

## ON THE APPROXIMATE NORMALITY OF EIGENFUNCTIONS OF THE LAPLACIAN

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ABSTRACT. The main result of this paper is a bound on the distance between the distribution of an eigenfunction of the Laplacian on a compact Riemannian manifold and the Gaussian distribution. If  $X$  is a random point on a manifold  $M$  and  $f$  is an eigenfunction of the Laplacian with  $L^2$ -norm one and eigenvalue  $-\mu$ , then

$$d_{TV}(f(X), Z) \leq \frac{2}{\mu} \mathbb{E} \left| \|\nabla f(X)\|^2 - \mathbb{E} \|\nabla f(X)\|^2 \right|.$$

This result is applied to construct specific examples of spherical harmonics of arbitrary (odd) degree which are close to Gaussian in distribution. A second application is given to random linear combinations of eigenfunctions on flat tori.

### 1. INTRODUCTION

A question which has arisen and been studied often throughout the history of probability has been to understand the value distributions of certain naturally occurring functions. The simplest example of such a question might be to understand the distribution of the number of times out of  $N$  trials that a fair coin lands on heads. A more geometric example (studied by Maxwell and Borel among others) is understanding the value distribution of the function  $f : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$  defined by  $f(x) = x_1$ . Value distribution in this context refers to endowing the sphere  $\mathbb{S}^{n-1}$  with a natural probability measure (normalized surface area) and considering the distribution of the random variable which is the push-forward of that natural measure on the sphere to  $\mathbb{R}$  by the function  $f$ ; i.e., if  $X$  is a random point on the sphere, the value distribution of  $f$  is the distribution of the random variable  $f(X)$  on  $\mathbb{R}$ .

A particular class of functions of interest is eigenfunctions of the Laplacian on Riemannian manifolds. The main result of this paper is to give a sufficient condition for the approximate normality of these functions, as follows.

**Theorem 1.** *Let  $M$  be a compact Riemannian manifold (without boundary), and let  $f$  be an eigenfunction for the Laplacian on  $M$  with eigenvalue  $-\lambda < 0$ , normalized so that  $\frac{1}{\text{vol}(M)} \int_M f^2 = 1$ . Let  $X$  be a random (i.e., distributed according to*

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normalized volume measure) point of  $M$ . Then

$$d_{TV}(f(X), Z) \leq \frac{2}{\lambda} \mathbb{E} \left| \|\nabla f(X)\|^2 - \mathbb{E}\|\nabla f(X)\|^2 \right|,$$

where  $d_{TV}$  denotes the total variation distance between random variables and  $Z$  is a standard Gaussian random variable on  $\mathbb{R}$ .

That is, the value distribution of  $f$  is close to Gaussian if the expected distance between the random variable  $\|\nabla f(X)\|^2$  and its mean is small, relative to the size of the eigenvalue. It should be pointed out that  $\mathbb{E}\|\nabla f(X)\|^2 = \lambda$  (this is an easy consequence of Stokes's theorem); thus the bound on the right-hand side is not always non-trivial. However, it will be seen in examples below that there is sometimes sufficient cancellation to obtain a normal limit.

Eigenfunctions of the Laplacian have arisen as important and natural objects in several branches of analysis; in particular, they are of central interest in quantum chaos, whose basic problem is the understanding of the quantization of a classical Hamiltonian system with chaotic dynamics. See [11] for a readable introduction. A key question in the chaotic case is whether individual eigenfunctions behave like random waves (see [2]). It is noted in [11] that, were this the case, it would imply that the distribution of high eigenfunctions would be approximately Gaussian. In connection with this conjecture, Hejhal and various coauthors (see, e.g., [7]) and Aurich and Steiner [1] have provided numerical evidence indicating that on certain hyperbolic manifolds, the distributions of eigenfunctions become close to Gaussian as the eigenvalue tends to infinity. Theorem 1 above provides a different perspective on the approximate normality of eigenfunctions. In the applications given below, asymptotic results are obtained as the dimension of the manifold in question tends to infinity. However, there is no *a priori* reason that Theorem 1 could not also be applied in the case of a fixed manifold, as the eigenvalue tends to infinity.

The results of this paper can be seen as an abstraction of earlier work of the author [9] and joint work with M. Meckes [10]. In those papers, it was shown that the value distributions of linear functions on the orthogonal group and on certain convex bodies are close to Gaussian, using similar methods to those used here. The approaches taken in those papers were based on the approach taken by Stein in [12], in studying the rate of convergence to Gaussian of traces of powers of random orthogonal matrices. The proof of Theorem 1 makes clear that eigenfunctions of the Laplacian are the natural candidates (at least for the techniques used here) to replace linear functions on more general Riemannian manifolds. The main result of [9] on the orthogonal group and the results of [10] in the case of the sphere can be recovered as applications of Theorem 1.

The contents of this paper are as follows. Notation and conventions are given in Section 1.1 below. Section 2 gives the proof of the main theorem, in fact proving a slightly more general version which treats linear combinations of eigenfunctions whose eigenvalues are close together. Section 3 gives applications of Theorem 1 to the distribution of certain eigenfunctions on the sphere  $\mathbb{S}^{n-1}$ , and Section 4 contains applications of the generalization of Theorem 1 from Section 2 to random eigenfunctions on the torus  $\mathbb{T}^n$ . The unifying idea in these two cases is that in situations in which eigenspaces have high dimension, or the closely related situation in which there are many low-dimensional eigenspaces whose eigenvalues are very close together, Gaussian distributions occur when taking linear combinations of many orthogonal eigenfunctions.

In the case of the sphere, we have the following result for the second-order eigenfunctions.

**Theorem 2.** *Let  $g(x) = \sum_{i,j} a_{ij}x_i x_j$ , where  $A = (a_{ij})_{i,j=1}^n$  is a symmetric matrix with  $\text{Tr}(A) = 0$ . Let  $f = Cg$ , where  $C$  is chosen such that  $\|f\|_2 = 1$  when  $f$  is considered as a function on  $S^{n-1}$ . Let  $W = f(X)$ , where  $X$  is a random point on  $S^{n-1}$ . If  $d$  is the vector in  $\mathbb{R}^n$  whose entries are the eigenvalues of  $A$ , then*

$$d_{TV}(W, Z) \leq \sqrt{6} \left( \frac{\|d\|_4}{\|d\|_2} \right)^2,$$

where  $\|d\|_p = (\sum_i |d_i|^p)^{1/p}$ .

This theorem gives a fairly complete picture of when second-order eigenfunctions of the spherical Laplacian have Gaussian value distributions. The bound on the right-hand side is of the order  $\frac{1}{\sqrt{n}}$  in the case that all of the  $d_i$  are of roughly the same size, e.g., when

$$d_i = \begin{cases} 1, & i \leq \frac{n}{2}, \\ -1, & \frac{n}{2} < i \leq n, \end{cases}$$

and  $n$  is even. In the opposite extreme case, e.g., for the function  $f(x) = c(x_1^2 - x_2^2)$  for a suitable normalization constant  $c$ , the bound is of order 1. This is what is expected in this case; since it is known that  $x_1$  and  $x_2$  are asymptotically distributed as independent Gaussians (as the dimension tends to infinity), the random variable  $f(X)$  is approximately distributed as  $Z_1^2 - Z_2^2$  for independent normal random variables  $Z_1$  and  $Z_2$  and  $X$  a random point of a high-dimensional sphere.

In the higher degree case, we have the following theorem.

**Theorem 3.** *There is a set  $\{p_\ell^{(i)}\}_{i=1}^n$  of orthonormal eigenfunctions of degree  $\ell$  on  $S^{n-1}$  and a constant  $c$  depending on  $\ell$  such that for  $p : S^{n-1} \rightarrow \mathbb{R}$  defined by*

$$\sum_{i=1}^n a_i p_\ell^{(i)}(x)$$

with  $\sum a_i^2 = 1$ ,

$$d_{TV}(p(X), Z) \leq c\|\mathbf{a}\|_4^2.$$

*If the vector of coefficients  $\{a_i\}$  is chosen to be random and uniformly distributed on the sphere, then there is another constant  $c'$  such that*

$$\mathbb{E}d_{TV}(p(X), Z) \leq \frac{c'}{\sqrt{n}}.$$

The functions  $p_\ell^{(i)}$  are given explicitly in Section 3. The second statement of the theorem says that the typical linear combination of the  $p_\ell^{(i)}$  has an approximately Gaussian value distribution. As in the second-degree case, the explicit form of the bound in the first statement of the theorem means that for specific choices of  $\mathbf{a}$  whose components all have roughly the same magnitude, e.g.,  $a_i = \frac{1}{\sqrt{n}}$  for each  $i$ , the bound is of order  $\frac{1}{\sqrt{n}}$ .

In the case of the torus, we have the following result on the distributions of random linear combinations of eigenfunctions.

**Theorem 4.** *Let  $B$  be a symmetric, positive definite matrix, and consider the flat torus  $\mathbb{R}^n/\mathbb{Z}^n$  with the metric given by  $(x, y)_B = \langle Bx, y \rangle$ . Define a random function  $f$  on  $(\mathbb{T}^n, B)$  as follows. Let  $\{a_v\}_{v \in \mathcal{V}}$  be a random point of the sphere of radius  $\sqrt{2}$  in  $\mathbb{R}^{\mathcal{V}}$ , where  $\mathcal{V}$  is a finite set of vectors of  $\mathbb{R}^n$ . Suppose that the set  $\mathcal{V}$  is such that if  $v \in \mathcal{V}$ , then  $Bv \in \mathbb{Z}^n$ . Define  $f$  by*

$$f(x) = \Re \left( \sum_{v \in \mathcal{V}} a_v e^{2\pi i \langle Bv, x \rangle} \right).$$

Then for  $\mu_v = (2\pi\|v\|_B)^2$  and  $\mu > 0$ ,

$$\begin{aligned} & \mathbb{E}d_{TV}(f(X), Z) \\ & \leq \frac{1}{|\mu|} \left[ \sqrt{\frac{8(2\pi)^4}{|\mathcal{V}|(|\mathcal{V}|+2)} \sum_{v, w \in \mathcal{V}} (v, w)_B^2} + (2\sqrt{2} + \sqrt{\pi}) \sqrt{\frac{1}{|\mathcal{V}|} \sum_{v \in \mathcal{V}} (\mu_v - \mu)^2} \right]. \end{aligned}$$

Section 4 contains a discussion of the bounds above and concrete examples in which they are small. The simplest possible case is that of the usual square-lattice torus, in which the matrix  $B = I$ . In that case, consider the set  $\mathcal{V}$  to be simply the standard basis vectors of  $\mathbb{R}^n$ . Then the fact that  $f$  as constructed above has a value distribution which is close to normal is essentially just the classical central limit theorem. In that case, the second term of the bound above is zero (take  $\mu = (2\pi)^2$ ), and the first term is  $\frac{2\sqrt{2}}{\sqrt{n}}$ .

In the case of more general  $B$  and  $\mathcal{V}$ , the first term is a weakening of the orthogonality in the previous case; it is small as long as  $\mathcal{V}$  is large but the elements of  $\mathcal{V}$  have directions which are fairly evenly distributed on the sphere. The second term is a weakening of the fact that, in the previous example, all of the summands in the linear combination were from the same eigenspace. By taking  $B$  to be a slight perturbation of  $I$ , the eigenspaces may become low- or even one-dimensional. However, the corresponding eigenvalues may still be clustered very close together, in which case the second term of the error above is small.

**1.1. Notation and conventions.** For random variables  $X$  and  $Y$ , the total variation distance between  $X$  and  $Y$  is defined by

$$d_{TV}(X, Y) = \sup_A |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)|,$$

where the supremum is taken over measurable sets  $A$ . This is equivalent to

$$d_{TV}(X, Y) = \frac{1}{2} \sup_f |\mathbb{E}f(X) - \mathbb{E}f(Y)|,$$

where the supremum is taken over functions  $f$  which are continuous and bounded by one.

Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold. Integration with respect to the normalized volume measure is denoted  $d\overline{\text{vol}}$ ; thus  $\int_M 1d\overline{\text{vol}} = 1$ . For coordinates  $\left\{ \frac{\partial}{\partial x_i} \right\}_{i=1}^n$  on  $M$ , define

$$(G(x))_{ij} = g_{ij}(x) = \left\langle \frac{\partial}{\partial x_i} \Big|_x, \frac{\partial}{\partial x_j} \Big|_x \right\rangle, \quad g(x) = \det(G(x)), \quad g^{ij}(x) = (G^{-1}(x))_{ij}.$$

Define the gradient  $\nabla f$  of  $f : M \rightarrow \mathbb{R}$  and the Laplacian  $\Delta_g f$  of  $f$  by

$$\nabla f(x) = \sum_{j,k} \frac{\partial f}{\partial x_j} g^{jk} \frac{\partial}{\partial x_k}, \quad \Delta_g f(x) = \frac{1}{\sqrt{g}} \sum_{j,k} \frac{\partial}{\partial x_j} \left( \sqrt{g} g^{jk} \frac{\partial f}{\partial x_k} \right).$$

Let  $\Phi^t(x, v)$  denote the geodesic flow on the tangent bundle  $TM$  and let  $\pi : TM \rightarrow M$  be the projection map of a tangent vector onto its base point.

2. THE MAIN RESULT

As mentioned above, the main result of this section is a slight generalization of Theorem 1 of the previous section. The idea is that for the method of proof used here, the functions considered need only be *approximate* eigenfunctions of  $\Delta_g$ ; in particular, linear combinations of eigenfunctions all of whose eigenvalues cluster about one value are close enough.

**Theorem 5.** *Let  $(M, g)$  be a compact Riemannian manifold without boundary, and let  $\{f_i\}_{i=1}^m$  be a sequence of mutually orthogonal eigenfunctions of the Laplacian  $\Delta_g$  on  $M$  with eigenvalues  $-\mu_i < 0$ . Assume that  $\int_M f_i^2 d\text{vol} = 1$  for each  $i$ . Define the function  $f : M \rightarrow \mathbb{R}$  by*

$$f(x) = \sum_{i=1}^m a_i f_i(x),$$

where the  $a_i$  are scalars such that  $\sum a_i^2 = 1$ . Let  $X$  be a random (i.e., distributed according to normalized volume measure) point of  $M$ . Then for  $\mu > 0$ ,

$$d_{TV}(f(X), Z) \leq \frac{2}{\mu} \left[ \mathbb{E} \left| \|\nabla f(X)\|^2 - \mathbb{E}\|\nabla f(X)\|^2 \right| + \left( 1 + \frac{\sqrt{\pi}}{2\sqrt{2}} \right) \sqrt{\sum_i a_i^2 (\mu_i - \mu)^2} \right],$$

where  $Z$  is a standard Gaussian random variable on  $\mathbb{R}$ .

The idea is to choose  $\mu = \frac{1}{m} \sum \mu_i$  and consider collections  $\{f_i\}$  of eigenfunctions such that the eigenvalues  $\mu_i$  are very close together. In particular, for manifolds on which eigenspaces of  $\Delta$  have high dimension, one can choose all the  $\mu_i$  to be equal, in which case the second error term drops out.

The proof is an application of the following abstract normal approximation theorem. A proof of the theorem was given in [9] in the case that  $E' = 0$ ; the proof of the theorem below is a trivial modification of the earlier case.

**Theorem 6.** *Suppose that  $(W, W_\epsilon)$  is a family of exchangeable pairs defined on a common probability space with  $\mathbb{E}W = 0$  and  $\mathbb{E}W^2 = \sigma^2$ . Suppose there are random variables  $E = E(W)$  and  $E' = E'(W)$ , deterministic functions  $h$  and  $k$  with*

$$\lim_{\epsilon \rightarrow 0} h(\epsilon) = \lim_{\epsilon \rightarrow 0} k(\epsilon) = 0$$

and functions  $\alpha$  and  $\beta$  with

$$\mathbb{E}|\alpha(\sigma^{-1}W)| < \infty, \quad \mathbb{E}|\beta(\sigma^{-1}W)| < \infty,$$

such that

(i)

$$\frac{1}{\epsilon^2} \mathbb{E} [W_\epsilon - W | W] = -\lambda W + E' + h(\epsilon)\alpha(W),$$

(ii)

$$\frac{1}{\epsilon^2} \mathbb{E} [(W_\epsilon - W)^2 | W] = 2\lambda\sigma^2 + E\sigma^2 + k(\epsilon)\beta(W),$$

(iii)

$$\frac{1}{\epsilon^2} \mathbb{E} |W_\epsilon - W|^3 = o(1),$$

where  $o(1)$  refers to the limit as  $\epsilon \rightarrow 0$  with implied constants depending on the distribution of  $W$ . Then

$$d_{TV}(W, Z) \leq \frac{1}{\lambda} \left[ \sqrt{\frac{\pi}{2}} \mathbb{E} |E'| + \mathbb{E} |E| \right],$$

where  $Z \sim \mathfrak{N}(0, \sigma^2)$ .

*Proof of Theorem 5.* To apply Theorem 6 above, start by constructing a family of exchangeable pairs of points in  $M$  parametrized by  $\epsilon$ . Let  $X$  be a random point of  $M$  and let  $\epsilon > 0$  be smaller than the injectivity radius of the exponential map at  $X$ . Since  $M$  is compact, there is a range of  $\epsilon$  small enough to work at every point. Now, choose a unit vector  $V \in T_X M$  at random, independent of  $X$ , and define

$$X_\epsilon = \exp_X(\epsilon V).$$

To see that this defines an exchangeable pair of random points on  $M$ , we show that

$$\int_{M \times M} g(x, x_\epsilon) d\mu = \int_{M \times M} g(x_\epsilon, x) d\mu$$

for all integrable  $g : M \times M \rightarrow \mathbb{R}$ , where  $\mu$  is the measure defined by the construction of the pair  $(X, X_\epsilon)$ .

Let  $L$  denote the normalized Liouville measure on  $SM$ , which is locally the product of the normalized volume measure on  $M$  and normalized Lebesgue measure on the unit spheres of  $T_x M$ . The measure  $L$  has the property that it is invariant under the geodesic flow  $\Phi^t(x, v)$ . See [3, Section 5.1] for a construction of  $L$  and proofs of its key properties. By construction of the measure  $\mu$ ,

$$\begin{aligned} \int_{M \times M} g(x, x_\epsilon) d\mu &= \int_{SM} g(x, \pi \circ \Phi^\epsilon(x, v)) dL(x, v) \\ &= \int_{SM} g(\pi \circ \Phi^{-\epsilon}(y, \eta), y) dL(y, \eta) \end{aligned}$$

by the substitution  $(y, \eta) = \Phi^\epsilon(x, v)$ , since Liouville measure is invariant under the geodesic flow. Now,

$$\Phi^{-\epsilon}(y, \eta) = -\Phi^\epsilon(y, -\eta),$$

as both correspond to going backwards on the same geodesic. It follows that

$$\begin{aligned} \int_{SM} g(x, x_\epsilon) dL(x, v) &= \int_{SM} g(\pi(-\Phi^\epsilon(y, -\eta)), y) dL(y, \eta) \\ &= \int_{SM} g(\pi(-\Phi^\epsilon(y, \eta)), y) dL(y, \eta) \\ &= \int_{SM} g(\pi(\Phi^\epsilon(y, \eta)), y) dL(y, \eta) \\ &= \int_{M \times M} g(y_\epsilon, y) d\mu, \end{aligned}$$

where the second line follows from the substitution  $-\eta \rightarrow \eta$ , under which Lebesgue measure on the tangent space is invariant, and the third line follows because  $\pi(-v) = \pi(v)$  (both have the same base point). Thus the pair  $(X, X_\epsilon)$  is exchangeable as required.

Now, let  $f(x) = \sum a_i f_i(x)$  as in the statement of the theorem. For notational convenience, let  $W = f(X)$  and  $W_\epsilon = f(X_\epsilon)$ . Since  $(X, X_\epsilon)$  is exchangeable,  $(W, W_\epsilon)$  is an exchangeable pair as well.

In order to verify the conditions of Theorem 6, first let  $\gamma : [0, \epsilon] \rightarrow M$  be a constant-speed geodesic such that  $\gamma(0) = X$ ,  $\gamma(\epsilon) = X_\epsilon$ , and  $\gamma'(0) = V$ . Then applying Taylor's theorem on  $\mathbb{R}$  to the function  $f \circ \gamma$  yields

$$\begin{aligned} (1) \quad f(X_\epsilon) - f(X) &= \epsilon \cdot \left. \frac{d(f \circ \gamma)}{dt} \right|_{t=0} + \frac{\epsilon^2}{2} \cdot \left. \frac{d^2(f \circ \gamma)}{dt^2} \right|_{t=0} + O(\epsilon^3) \\ &= \epsilon \cdot d_X f(V) + \frac{\epsilon^2}{2} \cdot \left. \frac{d^2(f \circ \gamma)}{dt^2} \right|_{t=0} + O(\epsilon^3), \end{aligned}$$

where the coefficient implicit in the  $O(\epsilon^3)$  depends on  $f$  and  $\gamma$  and  $d_x f$  denotes the differential of  $f$  at  $x$ . Recall that  $d_x f(v) = \langle \nabla f(x), v \rangle$  for  $v \in T_x M$  and the gradient  $\nabla f(x)$  defined as above. Now, for  $X$  fixed,  $V$  is distributed according to normalized Lebesgue measure on  $S_X M$  and  $d_X f$  is a linear functional on  $T_X M$ . It follows that

$$\mathbb{E} [d_X f(V) | X] = \mathbb{E} [d_X f(-V) | X] = -\mathbb{E} [d_X f(V) | X],$$

so

$$\mathbb{E} [d_X f(V) | X] = 0.$$

This implies that

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} \mathbb{E} [f(X_\epsilon) - f(X) | X]$$

exists and is finite. Indeed, it is well-known (see, e.g., Theorem 11.12 of [6]) that

$$(2) \quad \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} \mathbb{E} [f(X_\epsilon) - f(X) | X] = \frac{1}{2n} \Delta_g f(X)$$

for  $n = \dim(M)$ ; thus

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} \mathbb{E} [W_\epsilon - W | W] &= -\frac{1}{2n} \mathbb{E} \left[ \sum_{i=1}^m a_i \mu_i f_i(X) | W \right] \\ &= -\frac{\mu}{2n} W + \frac{1}{2n} \mathbb{E} \left[ \sum_{i=1}^m a_i (\mu - \mu_i) f_i(X) | W \right]. \end{aligned}$$

It follows that  $\lambda = \frac{\mu}{2n}$  and  $E' = \frac{1}{2n} \mathbb{E} [\sum_i a_i (\mu - \mu_i) f_i(X) | W]$ . The higher order terms in  $\epsilon$  of  $\mathbb{E} [W_\epsilon - W | W]$  satisfy the integrability requirement of Theorem 6

since  $f$  is smooth and  $M$  is compact. Finally,

$$\begin{aligned} \frac{1}{\lambda} \mathbb{E}|E'| &\leq \frac{1}{\mu} \mathbb{E} \left| \sum_i a_i (\mu_i - \mu) f_i(X) \right| \\ &\leq \frac{1}{\mu} \sqrt{\mathbb{E} \left[ \sum_i a_i (\mu_i - \mu) f_i(X) \right]^2} \\ &= \frac{1}{\mu} \sqrt{\sum_i a_i^2 (\mu_i - \mu)^2}, \end{aligned}$$

since the  $f_i$  are mutually orthogonal.

Consider next

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} \mathbb{E}[(W_\epsilon - W)^2 | X].$$

By the expansion (1),

$$\begin{aligned} \mathbb{E}[(W_\epsilon - W)^2 | X] &= \mathbb{E}[(f(X_\epsilon) - f(X))^2 | X] \\ &= \epsilon^2 \mathbb{E}[(d_X f(V))^2 | X] + O(\epsilon^3). \end{aligned}$$

Choose coordinates  $\left\{ \frac{\partial}{\partial x_i} \right\}_{i=1}^n$  in a neighborhood of  $X$  which are orthonormal at  $X$ . Then

$$\nabla f(X) = \sum_i \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i};$$

thus

$$\begin{aligned} [d_X f(v)]^2 &= [(\nabla f, v)]^2 \\ &= \sum_{i=1}^n \left( \frac{\partial f}{\partial x_i} \right)^2 v_i^2 + \sum_{i \neq j} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} v_i v_j. \end{aligned}$$

Since  $V$  is uniformly distributed on a Euclidean sphere,  $\mathbb{E}[V_i V_j] = \frac{1}{n} \delta_{ij}$ . Making use of this fact yields

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} \mathbb{E}[(d_X f(V))^2 | X] = \frac{1}{n} \|\nabla f\|^2 = \frac{\mu}{n} + \frac{1}{n} [\|\nabla f\|^2 - \mu];$$

thus condition (ii) is satisfied with

$$E = \frac{1}{n} \mathbb{E} [\|\nabla f(X)\|^2 - \mu | W].$$

By Stokes's theorem,

$$\begin{aligned} \mathbb{E} \|\nabla f(X)\|^2 &= -\mathbb{E}[f(X) \Delta_g f(X)] \\ &= \mathbb{E} \left[ f(X) \sum_i a_i \mu_i f_i(X) \right] \\ &= \mu \mathbb{E} f^2(X) + \mathbb{E} \left[ f(X) \sum_i a_i (\mu_i - \mu) f_i(X) \right] \\ &= \mu + \mathbb{E} \left[ f(X) \sum_i a_i (\mu_i - \mu) f_i(X) \right]. \end{aligned}$$



Thus

$$\begin{aligned}
 E &= \frac{1}{n} \mathbb{E} \left[ \|\nabla f\|^2 - \mu - \mathbb{E} \left[ f(X) \sum_i a_i (\mu_i - \mu) f_i(X) \right] \right. \\
 &\quad \left. + \mathbb{E} \left[ f(X) \sum_i a_i (\mu_i - \mu) f_i(X) \right] \middle| W \right] \\
 &= \frac{1}{n} \left[ \mathbb{E} [\|\nabla f\|^2 - \mathbb{E}\|\nabla f\|^2 | W] + \mathbb{E} \left[ f(X) \sum_i a_i (\mu_i - \mu) f_i(X) \right] \right],
 \end{aligned}$$

and so

$$\begin{aligned}
 \frac{1}{\lambda} \mathbb{E}|E| &\leq \frac{2}{\mu} \left[ \mathbb{E} \left| \|\nabla f(X)\|^2 - \mathbb{E}\|\nabla f(X)\|^2 \right| + \mathbb{E} \left| f(X) \sum_i a_i (\mu_i - \mu) f_i(X) \right| \right] \\
 &\leq \frac{2}{\mu} \left[ \mathbb{E} \left| \|\nabla f(X)\|^2 - \mathbb{E}\|\nabla f(X)\|^2 \right| + \sqrt{\mathbb{E}f^2(X)} \sqrt{\mathbb{E} \left( \sum_i a_i (\mu_i - \mu) f_i(X) \right)^2} \right] \\
 &\leq \frac{2}{\mu} \left[ \mathbb{E} \left| \|\nabla f(X)\|^2 - \mathbb{E}\|\nabla f(X)\|^2 \right| + \sqrt{\sum_i a_i^2 (\mu_i - \mu)^2} \right].
 \end{aligned}$$

Finally, (1) gives immediately that

$$\mathbb{E} [|W_\epsilon - W|^3 | W] = O(\epsilon^3). \quad \square$$

### 3. THE SPHERE

Eigenvalues and eigenfunctions of the Laplacian on the sphere  $\mathbb{S}^{n-1}$  are well-studied objects. In particular, it is discussed in Section 9.5 of [13] that the eigenfunctions of the spherical Laplacian are exactly the restrictions of homogeneous harmonic polynomials on  $\mathbb{R}^n$  to the sphere and that such polynomials of degree  $l$  have eigenvalue  $-l(l + n - 2)$ . It follows, then, that the simplest class of functions on the sphere to which one could try to apply Theorem 5 is the linear functions. As was mentioned in the introduction, understanding the value distribution of linear functions on the sphere is an old problem. A careful history is given in the paper of Diaconis and Freedman [4], in which the following theorem (and more general versions) are proved.

**Theorem 7** (Diaconis-Freedman). *Let  $X$  be a random point on the sphere  $\sqrt{n}\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ . Then*

$$d_{TV}(X_1, Z) \leq \frac{4}{n-1},$$

where  $Z \sim \mathfrak{N}(0, 1)$ .

It is straightforward to reproduce this result as a consequence of Theorem 5. In Section 3.1 below, Theorem 5 is applied to the next obvious example, the second order spherical harmonics. Section 3.2 treats more specialized examples of spherical harmonics of arbitrary (odd) degree.

The following lemma, taken from [5], for integrating polynomials over the sphere will be useful in applications.

**Lemma 8.** Let  $P(x) = |x_1|^{\alpha_1}|x_2|^{\alpha_2} \cdots |x_n|^{\alpha_n}$ . Then if  $X$  is uniformly distributed on  $S^{n-1}$ ,

$$\mathbb{E}[P(X)] = \frac{\Gamma(\beta_1) \cdots \Gamma(\beta_n)\Gamma(\frac{n}{2})}{\Gamma(\beta_1 + \cdots + \beta_n)\pi^{n/2}},$$

where  $\beta_i = \frac{1}{2}(\alpha_i + 1)$  for  $1 \leq i \leq n$  and

$$\Gamma(t) = \int_0^\infty s^{t-1}e^{-s} ds = 2 \int_0^\infty r^{2t-1}e^{-r^2} dr.$$

**3.1. Second order spherical harmonics.**

**Theorem 9.** Let  $g(x) = \sum_{i,j} a_{ij}x_i x_j$ , where  $A = (a_{ij})_{i,j=1}^n$  is a symmetric matrix with  $\text{Tr}(A) = 0$ . Let  $f = Cg$ , where  $C$  is chosen such that  $\|f\|_2 = 1$  when  $f$  is considered as a function on  $S^{n-1}$ . Let  $W = f(X)$ , where  $X$  is a random point on  $S^{n-1}$ . If  $d$  is the vector in  $\mathbb{R}^n$  whose entries are the eigenvalues of  $A$ , then

$$d_{TV}(W, Z) \leq \sqrt{6} \left( \frac{\|d\|_4}{\|d\|_2} \right)^2,$$

where  $\|d\|_p = (\sum_i |d_i|^p)^{1/p}$ .

*Proof.* To apply Theorem 5, first note that  $f$  is indeed an eigenfunction of the Laplacian on  $S^{n-1}$ ; since  $A$  is traceless,  $g$  is harmonic on  $\mathbb{R}^n$  and thus is an eigenfunction with eigenvalue  $\lambda = -2n$ .

Next, observe that  $g(x) = \langle x, Ax \rangle$  and  $A$  is a symmetric matrix, so  $A = U^*DU$  for a diagonal matrix  $D$  and an orthogonal matrix  $U$ . Thus

$$g(x) = \langle x, U^*DUx \rangle = \langle Ux, DUx \rangle.$$

Since the uniform distribution on the sphere is invariant under the action of the orthogonal group, this observation means that it suffices to prove the theorem in the case that  $A = \text{diag}(a_1, \dots, a_n)$ .

Since  $f$  is an eigenfunction, the second error term of Theorem 5 vanishes. By the theorem and an application of Hölder’s inequality,

$$d_{TV}(W, Z) \leq \frac{1}{|\lambda|} \sqrt{\text{Var}(\|\nabla f\|^2)},$$

and the main task of the proof is to estimate the right-hand side. A consequence of Stokes’s theorem is that

$$\int_{S^{n-1}} \|\nabla g\|^2 d\sigma = |\lambda| \int_{S^{n-1}} g^2 d\sigma,$$

so one can calculate  $\mathbb{E}\|\nabla g\|^2$  in order to determine  $C$ . The spherical gradient  $\nabla g$  is just the projection onto the hyperplane orthogonal to the radial direction of the usual gradient. Letting  $\sum'$  stand for summing over distinct indexes,

$$\begin{aligned} \mathbb{E}\|\nabla_{S^{n-1}}g(x)\|^2 &= \mathbb{E}\|\nabla_{\mathbb{R}^n}g(x)\|^2 - \mathbb{E}(x \cdot \nabla_{\mathbb{R}^n}g(x))^2 \\ &= \mathbb{E}\left[\sum_{i=1}^n 4a_i^2x_i^2\right] - 4\mathbb{E}\left[\sum_{i=1}^n a_i^2x_i^4 + \sum'_{i,j} a_ia_jx_i^2x_j^2\right] \\ &= \frac{4}{n}\sum_{i=1}^n a_i^2 - \frac{4}{n(n+2)}\left[3\sum_i a_i^2 + \sum'_{i,j} a_ia_j\right] \end{aligned}$$

$$\begin{aligned}
 &= \frac{4}{n} \sum_{i=1}^n a_i^2 - \frac{4}{n(n+2)} \left[ 2 \sum_i a_i^2 + \left( \sum_{i=1}^n a_i \right)^2 \right] \\
 &= \frac{4}{n+2} \sum_{i=1}^n a_i^2,
 \end{aligned}$$

where Lemma 8 has been used to get the third line and the fact that  $\text{Tr}(A) = 0$  is used to get the last line. From this computation it follows that the constant  $C$  in the statement of the theorem should be taken to be  $\sqrt{\frac{n(n+2)}{2\|a\|_2^2}}$ , where  $a = (a_i)_{i=1}^n$ .

Now,

$$\mathbb{E}\|\nabla_{S^{n-1}}g\|^4 \leq \mathbb{E}\|\nabla_{\mathbb{R}^n}g\|^4,$$

since  $\nabla_{S^{n-1}}g$  is a projection of  $\nabla_{\mathbb{R}^n}g$ .

$$\begin{aligned}
 \mathbb{E}\|\nabla_{\mathbb{R}^n}g\|^4 &= 16 \mathbb{E} \left[ \sum_{i=1}^n a_i^4 x_i^4 + \sum'_{i,j} a_i^2 a_j^2 x_i^2 x_j^2 \right] \\
 &= \frac{16}{n(n+2)} \left[ \left( \sum_i a_i^2 \right)^2 + 2 \sum_i a_i^4 \right],
 \end{aligned}$$

and so

$$\begin{aligned}
 \mathbb{E}\|\nabla_{S^{n-1}}f\|^4 &\leq \frac{n^2(n+2)^2}{4\|a\|_2^4} \mathbb{E}\|\nabla_{\mathbb{R}^n}g\|^4 \\
 &= \frac{4n(n+2)}{\|a\|_2^4} \left[ \left( \sum_i a_i^2 \right)^2 + 2 \sum_i a_i^4 \right] \\
 &= 4n(n+2) \left[ 1 + \frac{2\|a\|_4^4}{\|a\|_2^4} \right].
 \end{aligned}$$

This gives that

$$\text{Var}(\|\nabla f\|^2) \leq 8n^2 \left( \frac{\|a\|_4^4}{\|a\|_2^4} \right) + 8n \left[ 1 + \frac{2\|a\|_4^4}{\|a\|_2^4} \right];$$

thus

$$d_{TV}(W, Z) \leq \sqrt{\left(2 + \frac{4}{n}\right) \left(\frac{\|a\|_4}{\|a\|_2}\right)^4 + \frac{2}{n}} \leq \sqrt{4 + \frac{4}{n}} \left(\frac{\|a\|_4}{\|a\|_2}\right)^2,$$

since  $\|a\|_4 \geq n^{-1/4}\|a\|_2$ . □

*Remark.* For  $n$  even, consider the quadratic function

$$f(x) = \sqrt{\frac{n+2}{2}} \left[ \sum_{i=1}^{\frac{n}{2}} x_i^2 - \sum_{j=\frac{n}{2}+1}^n x_j^2 \right].$$

This  $f$  satisfies the conditions of the theorem and is normalized such that  $\mathbb{E}f^2(X) = 1$  for  $X$  a random point on  $S^{n-1}$ . Theorem 9 gives that

$$d_{TV}(f(X), Z) \leq \sqrt{\frac{6}{n}},$$

so in this case, Theorem 9 gives a rate of convergence to normal.

The cases in which the bound does not tend to zero as  $n \rightarrow \infty$  are those in which a small number of coordinates of  $X$  control the value of  $f(X)$ ; in those situations one would not expect  $f$  to be normally distributed. For example, if  $f(x) = cx_1^2 - cx_2^2$ , where  $c$  is the proper normalization constant, then because  $x_1$  and  $x_2$  are asymptotically independent Gaussian random variables (see [4]),  $f$  is asymptotically distributed as  $c(Z_1^2 - Z_2^2)$  for independent Gaussians  $Z_1$  and  $Z_2$ .

**3.2. Higher degree spherical harmonics.** Fix an odd integer  $\ell$ . A canonical example of an eigenfunction of the spherical Laplacian of degree  $\ell$  is the polynomial  $p_\ell(x)$  given by

$$p_\ell(x) = AC_\ell^{\frac{n-2}{2}}(x_n),$$

where  $A^2 = \frac{(n-3)! \ell!(n+2\ell-2)}{(n+\ell-3)!(n-2)}$  is a normalization constant chosen such that  $\mathbb{E}p_\ell^2(X) = 1$  for  $X$  a random point of  $S^{n-1}$  and

$$(3) \quad C_\ell^k(t) = \frac{2^\ell}{\Gamma(k)} \sum_{j=0}^{\lfloor \ell/2 \rfloor} \frac{(-1)^j \Gamma(k + \ell - j)}{2^{2j} j! (\ell - 2j)!} t^{\ell - 2j}$$

is a Gegenbauer polynomial. The eigenvalue of this eigenfunction of the Laplacian on  $S^{n-1}$  is  $\ell(n + \ell - 2)$ .

The reason that this particular eigenfunction is natural is that it arises in the representation theory of the orthogonal group. Let  $T^{n,\ell}$  be the irreducible representation of  $SO(n)$  in the space  $\mathfrak{R}^{n,\ell}/r^2\mathfrak{R}^{n,\ell-2}$ , where  $\mathfrak{R}^{n,\ell}$  is the space of homogeneous polynomials of degree  $\ell$  in  $n$  variables. The polynomial  $p_\ell(x)$  above is the zonal polynomial of degree  $\ell$  of the representation  $T^{n,\ell}$  relative to the subgroup  $SO(n-1)$ , where, as usual,  $SO(n-1)$  is identified with the subgroup of  $SO(n)$  fixing the north pole of  $S^{n-1}$ . See [13] for background on spherical harmonics and Gegenbauer polynomials, including all facts stated above.

Observe that since  $p_\ell^{(n)}(x) = AC_\ell^{\frac{n-2}{2}}(x_n)$  is an eigenfunction, so are  $p_\ell^{(k)}(x) = AC_\ell^{\frac{n-2}{2}}(x_k)$  for all  $1 \leq k \leq n$ . Let  $a_1, \dots, a_n \subseteq \mathbb{R}$  such that  $\sum a_i^2 = 1$ , and consider the degree  $\ell$  eigenfunction

$$p(x) = \sum_{k=1}^n a_k p_\ell^{(k)}(x).$$

Note that  $\mathbb{E}p^2(X) = 1$  for  $X$  a random point of  $S^{n-1}$ , because if  $\ell$  is odd, then the  $C_\ell^{\frac{n-2}{2}}(x_k)$  are orthogonal for different  $k$ . In this example, a bound on the total variation distance between the distribution of the random variable  $p(X)$  for

$X$  random on the sphere and a standard Gaussian random variable  $Z$  is obtained in terms of the  $a_i$ . Specifically,

**Theorem 10.** *There is a constant  $c$ , depending on  $\ell$ , such that for  $p : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$  defined by*

$$\sum_{i=1}^n a_i p_\ell^{(i)}(x)$$

with  $\sum a_i^2 = 1$ ,

$$d_{TV}(p(X), Z) \leq c \|\mathbf{a}\|_4^2.$$

If the vector of coefficients  $\{a_i\}$  is chosen to be random and uniformly distributed on the sphere, then there is another constant  $c'$  such that

$$\mathbb{E}d_{TV}(p(X), Z) \leq \frac{c'}{\sqrt{n}}.$$

*Remark.* The second statement of the theorem says that, typically, spherical harmonics obtained as linear combinations of the  $p_\ell^{(i)}(x)$  are approximately Gaussian in distribution. As in Theorem 9, it may also be helpful to consider a specific example. If  $|a_i| = \frac{1}{\sqrt{n}}$  for each  $i$ , then

$$d_{TV}(p(X), Z) \leq \frac{c}{\sqrt{n}}$$

for a constant  $c$  depending only on  $\ell$ . As discussed after the statement of Theorem 9, the bound  $\|\mathbf{a}\|_4^2$  is not small in the case that only a fixed small number  $k$  of  $a_i$  are non-zero. In such a case,  $p(X)$  is not distributed as a Gaussian random variable but as a degree  $\ell$  polynomial in  $k$  independent Gaussian random variables.

*Proof.* Recall that Theorem 5 gives

$$\begin{aligned} d_{TV}(p(X), Z) &\leq \frac{1}{\mu} \sqrt{\text{Var}(\|\nabla_{S^{n-1}} p(X)\|^2)} \\ (4) \qquad &= \frac{1}{\ell(n + \ell - 2)} \sqrt{\mathbb{E}\|\nabla_{S^{n-1}} p(X)\|^4 - \ell^2(n + \ell - 2)^2}, \end{aligned}$$

since

$$\mathbb{E}\|\nabla_{S^{n-1}} p(X)\|^2 = \ell(n + \ell - 2)$$

by Stokes's theorem. Thus to apply the theorem, we must compute (or bound)

$$\mathbb{E}\|\nabla_{S^{n-1}} p(X)\|^4.$$

Consider the function  $p(x) = \sum_{k=1}^n a_k p_\ell^{(k)}(x)$  on a neighborhood of  $S^{n-1}$ . Then for  $x \in S^{n-1}$ ,

$$\|\nabla_{S^{n-1}} p(x)\|^2 = \|\nabla p(x)\|^2 - \langle x, \nabla p(x) \rangle^2,$$

where  $\nabla p(x)$  denotes the usual  $\mathbb{R}^n$ -gradient of  $p$ . Thus

$$\mathbb{E}\|\nabla_{S^{n-1}} p(X)\|^4 \leq \mathbb{E}\|\nabla p(X)\|^4;$$

bounding the right-hand side turns out to be enough.

It is proved in [13] that the Gegenbauer polynomials have the property that

$$(5) \qquad \frac{d}{dt} C_m^p(t) = 2p C_{m-1}^{p+1}(t);$$

thus

$$\nabla p(x) = \left( A a_k (n - 2) C_{\ell-1}^{\frac{n}{2}}(x_k) \right)_{k=1}^n.$$

It is therefore necessary to estimate

$$\begin{aligned}
 (6) \quad & \mathbb{E} \left[ A^2(n-2)^2 \sum_{k=1}^n a_k^2 (C_{\ell-1}^{\frac{n}{2}}(x_k))^2 \right]^2 \\
 &= A^4(n-2)^4 \mathbb{E} \left[ \sum_{k=1}^n a_k^4 (C_{\ell-1}^{\frac{n}{2}}(x_k))^4 + \sum'_{k,j=1}^n a_k^2 a_j^2 (C_{\ell-1}^{\frac{n}{2}}(x_k))^2 (C_{\ell-1}^{\frac{n}{2}}(x_j))^2 \right],
 \end{aligned}$$

where  $\sum'$  stands for summing over distinct indices.

The proof hinges on the following key facts.

(i) For  $k \neq j$ ,

$$A^4(n-2)^4 \mathbb{E} \left[ (C_{\ell-1}^{\frac{n}{2}}(x_k))^2 (C_{\ell-1}^{\frac{n}{2}}(x_j))^2 \right] = \ell^2(n + O(1))^2.$$

(ii) There is a constant  $c$  such that for all  $k$ ,

$$A^4(n-2)^4 \mathbb{E} \left[ (C_{\ell-1}^{\frac{n}{2}}(x_k))^4 \right] \leq cn^2.$$

Using the normalization  $\sum a_k^2 = 1$ , these facts imply that

$$\mathbb{E} \|\nabla_{S^{n-1}} p(X)\|^4 \leq \ell^2 n^2 + c'n^2 \sum_{k=1}^n a_k^4 + c''n$$

for constants  $c', c''$ . Note that since  $\|\mathbf{a}\|_2 = 1$ ,  $\sum a_k^4 \geq \frac{1}{n}$  and so the last term can be absorbed into the middle term. The result is now immediate from (4).

It remains to verify the two key facts above. By (3),

$$\begin{aligned}
 & \mathbb{E} \left[ C_{\ell-1}^{\frac{n}{2}}(X_1) C_{\ell-1}^{\frac{n}{2}}(X_2) \right]^2 \\
 &= \left( \frac{2^{\ell-1}}{\Gamma(\frac{n}{2})} \right)^4 \\
 & \cdot \sum_{k,m,p,q=0}^{\frac{\ell-1}{2}} \left( \frac{(-1)^{k+m+p+q} \Gamma(\ell-1+\frac{n}{2}-k) \Gamma(\ell-1+\frac{n}{2}-m) \Gamma(\ell-1+\frac{n}{2}-p) \Gamma(\ell-1+\frac{n}{2}-q)}{2^{2k+2m+2p+2q} k! m! p! q! (\ell-1-2k)! (\ell-1-2m)! (\ell-1-2p)! (\ell-1-2q)!} \right) \\
 & \cdot \mathbb{E} \left[ X_1^{2\ell-2k-2m-2} X_2^{2\ell-2p-2q-2} \right].
 \end{aligned}$$

Lemma 8 gives

$$\begin{aligned}
 & \mathbb{E} \left[ X_1^{2\ell-2k-2m-2} X_2^{2\ell-2p-2q-2} \right] \\
 &= \frac{\Gamma(\ell-k-m-\frac{1}{2}) \Gamma(\ell-p-q-\frac{1}{2}) \Gamma(\frac{n}{2})}{\pi \Gamma(\frac{n}{2} + 2\ell - k - m - p - q - 2)} \\
 &= \frac{1}{\pi} \Gamma\left(\ell-k-m-\frac{1}{2}\right) \Gamma\left(\ell-p-q-\frac{1}{2}\right) \left(\frac{2}{n} + O\left(\frac{1}{n^2}\right)\right)^{2\ell-k-m-p-q-2},
 \end{aligned}$$

where here and throughout,  $O(\frac{1}{n^2})$ ,  $O(1)$ , etc., refer to the limit as  $n \rightarrow \infty$ , with  $\ell$  still fixed and the constants implicit in the  $O$  depending on  $\ell$ . It follows that

$$(7) \quad \mathbb{E} \left[ C_{\ell-1}^{\frac{n}{2}}(X_1) C_{\ell-1}^{\frac{n}{2}}(X_2) \right]^2 = \left( \frac{2^{2\ell-2}}{\Gamma^2(\frac{n}{2})} \sum_{k,m=0}^{\frac{\ell-1}{2}} \frac{(-1)^{k+m} \Gamma(\ell-1+\frac{n}{2}-k) \Gamma(\ell-1+\frac{n}{2}-m) \Gamma(\ell-k-m-\frac{1}{2})}{2^{2k+2m} k! m! (\ell-1-2k)! (\ell-1-2m)! \sqrt{\pi}} \right)^2 \cdot \left( \frac{2}{n} + O\left(\frac{1}{n^2}\right) \right)^{\ell-k-m-1}.$$

Now,

$$(8) \quad \Gamma\left(\ell-1+\frac{n}{2}-k\right) = \Gamma\left(\frac{n}{2}\right) \left(\frac{n}{2} + O(1)\right)^{\ell-k-1},$$

so the right-hand side of (7) reduces to

$$\left( (2n + O(1))^{\ell-1} \sum_{k,m=0}^{\frac{\ell-1}{2}} \frac{(-1)^{k+m} \Gamma(\ell-k-m-\frac{1}{2}) \pi^{-1/2}}{2^{2k+2m} k! m! (\ell-1-2k)! (\ell-1-2m)!} \right)^2.$$

Recall that

$$A^2 = \frac{(n-3)! \ell! (n+2\ell-2)}{(n+\ell-3)! (n-2)} = \ell! (n + O(1))^{-\ell};$$

thus

$$A^4 (n-2)^4 \mathbb{E} \left[ C_{\ell-1}^{\frac{n}{2}}(X_1) C_{\ell-1}^{\frac{n}{2}}(X_2) \right]^2 = (\ell!)^2 (n + O(1))^2 \left( 2^{\ell-1} \sum_{k,m=0}^{\frac{\ell-1}{2}} \frac{(-1)^{k+m} \Gamma(\ell-k-m-\frac{1}{2}) \pi^{-1/2}}{2^{2k+2m} k! m! (\ell-1-2k)! (\ell-1-2m)!} \right)^2.$$

*Claim.* For  $\ell \in \mathbb{N}$  odd,

$$2^{\ell-1} \sum_{k,m=0}^{\frac{\ell-1}{2}} \frac{(-1)^{k+m} \Gamma(\ell-k-m-\frac{1}{2}) \pi^{-1/2}}{2^{2k+2m} k! m! (\ell-1-2k)! (\ell-1-2m)!} = \frac{1}{(\ell-1)!}.$$

A proof of this claim is given in the appendix. Making use of the claim, it is immediate that

$$A^4 (n-2)^4 \mathbb{E} \left[ C_{\ell-1}^{\frac{n}{2}}(X_1) C_{\ell-1}^{\frac{n}{2}}(X_2) \right]^2 = \ell^2 (n + O(1))^2.$$

It remains to verify fact (ii): there is a constant  $c$  such that for all  $k$ ,

$$A^4 (n-2)^4 \mathbb{E} \left[ (C_{\ell-1}^{\frac{n}{2}}(x_k))^4 \right] \leq cn^2.$$

In fact, this is much easier than the previous part since the exact asymptotic constant is not needed. Proceeding as before,

$$\begin{aligned} & \mathbb{E} \left[ C_{\ell-1}^{\frac{n}{2}}(X_1) \right]^4 \\ &= \left( \frac{2^{\ell-1}}{\Gamma(\frac{n}{2})} \right)^4 \\ & \cdot \sum_{k,m,p,q=0}^{\ell-1} \left( \frac{(-1)^{k+m+p+q} \Gamma(\ell-1+\frac{n}{2}-k) \Gamma(\ell-1+\frac{n}{2}-m) \Gamma(\ell-1+\frac{n}{2}-p) \Gamma(\ell-1+\frac{n}{2}-q)}{2^{2k+2m+2p+2q} k!m!p!q!(\ell-1-2k)!(\ell-1-2m)!(\ell-1-2p)!(\ell-1-2q)!} \right. \\ & \left. \cdot \mathbb{E} \left[ X_1^{4\ell-2k-2m-2p-2q-4} \right] \right). \end{aligned}$$

From Lemma 8,

$$\begin{aligned} \mathbb{E} \left[ X_1^{4\ell-2k-2m-2p-2q-4} \right] &= \frac{\Gamma(2\ell-k-m-p-q-\frac{3}{2}) \Gamma(\frac{n}{2})}{\pi \Gamma(\frac{n}{2}+2\ell-k-m-p-q-2)} \\ &= \frac{1}{\pi} \Gamma\left(2\ell-k-m-p-q-\frac{3}{2}\right) \left(\frac{2}{n} + O\left(\frac{1}{n^2}\right)\right)^{2\ell-k-m-p-q-2}, \end{aligned}$$

and so

$$\begin{aligned} & \mathbb{E} \left[ C_{\ell-1}^{\frac{n}{2}}(X_1) \right]^4 \\ &= \left( 2^{4\ell-4} \sum_{k,m,p,q=0}^{\ell-1} \frac{(-1)^{k+m+p+q} \Gamma(2\ell-k-m-p-q-\frac{3}{2}) \left(\frac{n}{2} + O(1)\right)^{2\ell-2}}{2^{2k+2m+2p+2q} k!m!p!q!(\ell-1-2k)!(\ell-1-2m)!(\ell-1-2p)!(\ell-1-2q)!} \right), \end{aligned}$$

making use again of (8). Since

$$A^4 = (\ell!)^2 (n + O(1))^{-2\ell},$$

it follows that

$$A^4 (n-2)^4 \mathbb{E} \left[ C_{\ell-1}^{\frac{n}{2}}(X_1) \right]^4 = (n + O(1))^2 \cdot C,$$

where  $C$  (which depends on  $\ell$ ) is independent of  $n$ . □

#### 4. THE TORUS

This section deals with the value distributions of eigenfunctions on flat tori. The class of functions considered here are random functions; that is, they are linear combinations of eigenfunctions with random coefficients.

Let  $(M, g)$  be the torus  $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ , with the metric given by the symmetric positive-definite bilinear form  $B$ :

$$(x, y)_B = \langle Bx, y \rangle.$$

With this metric, the Laplacian  $\Delta_B$  on  $\mathbb{T}^n$  is given by

$$\Delta_B f(x) = \sum_{j,k} (B^{-1})_{jk} \frac{\partial^2 f}{\partial x_j \partial x_k}(x).$$

Eigenfunctions of  $\Delta_B$  are given by the real and imaginary parts of functions of the form

$$f_v(x) = e^{2\pi i \langle v, x \rangle_B} = e^{2\pi i \langle Bv, x \rangle},$$

for vectors  $v \in \mathbb{R}^n$  such that  $Bv$  has integer components, with corresponding eigenvalue  $-\mu_v = -(2\pi \|v\|_B)^2$ .



Consider a linear combination of eigenfunctions with random coefficients:

$$f(x) := \Re \left( \sum_{v \in \mathcal{V}} a_v e^{2\pi i \langle Bv, x \rangle} \right),$$

where  $\mathcal{V}$  is a finite collection of vectors  $v$  such that  $Bv$  has integer components for each  $v \in \mathcal{V}$  and  $\{a_v\}_{v \in \mathcal{V}}$  is a random vector on the sphere of radius  $\sqrt{2}$  in  $\mathbb{R}^{\mathcal{V}}$ . Assume further that  $v + w \neq 0$  for  $v, w \in \mathcal{V}$ . Then the following theorem holds.

**Theorem 11.** *For a random function  $f$  on  $(\mathbb{T}^n, B)$  defined as above and for  $\mu > 0$ ,*

$$\begin{aligned} & \mathbb{E}d_{TV}(f(X), Z) \\ & \leq \frac{1}{|\mu|} \left[ \sqrt{\frac{8(2\pi)^4}{|\mathcal{V}|(|\mathcal{V}| + 2)} \sum_{v, w \in \mathcal{V}} (v, w)_B^2} + (2\sqrt{2} + \sqrt{\pi}) \sqrt{\frac{1}{|\mathcal{V}|} \sum_{v \in \mathcal{V}} (\mu_v - \mu)^2} \right]. \end{aligned}$$

The idea is that the eigenvalues  $-\mu_v = -(2\pi\|v\|_B)^2$  corresponding to the summands be clustered about the value  $\mu$  and that the size of  $\mathcal{V}$  be large. In some cases, e.g., for  $B = I$ , it may be possible to eliminate the second error term by choosing a large set  $\mathcal{V}$  such that all elements have the same  $B$ -norm; that is, it may be that the eigenspaces of the Laplacian  $\Delta_B$  have large dimension. However, this need not be true in general. The presence of the second error term is to take into account that while it may not be the case that eigenspaces are large, it may be that there are many eigenspaces of low dimension whose eigenvalues are all close together.

Note that requiring the first term of the error to be small is a weakening of an orthogonality requirement on the elements of  $\mathcal{V}$ . By the Cauchy-Schwarz inequality,

$$\sum_{v, w \in \mathcal{V}} (v, w)_B^2 \leq \sum_{v, w \in \mathcal{V}} \|v\|_B^2 \|w\|_B^2 \leq \left( \max_{v \in \mathcal{V}} \|v\|_B^4 \right) |\mathcal{V}|^2.$$

If all of the vectors  $v \in \mathcal{V}$  have  $\mu_v \approx \mu$ , then this last expression is approximately  $\frac{\mu^2}{(2\pi)^4} |\mathcal{V}|^2$ , and the first error term is bounded by  $2\sqrt{2}$ . This is of course worthless, since total variation distance is always bounded by one. On the other hand, if the elements of  $\mathcal{V}$  are mutually orthogonal, then

$$\sum_{v, w \in \mathcal{V}} (v, w)_B^2 = \sum_{v \in \mathcal{V}} \|v\|_B^4 \approx \frac{\mu^2}{(2\pi)^4} |\mathcal{V}|,$$

and then the first error term is bounded by  $\frac{2\sqrt{2}}{\sqrt{|\mathcal{V}|}}$ , which is small as long as  $\mathcal{V}$  is large. The idea is to aim for a set  $\mathcal{V}$  somewhere in between these two extremes: not to require the elements of  $\mathcal{V}$  to be orthogonal, but to choose them to be fairly evenly distributed on the  $B$ -norm sphere on which they lie.

**Example.** (1) Let  $B = I$ . Choose  $\mathcal{V}$  to be the set of vectors in  $\mathbb{R}^n$  with two entries equal to one and all others zero. Then  $|\mathcal{V}| = \binom{n}{2}$  and  $\|v\|^2 = 2$  for each  $v \in \mathcal{V}$ . Given a fixed  $v \in \mathcal{V}$ , there are exactly  $2n - 3$  elements  $w \in \mathcal{V}$  (including  $v$  itself) such that  $\langle v, w \rangle \neq 0$ . Thus

$$\frac{1}{|\mathcal{V}|(|\mathcal{V}| + 2)} \sum_{v, w \in \mathcal{V}} \langle v, w \rangle^2 \leq \frac{1}{\binom{n}{2} \left( \binom{n}{2} + 2 \right)} \cdot \binom{n}{2} (4)(2n - 3) = \frac{16n - 12}{n^2 - n + 4};$$

thus in this case there is a constant  $c$  such that

$$\mathbb{E}d_{TV}(f(X), Z) \leq \frac{c}{\sqrt{n}}.$$

- (2) Let  $B = \text{diag}(1+\delta_1, \dots, 1+\delta_n)$ , with  $\delta_i$  small real numbers. Let  $\mathcal{V}$  be the set of vectors of the previous example. For  $v \in \mathcal{V}$ , let  $v' = \left(\frac{1}{1+\delta_1}v_1, \dots, \frac{1}{1+\delta_n}v_n\right)$ ;  $v'$  is constructed such that  $Bv'$  has integer components. Let  $\mathcal{V}' = \{v' : v \in \mathcal{V}\}$ .

Now, let  $\epsilon = \max_i \left|1 - \frac{1}{1+\delta_i}\right|$ . Then

$$\left| \|v'\|_B^2 - 2 \right| = \left| \sum_{i=1}^n \frac{1}{1+\delta_i} v_i^2 - \sum_{i=1}^n v_i^2 \right| \leq \epsilon \|v\|^2 = 2\epsilon.$$

By construction,  $(v', w')_B \neq 0$  if and only if  $\langle v, w \rangle \neq 0$ , and in those cases,

$$(v', w')_B^2 \leq \|v'\|_B^2 \|w'\|_B^2 \leq (2 + 2\epsilon)^2.$$

Taking  $\mu = 2(2\pi)^2$  in the theorem, it follows that for  $f$  defined as above using this matrix  $B$  and index set  $\mathcal{V}'$ , there is a constant  $C$  so that

$$\mathbb{E}d_{TV}(f(X), Z) \leq C \max\left(\frac{1}{\sqrt{n}}, \epsilon\right).$$

*Proof of Theorem 11.* To apply Theorem 5, an expression for  $\|\nabla_B f\|_B^2$  is needed.

$$\begin{aligned} \nabla_B f(x) &= \left\{ \Re \left( \sum_{j=1}^n \sum_{v \in \mathcal{V}} (2\pi i) a_v (Bv)_j (B^{-1})_{jk} e^{2\pi i \langle Bv, x \rangle} \right) \right\}_{k=1}^n \\ &= -\Im \left( \sum_{v \in \mathcal{V}} (2\pi) a_v e^{2\pi i \langle Bv, x \rangle} v \right), \end{aligned}$$

using the fact that  $B$  is symmetric.

It follows that

(9)

$$\begin{aligned} \|\nabla_B f(x)\|_B^2 &= \sum_{j,k} B_{jk} (\nabla_B f(x))_j (\nabla_B f(x))_k \\ &= \sum_{j,k} B_{jk} \Im \left( \sum_{v \in \mathcal{V}} (2\pi) a_v e^{2\pi i \langle Bv, x \rangle} v_j \right) \Im \left( \sum_{w \in \mathcal{V}} (2\pi) a_w e^{2\pi i \langle Bw, x \rangle} w_k \right) \\ &= \frac{1}{2} \Re \left[ \sum_{v,w \in \mathcal{V}} 4\pi^2 a_v a_w \left( \sum_{j,k} B_{jk} v_j w_k \right) \left( e^{2\pi i \langle Bv - Bw, x \rangle} - e^{2\pi i \langle Bv + Bw, x \rangle} \right) \right] \\ &= \frac{1}{2} \Re \left[ \sum_{v,w \in \mathcal{V}} 4\pi^2 a_v a_w (v, w)_B \left( e^{2\pi i \langle Bv - Bw, x \rangle} - e^{2\pi i \langle Bv + Bw, x \rangle} \right) \right]. \end{aligned}$$

Let  $X$  be a randomly distributed point on the torus. The first half of the error term in Theorem 5 is

$$\mathbb{E} \left| \|\nabla_B f(X)\|_B^2 - \mathbb{E}_X \|\nabla_B f(X)\|_B^2 \right| \leq \sqrt{\mathbb{E} \|\nabla_B f(X)\|_B^4 - \mathbb{E}(\mathbb{E}_X \|\nabla_B f(X)\|_B^2)^2}.$$

Start by computing  $\mathbb{E}\|\nabla_B f(X)\|_B^4$ . From above,

$$\begin{aligned} & \|\nabla_B f(x)\|_B^4 \\ &= 2\pi^4 \Re \left[ \sum_{v,w,v',w'} a_v a_w a_{v'} a_{w'} (v,w)_B (v',w')_B \right. \\ & \quad \cdot \left[ e^{2\pi i \langle Bv-Bw-Bv'+Bw',x \rangle} - e^{2\pi i \langle Bv-Bw-Bv'-Bw',x \rangle} + e^{2\pi i \langle Bv-Bw+Bv'-Bw',x \rangle} \right. \\ & \quad \left. - e^{2\pi i \langle Bv-Bw+Bv'+Bw',x \rangle} - e^{2\pi i \langle Bv+Bw-Bv'+Bw',x \rangle} + e^{2\pi i \langle Bv+Bw-Bv'-Bw',x \rangle} \right. \\ & \quad \left. \left. - e^{2\pi i \langle Bv+Bw+Bv'-Bw',x \rangle} + e^{2\pi i \langle Bv+Bw+Bv'+Bw',x \rangle} \right] \right]. \end{aligned}$$

Averaging over the coefficients  $\{a_v\}$  gives

$$\begin{aligned} & \mathbb{E}_a \|\nabla_B f(x)\|_B^4 \\ &= \frac{8\pi^4}{|\mathcal{V}|(|\mathcal{V}|+2)} \Re \left[ 3 \sum_{v \in \mathcal{V}} \|v\|_B^4 \left( 3 - e^{-4\pi i \langle Bv,x \rangle} - 3e^{4\pi i \langle Bv,x \rangle} + e^{8\pi i \langle Bv,x \rangle} \right) \right. \\ & \quad + \sum'_{v,v' \in \mathcal{V}} \|v\|_B^2 \|v'\|_B^2 \left( 2 - 2e^{4\pi i \langle Bv,x \rangle} - e^{-4\pi i \langle Bv',x \rangle} - e^{4\pi i \langle Bv',x \rangle} \right. \\ & \quad \left. \left. + e^{4\pi i \langle Bv-Bv',x \rangle} + e^{4\pi i \langle Bv+Bv',x \rangle} \right) \right. \\ & \quad \left. + 2 \sum'_{v,w \in \mathcal{V}} (v,w)_B^2 \left( 2 - 2e^{4\pi i \langle Bv,x \rangle} - e^{-4\pi i \langle Bw,x \rangle} - e^{4\pi i \langle Bw,x \rangle} \right. \right. \\ & \quad \left. \left. + e^{4\pi i \langle Bv-Bw,x \rangle} + e^{4\pi i \langle Bv+Bw,x \rangle} \right) \right]. \end{aligned}$$

Now averaging over a random point  $X$  of the torus gives

$$\begin{aligned} \mathbb{E}\|\nabla_B f(X)\|_B^4 &= \frac{8\pi^4}{|\mathcal{V}|(|\mathcal{V}|+2)} \left[ 9 \sum_{v \in \mathcal{V}} \|v\|_B^4 + 2 \sum'_{v,w \in \mathcal{V}} \|v\|_B^2 \|w\|_B^2 + 4 \sum'_{v,w \in \mathcal{V}} (v,w)_B^2 \right] \\ &= \frac{8\pi^4}{|\mathcal{V}|(|\mathcal{V}|+2)} \left[ 3 \sum_{v \in \mathcal{V}} \|v\|_B^4 + 2 \left( \sum_{v \in \mathcal{V}} \|v\|_B^2 \right)^2 + 4 \sum_{v,w \in \mathcal{V}} (v,w)_B^2 \right]. \end{aligned}$$

Next,

$$\begin{aligned} \mathbb{E} \left[ \mathbb{E}_X \|\nabla_B f(X)\|_B^2 \right]^2 &= \mathbb{E} \left[ 2\pi^2 \sum_{v \in \mathcal{V}} a_v^2 \|v\|_B^2 \right]^2 \\ &= \frac{(2\pi)^4}{|\mathcal{V}|(|\mathcal{V}|+2)} \left[ 3 \sum_{v \in \mathcal{V}} \|v\|_B^4 + \sum'_{v,w \in \mathcal{V}} \|v\|_B^2 \|w\|_B^2 \right] \\ &= \frac{(2\pi)^4}{|\mathcal{V}|(|\mathcal{V}|+2)} \left[ \left( \sum_{v \in \mathcal{V}} \|v\|_B^2 \right)^2 + 2 \sum_{v \in \mathcal{V}} \|v\|_B^4 \right]; \end{aligned}$$

thus

$$\begin{aligned} & \sqrt{\mathbb{E}\|\nabla_B f(X)\|_B^4 - \mathbb{E}(\mathbb{E}_X\|\nabla_B f(X)\|_B^2)^2} \\ & \leq \sqrt{\frac{(2\pi)^4}{|\mathcal{V}|(|\mathcal{V}|+2)} \left(2 \sum_{v,w \in \mathcal{V}} (v,w)_B^2 - \frac{1}{2} \sum_{v \in \mathcal{V}} \|v\|_B^4\right)} \\ & \leq \sqrt{\frac{2(2\pi)^4}{|\mathcal{V}|(|\mathcal{V}|+2)} \sum_{v,w \in \mathcal{V}} (v,w)_B^2}. \end{aligned}$$

Finally, the second error term of Theorem 5 is a multiple of

$$\sqrt{\sum_{v \in \mathcal{V}} a_v^2 (\mu_v - \mu)^2},$$

where  $\mu_v = (2\pi\|v\|_B)^2$  and  $\mu = \frac{1}{|\mathcal{V}|} \sum_{v \in \mathcal{V}} \mu_v$ . By Hölder’s inequality,

$$\mathbb{E} \sqrt{\sum_{v \in \mathcal{V}} a_v^2 (\mu_v - \mu)^2} \leq \sqrt{\frac{2}{|\mathcal{V}|} \sum_{v \in \mathcal{V}} (\mu_v - \mu)^2}.$$

This completes the proof. □

### 5. APPENDIX

This section contains the proof that for  $\ell$  odd,

$$(10) \quad 2^{\ell-1} \sum_{k,m=0}^{\frac{\ell-1}{2}} \frac{(-1)^{k+m} \Gamma(\ell - k - m - \frac{1}{2}) \pi^{-1/2} (\ell - 1)!}{2^{2k+2m} k! m! (\ell - 1 - 2k)! (\ell - 1 - 2m)!} = 1.$$

For  $n \in \mathbb{N}$ , let  $n!! = n(n - 2)(n - 4) \cdots (1, 2)$ , where the last factor is 1 or 2 depending on the parity of  $n$ . Using the fact that  $\Gamma(x) = (x - 1)\Gamma(x - 1)$  together with  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ , the left-hand side of (10) is

$$(11) \quad \begin{aligned} & \sum_{k,m=0}^{\frac{\ell-1}{2}} \frac{(-1)^{k+m} (2\ell - 2k - 2m - 3)!! (\ell - 1)!}{2^{k+m} k! m! (\ell - 2k - 1)! (\ell - 2m - 1)!} \\ & = \sum_{k,m=0}^{\frac{\ell-1}{2}} \frac{(-1)^{k+m} (2\ell - 2k - 2m - 2)!! (\ell - 1)!}{2^{k+m} k! m! (\ell - 2k - 1)! (\ell - 2m - 1)! (2\ell - 2k - 2m - 2)!!}. \end{aligned}$$

Next, let  $p = \frac{\ell-1}{2}$ . Then (11) becomes

$$\begin{aligned} & \sum_{k,m=0}^p \frac{(-1)^{k+m} (4p - 2k - 2m)! (2p)!}{2^{k+m} k! m! (2p - 2k)! (2p - 2m)! (4p - 2k - 2m)!!} \\ & = 2^{-2p} \sum_{k,m=0}^p \frac{(-1)^{k+m} (4p - 2k - 2m)! (2p)!}{k! m! (2p - 2k)! (2p - 2m)! (2p - k - m)!} \\ & = 2^{-2p} \sum_{k=0}^p \frac{(-1)^k (2p)!}{k! (2p - 2k)!} \left( \sum_{m=0}^p \frac{(-1)^m (4p - 2k - 2m)!}{m! (2p - 2m)! (2p - k - m)!} \right), \end{aligned}$$

and the goal is to prove that this last expression is equal to 1. We do this by showing

$$(12) \quad \sum_{m=0}^p \frac{(-1)^m (4p - 2k - 2m)!}{m!(2p - 2m)!(2p - k - m)!} = \begin{cases} 2^{2p}, & k = 0, \\ 0, & k = 1, \dots, p. \end{cases}$$

This can be done by a computer algebra system such as Maple; the following direct proof is due to Geir Helleloid [8].

Let  $(\alpha)_n$  be the shifted factorial

$$(\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1).$$

The following two straightforward identities involving shifted factorials will be useful:

$$(n - k)! = \frac{(-1)^k n!}{(-n)_k}$$

and

$$(a)_{2k} = 2^{2k} \left(\frac{a}{2}\right)_k \left(\frac{a+1}{2}\right)_k.$$

The hypergeometric series

$$\sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \cdot \frac{x^n}{n!}$$

is denoted

$${}_pF_q = (a_1, \dots, a_p; b_1, \dots, b_q; x),$$

with the  $x$  omitted if  $x = 1$ . The *Chu-Vandermonde* identity is

$${}_2F_1(-n, -b; c) = \frac{(c + b)_n}{(c)_n}.$$

The proof of (12) is as follows:

$$\begin{aligned} & \sum_{m=0}^p \frac{(-1)^m (4p - 2k - 2m)!}{m!(2p - 2m)!(2p - k - m)!} \\ &= \sum_{m=0}^p \frac{(-1)^m (-1)^{2m} (4p - 2k)! / (-4p + 2k)_{2m}}{m!((-1)^{2m} (2p)! / (-2p)_{2m})((-1)^m (2p - k)! / (-2p + k)_m)} \\ &= \frac{(4p - 2k)!}{(2p)!(2p - k)!} \sum_{m=0}^p \frac{(-2p)_{2m} (-2p + k)_m}{m!(-4p + 2k)_{2m}} \\ &= \frac{(4p - 2k)!}{(2p)!(2p - k)!} \sum_{m=0}^p \frac{2^{2m} (-p)_m \left(\frac{-2p+1}{2}\right)_m (-2p + k)_m}{m! 2^{2m} (-2p + k)_m \left(\frac{-4p+2k+1}{2}\right)_m} \\ &= \frac{(4p - 2k)!}{(2p)!(2p - k)!} \sum_{m=0}^p \frac{(-p)_m \left(\frac{-2p+1}{2}\right)_m}{m! \left(\frac{-4p+2k+1}{2}\right)_m} \\ &= \frac{(4p - 2k)!}{(2p)!(2p - k)!} \sum_{m=0}^{\infty} \frac{(-p)_m \left(\frac{-2p+1}{2}\right)_m}{m! \left(\frac{-4p+2k+1}{2}\right)_m} \\ &= \frac{(4p - 2k)!}{(2p)!(2p - k)!} {}_2F_1\left(-p, \frac{-2p + 1}{2}; \frac{-4p + 2k + 1}{2}\right) \end{aligned}$$

$$\begin{aligned}
&= \frac{(4p-2k)!}{(2p)!(2p-k)!} \cdot \frac{(k-p)_p}{\left(\frac{-4p+2k+1}{2}\right)_p} \\
&= \frac{(4p-2k)!}{(2p)!(2p-k)!} \cdot \frac{(k-p) \cdots (k-1)(-2)^p}{(4p-2k-1)(4p-2k-3) \cdots (2p-2k+1)}.
\end{aligned}$$

Now, for  $k = 1, \dots, p$ , this last expression clearly vanishes. For  $k = 0$ , the last line reduces to

$$\begin{aligned}
\frac{(4p)!(p)!2^p}{(2p)!(2p)!(4p-1)(4p-3) \cdots (2p+1)} &= \frac{(4p)!(p)!2^p(2p-1)!!}{(2p)!(2p)!(4p-1)!!} \\
&= \frac{2^{3p}(p)!(2p-1)!!}{(2p)!} \\
&= 2^{2p}.
\end{aligned}$$

This completes the proof.

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