_{\gamma\to\infty}\gamma^{\frac{1}{2}}\int_D u_{\lambda_0(\gamma,\mu)-\gamma}\mu dx=\frac{1}{\sqrt{2}}\int_{\partial D}\mu d\sigma, \tag{3.19}$$

and thus, from (3.13), we conclude that

$$\lim_{\gamma\to\infty}\gamma^{-\frac{1}{2}}\lambda_0(\gamma,\mu)=\frac{1}{\sqrt{2}}\int_{\partial D}\mu d\sigma.$$

We now return to prove (3.15). Fix a point $x_0\in\partial D$. We begin with a localization result. For small $\delta>0$, let $(\partial D)^\delta=\{x\in\partial D:\text{dist}(x,x_0)<\delta\}$. Let $U\subset D$ be a domain with $(\partial D)^\delta\subset\partial U$. For $\gamma>0$, let f be a continuous function on ∂U satisfying $f(x)=1$, for $x\in(\partial D)^\delta$, and $\sup_{x\in\partial U}|f(x)|<\infty$. Let $v_{-\gamma}$ solve the equation

$$\begin{aligned}&\frac{1}{2}\Delta v-\gamma v=0 \text{ in } U;\\ &v=f \text{ on } \partial U.\end{aligned} \tag{3.20}$$

Let g_γ denote the restriction of $u_{-\gamma}$ to ∂U. Let $W_\gamma = u_{-\gamma} - v_{-\gamma}$. Then W_γ satisfies the equation

$$\frac{1}{2}\Delta W - \gamma W = 0 \text{ in } U; \qquad W = g_\gamma - f \text{ on } \partial U.$$

Let $h(x)$ be a continuous function on ∂U satisfying $h(x) \geq \sup_{\gamma\geq 0} |g_\gamma(x) - f(x)|$, for $x \in \partial U$, and $h(x) = 0$, for $x \in (\partial D)^\delta$. By the maximum principle,

$$|W_\gamma| \leq \hat{W},$$

where $\hat{W}$ satisfies

$$\frac{1}{2}\Delta \hat{W} - \gamma \hat{W} = 0 \text{ in } U; \qquad \hat{W} = h \text{ on } \partial U. \tag{3.21}$$

Since $W_\gamma(x_0) = \hat{W}(x_0) = 0$, we have

$$|(\nabla W_\gamma \cdot N)(x_0)| = |\lim_{n\to\infty} \frac{W_\gamma(x_n)}{\text{dist}(x_n, x_0)}| \leq \lim_{n\to\infty} \frac{\hat{W}(x_n)}{\text{dist}(x_n, x_0)} = (\nabla \hat{W} \cdot N)(x_0) < \infty, \tag{3.22}$$

where x_n approaches x_0 from the normal direction. From (3.22) it follows that

$$\lim_{\gamma\to\infty} \gamma^{-\frac{1}{2}} (\nabla u_{-\gamma} \cdot N)(x_0) = \sqrt{2} \text{ if and only if } \lim_{\gamma\to\infty} \gamma^{-\frac{1}{2}} (\nabla v_{-\gamma} \cdot N)(x_0) = \sqrt{2}, \text{ where } v_{-\gamma} \text{ solves (3.20).} \tag{3.23}$$

We now prove a comparison result. Let D_i, $i = 1, 2$, be domains with smooth boundaries satisfying $D_1 \subset D \subset D_2$ and $x_0 \in \partial D_i$, $i = 1, 2$. Let $u^{(i)}_{-\gamma}(x) = \mathcal{E}_x \exp(-\gamma \tau_{D_i})$. Either by comparing the stochastic representations or by the maximum principle it follows that $u^{(2)}_{-\gamma}(x) \leq u_{-\gamma}(x) \leq u^{(1)}_{-\gamma}(x)$, for $x \in D^1$. Using this along with the fact that $u^{(i)}_{-\gamma}(x_0) = u_{-\gamma}(x_0) = 1$, for $i = 1, 2$, we obtain

$$(\nabla u^{(1)}_{-\gamma} \cdot N)(x_0) \leq (\nabla u_{-\gamma} \cdot N)(x_0) \leq (\nabla u^{(2)}_{-\gamma} \cdot N)(x_0). \tag{3.24}$$

In light of (3.24), to prove (3.15) it is enough to show that if the curvature of ∂D at $x_0 \in \partial D$ is given by $R \in (-\infty, \infty)$, then

$$\lim_{\gamma\to\infty} \gamma^{-\frac{1}{2}} (\nabla u_{-\gamma} \cdot N)(x_0) = \sqrt{2}, \text{ and the convergence is uniform over } R \text{ in any bounded set.} \tag{3.25}$$

In light of this and (3.23), it suffices to consider the following situation: for $R > 0$, we consider $\frac{du_{-\gamma}}{dr}(R)$, where $u_{-\gamma}$ is radially symmetric and satisfies (2.4) and (2.5) with $c = -\gamma$ and $D = A_{\frac{R}{2},R}(0) \equiv \{x \in R^d : \frac{R}{2} < |x| < R\}$; for $R < 0$, we consider $-\frac{du_{-\gamma}}{dr}(R)$, where $u_{-\gamma}$ is radially symmetric and satisfies (2.4) and (2.5) with $c = -\gamma$, and $D = A_{R,2R}(0) \equiv \{x \in R^d : R < |x| < 2R\}$; for $R = 0$, the boundary is flat, and without loss of generality we consider $-\frac{du_{-\gamma}(0)}{dx}$, where $u_{-\gamma}$ satisfies (2.4) and (2.5) with $c = -\gamma$, and $D = (0, 1)$.

The flat case has been reduced above to a constant coefficient second-order ODE; one solves explicitly and finds that $\lim_{\gamma\to\infty} \gamma^{-\frac{1}{2}}(-\frac{du_{-\gamma}(0)}{dx}) = \sqrt{2}$. We now turn to the case $R > 0$. Let

$$v_{-\gamma}(r) = \phi_\gamma(r)\exp(-\sqrt{2\gamma}(R-r)),\ \frac{R}{2} \le r \le R, \tag{3.26}$$

for some function ϕ_γ. For a radial function $v(r)$, one has $\Delta v = v'' + \frac{d-1}{r}v'$. Using this with (3.26) one finds that $v_{-\gamma}$ will solve the equation

$$\begin{aligned} &\frac{1}{2}\Delta v_{-\gamma} - \gamma v_{-\gamma} = 0,\ \frac{R}{2} < r < R, \\ &v_{-\gamma}(R) = 1,\ v'_{-\gamma}(R) = \sqrt{2\gamma}, \end{aligned} \tag{3.27}$$

if ϕ_γ solves the equation

$$\begin{aligned} &\frac{1}{2}\phi'' + (\sqrt{2\gamma} + \frac{d-1}{2r})\phi' + \frac{d-1}{2r}\sqrt{2\gamma}\phi = 0,\ \frac{R}{2} < r < R, \\ &\phi(R) = 1,\ \phi'(R) = 0. \end{aligned} \tag{3.28}$$

By the standard theory of linear ODE's, (3.28) has a unique solution. Furthermore, one has $\sup_{\gamma>0}|\phi_\gamma(\frac{R}{2})| < \infty$. This can be shown by making an appropriate comparison with the solution ψ_γ of a constant coefficient ODE of the form $\frac{1}{2}\psi'' + a(\gamma)\psi' + b(\gamma)\psi = 0$ with $\psi(R) = 1$ and $\psi'(R) = 0$, where $a(\gamma)$ and $b(\gamma)$ are of the order $\gamma^{\frac{1}{2}}$. One can calculate ψ_γ explicitly and show that $\sup_{\gamma>0}|\psi_\gamma(\frac{R}{2})| < \infty$. It follows then that $v_{-\gamma}(\frac{R}{2})$ is bounded in γ. Consequently, the $\{v_{-\gamma}\}_{\gamma>0}$ solve equations of the form (3.20). (The f in (3.20) will now depend on γ but is uniformly bounded in γ.)

Since $\gamma^{-\frac{1}{2}}\frac{dv_{-\gamma}}{dr}(R) = \sqrt{2}$ and since (3.27) is an equation of the form (3.20), it follows from (3.23) that $\lim_{\gamma\to\infty}\gamma^{-\frac{1}{2}}(\nabla u_{-\gamma}\cdot N)(R) = \sqrt{2}$. The difference $|(\nabla u_{-\gamma}\cdot N)(R) - (\nabla v_{-\gamma}\cdot N)(R)|$ is bounded from above by the right hand side of (3.22), where the function $\hat{W}$ solves an equation of the form (3.21) and depends on R. By taking the supremum of this quantity over R in a bounded set, one concludes that (3.25) holds. The case $R < 0$ is dealt with in an almost identical manner. □

4. Proof of Theorem 2

We can't use Proposition 1 and (2.6) directly to analyze $\lambda_0(\gamma,\mu)$ for small γ because $\mathcal{E}_x\exp(\lambda_0^D\tau_D) = \infty$. We make a renormalization. The proof of Proposition 1 showed that $\lambda_0(\gamma,\mu)$ solves (2.7) for λ. Recall the definition of F_0 in (1.3) and let

$$V_\gamma = w_{\lambda_0(\gamma,\mu)-\gamma} - \frac{F_0\phi_0}{\lambda_0^D + \gamma - \lambda_0(\gamma,\mu)}. \tag{4.1}$$

Using (2.1) and the fact that ϕ_0 is the principal eigenfunction corresponding to the principal eigenvalue λ_0^D for the Dirichlet Laplacian, one calculates that

$$\begin{aligned} &\frac{1}{2}\Delta V_\gamma + (\lambda_0(\gamma,\mu) - \gamma)V_\gamma = -1 + F_0\phi_0 \text{ in } D; \\ &V_\gamma = 0 \text{ on } \partial D; \\ &\int_D V_\gamma\phi_0 dx = 0. \end{aligned} \tag{4.2}$$

Using (4.1) and recalling the definition of $G_0(\mu)$ in (1.3), one can write (2.7) with $\lambda = \lambda_0(\gamma,\mu)$ in the form

$$\gamma\int_D V_\gamma d\mu + \frac{\gamma F_0 G_0(\mu)}{\lambda_0^D + \gamma - \lambda_0(\gamma,\mu)} = 1,$$

or equivalently,

$$\gamma \int_D V_\gamma d\mu + \frac{F_0 G_0(\mu)}{1 - \frac{\lambda_0(\gamma,\mu) - \lambda_0^D}{\gamma}} = 1. \tag{4.3}$$

From (4.2) we have $\lim_{\gamma \to 0} V_\gamma = V_0$, where V_0 is the solution to (1.4). Thus, it follows from (4.3) that $\lambda_0(\gamma,\mu)$ is differentiable from the right at $\gamma = 0$ and that $\frac{d^+\lambda_0}{d\gamma^+}(0,\mu) = 1 - F_0 G_0(\mu)$. In the case that $F_0 G_0(\mu) = 1$, it follows from (4.3) that if $\int_D V_0 d\mu < 0$ ($\int_D V_0 d\mu > 0$), then for $\gamma \ll 1$ one has $\lambda_0(\gamma,\mu) - \lambda_0^D > 0$ ($\lambda_0(\gamma,\mu) - \lambda_0^D < 0$). □

5. Proof of Theorem 4

A number $\lambda \in C$ will constitute an eigenvalue for $L_{\gamma,\mu}$ if and only if there exists a function u vanishing on ∂D satisfying

$$-\frac{1}{2}\Delta u + \gamma u - \gamma c = \lambda u, \tag{5.1}$$

where $c \equiv \int_D u d\mu$. (Note that since $L_{\gamma,\mu} u = \lambda u$ and $u \in C_0(\bar{D})$, the condition $L_{\gamma,\mu} u \in C_0(\bar{D})$ is fulfilled automatically.) Recalling (1.5), the function 1 is represented in $L^2(D)$ by $1 = \sum_{n=0}^\infty F_n \phi_n$. We represent a proposed eigenfunction $u \in L^2(D)$ by $u = \sum_{n=0}^\infty c_n \phi_n$. In order for u to be an eigenfunction, it must lie in the domain of the Dirichlet Laplacian; thus $-\frac{1}{2}\Delta u = \sum_{n=0}^\infty \lambda_n^D c_n \phi_n$. Substituting in (5.1) and equating coefficients, we find that

$$c_n(\gamma + \lambda_n^D) - c\gamma F_n = \lambda c_n,\ n = 0, 1, \ldots. \tag{5.2}$$

We first show that the condition $E_{\gamma,\mu}(\lambda) = 1$ is necessary and sufficient for $\lambda \notin \{\gamma + \Lambda_n^D\}_{n=0}^\infty$ to be an eigenvalue. Note that if u is an eigenfunction for $L_{\gamma,,\mu}$ and $\int_D u d\mu = 0$, then u is an eigenfunction for $-\frac{1}{2}\Delta$ and thus, for some n and m, $u = \phi_m$, $\lambda_m^D = \Lambda_n^D$ and u corresponds to the eigenvalue $\gamma + \Lambda_n^D$. Consequently we may assume that $c \neq 0$. From (5.2), one has

$$c_n = \frac{c\gamma F_n}{\lambda_n^D + \gamma - \lambda},\ n = 0, 1, \ldots.$$

Thus,

$$u = \sum_{n=0}^\infty \frac{c\gamma F_n}{\gamma + \lambda_n^D - \lambda}\phi_n. \tag{5.3}$$

Using the inner product if μ possesses an L^2-density, and using Assumption 1 and the bounded convergence theorem otherwise (recall from the remark after Assumption 1 that it always holds if $d = 1$ or 2), we have from (5.3) that

$$\int_D u d\mu = \sum_{n=0}^\infty \frac{c\gamma F_n G_n(\mu)}{\gamma + \lambda_n^D - \lambda}. \tag{5.4}$$

On the other hand, $0 \neq c = \int_D u d\mu$. Thus we conclude from (5.4) that $\lambda \notin \{\gamma + \lambda_n^D\}_{n=0}^\infty$ is an eigenvalue if and only if

$$\sum_{n=0}^\infty \frac{\gamma F_n G_n(\mu)}{\gamma + \lambda_n^D - \lambda} = 1. \tag{5.5}$$

Furthermore, it follows that such an eigenvalue is simple, since the corresponding eigenfunction has been uniquely specified (up to a multiplicative constant).

We now consider the possibility that $\lambda = \gamma + \Lambda^D_{n_0}$ is an eigenvalue, where n_0 is a nonnegative integer. Let S_{n_0} denote the d_{n_0}-dimensional eigenspace corresponding to the eigenvalue Λ_{n_0} of the Dirichlet Laplacian. Let $S^G_{n_0}(\mu) = \{v \in S_{n_0} : \int_D v d\mu = 0\}$ and let $S^F_{n_0} = \{v \in S_{n_0} : \int_D v dx = 0\}$. Clearly, each of these latter two spaces is either $(d_{n_0} - 1)$-dimensional or d_{n_0}-dimensional.

Consider first the case that $S^F_{n_0}$ is $(d_{n_0} - 1)$-dimensional. There exists an m_0 such that $\lambda^D_{m_0} = \Lambda^D_{n_0}$ and $F_{m_0} = \int_D \phi_{m_0} dx \neq 0$. But then (5.2) will hold with $n = m_0$ and $\lambda = \gamma + \Lambda^D_{n_0}$ if and only if $c = 0$. But if $c = 0$ and $\lambda = \gamma + \Lambda^D_{n_0}$, then the argument at the beginning of the second paragraph of the proof forces one to conclude that the eigenfunction u belongs to $S^G_{n_0}(\mu)$. Consequently, the multiplicity of $\Lambda^D_{n_0}$ will be either $d_{n_0} - 1$ or d_{n_0}, depending on which of these numbers is the dimension of $S^G_{n_0}(\mu)$. In particular, if $n_0 = 0$, then $d_{n_0} = 1$ and $S^F_{n_0} = S^G_{n_0}(\mu) = \{0\}$ since $\phi_0 > 0$. Thus, $\gamma + \lambda^D_0 = \gamma + \Lambda^D_0$ can never be an eigenvalue.

Now consider the case that $S^F_{n_0}$ is d_{n_0}-dimensional. In this case, $F_m = 0$, for all m such that $\lambda^D_m = \Lambda^D_{n_0}$. We first look for eigenfunctions for which $c \neq 0$. Solving (5.2) gives

$$\begin{cases} c_n = \frac{c\gamma F_n}{\lambda^D_n + \gamma - \lambda}, \text{ for all } n \text{ such that } \lambda^D_n \neq \Lambda^D_{n_0}; \\ c_n \text{ is arbitrary, for all } n \text{ such that } \lambda^D_n = \Lambda^D_{n_0}. \end{cases}$$

Writing $c_n = ck_n$, for n such that $\lambda^D_n = \Lambda^D_{n_0}$, and employing the same reasoning as in (5.3)–(5.5) yields

$$\sum_{n:\lambda^D_n \neq \Lambda^D_{n_0}} \frac{\gamma F_n G_n(\mu)}{\lambda^D_n - \Lambda^D_{n_0}} + \sum_{n:\lambda^D_n = \Lambda^D_{n_0}} k_n G_n(\mu) = 1. \tag{5.6}$$

There are two cases to consider: when $S^G_{n_0}(\mu)$ is $(d_{n_0} - 1)$-dimensional and when it is d_{n_0}-dimensional. In the latter case, $G_n(\mu) = 0$, for all n satisfying $\lambda^D_n = \Lambda^D_{n_0}$. Thus, (5.6) reduces to $E_{\gamma,\mu}(\gamma + \Lambda^D_{n_0}) = 1$. If this equation is satisfied, we obtain one eigenfunction with $c \neq 0$, and if it is not satisfied, we obtain no such eigenfunctions. Since $S^G_{n_0}(\mu)$ is d_{n_0}-dimensional, there are also d_{n_0} additional linearly independent eigenfunctions with $c = 0$. Thus, the multiplicity is either $d_{n_0} + 1$ or d_{n_0}, depending on whether or not $E_{\gamma,\mu}(\gamma + \Lambda^D_{n_0}) = 1$.

Now consider the case that $S^G_{n_0}(\mu)$ is $(d_{n_0} - 1)$-dimensional. Since we may choose the orthonormal basis $\{\phi_m\}_{\{m:\lambda^D_m = \Lambda^D_{n_0}\}}$ corresponding to the eigenspace S_{n_0} however we like, we may assume without loss of generality, that $G_m(\mu) = \int_D \phi_m d\mu = 0$, for all but one of the m for which $\lambda^D_m = \Lambda^D_{n_0}$. Denote the single m for which this is not true by m_0. Then (5.6) reduces to

$$\sum_{n:\lambda^D_n \neq \Lambda^D_{n_0}} \frac{\gamma F_n G_n(\mu)}{\lambda^D_n - \Lambda^D_{n_0}} + k_{m_0} G_{m_0}(\mu) = 1.$$

The above equation is uniquely solvable for k_{m_0} and thus yields one eigenfunction with $c \neq 0$. Since $S^G_{n_0}(\mu)$ is $(d_{n_0} - 1)$-dimensional, there are also $d_{n_0} - 1$ additional linearly independent eigenfunctions with $c = 0$; thus the multiplicity is d_{n_0}. □

6. Proof of Theorem 3

Part i. By Proposition 1 and Theorem 4 it follows that $\lambda_0(\gamma,\mu)$ is equal to the smallest root of the equation $E_{\gamma,\mu}(\lambda)=1$, where $E_{\gamma,\mu}$ is as in (1.6). The function 1 has the L^2-representation $1=\sum_{n=0}^{\infty}F_n\phi_n$. Since we are assuming that μ has an L^2-density, it follows that $\int_D 1d\mu=\sum_{n=0}^{\infty}F_nG_n(\mu)$; thus,

$$\sum_{n=0}^{\infty}F_nG_n(\mu)=\sum_{n=0}^{\infty}\int_D P_{\Lambda_n^D}1d\mu=1. \tag{6.1}$$

First consider the case that $\int_D P_{\Lambda_n^D}1d\mu=0$, for all $n\geq 1$. Then it follows from (6.1) that $\int_D P_{\Lambda_0^D}1d\mu=1$, and we have $E_{\gamma,\mu}(\lambda)=\frac{\gamma}{\gamma+\lambda_0^D-\lambda}$. Thus, $E_{\gamma,\mu}(\lambda_0^D)=1$, for all $\gamma>0$, and we conclude that $\lambda_0(\gamma,\mu)=\lambda_0^D$, for all $\gamma>0$.

Now consider the case that $\int_D P_{\Lambda_n^D}1d\mu\geq 0$, for all $n\geq 1$, and $\int_D P_{\Lambda_n^D}1d\mu>0$, for some $n\geq 1$. Recall that $\int_D P_{\Lambda_0^D}1d\mu>0$. By Proposition 1, one has $\lambda_0(\gamma,\mu)<\gamma+\lambda_0^D$. To prove that $\lambda_0(\gamma,\mu)$ is strictly monotone increasing in γ, it suffices to show that $E_{\gamma,\mu}(\lambda)$ is increasing as a function of $\lambda\in(0,\gamma+\lambda_0^D)$ and that

$$\frac{d\,(E_{\gamma,\mu}(\lambda))}{d\gamma}|_{\lambda=\lambda_0(\gamma,\mu)}<0,\ \text{for } \gamma>0. \tag{6.2}$$

Trivially, one has $\frac{dE_{\gamma,\mu}(\lambda)}{d\lambda}>0$, for $\lambda\in(0,\gamma+\lambda_0^D)$. It remains to show that (6.2) holds.

Differentiating $E_{\gamma,\mu}(\lambda)$ with respect to γ gives

$$\frac{d\,(E_{\gamma,\mu}(\lambda))}{d\gamma}|_{\lambda=\lambda_0(\gamma,\mu)}=\sum_{n=0}^{\infty}\frac{(\Lambda_n^D-\lambda_0(\gamma,\mu))\int_D P_{\Lambda_n^D}1d\mu}{(\gamma+\Lambda_n^D-\lambda_0(\gamma,\mu))^2}. \tag{6.3}$$

Subtracting the equation

$$1=E_{\gamma,\mu}(\lambda_0(\gamma,\mu))=\sum_{n=0}^{\infty}\frac{\gamma\int_D P_{\Lambda_n^D}1d\mu}{\gamma+\Lambda_n^D-\lambda_0(\gamma,\mu)}$$

from (6.1) gives

$$\sum_{n=0}^{\infty}\frac{(\Lambda_n^D-\lambda_0(\gamma,\mu))\int_D P_{\Lambda_n^D}1d\mu}{\gamma+\Lambda_n^D-\lambda_0(\gamma,\mu)}=0, \tag{6.4}$$

which can be rewritten as

$$\gamma\sum_{n=0}^{\infty}\frac{(\Lambda_n^D-\lambda_0(\gamma,\mu))\int_D P_{\Lambda_n^D}1d\mu}{(\gamma+\Lambda_n^D-\lambda_0(\gamma,\mu))^2}=-\sum_{n=0}^{\infty}\frac{(\Lambda_n^D-\lambda_0(\gamma,\mu))^2\int_D P_{\Lambda_n^D}1d\mu}{(\gamma+\Lambda_n^D-\lambda_0(\gamma,\mu))^2}. \tag{6.5}$$

Now (6.2) follows from (6.3), (6.5) and the assumption on $\{\int_D P_{\Lambda_n^D}1d\mu\}_{n=1}^{\infty}$.

We now turn to the case that $\int_D P_{\Lambda_n^D}1d\mu\leq 0$, for all $n\geq 1$, and $\int_D P_{\Lambda_n^D}1d\mu<0$, for some $n\geq 1$. To prove that $\lambda_0(\gamma,\mu)$ is strictly monotone decreasing in γ, it suffices to show that $E_{\gamma,\mu}(\lambda)$ is increasing as a function of $\lambda\in(0,\gamma+\lambda_0^D)$ and that

$$\frac{d\,(E_{\gamma,\mu}(\lambda))}{d\gamma}|_{\lambda=\lambda_0(\gamma,\mu)}>0,\ \text{for } \gamma>0. \tag{6.6}$$

We have

$$\frac{dE_{\gamma,\mu}}{d\lambda}(\lambda)=\sum_{n=0}^{\infty}\int_D\frac{\gamma P_{\Lambda_n^D}1d\mu}{(\gamma+\Lambda_n^D-\lambda)^2}. \tag{6.7}$$

By (6.1),

$$\int_D P_{\Lambda_0^D} 1 d\mu > \sum_{n=1}^{\infty} (-\int_D P_{\Lambda_n^D} 1 d\mu). \tag{6.8}$$

Since $\frac{\gamma}{(\gamma+\Lambda_n^D-\lambda)^2}$ is decreasing in n for $\lambda \in (0, \gamma+\Lambda_0^D)$, and since $-\int_D P_{\Lambda_n^D} 1 d\mu \geq 0$, for $n \geq 1$, it follows from (6.8) that

$$\frac{\gamma}{(\gamma+\Lambda_0^D-\lambda)^2} \int_D P_{\Lambda_0^D} 1 d\mu > \sum_{n=1}^{\infty} \frac{\gamma}{(\gamma+\Lambda_n^D-\lambda)^2} (-\int_D P_{\Lambda_n^D} 1 d\mu);$$

thus, the right hand side of (6.7) is positive.

The proof of (6.6) is almost identical to the proof of (6.2).

Part ii. By Theorem 4, we must show that the equation $E_{\gamma,\mu}(\lambda) = 1$, where $E_{\gamma,\mu}$ is as in (1.6), has no nonreal root λ. Writing $\lambda = \alpha + i\beta$ and multiplying each summand in $E_{\gamma,\mu}(\alpha + i\beta)$ by the conjugate of its denominator, we have

$$E_{\gamma,\mu}(\alpha + i\beta) = \sum_{n=0}^{\infty} \frac{\gamma(\int_D P_{\Lambda_n^D} 1 d\mu)\ (\gamma + \Lambda_n^D - \alpha + i\beta)}{(\gamma + \Lambda_n^D - \alpha)^2 + \beta^2}.$$

Thus, when $\beta \neq 0$, the equation $E_{\gamma,\mu}(\alpha + i\beta) = 1$ is equivalent to the equations

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{\gamma \Lambda_n^D (\int_D P_{\Lambda_n^D} 1 d\mu)}{(\gamma + \Lambda_n^D - \alpha)^2 + \beta^2} = 1; \\ &\sum_{n=0}^{\infty} \frac{\int_D P_{\Lambda_n^D} 1 d\mu}{(\gamma + \Lambda_n^D - \alpha)^2 + \beta^2} = 0. \end{aligned} \tag{6.9}$$

Consider first the case that $\int_D P_{\Lambda_n^D} 1 d\mu \geq 0$, for all $n \geq 1$. Since it is always true that $\int_D P_{\Lambda_0^D} 1 d\mu > 0$, the second equation in (6.9) cannot hold. Now consider the case that $\int_D P_{\Lambda_n^D} 1 d\mu \leq 0$ for all $n \geq 1$. If the second equation in (6.9) holds, then one has

$$\frac{\int_D P_{\Lambda_0^D} 1 d\mu}{(\gamma + \Lambda_0^D - \alpha)^2 + \beta^2} = \sum_{n=1}^{\infty} \frac{(-\int_D P_{\Lambda_n^D} 1 d\mu)}{(\gamma + \Lambda_n^D - \alpha)^2 + \beta^2}.$$

Using this and the fact that Λ_n^D is increasing in n, we obtain

$$\frac{\gamma \Lambda_0^D \int_D P_{\Lambda_0^D} 1 d\mu}{(\gamma + \Lambda_0^D - \alpha)^2 + \beta^2} < \sum_{n=1}^{\infty} \frac{\gamma \Lambda_n^D (-\int_D P_{\Lambda_n^D} 1 d\mu)}{(\gamma + \Lambda_n^D - \alpha)^2 + \beta^2};$$

thus the first equation in (6.9) cannot hold.

Part iii. By assumption, all of the eigenvalues $\{\lambda_n^D\}_{n=0}^{\infty}$ are distinct, and $F_n G_n(\mu) > 0$, for all n; thus, it follows from Theorem 4 that the set of eigenvalues of $L_{\gamma,\mu}$ coincides with the set of roots λ of the equation $E_{\gamma,\mu}(\lambda) = 1$. The condition $F_n G_n(\mu) > 0$, for all n, guarantees that $E_{\gamma,\mu}(\lambda)$ is increasing for $\lambda \in (0, \gamma + \lambda_0^D)$ and satisfies $E_{\gamma,\mu}(0) < 1$ and $E_{\gamma,\mu}((\gamma + \lambda_0^D)^-) = \infty$, and that for each $n \geq 0$, $E_{\gamma,\mu}(\lambda)$ is increasing for $\lambda \in (\gamma + \lambda_n^D, \gamma + \lambda_{n+1}^D)$ and satisfies $E_{\gamma,\mu}((\gamma + \lambda_n^D)^+) = -\infty$ and $E_{\gamma,\mu}((\gamma + \lambda_{n+1}^D)^-) = \infty$. Thus, there is exactly one root between 0 and $\gamma + \lambda_0^D$ and exactly one root between $\gamma + \lambda_n^D$ and $\gamma + \lambda_{n+1}^D$, for $n \geq 0$. Consequently, $\gamma + \lambda_{n-1}^D < \lambda_n(\gamma, \mu) < \gamma + \lambda_n^D$, for $n \geq 1$. □

Appendix A. The semigroup $T_t^{\gamma,\mu}$ and its connection to $L_{\gamma,\mu}$

We first show that $-L_{\gamma,\mu}$, defined on $C^2(\bar{D})\cap\{u: u, L_{\gamma,\mu}u\in C_0(\bar{D})\}$, and then extended appropriately, is the infinitesimal generator of $T_t^{\gamma,\mu}$. From the definition of the Brownian motion with random jumps we have

$$\begin{aligned}T_t^{\gamma,\mu}u(x) &= \exp(-\gamma t)\int_D p^D(t,x,y)u(y)dy\\ &+\int_0^t \gamma\exp(-\gamma s)\int_D p^D(t-s,\mu,y)u(y)dyds,\end{aligned} \tag{A.1}$$

where $p^D(t,x,y)$ is the transition sub-probability density for Brownian motion in D starting at $x\in D$ and killed at the boundary. A standard result for the semigroup corresponding to Brownian motion (without jumps) allows us to conclude that for $u\in C^2(\bar{D})\cap C_0(\bar{D})$, one has $\lim_{t\to 0}\frac{\int_D p^D(t,x,y)u(y)dy-u(x)}{t}=\frac{1}{2}(\Delta u)(x)$, and the convergence is uniform over $x\in\bar{D}$. Using this and (A.1), we obtain $\lim_{t\to 0}\frac{T_t^{\gamma,\mu}u(x)-u(x)}{t}=\frac{1}{2}(\Delta u)(x)-\gamma u(x)+\gamma\int_D ud\mu=-(L_{\gamma,\mu}u)(x)$, and the convergence is uniform over $x\in\bar{D}$. Furthermore, by assumption, $-L_{\gamma,\mu}u\in C_0(\bar{D})$. Thus, the infinitesimal generator of $T_t^{\gamma,\mu}$ is indeed an extension of $-L_{\gamma,\mu}$ on $C^2(\bar{D})\cap\{u: u, L_{\gamma,\mu}u\in C_0(\bar{D})\}$.

We now turn to compactness.

Proposition 2. *The semigroup $T_t^{\gamma,\mu}$ is compact.*

Proof. Let $\{Z_n\}_{n=1}^\infty$ be a sequence of IID random variables distributed according to the exponential distribution with parameter γ, and let $S_n=\sum_{j=1}^n Z_j$. Denote probabilities for these random variables by Q. From the definition of the Brownian motion with jumps, it follows that

$$\begin{aligned}T_t^{\gamma,\mu}f(x) &= \exp(-\gamma t)\int_D p^D(t,x,y)f(y)dy\\ &+\sum_{n=1}^\infty\int_0^t\int_D p^D(t-s,\mu,y)f(y)Q(S_n=s,S_{n+1}>t)dyds\\ &\equiv \exp(-\gamma t)\int_D p^D(t,x,y)f(y)dy+T_t^{\gamma,\mu;\infty}f(x).\end{aligned}$$

It is known [6] that for any $\epsilon>0$, $p^D(t,x,y)$ is continuous on $[\epsilon,\infty)\times\bar{D}\times\bar{D}$, and thus uniformly continuous on $[\epsilon,T]\times\bar{D}\times\bar{D}$, for $0<\epsilon<T<\infty$. Thus the transformation $f(x)\to\exp(-\gamma t)\int_D p^D(t,x,y)f(y)dy$ maps bounded sets in $C_0(\bar{D})$ to equicontinuous and bounded sets in $C_0(\bar{D})$; consequently, this transformation is compact. It remains to show that $T_t^{\gamma,\mu;\infty}$ is a compact map.

For $m\geq 1$, define the map $T_t^{\gamma,\mu;m}$ by

$$T_t^{\gamma,\mu;m}f(x)=\sum_{n=1}^m\int_0^t\int_D p^D(t-s,\mu,y)f(y)Q(S_n=s,S_{n+1}>t)dyds.$$

Then we have $|||T_t^{\gamma,\mu;\infty}-T_t^{\gamma,\mu;m}||\leq\sum_{n=m+1}^\infty Q(S_n\leq t,S_{n+1}>t)=Q(S_{m+1}\leq t)$. Thus, $T_t^{\gamma,\mu;m}$ converges in the operator norm to $T_t^{\gamma,\mu;\infty}$. Consequently, we need only show that the map U_n, defined by

$$U_nf(x)=\int_0^t\int_D p^D(t-s,\mu,y)f(y)Q(S_n=s,S_{n+1}>t)dyds,$$

is compact. Define the map $U_{n,\epsilon}$ by

$$U_{n,\epsilon}f(x) = \int_0^{t-\epsilon}\int_D p^D(t-s,\mu,y)f(y)Q(S_n = s, S_{n+1} > t)dyds.$$

Then $||U_n - U_{n,\epsilon}|| \leq Q(S_n \in [t-\epsilon, t])$. Thus, $U_{n,\epsilon}$ converges to U_n in the operator norm and consequently it suffices to show that $U_{n.\epsilon}$ is compact. By the above noted uniform continuity of $p^D(t,x,y)$, $U_{n,\epsilon}$ maps bounded sets in $C_0(\bar{D})$ to equicontinuous and bounded sets in $C_0(\bar{D})$; consequently it is compact. □

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