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THE TOPOLOGY OF SYMPLECTIC CIRCLE BUNDLES

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Abstract. We consider circle bundles over compact three-manifolds with symplectic total spaces. We show that the base of such a space must be irreducible or the product of the two-sphere with the circle. We then deduce that such a bundle admits a symplectic form if and only if it admits one that is invariant under the circle action in three special cases: namely, if the base is Seifert fibered, has vanishing Thurston norm, or if the total space admits a Lefschetz fibration.

1. Introduction

A conjecture due to Taubes states that if a closed, compact 4-manifold of the form $M \times S^1$ is symplectic, then M must fiber over S^1. A natural extension of this conjecture is to the case where $E \xrightarrow{\pi} M$ is a possibly nontrivial circle bundle. In [4] it was shown that an S^1-bundle admits an S^1-invariant symplectic form if and only if its base fibers over S^1 and the Euler class $e(E)$ of the total space pairs trivially with the fiber of some fibration. Thus based on the principle that an S^1-bundle should admit a symplectic form if and only if it admits an invariant one, one arrives at the following conjecture.

Conjecture 1 (Taubes)**.** *If a circle bundle $S^1 \to E \xrightarrow{\pi} M$ over a closed, oriented 3-manifold is symplectic, then there is a fibration $\Sigma \to M \xrightarrow{\phi} S^1$ such that $e(E)([\Sigma]) = 0$.*

If an oriented 3-manifold fibers over S^1 with fiber $\Sigma \neq S^2$, then it follows by the long exact homotopy sequence that M is in fact aspherical. So a necessary condition for Conjecture 1 to hold is that any M that is the base of an S^1-bundle, whose total space carries a symplectic form, must in fact be aspherical or $S^2 \times S^1$ in the case $\Sigma = S^2$. This observation provides the motivation for the following theorem, which is the main result of the first part of this paper.

Theorem 2. *Let M be an oriented, closed 3-manifold so that some circle bundle $S^1 \to E \xrightarrow{\pi} M$ admits a symplectic structure. Then, either M is diffeomorphic to $S^2 \times S^1$ and the bundle is trivial, or M is irreducible and aspherical.*

A similar statement was proved by McCarthy in [20] for the case $E = M \times S^1$. More precisely, McCarthy showed that if $M \times S^1$ admits a symplectic structure,

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then M decomposes as a connected sum $M = A\#B$, where the first Betti number $b_1(A) \geq 1$ and B has no nontrivial connected covering spaces. This can be refined quite substantially following Perelman's proof of Thurston's geometrisation conjecture (see [22], [23] or [21]), for one corollary of geometrisation is that the fundamental group of a closed 3-manifold is residually finite (see [11]), meaning that the B in McCarthy's theorem must have trivial fundamental group, and hence by the Poincaré Conjecture is diffeomorphic to S^3. Thus in fact M must be prime and hence irreducible and aspherical or $S^2 \times S^1$. Theorem 2 is then a generalisation of this more refined statement to the case of nontrivial S^1-bundles. Our argument will rely on a vanishing result of Kronheimer-Mrowka for the Seiberg-Witten invariants of a manifold that splits into two pieces along a copy of $S^2 \times S^1$, which in itself is of independent interest (cf. Proposition 1). One may also prove Theorem 2 by following the argument of [20]; see Remark 1 below.

In the remainder of this paper we will show that Conjecture 1 holds in various special cases. First we will verify the conjecture under certain additional assumptions on the topology of the base manifold M. In order to be able to do this we will need to understand when a manifold fibers over S^1. One gains significant insight into this problem by considering the Thurston norm $|| \, ||_T$ on $H^1(M, \mathbb{R})$, which was introduced by Thurston in [27]. The Thurston norm enables one to see which integral classes $\alpha \in H^1(M, \mathbb{Z})$ can be represented by closed, nonvanishing 1-forms, which in turn induce fibrations of M by compact surfaces.

In [5] Friedl and Vidussi showed that if $E = M \times S^1$ admits a symplectic form and $|| \, ||_T \equiv 0$ or M is Seifert fibered, then M must fiber over S^1. In Corollary 2 below we will show that in fact Conjecture 1 holds in these two cases. The argument will be based on understanding the Seiberg-Witten invariants of the total space E given that M has vanishing Thurston norm and the Seifert case will be deduced as a corollary of this. Indeed, if M has vanishing Thurston norm and $S^1 \to E \xrightarrow{\pi} M$ is symplectic, then the canonical class of E must be trivial. This combined with the restrictions on Seiberg-Witten basic classes of a symplectic manifold as proved by Taubes in [26] means that $K = 0$ is the only Seiberg-Witten basic class and the result then follows by an application of a vanishing result of Lescop (cf. [17] or [28]).

Another special case of the Taubes conjecture is when the total space E admits a Lefschetz fibration, as was considered in [2] and [3] for a trivial bundle. In view of Corollary 2 we will be able to give a comparatively simple proof of the following result.

Theorem 9. *Let $S^1 \to E \xrightarrow{\pi} M$ be a symplectic circle bundle over an irreducible base M. If E admits a Lefschetz fibration, then M fibers over S^1.*

It then follows by considering the Kodaira classification of complex surfaces that Conjecture 1 holds under the assumption that the total space admits a complex structure.

Outline of paper. In Section 2 we will state the relevant vanishing result of Kronheimer-Mrowka in order to prove Theorem 2. In Section 3 we recall the definition of the Thurston norm and quote some well-known facts about it. In Section 4 we will use our knowledge of the Thurston norm to verify Conjecture 1 under the assumption that the base is Seifert fibered or has vanishing Thurston norm. Finally in Section 5 we will define Lefschetz fibrations and prove that M is fibered when one has a Lefschetz fibration on the total space E.

2. Asphericity of the base M

Throughout this article all manifolds will be closed, connected and oriented and M will always denote a manifold of dimension 3. In addition we will make the convention that all (co)homology groups will be taken with integral coefficients unless otherwise stated.

In [20] it was shown that if $M \times S^1$ is symplectic, then M must be irreducible and aspherical or $S^2 \times S^1$. We extend this to the case of a nontrivial S^1-bundle. We first collect some relevant lemmas.

Lemma 1. *Let $M = M_1 \# M_2$ be a nontrivial connect sum decomposition with $b_1(M) \geq 1$. Then there is a finite covering N of M that decomposes as a direct sum $N = N_1 \# N_2$ where $b_1(N_i) \geq k$ for any given k.*

Proof. It follows from Mayer-Vietoris that the Betti numbers are additive for a connect sum; hence by assumption we may assume that $b_1(M_1) \geq 1$. By the proof of geometrisation it follows that the fundamental group of a 3-manifold is residually finite (cf. [11]) and hence M_2 has a nontrivial d-fold cover $\tilde{M}_2$, with $d \geq 2$. By removing a ball from M_2 and its disjoint lifts from $\tilde{M}_2$ and then gluing in d copies of M_1 we obtain a cover $\tilde{M}$ of $M = M_1 \# M_2$, and by construction $\tilde{M}$ has a connect sum decomposition as $\tilde{M} = M_1 \# P$, where $b_1(P) \geq 1$. We may now take a k-fold cover associated to some surjective homomorphism of $\pi_1(M_1) \to \mathbb{Z}_k$ and glue in copies of P to get a cover of $\tilde{M}$ (and hence of M), which decomposes in two pieces one of which has first Betti number at least k. One more application of this procedure gives the desired result. □

Lemma 2. *Let $S^1 \to E \xrightarrow{\pi} M$ be a circle bundle, whose Euler class we denote by $e(E) \in H^2(M)$. Then*

(1) $b_2(E) = \begin{cases} 2b_1(M) - 2 & \textit{if } e(E) \textit{ is not torsion,} \\ 2b_1(M) & \textit{if } e(E) \textit{ is torsion.} \end{cases}$

(2) $b_2^+(E) = b_2^-(E) \geq b_1(M) - 1.$

Proof. We consider the Gysin sequence

$$H^0(M) \xrightarrow{\cup e} H^2(M) \xrightarrow{\pi^*} H^2(E) \xrightarrow{\pi_*} H^1(M) \xrightarrow{\cup e} H^3(M),$$

where $e \in H^2(M)$ denotes the Euler class of the bundle. By Poincaré duality $H^0(M) = H^3(M) = \mathbb{Z}$ and $b_1(M) = b_2(M)$, so we conclude by exactness that $b_2(E) = 2b_1(M) - 2$ if e is not torsion and $b_2(E) = 2b_1(M)$ if e is torsion. Furthermore since E bounds its associated disc bundle, it has zero signature and hence

$$b_2^+(E) = b_2^-(E) \geq b_1(M) - 1.$$ □

We will need to appeal to a vanishing result for the Seiberg-Witten invariants of manifolds that decompose along $S^2 \times S^1$, which we take from [16]. For this we will need to define a relative notion of b_2^+ for an oriented 4-manifold X with boundary. This is done by considering the symmetric form induced on rational cohomology that is obtained as the composition

$$H^2(X, \partial X) \times H^2(X, \partial X) \overset{i^* \times Id}{\to} H^2(X) \times H^2(X, \partial X) \xrightarrow{\cup} \mathbb{Q} .$$

Here the map i^* is the map coming from the long exact sequence of the pair $(X, \partial X)$ and the second map is nondegenerate by Poincaré duality. This is then a symmetric, possibly degenerate, form on $H^2(X, \partial X)$ and we define $b_2^+(X)$ to be the dimension of a maximal positive definite subspace.

Theorem 1 (Kronheimer-Mrowka, [16])**.** *Let $X = X_1 \cup_{\partial X_1 = \partial X_2} X_2$ where $\partial X_1 = -\partial X_2 = S^2 \times S^1$ and $b_2^+(X_1), b_2^+(X_2) \geq 1$. Then for all $Spin^c$-structures ξ,*

$$\sum_{\xi^* - \xi \in \ Tor} \mathrm{SW}(\xi^*) = 0.$$

Although it is not explicitly stated in the book [16], Theorem 1 can be deduced as follows: formula 3.27 (p. 75) allows one to compute the sum of the SW invariants of all $Spin^c$-structures that differ by torsion as a pairing of certain Floer groups. However these groups are zero for $S^2 \times S^1$ by Proposition 3.10.3 in the case of an untwisted coefficient system and by Proposition 3.10.4 in the twisted case, and thus this sum must vanish.

Theorem 1 then implies certain restrictions on the decomposition of symplectic manifolds along a copy of $S^2 \times S^1$.

Proposition 1. *A symplectic manifold X cannot be decomposed as $X = X_1 \cup_{\partial X_1 = \partial X_2} X_2$, where $\partial X_1 = -\partial X_2 = S^2 \times S^1$ and $b_2^+(X_1), b_2^+(X_2) \geq 1$.*

Proof. By the hypotheses of the proposition, we conclude from Theorem 1 that for every $Spin^c$-structure $\xi \in Spin^c(X)$,

$$\sum_{\xi^* - \xi \in \ Tor} \mathrm{SW}(\xi^*) = 0.$$

However as X is symplectic and

$$b_2^+(X) \geq b_2^+(X_1) + b_2^+(X_2) \geq 2$$

the nonvanishing result of Taubes implies $\mathrm{SW}(\xi_{can}) = \pm 1$, where ξ_{can} denotes the canonical $Spin^c$-structure associated to the symplectic structure on E (cf. [25]). Moreover it follows from the constraints on SW basic classes of a symplectic manifold of [26] that if ξ^* is another $Spin^c$-structure with nontrivial SW invariant and $\xi_{can} - \xi^* \in Tor$, then in fact $\xi_{can} = \xi^*$. Hence

$$\sum_{\xi^* - \xi_{can} \in \ Tor} \mathrm{SW}(\xi^*) = \pm 1,$$

which is a contradiction. □

Theorem 2. *Let M be an oriented, closed 3-manifold, so that some circle bundle $S^1 \to E \xrightarrow{\pi} M$ admits a symplectic structure. Then M is irreducible and aspherical or $M = S^2 \times S^1$ and the bundle is trivial.*

Proof. We first show that M must be prime. Since E is symplectic it follows from Lemma 2 that $b_1(M) \geq 1$. Assume that $M = M_1 \# M_2$ is a nontrivial connected sum. Then by taking a suitable covering as in Lemma 1 and pulling back E and its symplectic form we may assume without loss of generality that $b_1(M_i) \geq 2$. We let S denote the gluing sphere of the connected sum. Then as S is nullhomologous the bundle restricted to S is trivial. Thus the connect sum decomposition induces a decomposition $E = E_1 \cup_{S^2 \times S^1} E_2$. Since the bundles $E_i \to M_i \setminus B^3$ are trivial on the boundary we may extend them to bundles $\tilde{E}_i \to M_i$ and as $b_1(M_i) \geq 2$, Lemma 2 implies that $b_2^+(\tilde{E}_i) \geq 1$. Further, since $E_i \simeq \tilde{E}_i \setminus (S^1 \times pt)$ we have that

$$b_2^+(E_i) \geq b_2^+(\tilde{E}_i) \geq 1,$$

which then contradicts Proposition 1. Hence M is prime, and thus irreducible or $S^2 \times S^1$.

We assume that M is irreducible. Then by the sphere theorem, $\pi_2(M) = 0$. Since $b_1(M) \geq 1$, we have that $\pi_1(M)$ is infinite, so the universal cover $\tilde{M}$ of M is not compact and has $\pi_i(\tilde{M})$ trivial for $i = 1, 2$. The Hurewicz theorem then implies that the first nontrivial $\pi_i(\tilde{M})$ is isomorphic to $H_i(\tilde{M})$. But since $\tilde{M}$ is not compact, $H_3(\tilde{M}) = 0$ and as $\tilde{M}$ is 3-dimensional, $H_i(\tilde{M}) = 0$ for all $i \geq 4$. Hence $\pi_i(\tilde{M}) = 0$ for all $i \geq 1$ and it follows from Whitehead's Theorem that $\tilde{M}$ is contractible, that is, M is aspherical.

In the case where $M = S^2 \times S^1$, any symplectic bundle must be trivial by Lemma 2. □

Remark 1. One may also give a proof of Theorem 2 that uses the covering construction of [20]. In order to do this one first takes finite coverings on each of the two pieces in the connect sum decomposition. Then one glues these together to find a covering $\tilde{M}$ where the sphere of the connect sum lifts to a sphere that is nontrivial in real cohomology. This sphere then lifts to the total space of the pullback bundle $\tilde{E}$ over $\tilde{M}$. One may also assume by Lemma 1 that $b_1(\tilde{M})$ is large and hence $b_2^+(\tilde{E})$ is large. Then a standard vanishing theorem for the SW invariants (cf. [13]) implies that all invariants are zero, which then contradicts Taubes' result if E and hence $\tilde{E}$ is symplectic.

By considering the long exact homotopy sequence we have the following corollary that was first proved by Kotschick in [14].

Corollary 1. *Let $S^1 \to E \xrightarrow{\pi} M$ be a symplectic circle bundle over an oriented 3-manifold M. Then the map $\pi_1(S^1) \to \pi_1(E)$ induced by the inclusion of the fiber is injective. In particular a fixed point free circle action on a symplectic 4-manifold can never have contractible orbits.*

3. The Thurston norm

In this section we will define and collect several relevant facts about the Thurston norm. We first define the negative Euler characteristic or *complexity* of a possibly disconnected, orientable surface $\Sigma = \bigsqcup_i \Sigma_i$ to be

$$\chi_-(\Sigma) = \sum_{\chi(\Sigma_i) \leq 0} -\chi(\Sigma_i),$$

where χ denotes the Euler characteristic of the surface.

Next we define the Thurston norm $||\ ||_T$ as a map on $H^1(M)$ by

$$||\sigma||_T = \min\{\chi_-(\Sigma) \mid PD(\Sigma) = \sigma\}.$$

It is a basic fact that this map extends uniquely to a (semi)norm on $H^1(M, \mathbb{R})$, which we will denote again by $||\ ||_T$. One particularly important property of the Thurston norm is that its unit ball, which we denote by B_T, is a (possibly noncompact) convex polytope with finitely many faces. If B_{T^*} denotes the unit ball in the dual space we have the following characterisation of B_T.

Theorem 3 ([27], p. 106)**.** *The unit ball B_{T^*} is a polyhedron whose vertices are integral lattice points, $\pm\beta_1, ..., \pm\beta_k$ and the unit ball B_T is defined by the following inequalities:*

$$B_T = \{\alpha \mid |\beta_i(\alpha)| \leq 1,\ 1 \leq i \leq k\}.$$

We are interested in understanding how a manifold fibers over S^1, and the following theorem says that the Thurston norm determines precisely which cohomology classes can be represented by fibrations.

Theorem 4 ([27], p. 120)**.** *Let M be a compact, oriented 3-manifold. The set F of cohomology classes in $H^1(M,\mathbb{R})$ representable by nonsingular closed 1-forms is the union of the open cones on certain top-dimensional open faces of B_T, minus the origin. The set of elements in $H^1(M,\mathbb{Z})$ whose Poincaré dual is represented by the fiber of some fibration consists of the set of lattice points in F.*

We call a top-dimensional face of the unit ball B_T *fibered* if some integral class, and hence all, in the cone over its interior can be represented by a fibration. One also understands how the Thurston norm behaves under finite covers by the following result of Gabai.

Theorem 5 ([7], Cor. 6.13)**.** *Let $\tilde{M} \xrightarrow{p} M$ be a finite connected d-sheeted covering. Then for $\sigma \in H^1(M,\mathbb{R})$ we have*

$$||\sigma||_T = \frac{1}{d}||p^*\sigma||_T.$$

These facts then allow us to completely characterise the Thurston norm of an irreducible Seifert fibered manifold.

Proposition 2. *If M is irreducible and Seifert fibered, then either the Thurston norm of M vanishes identically or M fibers over S^1 and*

$$||\sigma||_T = m.|\sigma(\gamma)|$$

for some class $\gamma \in H_1(M)$.

Proof. Since M is irreducible and Seifert fibered, either M has a horizontal surface, i.e. a closed surface transverse to all fibers, or every embedded surface is isotopic to a vertical surface, i.e. a surface that is a union of fibers (cf. [9], Prop 1.11) and is hence a union of tori, so the Thurston norm is identically zero. If M has a horizontal surface F, which we may assume to be connected, then M is a mapping torus with monodromy $\phi \in \mathrm{Diff}^+(F)$ so that $\phi^n = Id$ for some n. This means that M is covered by $\tilde{M} = F \times S^1$. If $\tilde{\gamma} = pt \times S^1$, then the Thurston norm of $\tilde{M} \xrightarrow{p} M$ is given by

$$||\sigma||_T = \chi_-(F)|\sigma(\tilde{\gamma})|.$$

We let $\gamma = p_*(\tilde{\gamma})$. Then by Theorem 5 the norm on M is given by

$$||\sigma||_T = \frac{1}{n}||p^*\sigma||_T = \frac{\chi_-(F)}{n}|p^*\sigma(\tilde{\gamma})| = \frac{\chi_-(F)}{n}|\sigma(\gamma)| = m.|\sigma(\gamma)|.$$

□

Example 1 (Seifert fibered spaces with horizontal surfaces)**.** We note that in the second case of Proposition 2 the Thurston ball B_T consists of two (noncompact) faces that are both fibered and that the Thurston norm is identically zero on a codimension one subspace K. Thus by [4] any bundle over such an M will admit an S^1-invariant symplectic form except possibly in the case where the Euler class $e(E)$ pairs trivially with all elements in K; that is, $e(E)$ pairs trivially with all tori in M. By taking the pullback bundle of the cover $\tilde{M} = F \times S^1 \to M$ we may assume that we have a bundle E over $F \times S^1$ that is symplectic and has Euler class that again pairs trivially with embedded tori in $\tilde{M}$ and is thus a nonzero multiple of

$PD(\tilde{\gamma})$. This in turn has a covering $\bar{E}$ that is an S^1-bundle with Euler class equal to $PD(\tilde{\gamma})$. Now if we let $T = \tilde{\gamma} \times S^1$ and $X = \tilde{M} \times S^1$, then the SW polynomial of X can be computed to be

$$\mathcal{SW}_X^4 = (t_T - t_T^{-1})^{2g-2},$$

where g is the genus of F. Then by the formula of Baldridge in [1], it follows that all the SW invariants of $\bar{E}$ are zero, contradicting Taubes' nonvanishing result for the SW invariants of a symplectic manifold. So in fact Conjecture 1 holds for Seifert fibered spaces that have horizontal surfaces.

4. The case of vanishing Thurston norm

In [5] Friedl and Vidussi showed that if $E = M \times S^1$ admits a symplectic form and $|| \, ||_T \equiv 0$ or M is Seifert fibered, then M must fiber over S^1. In this section we shall extend this to the case of a nontrivial S^1-bundle and then show that Conjecture 1 holds in both of these cases. From now on we shall assume that M is irreducible, which in view of Theorem 2 only excludes the case where $M = S^2 \times S^1$ and the bundle is trivial. Our argument will be based on that of [5], and we begin with the following lemma.

Lemma 3. *If $S^1 \to E \xrightarrow{\pi} M$ is a bundle over an M that has vanishing Thurston norm, then*

$$H^2(E)/Tor = V \oplus W,$$

where V, W are isotropic subspaces that admit a basis of embedded tori.

Proof. We consider the Gysin sequence

$$\mathbb{Z} \longrightarrow H^2(M) \xrightarrow{\pi^*} H^2(E) \underset{s}{\overset{\pi_*}{\rightleftarrows}} H^1(M) \longrightarrow \mathbb{Z}.$$

Here s is a section defined on the image of π_* as follows: we represent an element $\sigma \in H^1(M)$ by an embedded surface Σ. By exactness, σ will be in $Im(\pi_*)$ precisely when the bundle is trivial on Σ and in this case we may lift Σ to some $\tilde{\Sigma}$ in E. As $H^1(M)$ is free, we define s on a $\mathbb{Z}$-basis $\{\sigma_i\}$ by $s(\sigma_i) = \tilde{\Sigma}_i$. We set $V = \pi^*(H^2(M))$ and $W = s(H^1(M))$. Then V is clearly spanned by embedded tori and the statement for W is precisely the assumption on the Thurston norm. □

Proposition 3. *Let $S^1 \to E \xrightarrow{\pi} M$ be an S^1-bundle with torsion Euler class $e(E)$. Then there is a finite cover $\tilde{M} \xrightarrow{p} M$ such that the pullback bundle $p^*E \to \tilde{M}$ is trivial.*

Proof. We choose a splitting of $H_1(M) = F \oplus T$, where T is the torsion subgroup and F is any free complement. We take the cover $\tilde{M} \xrightarrow{p} M$ associated to the kernel of the composition

$$\pi_1(M) \to H_1(M) \xrightarrow{\phi} T,$$

where ϕ is the projection with kernel F. Note that the composition $H_1(\tilde{M}) \xrightarrow{p_*} H_1(M) \xrightarrow{\phi_*} T$ is zero. Then by the Universal Coefficient Theorem we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Ext(H_1(\tilde{M}),\mathbb{Z}) & \longrightarrow & H^2(\tilde{M}) & \longrightarrow & Hom(H_2(\tilde{M}),\mathbb{Z}) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle (p_*)^*} & & \uparrow{\scriptstyle p^*} & & \uparrow{\scriptstyle (p_*)^*} & & \\
0 & \longrightarrow & Ext(H_1(M),\mathbb{Z}) & \longrightarrow & H^2(M) & \longrightarrow & Hom(H_2(M),\mathbb{Z}) & \longrightarrow & 0\,. \\
 & & \uparrow{\scriptstyle (\phi_*)^*}\,\cong & & & & & & \\
 & & Ext(T,\mathbb{Z}) & & & & & &
\end{array}$$

This implies that p^* is zero on torsion in $H^2(M)$, so the pullback bundle is indeed trivial. □

Theorem 6. *If $S^1 \to E \xrightarrow{\pi} M$ is a symplectic circle bundle over an irreducible manifold for which $||\ ||_T$ is identically zero, then M fibers over S^1.*

After this paper had been submitted the author was informed that Friedl and Vidussi have independently proved this result (cf. [6]).

Proof. Since E is symplectic it has an associated canonical $Spin^c$-structure ξ_{can} and canonical class that we denote by K. We claim that our assumption on the Thurston norm of the base implies that K must be torsion. For by Taubes' nonvanishing result ξ_{can} has nontrivial SW invariant. If $\alpha \in H^2(E)$, the adjunction inequality (see [15]) and Lemma 3 imply that

$$|\alpha.K| = 0.$$

This also holds in the case $b_2^+(E) = 1$ (cf. [18], Theorem E and [19], Theorem B). As M is irreducible and $b_2(M) \geq 1$ the assumption on the vanishing of the Thurston norm implies that M contains an embedded, incompressible torus $T \hookrightarrow M$. Then by Proposition 7 of [12] either T is the fiber of some fibration or there is a finite cover $\bar{M} \xrightarrow{p} M$ with large b_1, say $b_1(\bar{M}) \geq 4$. We assume that the latter holds. Then the pullback $\bar{E} = p^*E$ will be symplectic with canonical class $\bar{K} = p^*K$, with symplectic form $\bar{\omega} = p^*\omega$ and $b_2^+(\bar{E}) \geq 2$. Then for any $Spin^c$-structure $\xi_{can} \otimes F$ that has nontrivial SW invariant we have by [26],

$$0 \leq F.[\bar{\omega}] \leq \bar{K}.[\bar{\omega}].$$

Moreover, since $\bar{K}$ is torsion and equality on the left implies $F = 0$, we conclude that in fact $\bar{K} = 0$. Thus $\bar{K} = 0$, so $\bar{\xi}_{can}$ is trivial and this is the only $Spin^c$-structure with nonzero SW invariant. We now need to consider two cases. We first assume that $e(E)$ and hence $e(\bar{E})$ is nontorsion. In this case we compute

$$\pm 1 = \sum_{\xi^* \in Spin^c(\bar{E})} SW^4_{\bar{E}}(\xi^*) = \sum_{\xi^* \in Spin^c(\bar{E})} \sum_{\xi^* \equiv\ \xi \ mod\ \bar{e}} SW^3_{\bar{M}}(\xi) = \sum_{\xi \in Spin^c(\bar{E})} SW^3_{\bar{M}}(\xi),$$

where the second inequality follows from Theorem 1 in [1]. However as $b_1(\bar{M}) \geq 4$ this sum is zero (cf. [28], p. 114), a contradiction. If the Euler class is torsion we may assume by Proposition 3 that it is indeed zero and the above calculation reduces to

$$\pm 1 = \sum_{\xi \in Spin^c(\bar{E})} SW^4_{\bar{E}}(\xi) = \sum_{\xi \in Spin^c(\bar{E})} SW^3_{\bar{M}}(\xi) = 0.$$

In either case we obtain a contradiction and hence M must fiber over S^1. □

As a consequence of this theorem we conclude that Conjecture 1 holds if M has vanishing Thurston norm or is Seifert fibered.

Corollary 2. *Conjecture* 1 *holds if* M *is Seifert fibered or* $|| \, ||_T \equiv 0$.

Proof. If M has vanishing Thurston norm, then by Theorem 4 we conclude that if one class in $H^1(M)$ can be represented by a fibration, then so can all classes and by the construction of [4] every bundle over M admits an S^1-invariant symplectic form. If M is Seifert fibered it either has vanishing Thurston norm by Proposition 2 and we proceed as in the previous case or M has a horizontal surface and the claim follows by Example 1 above. □

5. The case where E admits a Lefschetz fibration

In [2] Chen and Matveyev showed that if $S^1 \times M$ admits a symplectic Lefschetz fibration, then M fibers over S^1. This was extended by Etgü in [3] to the case where the fibration may or may not be symplectic. In this section we shall show that the same statement holds for arbitrary S^1-bundles. Let us begin with some definitions and basic facts concerning Lefschetz fibrations.

Definition 7. Let E be a compact, connected, oriented smooth 4-manifold. A Lefschetz fibration is a map $E \xrightarrow{p} B$ to an orientable surface so that any critical point has an oriented chart on which $p(z_1, z_2) = z_1^2 + z_2^2$.

We list some basic properties of Lefschetz fibrations (for proofs see [8]).

(1) There are finitely many critical points, so the generic preimage of a point will be a surface and we may assume that this is connected. To each critical point one associates a vanishing cycle in the fiber.
(2) A Lefschetz fibration admits a symplectic form so that the fiber is a symplectic submanifold if the class $[F]$ of the fiber is nontorsion in $H_2(E)$. Moreover this is always true if $\chi(F) \neq 0$.
(3) We have a formula for the Euler characteristic given by
$$\chi(E) = \chi(B).\chi(F) + \#\{\textit{critical points}\}.$$

We will first show that for a symplectic circle bundle any Lefschetz fibration will actually be a proper fibration, i.e. cannot have any critical points. The following lemma is essentially Lemma 3.4 of [2].

Lemma 4. *Let* $S^1 \to E \xrightarrow{\pi} M$ *be a circle bundle that admits a Lefschetz fibration* $E \xrightarrow{p} B$. *Then* p *has no critical points.*

Proof. We first consider the case where $F = S^2$. Since E is spin, it has an even intersection form and thus all vanishing cycles are nonseparating in the fiber $F = S^2$. However this means there cannot be any since S^2 is simply connected and hence p has no critical points.

If $F = T^2$, the equation
$$0 = \chi(E) = \chi(B).\chi(F) + \#\{\textit{critical points}\}$$
implies that E has no critical points.

We now consider the case when F has genus *greater* than 1. We know that E admits a symplectic Lefschetz fibration by (2) above. Thus by the adjunction formula for symplectic surfaces we see that

$$K.F = \chi_-(F) \neq 0, \qquad (1)$$

where K is the canonical class on E. If $b_2^+ > 1$, then it follows from Taubes' result that K is a basic class and thus the adjunction inequality holds. In the case where $b_2^+(E) = 1$ we may apply the adjunction inequality exactly as in the case of $b_2^+ > 1$ by ([18], Theorem E). Now we assume that our fibration has a critical point and hence a vanishing cycle γ. Then we know that this is nonseparating, so the fiber F is homologous to a surface obtained by collapsing γ to a point and this can in turn be thought of as the image of a map $F' \xrightarrow{f} E$ where $\chi_-(F') < \chi_-(F)$. Hence the image $\pi_*[F]$ may be represented by a surface of complexity at most $\chi_-(F')$ (see [7]). We know that any basic class of a circle bundle is a pullback of a class on the base (see [1]). Thus by the adjunction inequality (which still holds for $b_2^+ = 1$) and equation (1),

$$\chi_-(F) = |K.F| = |K.\pi_*F| \leq ||\pi_*F||_T \leq \chi_-(F') < \chi_-(F),$$

which is a contradiction. □

Our proof of Theorem 9 below, which differs from those of [2] and [3], will rely on a theorem of Stallings that characterises fibered 3-manifolds in terms of their fundamental group.

Theorem 8 (Stallings [24])**.** *Let M be a compact, irreducible 3-manifold and suppose there is an extension*

$$1 \to G \to \pi_1(M) \to \mathbb{Z} \to 1,$$

where G is finitely generated and $G \neq \mathbb{Z}_2$. Then M fibers over S^1.

We now come to the main result of this section.

Theorem 9. *Let $S^1 \to E \xrightarrow{\pi} M$ be a symplectic circle bundle over an irreducible base M. If E admits a Lefschetz fibration, then M fibers over S^1.*

Proof. First of all by Lemma 4 we have that E actually admits a fibration $F \to E \xrightarrow{p} B$. In addition we note that the fiber γ of any oriented circle bundle lies in the centre of the fundamental group of the total space. We shall have to consider two distinct cases according to whether γ is in the kernel of p_* or not.

Case 1. $p_*(\gamma) \neq 1$.

Since γ was central in the fundamental group of E, the fact that $p_*(\gamma)$ is nontrivial in $\pi_1(B)$ means that B must be a torus. Hence the long exact homotopy sequence of the fibration gives the following short exact sequence:

$$1 \to \pi_1(F) \to \pi_1(E) \xrightarrow{p_*} \pi_1(T^2) = \mathbb{Z}^2 \to 1. \tag{2}$$

Since M is assumed to be irreducible and hence aspherical, we also have the following exact sequence from the homotopy exact sequence of the fibration $S^1 \to E \xrightarrow{\pi} M$:

$$1 \to \pi_1(S^1) = \langle\gamma\rangle \to \pi_1(E) \xrightarrow{\pi_*} \pi_1(M) \to 1. \tag{3}$$

Because γ is central in $\pi_1(E)$, the sequence (2) gives the following exact sequence:

$$1 \to \pi_1(F) \to \pi_1(E)/\langle\gamma\rangle \xrightarrow{p_*} \mathbb{Z}^2/\langle p_*(\gamma)\rangle \to 1.$$

Moreover since $p_*(\gamma) \neq 1$ we have that $\mathbb{Z}^2/\langle p_*(\gamma)\rangle = \mathbb{Z} \oplus \mathbb{Z}_k$ for some k. If we let $H = p_*^{-1}(\mathbb{Z}_k)$ we see that H has $\pi_1(F)$ as a finite index subgroup and is thus also finitely generated. Then by taking the projection to $\mathbb{Z}$ in the above sequence we obtain

$$1 \to H \to \pi_1(E)/\langle\gamma\rangle = \pi_1(M) \xrightarrow{p_*} \mathbb{Z} \to 1.$$

This is exact and $H \neq \mathbb{Z}_2$ since it contains $\pi_1(F)$. As M is irreducible, the hypotheses of Theorem 8 are satisfied and we conclude that M fibers over S^1.

Case 2. $p_*(\gamma) = 1$.

In this case, $\langle\gamma\rangle \subset \pi_1(F)$ and hence $F = T^2$. Thus sequence (2) above yields the following:

$$1 \to \mathbb{Z}^2 \to \pi_1(E) \xrightarrow{p_*} \pi_1(B) \to 1$$

and $\langle\gamma\rangle \subset \mathbb{Z}^2$. Again by taking the quotient by $\langle\gamma\rangle$ we obtain the following short exact sequence:

$$1 \to \mathbb{Z} \oplus \mathbb{Z}_k = \mathbb{Z}^2/\langle\gamma\rangle \to \pi_1(E)/\langle\gamma\rangle = \pi_1(M) \xrightarrow{p_*} \pi_1(B) \to 1.$$

However since M is irreducible and hence prime and $\pi_1(M)$ is infinite, it follows from ([10], Corollary 9.9) that $\pi_1(M)$ is torsion free. Hence $k = 0$ and $\pi_1(M)$ contains an infinite cyclic normal subgroup; thus by ([10], Corollary 12.8) it is in fact Seifert fibered and the result follows from Corollary 2 above. □

Theorem 9 then allows us to prove Conjecture 1 under the assumption that the total space is a complex manifold.

Corollary 3. *Conjecture* 1 *holds in the case that E is a complex manifold.*

Proof. By considering the Kodaira classification and noting that E is spin, symplectic and has $\chi(E) = 0$ one concludes that one of the following must hold (cf. [3] Theorem 5.1):

(1) $E = S^2 \times T^2$,
(2) E is a T^2-bundle over T^2,
(3) E is a Seifert fibration over a hyperbolic orbifold.

If $E = S^2 \times T^2$, then $M = S^2 \times S^1$ and one clearly has an S^1-invariant symplectic form. In the second case it follows from the argument above that M is a T^2-bundle over S^1 and hence has vanishing Thurston norm. In the final case M must be Seifert fibered as in Case 2 in the proof of Theorem 9 and hence the claim holds in the latter two cases by Corollary 2. □

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