

## INTEGRABLE MODULES FOR AFFINE LIE SUPERALGEBRAS

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ABSTRACT. Irreducible nonzero level modules with finite-dimensional weight spaces are discussed for nontwisted affine Lie superalgebras. A complete classification of such modules is obtained for superalgebras of type  $A(m, n)$  and  $C(n)$  using Mathieu’s classification of cuspidal modules over simple Lie algebras. In other cases the classification problem is reduced to the classification of cuspidal modules over finite-dimensional cuspidal Lie superalgebras described by Dimitrov, Mathieu and Penkov. Based on these results a complete classification of irreducible integrable (in the sense of Kac and Wakimoto) modules is obtained by showing that any such module is of highest weight, in which case the problem was solved by Kac and Wakimoto.

### 1. INTRODUCTION

Let  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$  be a Lie superalgebra over  $\mathbb{C}$ , i.e.

- $[\mathfrak{g}_\varepsilon, \mathfrak{g}_{\varepsilon'}] \subset \mathfrak{g}_{\varepsilon+\varepsilon'(\bmod 2)}$ ;
- $[x, y] = -(-1)^{|x||y|}[y, x]$ ;
- $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$ ,

where  $|x| = \varepsilon$  if  $x \in \mathfrak{g}_\varepsilon$ . We will assume that  $\mathfrak{g}$  is a finite-dimensional basic classical Lie superalgebra, i.e. a simple Lie superalgebra with a nondegenerate even supersymmetric invariant bilinear form and with reductive  $\mathfrak{g}_0$ . The classification of such algebras was obtained in [12].

Let  $V = V_0 \oplus V_1$  be a  $\mathbb{Z}_2$ -graded vector space,  $\dim V_0 = m$ ,  $\dim V_1 = n$ . Then the endomorphism algebra  $EndV$  is an associative superalgebra with a natural  $\mathbb{Z}_2$ -gradation. Defining the Lie bracket as  $[A, B] = AB - (-1)^{|A||B|}BA$  we make  $EndV$  into a Lie superalgebra  $\mathfrak{gl}(m, n)$ . The *supertrace* is a linear function  $str : \mathfrak{gl}(m, n) \rightarrow \mathbb{C}$  such that  $str(id_V) = m - n$  and  $str[A, B] = 0$ ,  $A, B \in \mathfrak{gl}(m, n)$ . Denote

$$\mathfrak{sl}(m+1, n+1) = \{A \in \mathfrak{gl}(m+1, n+1) \mid str A = 0\}.$$

If  $m \neq n$ , this is the superalgebra of type  $A(m, n)$ . In the case  $m = n$ ,  $\mathfrak{sl}(n+1, n+1)$  has a one-dimensional ideal consisting of scalar matrices. Its quotient  $\mathfrak{psl}(n+1, n+1)$  is a basic classical Lie superalgebra of type  $A(n, n)$ . Let  $\dim V_1 = 2n$ , and let  $F$  be a non-degenerate consistent supersymmetric bilinear form on  $V$ , so that  $V_0$  and

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$V_1$  are orthogonal and the restriction of  $F$  to  $V_0$  is symmetric and to  $V_1$  is a skew-symmetric form. Then ([12])  $\mathfrak{osp}(m, 2n) = \mathfrak{osp}(m, 2n)_0 \oplus \mathfrak{osp}(m, 2n)_1$ , where

$$\mathfrak{osp}(m, 2n)_a = \{A \in \mathfrak{gl}(m, 2n)_a \mid F(Ax, y) = -(-1)^{a \deg x} F(x, Ay)\}, a = 0, 1.$$

Let

$$\begin{aligned} B(m, n) &= \mathfrak{osp}(2m + 1, 2n), \\ C(n) &= \mathfrak{osp}(2, 2n), \\ D(m, n) &= \mathfrak{osp}(2m, 2n), m > 1. \end{aligned}$$

The series  $A(m, n)$ ,  $B(m, n)$ ,  $D(m, n)$ ,  $C(n)$  and special Lie superalgebras  $D(2, 1; a)$  ( $a \neq 0, -1$ ),  $G(3)$ ,  $F(4)$  (cf. Section 5) form a complete list of basic classical Lie superalgebras.

For the purposes of this paper we will consider the Lie superalgebra  $\mathfrak{pgl}(n + 1, n + 1)$  instead of  $\mathfrak{psl}(n + 1, n + 1)$  as a superalgebra of type  $A(n, n)$ .

Let  $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1$  be the corresponding nontwisted affine Lie superalgebra, i.e. a 1-dimensional central extension of the loop superalgebras  $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$  with a degree derivation  $d$ ,  $d(x \otimes t^n) = n(x \otimes t^n)$ . Denote by  $\mathcal{H}$  a Cartan subalgebra of  $\mathfrak{G}_0$  and let  $c \in \mathcal{H}$  be the central element of  $\mathfrak{G}$ . Also let  $\Delta$  be the root system of  $\mathfrak{G}$  with respect to  $\mathcal{H}$  and  $Q = \sum_{\alpha \in \Delta} \mathbb{Z}\alpha$ .

If  $V$  is a  $\mathfrak{G}$ -module and  $\lambda \in \mathcal{H}^*$  we set  $V_\lambda = \{v \in V \mid hv = \lambda(h)v, \text{ for all } h \in \mathcal{H}\}$ . If  $V_\lambda$  is nonzero, then  $\lambda$  is a *weight* of  $V$  and  $V_\lambda$  is the corresponding weight space. We denote by  $P(V)$  the set of all weights of  $V$ . A  $\mathfrak{G}$ -module is a *weight* (with respect to  $\mathcal{H}$ ) if  $V = \bigoplus V_\lambda$ ,  $\lambda \in P(V)$ . This definition naturally extends to any subsuperalgebra of  $\mathfrak{G}$  containing  $\mathcal{H}$ , in particular its parabolic subsuperalgebras.

Similarly one defines weight modules for finite-dimensional reductive Lie algebras and Lie superalgebras. A weight module for a reductive Lie algebra  $\mathcal{B}$  is *cuspidal* if all its root elements act injectively. In this case all simple components of  $\mathcal{B}$  are of type  $A$  and  $C$  [8]. Irreducible cuspidal modules with finite-dimensional weight spaces were classified by Mathieu [15]. The concept of cuspidality has been extended to weight irreducible modules over finite-dimensional Lie superalgebras by Dimitrov, Mathieu and Penkov in [5]. Such a module  $V$  is cuspidal if the monoid generated by the roots with the injective action of the corresponding root elements is a subgroup of finite index in  $Q$ , which is equivalent to the fact that  $V$  is not parabolically induced (cf. [5], Corollary 3.7).

If  $c$  acts as a scalar on a  $\mathfrak{G}$ -module  $V$ , then this scalar is called the level of  $V$ . Denote by  $K(\mathfrak{G})$  the category of weight  $\mathfrak{G}$ -modules of nonzero levels with finite-dimensional weight spaces. The first goal of the present paper is to describe the irreducible modules in the category  $K(\mathfrak{G})$ . For an affine Lie algebra  $\mathcal{B}$  the classification of irreducible modules in  $K(\mathcal{B})$  was obtained in [11]. It was shown that any such module is a quotient of a module induced from an irreducible cuspidal module over a finite-dimensional reductive Lie subalgebra.

We show that in the case of an affine Lie superalgebra  $\mathfrak{G}$  the classification of irreducible  $\mathfrak{G}$ -modules in  $K(\mathfrak{G})$  is reduced to the classification of irreducible cuspidal modules over *cuspidal* Levi subsuperalgebras. The cuspidal Levi subsuperalgebras of affine Lie superalgebras are essentially the same as finite-dimensional classical Lie superalgebras, where they were described in [5] and [6].

A subset  $\mathcal{P} \subset \Delta$  is called a *parabolic subset* if  $\mathcal{P}$  is additively closed and  $\mathcal{P} \cup -\mathcal{P} = \Delta$ . This concept in the affine Lie algebras setting was introduced in [10]. For a

parabolic subset  $\mathcal{P}$  denote  $\mathcal{P}^\pm = \pm(\mathcal{P} \setminus (-\mathcal{P}))$  and  $\mathcal{P}^0 = \mathcal{P} \cap (-\mathcal{P})$ . This induces the corresponding triangular decomposition of  $\mathfrak{G}$ :  $\mathfrak{G} = \mathfrak{G}_\mathcal{P}^- \oplus (\mathfrak{G}_\mathcal{P}^0 + \mathcal{H}) \oplus \mathfrak{G}_\mathcal{P}^+$ . Denote  $\mathfrak{G}_\mathcal{P} = (\mathfrak{G}_\mathcal{P}^0 + \mathcal{H}) \oplus \mathfrak{G}_\mathcal{P}^+$ .

Now we state our main result.

**Theorem A.** *Let  $V$  be an irreducible module in  $K(\mathfrak{G})$ . Then there exists a parabolic subset  $\mathcal{P} \subset \Delta$  and an irreducible weight module  $N$  over the corresponding parabolic subalgebra  $\mathfrak{G}_\mathcal{P}$  such that  $V$  is parabolically induced (cf. Section 3) from  $\mathfrak{G}_\mathcal{P}$ . Moreover,  $N$  is an irreducible cuspidal module over the Levi subsuperalgebra of  $\mathfrak{G}_\mathcal{P}$  and  $\dim \mathfrak{G}_\mathcal{P}^0 < \infty$ .*

If  $\mathfrak{G}$  is of type  $A(m, n)^\wedge$  or  $C(n)^\wedge$ , then by combining Theorem A with Mathieu’s classification of irreducible cuspidal modules for Lie algebras of types  $A$  and  $C$  we obtain a classification of irreducible modules in  $K(\mathfrak{G})$  (up to the Weyl group action). In all other cases the classification is reduced to a finite indeterminacy by [5], Proposition 6.3. We should point out that there might be different parabolic subalgebras that give the same irreducible module. It would be interesting to study this question in order to get a complete classification.

A parabolic subset  $\mathcal{P}$  is *standard* (with respect to some basis  $\pi$  of  $\Delta$ ) if  $\mathcal{P}^+$  lies in the subset of positive roots with respect to  $\pi$  and  $\mathcal{P}^0$  is the closure of  $S$  in  $\Delta$  for some  $S \subset \pi$ . In this case we write  $\mathcal{P} = \mathcal{P}_S$  and  $\mathfrak{G}_\mathcal{P}^0 = \mathfrak{G}(S)$ . In Section 6 we prove the following stronger version of Theorem A.

**Theorem B.** *Modules  $L_\mathcal{P}(N)$  with an irreducible cuspidal  $\mathfrak{G}_\mathcal{P}^0$ -module  $N$  exhaust all irreducible objects in  $K(\mathfrak{G})$ . Moreover  $\mathcal{P}$  can be chosen in the following way:*

- (1) *If  $\mathfrak{G}$  is of type  $A(m, n)^\wedge$ , then  $\mathcal{P} = \mathcal{P}_S$  for some choice of basis  $\pi$ ,  $S$  is any subset of even roots in  $\pi$ , and  $\mathfrak{G}(S)$  is isomorphic to a semisimple Lie subalgebra of  $\mathfrak{sl}(m) \oplus \mathfrak{sl}(n)$ .*
- (2) *If  $\mathfrak{G}$  is of type  $C(n)^\wedge$  ( $n > 2$ ), then either  $\mathcal{P} = \mathcal{P}_S$  for some choice of basis  $\pi$ ,  $S$  is any subset of even roots in  $\pi$  and  $\mathfrak{G}_\mathcal{P}^0 = \mathfrak{G}(S)$  is isomorphic to a semisimple Lie subalgebra of  $\mathfrak{sp}(2n - 2)$ , or  $\mathfrak{G}_\mathcal{P}^0$  is isomorphic to a semisimple subalgebra of  $\mathfrak{sp}(2k - 2) \oplus \mathfrak{sp}(2m - 2)$  with  $k + m = n$  (in this case  $\mathfrak{G}_\mathcal{P}^0$  is not necessarily of  $\mathfrak{G}(S)$  type).*
- (3) *If  $\mathfrak{G}$  is of type  $B(m, n)^\wedge$  ( $m > 0$ ), then  $\mathfrak{G}_\mathcal{P}^0$  is isomorphic to one of the following (super)algebras or their direct summands:*

$$\begin{aligned} & \mathfrak{A}_n \oplus \mathfrak{B}_m; \mathfrak{osp}(4, 2n) \oplus \mathfrak{B}_{m-2}; \mathfrak{osp}(6, 2n) \oplus \mathfrak{B}_{m-3}; \\ & \mathfrak{A}_k \oplus \mathfrak{osp}(2m + 1, 2(n - k)), k = 0, \dots, n - 1, m = 1, 2; \\ & \mathfrak{sp}(2i) \oplus \mathfrak{sp}(2n - 2i) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{B}_{m-2}; \\ & \mathfrak{sp}(2i) \oplus \mathfrak{sp}(2n - 2i) \oplus \mathfrak{sl}(3) \oplus \mathfrak{B}_{m-3}, \mathfrak{sp}(2i) \oplus \mathfrak{sp}(2n - 2i) \oplus \mathfrak{B}_m, \end{aligned}$$

$i = 0, \dots, n - 1$ , where  $\mathfrak{A}_k$  is a semisimple subalgebra of  $\mathfrak{sp}(2k)$  and  $\mathfrak{B}_k$  is a semisimple subalgebra of  $\mathfrak{so}(2k + 1)$  with simple components of types  $A$  and  $C_2$ .

- (4) *If  $\mathfrak{G}$  is of type  $B(0, n)^\wedge$ , then  $\mathcal{P} = \mathcal{P}_S$  for some choice of basis  $\pi$  and  $\mathfrak{G}(S)$  is a semisimple Lie subalgebra of  $\mathfrak{sp}(2n)$  if  $S$  contains only even roots, or  $\mathfrak{G}(S) = \mathfrak{A}_i \oplus \mathfrak{osp}(1, 2(n - i))$  for some  $i \in \{0, \dots, n - 1\}$ , otherwise, where  $\mathfrak{A}_i$  is a semisimple subalgebra of  $\mathfrak{sp}(2i)$ .*

- (5) If  $\mathfrak{G}$  is of type  $D(m, n)$ , then  $\mathfrak{G}_{\mathcal{P}}^0$  is isomorphic to one of the following (super)algebras or their direct summands:

$$\begin{aligned} & \mathfrak{A}_n \oplus \mathfrak{D}_m; \mathfrak{osp}(4, 2n) \oplus \mathfrak{D}_{m-2}; \mathfrak{osp}(6, 2n) \oplus \mathfrak{D}_{m-3}; \\ & \mathfrak{A}_k \oplus \mathfrak{osp}(2m, 2(n-k)), k = 0, \dots, n-1, m = 2, 3; \\ & \mathfrak{sp}(2i) \oplus \mathfrak{sp}(2n-2i) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{D}_{m-2}; \\ & \mathfrak{sp}(2i) \oplus \mathfrak{sp}(2n-2i) \oplus \mathfrak{sl}(3) \oplus \mathfrak{D}_{m-3}, \mathfrak{sp}(2i) \oplus \mathfrak{sp}(2n-2i) \oplus \mathfrak{D}_m, \end{aligned}$$

$i = 0, \dots, n-1$ , where  $\mathfrak{A}_k$  is a semisimple subalgebra of  $\mathfrak{sp}(2k)$  and  $\mathfrak{D}_k$  is a semisimple subalgebra of  $\mathfrak{so}(2k)$  with simple components of type  $A$ .

- (6) If  $\mathfrak{G}$  is of type  $D(2, 1; a)$ , then  $\mathcal{P} = \mathcal{P}_S$ ,

$$\mathfrak{G}(S) \in \{\mathfrak{sl}(2), \mathfrak{sl}(2) \oplus \mathfrak{sl}(2), \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2), D(2, 1; a)\}.$$

- (7) If  $\mathfrak{G}$  is of type  $G(3)$ , then  $\mathfrak{G}_{\mathcal{P}}^0$  is isomorphic to one of the following (super)algebras:

$$\begin{aligned} & \{\mathfrak{sl}(2), \mathfrak{sl}(2) \oplus \mathfrak{sl}(2), \mathfrak{sl}(2) \oplus \mathfrak{sl}(3), \mathfrak{osp}(1, 2), \mathfrak{sl}(2) \oplus \mathfrak{osp}(1, 2), \\ & \mathfrak{sl}(3) \oplus \mathfrak{osp}(1, 2), \mathfrak{osp}(3, 2), \mathfrak{osp}(4, 2), \mathfrak{sl}(2) \oplus \mathfrak{osp}(3, 2)\}. \end{aligned}$$

- (8) If  $\mathfrak{G}$  is of type  $F(4)$ , then

$$\begin{aligned} \mathfrak{G}_{\mathcal{P}}^0 \in & \{\mathfrak{sl}(2), \mathfrak{sl}(2) \oplus \mathfrak{sl}(2), \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2), \mathfrak{so}(5), \mathfrak{sl}(2) \oplus \mathfrak{so}(5), \\ & \mathfrak{sl}(3), \mathfrak{sl}(2) \oplus \mathfrak{sl}(3), \mathfrak{sl}(2) \oplus \mathfrak{sl}(4), \mathfrak{sl}(4), \mathfrak{osp}(4, 2), \mathfrak{sl}(2) \oplus \mathfrak{osp}(4, 2)\}. \end{aligned}$$

The next question we address in this paper is the integrability of irreducible modules in the category  $K(\mathfrak{G})$ . In the case when  $\mathfrak{G}$  is an affine Lie algebra the classification of all irreducible integrable modules with finite-dimensional weight spaces was obtained by Chari in [3]. If the central element acts nontrivially, then such a module is of highest weight (for some choice of positive roots); otherwise it is a *loop* module introduced by Chari and Pressley in [4].

Analogous of loop modules for affine Lie superalgebras were constructed by Dimitrov and Penkov in [7]. For  $\mathfrak{G}$  different from  $A(m, n)$  and  $C(n)$  the irreducible integrable loop modules were classified Rao and Zhao in [17]. Moreover only trivial integrable modules exist if  $\mathfrak{G}$  is not of type  $A(0, n)$ ,  $B(0, n)$ ,  $n \geq 1$ ,  $C(n)$ ,  $n \geq 3$ , if the central element acts nontrivially. Also, if  $\mathfrak{G}$  has one of these types, then an irreducible integrable module is of highest weight with respect to some choice of a Borel subalgebra. Integrable highest weight modules were classified by Kac and Wakimoto [13], [14].

It was shown in [13] that the usual definition of integrability for most affine Lie superalgebras allows only trivial highest weight modules. Therefore, following [13] we introduce a concept of *weak integrability* for  $\mathfrak{G}$ -modules. Weakly integrable irreducible highest weight modules of nontwisted affine Lie superalgebras were classified by Kac and Wakimoto in [14]. We show that they are the only weakly integrable irreducible modules in the category  $K(\mathfrak{G})$ . Namely we prove the following

**Theorem C.** *Let  $V$  be a weakly integrable irreducible module in  $K(\mathfrak{G})$ . Then  $V$  is a highest weight module.*

Theorem C together with the classification of weakly integrable highest weight modules in [14] gives a classification of weakly integrable irreducible modules in  $K(\mathfrak{G})$ .

2. PRELIMINARIES

Let  $\mathcal{U}(\mathfrak{G})$  be the universal enveloping algebra of  $\mathfrak{G}$ ,  $\Delta$  the root system of  $\mathfrak{G}$ , and  $\Delta = \Delta_0 \cup \Delta_1$ , where  $\Delta_0$  (resp.  $\Delta_1$ ) consists of all even (resp. odd) roots. Let  $\pi = \pi_0 \cup \pi_1$  be a basis of  $\Delta$  (set of simple roots), where  $\pi_i \subset \Delta_i$ ,  $i = 0, 1$ . We will always assume that  $\pi$  contains only *one* simple odd root of the underlined finite-dimensional Lie superalgebra  $\mathfrak{g}$ , which is possible due the choice of considered superalgebras [12]. For a subset  $S \subset \pi$  denote by  $\Delta(S)$  the additive closure of  $S$  in  $\Delta$ ,  $\Delta(S) = \mathbb{Z}S \cap \Delta$ . Then  $\Delta_+ = \Delta_+(\pi)$  (respectively  $\Delta_- = \Delta_-(\pi)$ ) is the set of all positive (resp., negative) roots in  $\Delta$  with respect to  $\pi$ . Set  $\Delta_0^\pm = \Delta_0 \cap \Delta_\pm$ ,  $\Delta_1^\pm = \Delta_1 \cap \Delta_\pm$ .

Let  $\Delta^{\text{re}}$  (respectively  $\Delta^{\text{im}}$ ) be the set of real (respectively imaginary) roots. Set  $\Delta_\pm^{\text{re}} = \Delta^{\text{re}} \cap \Delta_\pm$ ,  $\Delta_\pm^{\text{im}} = \Delta^{\text{im}} \cap \Delta_\pm$ . Then  $\Delta_\pm^{\text{im}} = \{k\delta \mid k \in \mathbb{Z}_\pm \setminus \{0\}\}$ , where  $\delta$  is the indivisible imaginary root.

Let  $\dot{\Delta} \subset \Delta$  be the root system of  $\mathfrak{g}$ ,  $\dot{\Delta} = \dot{\Delta}_0 \cup \dot{\Delta}_1$ ,  $\dot{\Delta}_0^\pm = \dot{\Delta}_0 \cap \Delta_\pm$ ,  $\dot{\Delta}_1^\pm = \dot{\Delta}_1 \cap \Delta_\pm$ ,  $\dot{\Delta}_+ = \dot{\Delta}_0^+ \cup \dot{\Delta}_1^+$ . Then  $\Delta^{\text{re}} = \dot{\Delta} + \mathbb{Z}\delta$ ,  $\Delta_+^{\text{re}} = \{\dot{\Delta}_+ + \mathbb{Z}_{\geq 0}\delta\} \cup \{-\dot{\Delta}_+ + \mathbb{Z}_{> 0}\delta\}$ ,  $\Delta_0 = (\dot{\Delta}_0 + \mathbb{Z}\delta) \cup \Delta^{\text{im}}$ . If  $\pi$  is fixed, then  $\dot{\pi} \subset \pi$  will denote the basis of  $\dot{\Delta}$ .

For  $\alpha \in \Delta$  denote by  $\mathfrak{G}_\alpha$  the root subspace corresponding to  $\alpha$ .

Let  $G = \mathbb{C}c \oplus \sum_{k \in \mathbb{Z} \setminus \{0\}} \mathfrak{G}_{k\delta} \subset \mathfrak{G}$ ,  $G_\pm = \sum_{\pm k > 0} \mathfrak{G}_{k\delta}$ . For  $a \in \mathbb{C}$  denote by  $M_+(a)$

(resp.  $M_-(a)$ ) the Verma module for  $G$  generated by a highest (lowest) vector  $v$  such that  $G_+v = 0$  (resp.  $G_-v = 0$ ) and  $cv = av$ . It is known ([9], Proposition 4.5, [11], Theorem 2.3) that any finitely generated  $\mathbb{Z}$ -graded  $G$ -module  $V$  with  $\dim V_i < \infty$  for at least one  $i \in \mathbb{Z}$  and with  $cV = aV$ ,  $a \neq 0$ , is completely reducible. Moreover, all irreducible components of  $V$  are isomorphic  $M_+(a)$  or  $M_-(a)$  up to a shift of the gradation.

For a  $\mathfrak{G}$ -module  $V$  in  $K(\mathfrak{G})$  denote  $V^\pm = \{v \in V \mid G_{\pm n\delta}v = 0 \text{ for all } n \geq 1\}$ . Following [11] we define maximal and minimal elements in  $V$ . A nonzero element  $v \in V_\lambda$ ,  $\lambda \in \mathcal{H}^*$ , is *maximal* (resp. *minimal*) if  $v \in V^+$  (resp.  $v \in V^-$ ) and  $V_{\lambda+k\delta} \cap V^+ = \{0\}$  for all  $k > 0$  (resp.  $V_{\lambda+k\delta} \cap V^- = \{0\}$  for all  $k < 0$ ).

**Proposition 2.1.** *Let  $V$  be a module in  $K(\mathfrak{G})$ ,  $\lambda \in P(V)$ .*

- (i) *The  $G$ -module  $V' = \sum_{k \in \mathbb{Z}} V_{\lambda+k\delta}$  contains a maximal or a minimal element.*
- (ii) *Let  $v$  be a nonzero element in  $V^+$ . Then for any  $\alpha \in \Delta$  there exists  $m_0 \in \mathbb{Z}$  such that  $\mathfrak{G}_{\alpha+m\delta}v = 0$  for all  $m \geq m_0$ .*
- (iii) *If  $V$  is irreducible, then  $P(V)$  is a proper subset of a  $Q$ -coset.*

*Proof.* Statement (i) is an analog of Proposition 3.1 in [11]. Consider the  $G$ -module  $V''$  of  $V'$  generated by  $V_\lambda$ . Then the module  $V''$  is completely reducible, and all its irreducible submodules are of type  $M_\pm(\lambda(c))$  up to the shift of gradation. Suppose that  $V^+ \cap V' \neq 0$ . Since  $(V'/V'')_\lambda = 0$ , then  $V^+ \cap V_{\lambda+k\delta} \subset V''$  for all  $k \geq 0$  and there exists only a finite number of integers  $k \geq 0$  for which  $V^+ \cap V_{\lambda+k\delta} \neq 0$ . Hence  $V'$  contains a maximal element. If  $V^+ \cap V' = 0$ , then clearly  $V^- \cap V' \neq 0$ , and the same arguments as above show that  $V'$  contains a minimal element.

Statement (ii) generalizes Lemma 3.2 in [11]. Assume that  $\alpha \in \Delta^{\text{re}}$ . Let  $v \in V_\lambda \cap V^+$ . For each  $\varphi \in \Delta^{\text{re}}$  fix a nonzero element  $X_\varphi \in \mathfrak{G}_\varphi$ . Suppose that  $w_m = X_{\alpha+m\delta}v \neq 0$  for all  $m \in \mathbb{Z}_+$ . Since  $[\mathfrak{G}_{m\delta}, \mathfrak{G}_\alpha] = \mathfrak{G}_{\alpha+m\delta}$ , then a  $G$ -submodule  $N$  generated by  $w_0$  contains all  $w_m$ ,  $m \in \mathbb{Z}_+$ . The module  $N$  is completely reducible with irreducible submodules isomorphic to  $M_\pm(\lambda(c))$ . Since  $\dim V_\lambda < \infty$ , there exists  $m_0 \in \mathbb{Z}_+$  such that the  $G_+$ -submodule  $\mathcal{U}(G_+)w_{m_0}$  is  $\mathcal{U}(G_+)$ -free. On the

other hand,

$$[\mathfrak{G}_{2\delta}, \mathfrak{G}_{\alpha+m_0\delta}] = [\mathfrak{G}_\delta, [\mathfrak{G}_\delta, \mathfrak{G}_{\alpha+m_0\delta}]] = \mathfrak{G}_{\alpha+(m_0+2)\delta},$$

and hence  $xw_{m_0}$  is proportional to the element  $y^2w_{m_0}$  for all  $x \in \mathfrak{G}_{2\delta}$ ,  $y \in \mathfrak{G}_\delta$  since the real root spaces are 1-dimensional. Since  $v \in V^+$ , then the obtained contradiction shows that  $\mathfrak{G}_{\alpha+m\delta}v = 0$  for all  $m$  sufficiently large. This completes the proof of (ii).

Statement (iii) is an analog of Proposition 3.4 in [11]. Suppose  $V$  is irreducible. Without loss of generality we may assume by (i) that  $V$  contains a maximal element  $v \in V_\lambda \setminus \{0\}$ . Let  $\mu \in P(V)$ . Since  $V$  is irreducible, there exists  $u \in \mathcal{U}(\mathfrak{G})$  such that  $0 \neq uv \in V_\mu$ . Applying (ii) we conclude that  $\mathfrak{G}_{n\delta}uv = 0$  for sufficiently large  $n$ . From here we deduce that a subspace  $V' = \sum_{k \in \mathbb{Z}_+} V_{\mu+k\delta}$  has a nonzero intersection with  $V^+$ . The  $G$ -submodule  $\mathcal{U}(G)uv$  is completely reducible with irreducible components of type  $M_\pm(\lambda(c))$ ; moreover there exists an irreducible component  $N$  of type  $M_+(\lambda(c))$  such that  $N \cap V_\mu \neq 0$ . As in the proof of (i) we can show the existence of a maximal element in  $V'$ . Now suppose that  $n > 0$  and  $\lambda + n\delta \in P(V)$ . Applying the same argument as above to the weight  $\mu = \lambda + n\delta$ , we conclude that

$$V^+ \cap \left( \sum_{k \in \mathbb{Z}_+} V_{\lambda+(k+n)\delta} \right) \neq 0.$$

But this contradicts our assumption on the maximality of  $v$ , which shows that  $\lambda + n\delta \notin P(V)$  for all  $n > 0$ . Hence  $P(V)$  is a proper subset of a  $Q$ -coset.  $\square$

*Remark 2.2.* Proposition 2.1 also holds for  $\mathfrak{g} = \mathfrak{psl}(n, n)$ . Indeed all arguments are valid for  $n > 2$ . If  $n = 2$ , then all root spaces are 2-dimensional. In this case the proof of item (ii) can be modified as follows. Consider the element  $w_{m_0}$  from the proof of Proposition 2.1(ii). For each  $m > 0$  choose a nonzero  $x_m \in \mathfrak{G}_{m\delta}$ . Then for any positive  $s$  the vectors

$$\{x_{i_1} \dots x_{i_r} w_{m_0} \mid i_1 + \dots + i_r = s, i_j > 0, j = 1, \dots, r\}$$

are linearly independent. On the other hand

$$[\mathfrak{G}_{i_r\delta}, [\dots [\mathfrak{G}_{i_1\delta}, \mathfrak{G}_{\alpha+m_0\delta}] \dots]] = \mathfrak{G}_{\alpha+(m_0+s)\delta},$$

and hence the space spanned by all  $x_{i_1} \dots x_{i_r} w_{m_0}$ ,  $i_1 + \dots + i_r = s$ , is at most 2-dimensional, which is a contradiction.

Without loss of generality we will always assume that modules of  $K(\mathfrak{G})$  contain a maximal element.

### 3. CUSPIDAL LEVI SUBSUPERALGEBRAS

Let  $\mathcal{P}$  be a parabolic subset of  $\Delta$ ,  $\mathfrak{G} = \mathfrak{G}_{\mathcal{P}}^- \oplus (\mathfrak{G}_{\mathcal{P}}^0 + \mathcal{H}) \oplus \mathfrak{G}_{\mathcal{P}}^+$ , where  $\mathfrak{G}_{\mathcal{P}}^\pm = \sum_{\alpha \in \mathcal{P}} \mathfrak{G}_{\pm\alpha}$  and  $\mathfrak{G}_{\mathcal{P}}^0$  is generated by the subspaces  $\mathfrak{G}_\alpha$  with  $\alpha \in \mathcal{P}^0$ . The subsuperalgebra  $\mathfrak{G}_{\mathcal{P}}^0$  is called a *Levi subsuperalgebra* of  $\mathfrak{G}$ . Let  $N$  be an irreducible weight module over a parabolic subalgebra  $\mathfrak{G}_{\mathcal{P}} = (\mathfrak{G}_{\mathcal{P}}^0 + \mathcal{H}) \oplus \mathfrak{G}_{\mathcal{P}}^+$  with a trivial action of  $\mathfrak{G}_{\mathcal{P}}^+$ , and let

$$M_{\mathcal{P}}(N) = \text{ind}(\mathfrak{G}_{\mathcal{P}}, \mathfrak{G}; N)$$

be the induced  $\mathfrak{G}$ -module. The module  $M_{\mathcal{P}}(N)$  has a unique irreducible quotient  $L_{\mathcal{P}}(N)$ . If  $\mathfrak{G}_{\mathcal{P}}^0 \neq \mathfrak{G}$ , then  $L_{\mathcal{P}}(N)$  is said to be *parabolically induced*. An irreducible weight  $\mathfrak{G}$ -module is called *cuspidal* if is not parabolically induced. A Levi subsuperalgebra of  $\mathfrak{G}$  is *cuspidal* if it admits a weight cuspidal module with finite-dimensional

weight spaces. All cuspidal Levi subalgebras of reductive Lie algebras were classified by Fernando [8]. They are the subalgebras with simple components of type  $A$  and  $C$ . Cuspidal Levi subsuperalgebras of finite-dimensional Lie superalgebras were described by Dimitrov, Mathieu and Penkov in [5] and [6]. Here we present their list. Besides the cuspidal Levi subalgebras of types  $A$  and  $C$  it includes the superalgebras of types  $\mathfrak{osp}(1, 2)$ ,  $\mathfrak{osp}(1, 2) \oplus \mathfrak{sl}(2)$ ,  $\mathfrak{osp}(n, 2m)$  with  $2 < n \leq 6$  and  $D(2, 1; a)$ . Note that the notion of a cuspidal Levi subsuperalgebra in [5] is more general than ours since it refers to generalized weight modules. We will classify all cuspidal subsuperalgebras of affine Lie superalgebras in Section 6.

4. PARABOLICALLY INDUCED MODULES

Let  $\mathcal{P} \subset \Delta$  be a parabolic subset. A nonzero element  $v$  of a  $\mathfrak{G}$ -module  $V$  is called  $\mathcal{P}$ -primitive if  $\mathfrak{G}_{\mathcal{P}}^+ v = 0$ . The module  $M_{\mathcal{P}}(N)$  has a  $\mathfrak{G}_{\mathcal{P}}^0 \oplus \mathfrak{G}_{\mathcal{P}}^+$ -submodule isomorphic to  $N$ , which consists of  $\mathcal{P}$ -primitive elements. Let  $Q_{\mathcal{P}}$  be the free abelian group generated by  $\mathcal{P}^0$ . The universality of the module  $M_{\mathcal{P}}(N)$  is clear from the following standard statement (cf. [11], Proposition 2.2).

**Proposition 4.1.** *Let  $\mathcal{P}^0 \neq \Delta$ ,  $V$  be an irreducible weight  $\mathfrak{G}$ -module with a  $\mathcal{P}$ -primitive element of weight  $\lambda$ ,  $N = \sum_{\nu \in Q_{\mathcal{P}}} V_{\lambda+\nu}$ . Then  $N$  is an irreducible  $\mathfrak{G}_{\mathcal{P}}^0 \oplus \mathfrak{G}_{\mathcal{P}}^+$ -module and  $V$  is isomorphic to  $L_{\mathcal{P}}(N)$ .*

*Remark 4.2.* Suppose  $\dim \mathfrak{G}_{\mathcal{P}}^0 < \infty$ . Then the module  $N$  in Proposition 4.1 is an irreducible weight module over the finite-dimensional Lie superalgebra. By [5], Theorem 6.1, any such module is parabolically induced from a cuspidal module over a subsuperalgebra of  $\mathfrak{G}_{\mathcal{P}}^0$ . Hence in this case the classification of all irreducible weight  $\mathfrak{G}$ -modules with a  $\mathcal{P}$ -primitive element is reduced to the classification of cuspidal modules over finite-dimensional Lie superalgebras.

If a parabolic subset  $\mathcal{P}$  is standard (with respect to  $\pi$ ), i.e.  $\mathcal{P}^+ \subset \Delta_+$ ,  $\mathcal{P}^0 = \Delta(S)$  for some  $S \subset \pi$ , then we write  $L_{\mathcal{P}}(N) = L^S(N)$ . If  $\mathcal{B} \subset \mathfrak{G}$  is a subsuperalgebra, then we will also consider irreducible modules  $L_{\mathcal{B}}^T(K)$  obtained in the same manner, where  $T$  is a subset of a basis of the root system of  $\mathcal{B}$  and  $K$  is an irreducible  $\mathcal{B}(T)$ -module.

A  $\mathcal{P}$ -primitive element with  $\mathcal{P} = \mathcal{P}_S$  will be called  $S$ -primitive. If  $V$  is generated by a  $\emptyset$ -primitive element  $v \in V_{\lambda}$ , then  $V$  is a highest weight module with highest weight  $\lambda$ .

The situation is especially pleasant when  $S \subset \pi_0$ . In this case  $\mathfrak{G}(S)$  is a finite-dimensional semisimple Lie algebra, and we have the following stronger version of Proposition 4.1.

**Proposition 4.3.** *Let  $S \subset \pi_0$ ,  $V$  be an irreducible weight  $\mathfrak{G}$ -module with an  $S$ -primitive element. Then there exist a basis  $\pi'$  of  $\Delta$ , a subset  $S' \subset \pi'_0$  and an irreducible cuspidal  $\mathfrak{G}(S') + \mathcal{H}$ -module  $N$  such that  $V \simeq L^{S'}(N)$ . Moreover, if  $V$  has finite-dimensional weight spaces, then all simple components of  $\mathfrak{G}(S')$  are of type  $A$  and  $C$ .*

In particular, Proposition 4.3 together with [15] describes all irreducible modules with  $S$ -primitive elements in the category  $K(\mathfrak{G})$ . Moreover, in the case when  $\mathfrak{G}$  is an affine Lie algebra the modules of type  $L^S(N)$  with cuspidal  $N$  exhaust all irreducible modules in  $K(\mathfrak{G})$ . Namely, the following result was proved in [11].

**Theorem 4.4.** *Let  $\mathfrak{G}$  be an affine Lie algebra, and let  $V$  be a  $\mathfrak{G}$ -module in  $K(\mathfrak{G})$ .*

- (i) ([11], Proposition 4.3) *There exist a basis  $\pi$  of  $\Delta$  and  $\alpha \in \pi$  such that  $V$  contains an  $S$ -primitive element,  $S = \pi \setminus \{\alpha\}$ .*
- (ii) ([11], Theorem 4.1) *If  $V$  is irreducible, then there exist a basis  $\pi$  of  $\Delta$ , a proper subset  $S \subset \pi$  and an irreducible weight cuspidal  $\mathfrak{G}(S) + \mathcal{H}$ -module  $N$  such that  $V \simeq L^S(N)$ .*

5. EXISTENCE OF  $\mathcal{P}$ -PRIMITIVE ELEMENTS IN OBJECTS OF  $K(\mathfrak{G})$

Recall that  $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1$  and  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ . Here  $\mathfrak{g}_0$  is a reductive Lie algebra with at most three simple components. We will denote by  $\mathcal{B}_0$  the semisimple part of  $\mathfrak{g}_0$  and by  $\mathcal{B}$  its affinization. Let  $W_0$  be the Weyl group of  $\mathfrak{G}_0$ .

In this section we establish the following key result for the proof of Theorem A.

**Proposition 5.1.** *Let  $V$  be a nonzero  $\mathfrak{G}$ -module in  $K(\mathfrak{G})$ . Then  $V$  contains a  $\mathcal{P}$ -primitive element for some proper parabolic subset  $\mathcal{P}$  of  $\Delta$ .*

We start with the following

**Lemma 5.2.** *Let  $V$  be a nonzero module in  $K(\mathfrak{G})$  and  $V_{\mathcal{B}}$  the restriction of  $V$  on  $\mathcal{B}$ . Also let  $\mathcal{B}_0 = \mathcal{B}_0^1 \oplus \mathcal{B}_0^2$  and denote by  $\mathcal{B}^1$  and  $\mathcal{B}^2$  the corresponding affine Lie algebras. Then there exist a basis  $\pi_{\mathcal{B}} = \pi_{\mathcal{B}}^1 \cup \pi_{\mathcal{B}}^2$  of the root system of  $\mathcal{B}$  and  $\alpha_i \in \pi_{\mathcal{B}}^i$  such that  $V_{\mathcal{B}}$  contains an  $S$ -primitive element  $v \in V_{\mathcal{B}}$ ,  $S = \pi_{\mathcal{B}} \setminus \{\alpha_1, \alpha_2\}$ , where  $\pi_{\mathcal{B}}^i$  is a basis of the root system of  $\mathcal{B}_i$ ,  $i = 1, 2$ .*

*Proof.* Let  $W_i \subset W_0$  be the Weyl group of  $\mathcal{B}^i$ ,  $i = 1, 2$ . Consider  $V_{\mathcal{B}}$  as a  $\mathcal{B}^1$ -module and apply Theorem 4.4(i). Then there exist  $w \in W_1$ , a basis  $\tilde{\pi}_{\mathcal{B}}^1 = w\pi_{\mathcal{B}}^1$  and a root  $\alpha_1 \in \tilde{\pi}_{\mathcal{B}}^1$  such that  $V$  contains an  $S_1$ -primitive element  $v'$  for  $\mathcal{B}_0^1$ ,  $S_1 = \tilde{\pi}_{\mathcal{B}}^1 \setminus \{\alpha_1\}$ . Consider a basis  $w\pi$  of  $\Delta$ ; note that  $w\pi_{\mathcal{B}}^2 = \pi_{\mathcal{B}}^2$ . Denote by  $V'$  a subspace of  $V^+$  consisting of  $S_1$ -primitive elements for  $\mathcal{B}_0^1$ ,  $v' \in V'$ . Clearly,  $V'$  is a  $\mathcal{B}_0^2$ -module. Again applying Theorem 4.4(i) to  $V'$  we find  $w' \in W_2$ , a basis  $\tilde{\pi}_{\mathcal{B}}^2 = w'\pi_{\mathcal{B}}^2$  and a root  $\alpha_2 \in \tilde{\pi}_{\mathcal{B}}^2$  such that  $V'$  contains an  $S_2$ -primitive element  $v$  for  $\mathcal{B}_0^2$ ,  $S_2 = \tilde{\pi}_{\mathcal{B}}^2 \setminus \{\alpha_2\}$ . Then  $\tilde{\pi}_{\mathcal{B}}^1 \cup \tilde{\pi}_{\mathcal{B}}^2$  is a required basis of the root system of  $\mathcal{B}$  and  $v$  is an  $S = S_1 \cup S_2$ -primitive element. □

Let  $V$  be a  $\mathfrak{G}$ -module in  $K(\mathfrak{G})$ . For  $v \in V \setminus \{0\}$  denote by  $\Omega_v$  the set of all roots  $\alpha \in \Delta_+$  such that  $\mathfrak{G}_{\alpha}v = 0$ . In particular,  $\Delta_+^{\text{im}} \subset \Omega_v$  if  $v \in V^+$ . Moreover, in this case  $\alpha + N\delta \in \Omega_v$  for any  $\alpha \in \dot{\Delta}$  and any  $N$  sufficiently large by Proposition 2.1(ii).

Given  $\alpha \in \Delta^{\text{re}}$  denote

$$\Sigma_{\alpha} = \{\alpha + n\delta, n \in \mathbb{Z}_{\geq 0}\} \cup \{-\alpha + m\delta, m \in \mathbb{Z}_{> 0}\}.$$

If  $\alpha \in \Delta_0^+$ , then  $\Sigma_{\alpha}$  consists of positive real roots of the affine  $\mathfrak{sl}(2)$ -subalgebra of  $\mathfrak{G}$  corresponding to the root  $\alpha$ . Note that  $\Sigma_{\alpha} = \Sigma_{-\alpha+\delta}$ . If  $V$  is a  $\mathfrak{G}$ -module, then an element  $v \in V^+ \setminus \{0\}$  is said to be  $(\alpha, \pi)$ -admissible if  $\Sigma_{\alpha} \setminus \{\alpha\} \subset \Omega_v$ . If  $S \subset \dot{\pi}$ , then it follows immediately from the definition that  $v \in V^+$  is  $S$ -primitive if and only if it is  $(\alpha, \pi)$ -admissible for all  $\alpha \in \Delta(S) \cap \Delta_+$  and  $(-\beta + \delta, \pi)$ -admissible for all  $\beta \in \dot{\Delta}_+ \setminus \Delta(S)$ . We will say that  $v \in V^+$  is  $\pi$ -admissible if it is  $(\beta, \pi)$ -admissible or  $(-\beta + \delta, \pi)$ -admissible for all  $\beta \in \dot{\Delta}_0^+$ .

Recall that an odd simple root  $\gamma \in \dot{\pi}$  is isotropic if  $(\gamma, \gamma) = 0$ , which is if and only if  $[\mathfrak{G}_{\gamma}, \mathfrak{G}_{\gamma}] = 0$ .



- Lemma 5.3.** (i) *Let  $V$  be a weight  $\mathfrak{G}$ -module that contains a  $\pi$ -admissible element. Then there exists a  $\pi$ -admissible  $v \in V^+$  which is  $(\varepsilon, \pi)$ -admissible for all  $\varepsilon \in \dot{\Delta}_1^+$ . Moreover, if  $v$  is  $(\beta, \pi)$ -admissible for all  $\beta \in \dot{\Delta}_0^+$  and  $[\mathfrak{G}_\varepsilon, \mathfrak{G}_{\varepsilon'}]v = 0$  for all  $\varepsilon, \varepsilon' \in \dot{\Delta}_1^+$ , then  $V$  contains a  $\dot{\pi} \setminus \{\gamma\}$ -primitive element, where  $\gamma$  is the odd simple root in  $\dot{\pi}$ .*
- (ii) *Let  $\gamma \in \dot{\pi}$  be an isotropic root. If  $[\mathfrak{g}_\varepsilon, \mathfrak{g}_{\varepsilon'}]v = 0$  for all  $\varepsilon, \varepsilon' \in \dot{\Delta}_1^+$ , then any weight  $\mathfrak{g}$ -module  $V$  contains an  $S$ -primitive element,  $S = \dot{\pi} \setminus \{\gamma\}$ .*
- (iii) *Let  $V$  be a weight  $\mathfrak{G}$ -module and  $v$  a nonzero element of  $V$  such that  $\Delta_0^+ \subset \Omega_v$ . Then  $V$  contains a  $\mathcal{P}$ -primitive element, where  $\mathcal{P} = \mathcal{P}_\theta = \Delta_+$ ; thus  $V$  is a highest weight module.*

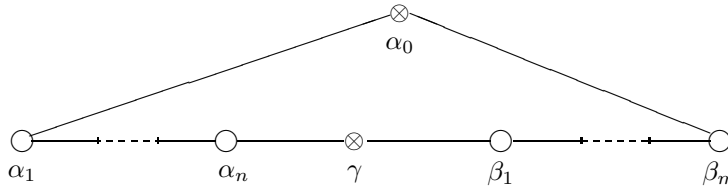
*Proof.* Let  $V' \subset V$  be a subspace of  $\pi$ -admissible elements in  $V$  and  $v \in V' \setminus \{0\}$ . By Proposition 2.1(ii) there exists  $N > 0$  such that  $\varepsilon + n\delta \in \Omega_v$  for all  $n \geq N$  and all  $\varepsilon \in \dot{\Delta}_1^+$ . Let  $N_0$  be the least nonnegative integer such that  $\varepsilon + N_0\delta \in \Omega_v$  for all  $\varepsilon \in \dot{\Delta}_1^+$  and there exists  $\varepsilon' \in \dot{\Delta}_1^+$  such that  $\varepsilon' + (N_0 - 1)\delta \notin \Omega_v$ . Suppose  $N_0 > 1$  and choose  $\varepsilon_0 \in \dot{\Delta}_1^+$  of maximal height with  $\varepsilon_0 + (N_0 - 1)\delta \notin \Omega_v$ . Consider  $w = X_{\varepsilon_0 + (N_0 - 1)\delta}v \neq 0$  which obviously belongs to  $V'$ . Moreover,  $\varepsilon_0 + (N_0 - 1)\delta \in \Omega_w$ , and hence  $\varepsilon_0 + k\delta \in \Omega_w$  for all  $k \geq N_0 - 1$ . Since  $\dot{\Delta}_1$  is finite we can find a nonzero  $v' \in V'$  such that  $\varepsilon + k\delta \in \Omega_{v'}$  for all  $\varepsilon \in \dot{\Delta}_1^+$  and  $k > 0$ . Now considering the roots  $-\varepsilon + n\delta$ ,  $\varepsilon \in \dot{\Delta}_1^+$ ,  $n > 0$ , we choose  $N_0$  as above and  $\varepsilon_0$  of minimal height such that  $\varepsilon_0 + (N_0 - 1)\delta \notin \Omega_{v'}$ . Then the same argument as above shows the existence of  $w \in V' \setminus \{0\}$  such that  $\alpha + k\delta \in \Omega_w$  for all  $\alpha \in \dot{\Delta}_1^+$  and  $k > 0$ . Hence  $w$  is  $(\varepsilon, \pi)$ -admissible for all  $\varepsilon \in \dot{\Delta}_1^+$ .

Suppose now that  $v$  is  $(\beta, \pi)$ -admissible for all  $\beta \in \dot{\Delta}_0^+$  and  $[\mathfrak{G}_\varepsilon, \mathfrak{G}_{\varepsilon'}]v = 0$  for all  $\varepsilon, \varepsilon' \in \dot{\Delta}_1^+$ . Then in particular  $v$  is  $\dot{\pi}$ -primitive. Moreover, in this case using the same arguments as above we find a nonzero  $w \in V'$  which is  $(\varepsilon, \pi)$ -admissible for all  $\varepsilon \in \dot{\Delta}_1^+$  and  $\dot{\Delta}_1^+ \subset \Omega_w$ . Therefore  $\mathfrak{G}_\beta w = 0$  for all  $\beta \in \Delta_+ \setminus \Delta(\dot{\pi} \setminus \{\gamma\})$ , which proves (i). The proof of (ii) and (iii) is similar.  $\square$

*Remark 5.4.* The idea of the proof of Lemma 5.3 will be used in various cases in this section. We will leave the details out since the exposition is similar to the one above.

We fix a nonzero  $\mathfrak{G}$ -module  $V$  in  $K(\mathfrak{G})$ .

5.1. **Case  $A(m, n)$ .** Let  $\pi_{01} = \{\alpha_1, \dots, \alpha_n\}$ ,  $\pi_{02} = \{\beta_1, \dots, \beta_m\}$ ,  $\pi_0 = \pi_{01} \cup \pi_{02}$ , and  $\pi_1 = \{\gamma, \alpha_0\}$  according to the following diagram:



Let  $\mathfrak{g}_{01}$  and  $\mathfrak{g}_{02}$  be the corresponding simple components of  $\mathfrak{g}_0$  with the root systems  $\dot{\Delta}_{01}$  and  $\dot{\Delta}_{02}$  respectively. Set  $\theta = \alpha_1 + \dots + \alpha_n$ .

- Lemma 5.5.** (i) *For any  $\beta \in \pi_{01}$  there exists a basis  $\pi'$  of  $\Delta$  such that  $\pi'_0 = \pi'_{01} \cup \pi'_{02}$ ,  $\pi'_{02} = \pi_{02}$ ,  $\pi'_{01} = (\pi_{01} \setminus \{\beta\}) \cup \{-\theta + \delta\}$ .*

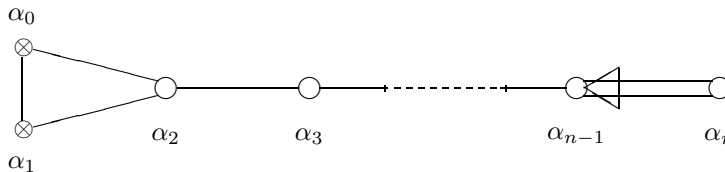
- (ii) Let  $\beta \in \pi_{01}$ . If  $v$  is  $(\alpha, \pi)$ -admissible for all  $\alpha \in \Delta(\pi_{01} \setminus \{\beta\}) \cap \Delta_0^+$  and  $(-\varepsilon + \delta, \pi)$ -admissible for all  $\varepsilon \in \dot{\Delta}_{01}^+ \setminus \Delta(\pi_{01} \setminus \{\beta\})$ , then there exists a basis  $\pi'$  of  $\Delta$  such that  $\pi'_0 = \pi'_{01} \cup \pi'_{02}$ ,  $\pi'_{02} = \pi_{02}$  and  $v$  is  $(\alpha, \pi')$ -admissible for  $\alpha \in \Delta(\pi'_{01}) \cap \Delta_0^+(\pi')$ .

*Proof.* For any  $i = 1, \dots, n$  there exists a new basis  $\pi'$  of  $\Delta$  such that  $\pi'_0 = \pi'_{01} \cup \pi'_{02}$ ,  $\pi'_{01} = \{\alpha'_1, \dots, \alpha'_n\}$ , where  $\alpha'_k = \alpha_{i-k}$ ,  $k = 1, \dots, i-1$ ,  $\alpha'_i = -\theta + \delta = \gamma + \beta_1 + \dots + \beta_m + \alpha_0$ ,  $\alpha'_{i+k} = \alpha_{n-k+1}$ ,  $k = 1, \dots, n-i$ ,  $\pi'_{02} = \{\beta_1, \dots, \beta_m\}$ , and  $\pi'_1 = \{-\alpha_{i+1} - \dots - \alpha_n - \gamma - \beta_1 - \dots - \beta_m, \gamma + \alpha_n + \dots + \alpha_i\}$ , which implies (i). Given  $\beta \in \pi'_{01}$  choose a new basis  $\pi'$  such that  $\pi'_{01} = (\pi_{01} \setminus \{\beta\}) \cup \{-\theta + \delta\}$ ,  $\pi'_{02} = \pi_{02}$  which exists by (i). Since  $\Delta(\pi_{01} \setminus \{\beta\}) \subset \Delta(\pi'_{01})$  and  $-\varepsilon + \delta \in \Delta(\pi'_{01})$  for all  $\varepsilon \in \dot{\Delta}_{01}^+ \setminus \Delta(\pi_{01} \setminus \{\beta\})$ , we conclude that  $\pi'$  is a required basis which proves (ii).  $\square$

Denote by  $\mathfrak{G}_{01}$  and  $\mathfrak{G}_{02}$  the affine Lie algebras with underlying finite-dimensional algebras  $\mathfrak{g}_{01}$  and  $\mathfrak{g}_{02}$ , and with the root systems  $\Delta_{01}$  and  $\Delta_{02}$  respectively. Then  $\tilde{\pi}_{01} = \pi_{01} \cup \{-\theta + \delta\}$  is a basis of  $\Delta_{01}$ . Let  $W_{0i} \subset W_0$  be the Weyl group of  $\Delta_{0i}$ ,  $i = 1, 2$ . As in the proof of Lemma 5.2 consider  $V$  as a  $\mathfrak{G}_{01}$ -module and apply Theorem 4.4(i). Then there exist  $w \in W_{01}$ , a basis  $\tilde{\pi}'_{01} = w\tilde{\pi}_{01}$  of  $\Delta_{01}$  and a root  $\alpha \in \tilde{\pi}'_{01}$  such that  $V$  contains an  $S$ -primitive element  $v' \in V$ ,  $S = \tilde{\pi}'_{01} \setminus \{\alpha\}$ . Consider a basis  $w\pi$  of  $\Delta$ ; note that  $w\pi_{02} = \pi_{02}$ . Applying Lemma 5.5(ii) to  $\alpha$  we conclude that there exists a basis  $\pi'$  of  $\Delta$  such that  $\pi'_0 = \pi'_{01} \cup \pi_{02}$  and  $v'$  is  $(\beta, \pi')$ -admissible for all  $\beta \in \Delta(\pi'_{01}) \cap w\Delta_0^+$ . Denote by  $V'$  a subspace of  $V^+$  consisting of elements with such property ( $v' \in V'$ ). Clearly,  $V'$  is a  $\mathfrak{G}_{02}$ -module. Subsequently applying Theorem 4.4(i) and Lemma 5.5(ii) to  $V'$  we conclude that there exists a basis  $\pi''$  of  $\Delta$  and a nonzero  $\tilde{v} \in V'$  which is  $(\alpha, \pi'')$ -admissible for all  $\alpha \in \Delta_+(\pi''_0)$ . Let  $\tilde{V}$  be a subspace of  $V'$  consisting of elements with such property and apply Lemma 5.3(i) to  $\tilde{V}$ . Since  $[\mathfrak{G}_\varepsilon, \mathfrak{G}_{\varepsilon'}] = 0$  for all  $\varepsilon, \varepsilon' \in \dot{\Delta}(\pi'')_1^+$ , there exists a nonzero  $\bar{v} \in \tilde{V}$  which is  $\pi''_0$ -primitive. If  $\gamma \in \pi''$  is an odd root, then in particular,  $\bar{v}$  is  $S$ -primitive, and  $S = \pi'' \setminus \{\gamma\}$ , which completes the proof of Proposition 5.1 for  $A(m, n)$ .

Consider the restriction  $V_{\mathcal{B}}$  of  $V$  on  $\mathcal{B} = \sum_{i \in I} \mathcal{B}_i$ . Then by Lemma 5.2 there exists a basis  $\pi_{\mathcal{B}} = \bigcup_{i \in I} \pi_{\mathcal{B}}^i$  of the root system of  $\mathcal{B}$  and  $\alpha_i \in \pi_{\mathcal{B}}^i$  for each  $i \in I$  such that  $V_{\mathcal{B}}$  contains an  $S_{\mathcal{B}}$ -primitive element  $v \in V_{\mathcal{B}}$ ,  $S_{\mathcal{B}} = \pi_{\mathcal{B}} \setminus \{\alpha_i, i \in I\}$ . In particular,  $v$  is  $\pi'$ -admissible, where  $\pi'$  is a basis of  $\Delta$  containing  $\pi_{\mathcal{B}}$ . In all subsequent cases we fix such an element  $v \in V_{\mathcal{B}}$  and assume without loss of generality that  $\pi' = \pi$ .

**5.2. Case  $C(n)$ ,  $n \geq 3$ .** Let  $\mathfrak{G}$  be of type  $C(n)$ , and let  $\pi = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$  be a basis of  $\Delta$  ordered as follows:



Here  $\tilde{\pi} = \{\alpha_1, \dots, \alpha_n\}$  is a basis of  $\dot{\Delta}$  and  $\pi_0 = \{\alpha_2, \dots, \alpha_n\}$  is a basis of  $\dot{\Delta}_0$ . Note that  $\alpha_0 + \alpha_1 \in \Delta_0 \setminus \dot{\Delta}_0$ . Then  $\pi_0$  is a basis of  $\mathcal{B}_0$  which has type  $C_{n-1}$ , and  $\tilde{\pi} = \pi_0 \cup \{-\theta + \delta\}$  is a basis of  $\mathcal{B}$  of type  $C_n^{(1)}$ , where  $\theta$  is the highest positive root in  $\Delta(\pi_0)$ .

By Lemma 5.2,  $v$  is an  $S_{\mathcal{B}}$ -primitive element, where  $S_{\mathcal{B}} = \tilde{\pi} \setminus \{\alpha\}$  for some  $\alpha \in \tilde{\pi}$ . Hence we have the following cases.

*Case 1.* Suppose  $\alpha = -\theta + \delta$ . Then  $v$  is  $(\beta, \pi)$ -admissible for any  $\beta \in \Delta_+(\pi_0)$ . Since  $[\mathfrak{G}_\varepsilon, \mathfrak{G}_{\varepsilon'}] = 0$  for all  $\varepsilon, \varepsilon' \in \dot{\Delta}_1^+$ , we can apply Lemma 5.3(i) to find a  $\pi_0$ -primitive element in  $V$ .

*Case 2.* Suppose now that  $\alpha = \alpha_n$ . Hence  $v$  is  $(\beta, \pi)$ -admissible for all

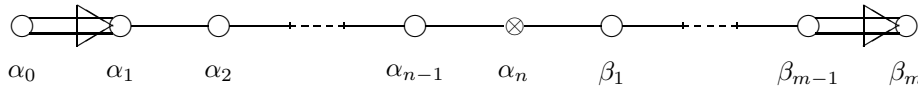
$$\beta \in \Delta_+(\pi_0 \setminus \{\alpha_n\}) \cup \{-\theta + \delta\}.$$

Choose a new basis  $\pi' = \{\alpha'_0, \alpha'_1, \dots, \alpha'_n\}$ , where  $\alpha'_0 = -\alpha_1 - \dots - \alpha_{n-1}$ ,  $\alpha'_1 = \alpha_1 + \dots + \alpha_n$ ,  $\alpha'_k = \alpha_{n-k+1}$ ,  $k = 2, \dots, n-1$ , and  $\alpha'_n = \alpha_0 + \alpha_1$ . Then  $v$  is  $(\beta, \pi')$ -admissible for any  $\beta \in \Delta_+(\pi'_0)$  and we are back to Case 1.

*Case 3.* Let  $\alpha = \alpha_k$ ,  $2 \leq k \leq n-1$ . Hence  $v$  is  $(\beta, \pi)$ -admissible for any  $\beta \in \Delta_+(\pi_0 \cup \{\alpha_0 + \alpha_1\}) \setminus (\Delta_+(\pi_0) \setminus \Delta_+(\pi_0 \setminus \{\alpha\}))$ . Denote a subspace of such elements by  $V'$ , hence  $V' \neq 0$ . Consider the set  $\Delta_{1,\alpha}^+$  of positive roots in  $\Delta_1$  that have  $\alpha$  in their decomposition in simple roots. Then applying the same argument as in the proof of Lemma 5.3(i), we find a nonzero  $v' \in V'$  such that  $\gamma \in \Omega_{v'}$  for all  $\gamma \in \Delta_{1,\alpha}^+$  and  $k > 0$ . Hence  $v'$  is  $S$ -primitive, where  $S = \pi \setminus \{\alpha\}$ . This completes the proof of Proposition 5.1 for  $C(n)$ .

We will say that a root  $\alpha$  “contains” a simple (with respect to some fixed basis  $\pi$ ) root  $\beta$  if it has  $\beta$  in the decomposition into a sum of simple roots.

**5.3. Case  $B(m, n)$ ,  $m > 0$ .** Let  $\mathfrak{G}$  be of type  $B(m, n)$ ,  $m > 0$ ,  $\pi = \{\alpha_0, \dots, \alpha_n, \beta_1, \dots, \beta_m\}$ ,  $\tilde{\pi}_0 = \{\alpha_1, \dots, \alpha_{n-1}, \beta_1, \dots, \beta_m\}$ ,  $\tilde{\pi}_1 = \{\alpha_n\}$ , and  $\delta = \alpha_0 + 2 \sum_{i=1}^n \alpha_i + 2 \sum_{j=1}^m \beta_j$ .



Since  $\mathcal{B}_0 \simeq C_n \oplus B_m$  then  $\mathcal{B} \simeq \mathfrak{A}_1 + \mathfrak{A}_2$ , where  $\mathfrak{A}_1$  is an affine Lie algebra of type  $C_n^{(1)}$ , while  $\mathfrak{A}_2$  is an affine Lie algebra of type  $B_m^{(1)}$  for  $m > 2$ , of type  $C_2^{(1)}$  for  $m = 2$  and of type  $A_1^{(1)}$  for  $m = 1$ . Denote  $\gamma_1 = 2\alpha_n + 2 \sum_{j=1}^m \beta_j$  and  $\gamma_2 = -\beta_1 - 2 \sum_{j=2}^m \beta_j + \delta$ . Then  $\{\alpha_0, \alpha_1, \dots, \alpha_{n-1}, \gamma_1\}$  is a basis of the root system of  $\mathfrak{A}_1$  and  $\{\beta_1, \dots, \beta_m, \gamma_2\}$  is a basis of the root system of  $\mathfrak{A}_2$ . Since a simple Lie algebra of type  $D_r$ ,  $r > 4$ , is not cuspidal by [8], then we have the following cases up to a change of the basis of  $\Delta$ .

*Case 1.* Let  $\gamma_i \in \Omega_v$ ,  $i = 1, 2$ . Then in particular  $[\mathfrak{G}_\varepsilon, \mathfrak{G}_{\varepsilon'}]v = 0$  for all  $\varepsilon, \varepsilon' \in \dot{\Delta}_1^+$ . It follows from the proof of Lemma 5.3(i) that in this case there exists a nonzero element  $v' \in V$  which is  $S$ -primitive for  $S = \pi \setminus \{\alpha_n\}$ .

*Case 2.* Let  $m > 1$ . Suppose  $\gamma_1 \in \Omega_v$ ,  $\gamma_2 \notin \Omega_v$  and  $\beta_2 \in \Omega_v$ . Then in particular all roots from  $\dot{\Delta}_0^+$  containing  $\beta_2$  belong to  $\Omega_v$ . As in the proof of Lemma 5.3(i) we can show that there exists a nonzero element  $v' \in V$  such that  $\Omega_v \subset \Omega_{v'}$  and  $\Sigma_\beta \subset \Omega_{v'}$  for all  $\beta \in \dot{\Delta}_1^+ \setminus \{\alpha_n + \dots + \alpha_i, \beta_1 + \alpha_n + \dots + \alpha_i, i = 1, \dots, n\}$ . Hence  $v'$  is  $S$ -primitive for  $S = \pi \setminus \{\beta_2\}$ .

*Case 3.* Let  $m > 1$ . Suppose  $\gamma_1, \beta_3 \in \Omega_v$  and  $\gamma_2, \beta_2 \notin \Omega_v$ . Then as in the previous case there exists a nonzero element  $v' \in V$  such that  $\Omega_v \subset \Omega_{v'}$  and  $\Sigma_\beta \subset \Omega_{v'}$  for all  $\beta \in \dot{\Delta}_1^+ \setminus \{\alpha_n + \dots + \alpha_i, \beta_1 + \alpha_n + \dots + \alpha_i, \beta_2 + \beta_1 + \alpha_n + \dots + \alpha_i, i = 1, \dots, n\}$ . Hence  $v'$  is  $S$ -primitive for  $S = \pi \setminus \{\beta_3\}$ .

*Case 4.* Suppose  $\gamma_2 \in \Omega_v, \gamma_1 \notin \Omega_v$  and  $\alpha_i \in \Omega_v$  for some  $i = 0, \dots, n - 1$ . In this case there exists an  $S$ -primitive element with  $S = \pi \setminus \{\alpha_i\}$ .

*Case 5.* Let  $m > 1$ . Suppose  $\gamma_1, \gamma_2 \notin \Omega_v, \beta_2 \in \Omega_v$  and  $\alpha_k \in \Omega_v$  for some  $k = 0, \dots, n - 1$ . Then in particular all roots from  $\dot{\Delta}_0^+$  containing  $\beta_2$  belong to  $\Omega_v$ . As in the proof of Lemma 5.3 one can find a nonzero element  $v' \in X$  which is  $(\varepsilon, \pi)$ -admissible for all  $\varepsilon \in \dot{\Delta}_1^+$ . Now consider the following odd roots:  $z_{1i} = \alpha_n + \dots + \alpha_i$  and  $z_{2i} = \beta_1 + \alpha_n + \dots + \alpha_i, i = 1, \dots, n$ . Define nonzero elements  $v_{ji}$  by:  $v_{ji} = v_{j,i-1}$  if  $X_{-z_{ji}}v_{j,i-1} = 0, v_{ji} = X_{-z_{ji}}v_{j,i-1}$  otherwise,  $j = 1, 2, i = 1, \dots, n, v_{11} = v', v_{21} = v_{1n}$ . Then  $v_{2n}$  is a  $\mathcal{P}$ -primitive element where  $\mathcal{P}^0$  is spanned by  $\pm\alpha_0, \dots, \pm\alpha_{k-1}, \pm\alpha_{k+1}, \dots, \pm\alpha_{n-1}, \pm\gamma_1, \pm\beta_1, \pm\beta_3, \dots, \pm\beta_k, \pm\gamma_2$  and

$$\mathcal{P}^+ = (\Delta_+ \setminus (\mathcal{P}^0 \cup \{z_{ji}, j = 1, 2, i = 1, \dots, n\})) \cup \{-z_{ji}, j = 1, 2, i = 1, \dots, n\}.$$

*Case 6.* Let  $m > 1$ . Suppose  $\gamma_1, \gamma_2, \beta_2 \notin \Omega_v, \beta_3 \in \Omega_v$  and  $\alpha_k \in \Omega_v$  for some  $k \in \{0, \dots, n - 1\}$ . As in Case 5, there exists a  $\mathcal{P}$ -primitive element, where  $\mathcal{P}^0$  is spanned by  $\pm\alpha_0, \dots, \pm\alpha_{k-1}, \pm\alpha_{k+1}, \dots, \pm\alpha_{n-1}, \pm\gamma_1, \pm\beta_1, \pm\beta_2, \pm\beta_4, \dots, \pm\beta_k, \pm\gamma_2$  and

$$\mathcal{P}^+ = (\Delta_+ \setminus (\mathcal{P}^0 \cup \{d_{ji}, j = 1, 2, 3, i = 1, \dots, n\})) \cup \{-d_{ji}, j = 1, 2, 3, i = 1, \dots, n\},$$

$$d_{1i} = \alpha_n + \dots + \alpha_i, d_{2i} = \beta_1 + \alpha_n + \dots + \alpha_i, d_{3i} = \beta_2 + \beta_1 + \alpha_n + \dots + \alpha_i, i = 1, \dots, n.$$

*Case 7.* Suppose  $\gamma_1, \beta_1 \in \Omega_v$ . Then there exists an  $S$ -primitive element,  $S = \pi \setminus \{\beta_1\}$ .

*Case 8.* Suppose  $\gamma_1 \notin \Omega_v, \beta_1 \in \Omega_v$  and  $\alpha_k \in \Omega_v$  for some  $k \in \{0, \dots, n - 1\}$ . Then there exists a  $\mathcal{P}$ -primitive element, where  $\mathcal{P}^0$  is spanned by

$$\pm\alpha_0, \dots, \pm\alpha_{k-1}, \pm\alpha_{k+1}, \dots, \pm\alpha_{n-1}, \pm\gamma_1, \pm\beta_2, \dots, \pm\beta_k, \pm\gamma_2.$$

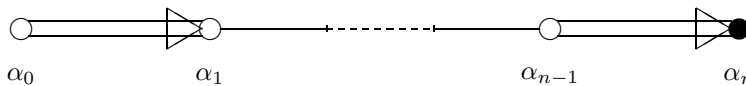
*Case 9.* Let  $m = 1, \gamma_1$  be as before and  $\gamma_2 = -\beta_1 + \delta$ . Changing the basis of  $\Delta$  if necessary, we can assume that we have one of the following subcases:

*Case 9.1.* Let  $\gamma_i \in \Omega_v, i = 1, 2$ . Then there exists an  $S$ -primitive element  $v' \in V$  such that  $S = \pi \setminus \{\alpha_n\}$ .

*Case 9.2.* Let  $\gamma_1 \notin \Omega_v$  and  $\gamma_2, \alpha_i \in \Omega_v$  for some  $i \in \{0, \dots, n - 1\}$ . As in Case 4 there exists an  $S$ -primitive element with  $S = \pi \setminus \{\alpha_i\}$ .

This completes the proof of Proposition 5.1 for  $B(m, n)$ .

5.4. **Case  $B(0, n)$ .** Let  $\mathfrak{G}$  be of type  $B(0, n), \pi = \{\alpha_0, \dots, \alpha_n\}, \dot{\pi}_0 = \{\alpha_1, \dots, \alpha_{n-1}\}, \dot{\pi}_1 = \{\alpha_n\}, \dot{\Delta}_0^+ = \{\alpha_i + \dots + \alpha_j, \alpha_i + \dots + \alpha_j + 2\alpha_{j+1} + \dots + 2\alpha_n, 1 \leq i \leq j \leq n - 1\} \cup \{2\alpha_i + \dots + 2\alpha_n, 1 \leq i \leq n - 1\}, \dot{\Delta}_1^+ = \{\alpha_i + \dots + \alpha_n, 1 \leq i \leq n\}$ .



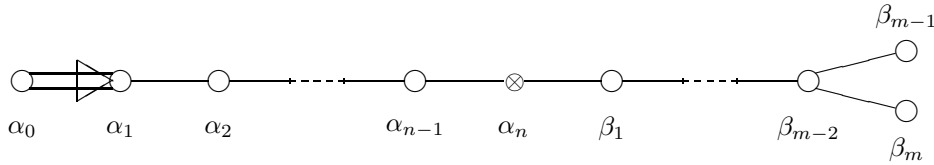
In this case  $\mathcal{B}_0 \simeq C_n$ . Then  $\{\alpha_0, \alpha_1, \dots, \alpha_{n-1}, 2\alpha_n\}$  is a basis of the root system of  $\mathcal{B}$ . Then we have the following cases.

*Case 1.* Let  $2\alpha_n \in \Omega_v$ . Then in particular  $\Omega_v$  contains all roots from  $\hat{\Delta}_0^+$  of the form  $\beta + 2\alpha_n$  and  $[\mathfrak{G}_\varepsilon, \mathfrak{G}_{\varepsilon'}]v = 0$  for all  $\varepsilon, \varepsilon' \in \hat{\Delta}_1^+$ . It follows from the proof of Lemma 5.3(i) that there exists a nonzero element  $v' \in V$  which is  $S$ -primitive for  $S = \pi \setminus \{\alpha_n\}$ .

*Case 2.* Suppose  $2\alpha_n \notin \Omega_v$ ,  $\alpha_0 = -2\sum_{i=1}^n \alpha_i + \delta \notin \Omega_v$  and  $\alpha_i \in \Omega_v$  for some  $1 \leq i \leq n - 1$ . Then in particular all roots from  $\hat{\Delta}_0^+$  containing  $\alpha_i$  belong to  $\Omega_v$ . The same arguments as in the proof of Lemma 5.3(i) show that there exists a nonzero element  $v' \in V$  such that  $\Omega_v \subset \Omega_{v'}$  and  $\Sigma_\beta \subset \Omega_{v'}$  for all  $\beta \in \hat{\Delta}_1^+ \setminus \{\alpha_n, \alpha_n + \alpha_{n-1}, \dots, \alpha_n + \dots + \alpha_{i+1}\}$ . Hence  $v'$  is  $S$ -primitive for  $S = \pi \setminus \{\alpha_i\}$ .

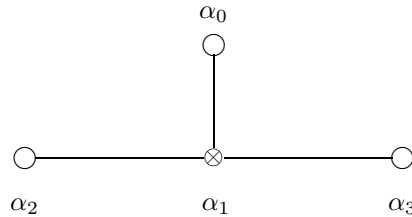
All other cases can be reduced to the above by the change of a basis of  $\Delta$ . This completes the proof of Proposition 5.1 for  $B(0, n)$ .

**5.5. Case  $D(m, n)$ .** Let  $\mathfrak{G}$  be of type  $D(m, n)$ ,  $m \geq 2$ ,  $\pi = \{\alpha_0, \dots, \alpha_n, \beta_1, \dots, \beta_m\}$ ,  $\hat{\pi}_0 = \{\alpha_1, \dots, \alpha_{n-1}, \beta_1, \dots, \beta_m\}$ ,  $\hat{\pi}_1 = \{\alpha_n\}$ ,  $\delta = \alpha_0 + 2\sum_{i=1}^n \alpha_i + 2\sum_{j=1}^{m-2} \beta_j + \beta_{m-1} + \beta_m$ .



Since  $\mathcal{B}_0 \simeq C_n \oplus D_m$  we have that  $\mathcal{B} \simeq \mathfrak{A}_1 + \mathfrak{A}_2$ , where  $\mathfrak{A}_1$  is an affine Lie algebra of type  $C_n^{(1)}$ , while  $\mathfrak{A}_2$  is an affine Lie algebra of type  $D_m^{(1)}$  for  $m > 3$ , of type  $A_3^{(1)}$  for  $m = 3$  and of type  $A_2^{(1)}$  for  $m = 2$ . Denote  $\gamma_1 = 2\alpha_n + 2\sum_{j=1}^{m-2} \beta_j + \beta_{m-1} + \beta_m$  and  $\gamma_2 = -\beta_1 - 2\sum_{j=2}^{m-2} \beta_j - \beta_{m-1} - \beta_m + \delta$ . Then  $\{\alpha_0, \alpha_1, \dots, \alpha_{n-1}, \gamma_1\}$  is a basis of the root system of  $\mathfrak{A}_1$  and  $\{\beta_1, \dots, \beta_m, \gamma_2\}$  is a basis of the root system of  $\mathfrak{A}_2$ . The proof of Proposition 5.1 in this case is fully analogous to the case of  $B(m, n)$ , and we leave out the details.

**5.6. Case  $D(2, 1; a)$ .** Let  $\mathfrak{G}$  be of type  $D(2, 1; a)$ ,  $\pi = \{\alpha_0, \dots, \alpha_3\}$ ,  $\hat{\pi}_0 = \{\alpha_2, \alpha_3\}$ ,  $\hat{\pi}_1 = \{\alpha_1\}$ ,  $\hat{\Delta}_0^+ = \{\alpha_2, \alpha_3, \alpha_2 + 2\alpha_1 + \alpha_3\}$ ,  $\hat{\Delta}_1^+ = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$ .



Note that  $\mathcal{B}_0 \simeq \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ . Consider the following cases.

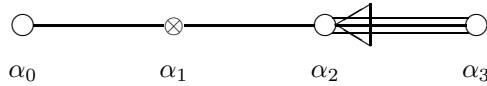
*Case 1.* Suppose that  $\alpha_2, \alpha_3$  and  $\alpha_2 + 2\alpha_1 + \alpha_3$  are not in  $\Omega_v$ . Then by Lemma 5.3(i) there exists a nonzero element  $v' \in V$  which is  $(\beta, \pi)$ -admissible for all  $\beta \in \hat{\Delta}_1^+$ . Therefore  $v'$  is  $S$ -primitive for  $S = \pi \setminus \{\alpha_0\}$ .

*Case 2.* Suppose now that  $\alpha_2, \alpha_3, \alpha_0 \notin \Omega_v$ , where  $\alpha_0 = -\alpha_2 - 2\alpha_1 - \alpha_3 + \delta$ . Then by Lemma 5.3(i) there exists a nonzero element  $v' \in V$  which is  $\pi$ -admissible and

$(\beta, \pi)$ -admissible for all  $\beta \in \dot{\Delta}_1^+$ . Denote the subspace of all such elements by  $V'$ . Consider  $w = X_{\alpha_1}v'$ . Suppose  $w \neq 0$ . Then  $w \in V'$  and  $X_{\alpha_1}w = 0$ . Next consider  $\alpha \in \dot{\Delta}_1^+$  of the least height such that  $X_\alpha w \neq 0$  and repeat the argument above. Note that  $X_\alpha w \in V'$  and  $X_\beta X_\alpha w = 0$  for any  $\beta \in \dot{\Delta}_1^+$ . Eventually we will find a nonzero  $\tilde{v} \in V'$  such that  $\dot{\Delta}_1^+ \subset \Omega_{\tilde{v}}$ , and hence  $\tilde{v}$  is  $S$ -primitive for  $S = \pi \setminus \{\alpha_1\}$ .

All other cases can be reduced to Case 1 or Case 2 by a change of basis of  $\Delta$ . This completes the proof of Proposition 5.1 for  $D(2, 1; a)$ .

5.7. **Case  $G(3)$ .** Suppose that  $\mathfrak{G}$  is of type  $G(3)$  with Dynkin diagram



Let  $\pi = \{\alpha_0, \dots, \alpha_3\}$ ,  $\dot{\pi}_0 = \{\alpha_2, \alpha_3\}$ ,  $\dot{\pi}_1 = \{\alpha_1\}$ ,  $\dot{\Delta}_0^+ = \{\alpha_2, \alpha_3, \alpha_2 + \alpha_3, 2\alpha_2 + \alpha_3, 3\alpha_2 + \alpha_3, 3\alpha_2 + 2\alpha_3, 2\alpha_1 + 4\alpha_2 + 2\alpha_3\}$ ,  $\dot{\Delta}_1^+ = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + 3\alpha_2 + \alpha_3, \alpha_1 + 3\alpha_2 + 2\alpha_3, \alpha_1 + 4\alpha_2 + 2\alpha_3\}$ . In this case  $\mathcal{B}_0 \simeq \mathfrak{sl}(2) \oplus \mathbf{G}_2$ .

Denote  $\beta = \alpha_0 + 2\alpha_1 + \alpha_2 = -3\alpha_2 - 2\alpha_3 + \delta$ . Then  $\beta, \alpha_2, \alpha_3$  is a basis of the root system of the affinization of  $\mathbf{G}_2$ . Consider the following cases.

*Case 1.* Suppose that  $\beta \in \Omega_v$  and  $-\alpha_0 + \delta \in \Omega_v$ . Then  $[\mathfrak{G}_\varepsilon, \mathfrak{G}_{\varepsilon'}]v = 0$  for all  $\varepsilon, \varepsilon' \in \dot{\Delta}_1^+$ . It is not difficult to modify the proof of Lemma 5.3(i) to show the existence of a nonzero element  $v' \in V$  which is  $S$ -primitive for  $S = \pi \setminus \{\alpha_1\}$ .

*Case 2.* Suppose now that  $\beta \in \Omega_v$  but  $-\alpha_0 + \delta \notin \Omega_v$ . Then in particular  $v$  is  $(\beta, \pi)$ -admissible for all  $\beta \in \dot{\Delta}_0^+$ . Again using the same arguments as in the proof of Lemma 5.3(i) we can find a nonzero element  $v' \in V$  which is  $(\beta, \pi)$ -admissible for all  $\beta \in \dot{\Delta} \cap \Delta_+$ . Hence  $v'$  is  $S$ -primitive, where  $S = \pi \setminus \{\alpha_0\}$ .

*Case 3.* Suppose that  $\alpha_2 \in \Omega_v$  and  $-\alpha_0 + \delta \in \Omega_v$ . Denote  $Y = \{\alpha_0, \alpha_3, \alpha_0 + 2\alpha_1 + \alpha_2, \alpha_0 + 2\alpha_1 + \alpha_2 + \alpha_3\}$ . Then  $\Omega_v$  contains all positive even roots except those in  $Y$ . Let  $X$  be a subspace of  $V^+$  of all elements  $w$  such that  $(\Delta_0^+ \setminus Y) \subset \Omega_w$ . Using the same algorithm as in the proof of Lemma 5.3 we can find a nonzero element  $v_0 \in X$  which is  $(\varepsilon, \pi)$ -admissible for all  $\varepsilon \in \dot{\Delta}_1^+$ . Finally, consider the following odd roots  $\beta_1 = \alpha_1 + 4\alpha_2 + 2\alpha_3$ ,  $\beta_2 = \alpha_1 + 3\alpha_2 + 2\alpha_3$ ,  $\beta_3 = \alpha_1 + 3\alpha_2 + \alpha_3$ ,  $\beta_4 = \alpha_1 + 2\alpha_2 + \alpha_3$ ,  $\beta_5 = -\alpha_1$ ,  $\beta_6 = \alpha_1 + \alpha_2$ ,  $\beta_7 = \alpha_1 + \alpha_2 + \alpha_3$ ,  $\beta_8 = -\alpha_0 - \alpha_1$  and define nonzero elements  $v_i$  by:  $v_i = v_{i-1}$  if  $X_{\beta_i}v_{i-1} = 0$  and  $v_i = X_{\beta_i}v_{i-1}$  otherwise,  $i = 1, \dots, 8$ . Then  $v_8 \neq 0$  is a  $\mathcal{P}$ -primitive element, where  $\mathcal{P}^0 = -Y \cup Y$  and

$$\mathcal{P}^+ = (\Delta_+ \setminus (Y \cup \{\alpha_1, \alpha_0 + \alpha_1\})) \cup \{-\alpha_1, -\alpha_0 - \alpha_1\}.$$

*Case 4.* Suppose that  $\alpha_2 \in \Omega_v$  and  $-\alpha_0 + \delta \notin \Omega_v$ . Denote  $Y = \{2\alpha_1 + 4\alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_3, \alpha_0 + 2\alpha_1 + \alpha_2, \alpha_0 + 2\alpha_1 + \alpha_2 + \alpha_3\}$ . As in the previous case we find a nonzero  $\mathcal{P}$ -primitive element, where  $\mathcal{P}^0 = -Y \cup Y$  and

$$\mathcal{P}^+ = (\Delta_+ \setminus (Y \cup \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3\})) \cup \{-\alpha_1, -\alpha_1 - \alpha_2, -\alpha_1 - \alpha_2 - \alpha_3\}.$$

*Case 5.* Suppose that  $\alpha_3 \in \Omega_v$  and  $-\alpha_0 + \delta \in \Omega_v$ . In this case there exists a nonzero  $S$ -primitive element, where  $S = \pi \setminus \{\alpha_3\}$ .

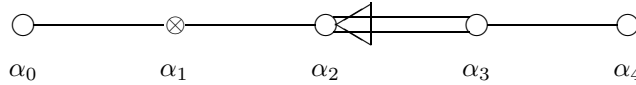
*Case 6.* Suppose that  $\alpha_3 \in \Omega_v$  and  $-\alpha_0 + \delta \notin \Omega_v$ . Denote  $Y = \{2\alpha_1 + 4\alpha_2 + 2\alpha_3, \alpha_2, \alpha_0 + 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + 3\alpha_2 + \alpha_3\}$ . In this case

there exists a nonzero  $\mathcal{P}$ -primitive element, where  $\mathcal{P}^0 = -Y \cup Y$  and

$$\mathcal{P}^+ = (\Delta_+ \setminus (Y \cup \{\alpha_1, \alpha_1 + \alpha_2\})) \cup \{-\alpha_1, -\alpha_1 - \alpha_2\}.$$

This completes the proof of Proposition 5.1 for  $G(3)$ .

5.8. **Case  $F(4)$ .** Let  $\mathfrak{G}$  be of type  $F(4)$  with Dynkin diagram



Let  $\pi = \{\alpha_0, \dots, \alpha_4\}$ ,  $\dot{\pi}_0 = \{\alpha_2, \alpha_3, \alpha_4\}$ ,  $\dot{\pi}_1 = \{\alpha_1\}$ ,  $\dot{\Delta}_0^+ = \{\alpha_2, \alpha_3, \alpha_4, \alpha_2 + \alpha_3, 2\alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, 2\alpha_2 + \alpha_3 + \alpha_4, 2\alpha_2 + 2\alpha_3 + \alpha_4, 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4\}$ ,  $\dot{\Delta}_1^+ = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4\}$ . In this case  $\mathcal{B}_0 \simeq \mathfrak{sl}(2) \oplus B_3$ .

Let  $\beta = \alpha_0 + 2\alpha_1 + \alpha_2$ . Then  $\{\alpha_0, -\alpha_0 + \delta\} \cup \{\alpha_2, \alpha_3, \beta\}$  is a basis of  $\mathcal{B}$  and at least one element from each set is in  $\Omega_v$ . Hence we have the following cases.

*Case 1.* If  $-\alpha_0 + \delta \in \Omega_v$  and  $\beta \in \Omega_v$ , then  $[\mathfrak{G}_\varepsilon, \mathfrak{G}_{\varepsilon'}]v = 0$  for all  $\varepsilon, \varepsilon' \in \dot{\Delta}_1^+$ . It follows from the proof of Lemma 5.3(i) that there exists a nonzero element  $v' \in V$  which is  $S$ -primitive for  $S = \pi \setminus \{\alpha_1\}$ .

*Case 2.* If  $\alpha_0 \in \Omega_v$  and  $\beta \in \Omega_v$ , then  $v$  is  $(\beta, \pi)$ -admissible for all  $\beta \in \dot{\Delta}_0^+$ . The proof of Lemma 5.3(i) implies in this case that there exists a nonzero element  $v' \in V$  which is  $S$ -primitive for  $S = \pi \setminus \{\alpha_0\}$ .

*Case 3.* Let  $-\alpha_0 + \delta \in \Omega_v$  and  $\alpha_3 \in \Omega_v$ . Again, as in the proof of Lemma 5.3(i) we can find a nonzero element  $v' \in V$  which is  $S$ -primitive for  $S = \pi \setminus \{\alpha_3\}$ .

*Case 4.* Let  $-\alpha_0 + \delta \in \Omega_v$  and  $\alpha_2 \in \Omega_v$ . Denote  $Y = \{\alpha_3, \alpha_4, \alpha_3 + \alpha_4, \beta, \beta + \alpha_3, \beta + \alpha_3 + \alpha_4\}$ . In this case there exists a nonzero  $\mathcal{P}$ -primitive element, where  $\mathcal{P}^0 = -Y \cup Y$  and  $\mathcal{P}^+ = (\Delta_+ \setminus (Y \cup \{\alpha_1, -\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4 + \delta\})) \cup \{-\alpha_1, \alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 - \delta\}$ .

*Case 5.* Let  $\alpha_0 \in \Omega_v$  and  $\alpha_3 \in \Omega_v$ . Denote  $Y = \{\alpha_2, \alpha_4, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4, -2\alpha_2 - 2\alpha_3 - \alpha_4 + \delta, 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4\}$ . In this case there exists a nonzero  $\mathcal{P}$ -primitive element, where  $\mathcal{P}^0 = -Y \cup Y$  and  $\mathcal{P}^+ = (\Delta_+ \setminus (Y \cup \{\alpha_1, \alpha_1 + \alpha_2\})) \cup \{-\alpha_1, -\alpha_1 - \alpha_2\}$ .

*Case 6.* Let  $\alpha_0 \in \Omega_v$  and  $\alpha_2 \in \Omega_v$ . Denote  $Y = \{\alpha_3, \alpha_4, \alpha_3 + \alpha_4, \alpha_0 + 2\alpha_1 + \alpha_2, \alpha_0 + 2\alpha_1 + \alpha_2 + \alpha_3, \alpha_0 + 2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4\}$ . In this case there exists a nonzero  $\mathcal{P}$ -primitive element, where  $\mathcal{P}^0 = -Y \cup Y$  and  $\mathcal{P}^+ = (\Delta_+ \setminus (Y \cup \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\})) \cup \{-\alpha_1, -\alpha_1 - \alpha_2, -\alpha_1 - \alpha_2 - \alpha_3, -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4\}$ . This completes the proof of Proposition 5.1 for  $F(4)$ .

**Corollary 5.6.** *Let  $V$  be a nonzero  $\mathfrak{G}$ -module in  $K(\mathfrak{G})$ . Then  $V$  contains a  $\mathcal{P}$ -primitive element for some proper parabolic subset  $\mathcal{P} \subset \Delta$  with finite-dimensional  $\mathfrak{G}_{\mathcal{P}}^0$ .*

*Proof.* Follows immediately from the proof of Proposition 5.1. □

6. IRREDUCIBLE MODULES IN  $K(\mathfrak{G})$ 

**Proof of Theorem A.** Note that the category  $K(\mathfrak{G})$  is closed under taking submodules and quotients. Let  $V$  be an irreducible module in  $K(\mathfrak{G})$ . Then  $V$  contains a  $\mathcal{P}$ -primitive element for some parabolic subset  $\mathcal{P} \subset \Delta$  by Proposition 5.1. Therefore  $V \simeq L_{\mathcal{P}}(N')$  by Proposition 4.1 for some irreducible module  $N'$  over a Lie subsuperalgebra  $\mathfrak{G}_{\mathcal{P}}^0$ . Moreover,  $\mathfrak{G}_{\mathcal{P}}^0$  is finite-dimensional by Corollary 5.6. Hence  $N'$  is parabolically induced from a cuspidal module by [5], Theorem 6.1. That is, there exist a parabolic subset  $\mathcal{P}'$  of the root system of  $\mathbf{B} = \mathfrak{G}_{\mathcal{P}}^0$  and an irreducible cuspidal  $\mathbf{B}$ -module  $N$  such that  $N'$  is a quotient of  $\text{ind}(\mathbf{B}_{\mathcal{P}'}^0 \oplus \mathbf{B}_{\mathcal{P}'}^+, \mathbf{B}; N)$ . Combining the parabolic subsets  $\mathcal{P}$  and  $\mathcal{P}'$  we obtain a parabolic subset  $\tilde{\mathcal{P}}$  of  $\Delta$  such that  $\mathfrak{G}_{\tilde{\mathcal{P}}}^0 = \mathbf{B}_{\mathcal{P}'}^0$ ,  $\mathfrak{G}_{\tilde{\mathcal{P}}}^{\pm} = \mathbf{B}_{\mathcal{P}'}^{\pm} \oplus \mathfrak{G}_{\mathcal{P}}^{\pm}$ . Hence  $V \simeq L_{\tilde{\mathcal{P}}}(N)$ , a unique irreducible quotient of  $\text{ind}(\mathfrak{G}_{\tilde{\mathcal{P}}}^0, \mathfrak{G}; N)$ . This completes the proof of Theorem A.

Theorem A reduces the classification of irreducible modules in  $K(\mathfrak{G})$  to the classification of irreducible cuspidal modules over cuspidal Lie subsuperalgebras.

**Proof of Theorem B.** Let  $V$  be an irreducible module in  $K(\mathfrak{G})$ . Then  $V$  is isomorphic to  $L_{\mathcal{P}}(M)$  by Theorem A, where  $\mathcal{P}$  is a parabolic subset of  $\Delta$  and  $M$  is an irreducible  $\mathfrak{G}_{\mathcal{P}}^0$ -module.

If  $\mathfrak{G}$  is of type  $A(m, n)$ , then it follows from the proof of Proposition 5.1 that  $V \simeq L^{\pi_0}(M)$ , where  $M$  is an irreducible  $\mathfrak{g}_0$ -module. Since  $\mathfrak{g}_0$  is finite-dimensional reductive Lie algebra, then  $M \simeq L_{\mathfrak{g}_0}^S(N)$  for some subset  $S \subset \pi_0$  and some irreducible cuspidal  $\mathfrak{g}(S)$ -module  $N$  by [8], implying (1).

Suppose that  $\mathfrak{G}$  is of type  $C(n)$ . It follows from the proof of Proposition 5.1 that  $V$  contains an  $S$ -primitive element of weight  $\lambda$  where either  $S = \pi_0$  or  $S = \pi \setminus \{\alpha_k\}$  for some  $2 \leq k \leq n-1$ . In the first case  $V \simeq L^{\pi_0}(M)$ , where  $M$  is an irreducible weight module over  $\mathfrak{g}_0 \simeq \mathfrak{sp}(2n-2) \oplus \mathbb{C}$ . Hence, by [8], there exists  $S \subset \pi_0$  and an irreducible cuspidal  $\mathfrak{g}(S)$ -module  $N$  such that  $M \simeq L_{\mathfrak{g}_0}^S(N)$  and  $V \simeq L^S(N)$ . Now consider the case  $S = \pi \setminus \{\alpha_k\}$ . By Proposition 2.1,  $V \simeq L^S(M)$ , where  $M = \sum_{\nu \in Q_S} V_{\nu+\lambda}$  is an irreducible module over a finite-dimensional Lie subsuperalgebra  $\mathfrak{G}(S)$ . The superalgebra  $\mathfrak{G}(S)$  is a direct sum of  $G_1$  and  $G_2$ , where  $G_1 \simeq \mathfrak{osp}(2, 2k-2)$  is a Lie superalgebra generated by  $\mathfrak{G}_{\beta}$ ,  $\beta \in \{\alpha_0, \alpha_1, \dots, \alpha_{k-1}\}$  and  $G_2 \simeq \mathfrak{sp}(2n-2k-2)$  is a Lie algebra generated by  $\mathfrak{G}_{\beta}$ ,  $\beta \in \{\alpha_{k+1}, \dots, \alpha_n\}$ . Then  $M$  is isomorphic to a tensor product  $N_1 \otimes N_2$ , where  $N_i$  is an irreducible  $G_i$ -module,  $i = 1, 2$ . Module  $N_2$  is parabolically induced from a cuspidal module over a subalgebra of  $G_2$ . Let  $\tilde{\pi}$  be a basis of the root system of  $G_1$ . Then  $N_1$  contains a  $\tilde{\pi}_0$ -primitive element by Lemma 5.3(ii), and therefore  $N_1$  is parabolically induced from a cuspidal module over a subalgebra of the even part of  $G_1$ , implying (2).

Let  $\mathfrak{G}$  be of type  $B(m, n)$  ( $m > 0$ ). By Proposition 5.1, one can choose  $\mathcal{P}$  in such a way that  $\mathfrak{G}_{\mathcal{P}}^0$  is isomorphic to one of the (super)algebras of the following types:  $C_n \oplus B_m$ ,  $D(2, n) \oplus B_{m-2}$ ,  $D(3, n) \oplus B_{m-3}$ ,  $C_k \oplus B(m, n-k)$ ,  $k = 0, \dots, n-1$ ,  $C_i \oplus C_{n-i} \oplus A_1 \oplus A_1 \oplus B_{m-2}$ ,  $C_i \oplus C_{n-i} \oplus A_2 \oplus B_{m-3}$ ,  $C(n+1) \oplus B_{m-1}$  and  $C_i \oplus C(n-i) \oplus B_m$ . The only simple cuspidal subalgebras of  $B_m$  are those of types  $A$  and  $C_2$  by [8]. The superalgebras  $D(2, n)$ ,  $D(3, n)$  and  $B(m, n-k)$  with  $m = 1, 2$  are cuspidal by [5], 7.3. Note that  $\mathfrak{C} = B(m, n-k)$  is not cuspidal if  $m > 2$ . The reason is that in this case  $\mathfrak{C}$  contains a subalgebra of type  $B_m$  which is not cuspidal, and a subgroup generated by the even roots of  $\mathfrak{C}$  has a finite index in the root lattice. Also note that  $C(k)$  is not cuspidal by [6], 7.3 (also by (2)). Hence statement (3) follows.



Suppose that  $\mathfrak{G}$  is of type  $B(0, n)$ . Then  $V$  contains an  $S$ -primitive element, where  $S = \pi \setminus \{\alpha\}$ ,  $\alpha \in \pi$ , by the proof of Proposition 5.1. If  $\alpha \in \pi_0$ , then  $\mathfrak{G}(S) \simeq \mathfrak{sp}(2i) \oplus \mathfrak{osp}(1, 2(n-i))$ , where  $\alpha = \alpha_i$ ,  $i \in \{0, \dots, n-1\}$ . If  $\alpha$  is an odd root, then  $\mathfrak{G}(S) \simeq \mathfrak{sp}(2n)$ . Since  $\mathfrak{osp}(1, 2(n-i))$  is cuspidal by [5], 7.3, and any semisimple subalgebra of  $\mathfrak{sp}(2n)$  is cuspidal by [8], then statement (4) follows.

The proof of (5) is similar to the proof of (3). Just note that  $\mathfrak{osp}(2m, 2l)$  is cuspidal only if  $m = 2$  or  $m = 3$ , and  $so(2k)$  is cuspidal only when  $k \leq 3$ .

The statement (6) is immediate.

If  $\mathfrak{G}$  is of type  $G(3)$ , then it follows from the proof of Proposition 5.1 that  $\mathcal{P}$  can be chosen in such a way that  $\mathfrak{G}_{\mathcal{P}}^0$  is isomorphic to one of the following:

$$\mathfrak{sl}(2) \oplus \mathbf{G}_2, \mathfrak{sl}(2) \oplus \mathfrak{sl}(3), \mathfrak{sl}(3) \oplus \mathfrak{osp}(1, 2), \mathfrak{osp}(4, 2), \mathfrak{sl}(2) \oplus \mathfrak{osp}(3, 2), G(3).$$

Simple cuspidal subsuperalgebras of  $G(3)$  are isomorphic to  $\mathfrak{sl}(2)$ ,  $\mathfrak{osp}(1, 2)$  and  $\mathfrak{osp}(3, 2)$  by [5], 7.7, and [6], while any cuspidal subalgebra of  $\mathbf{G}_2$  is isomorphic to  $\mathfrak{sl}(2)$  by [8]. Since  $\mathfrak{osp}(4, 2)$  is cuspidal by [5], 7.8, statement (7) follows.

Let  $\mathfrak{G}$  be of type  $F(4)$ . It follows from the proof of Proposition 5.1 that  $\mathcal{P}$  can be chosen such that  $\mathfrak{G}_{\mathcal{P}}^0$  is isomorphic to one of the following:

$$\mathfrak{sl}(2) \oplus \mathfrak{so}(7), \mathfrak{sl}(2) \oplus \mathfrak{osp}(4, 2), \mathfrak{sl}(2) \oplus \mathfrak{sl}(4), F(4).$$

Hence statement (8) follows from [8] and [5], 7.8. This completes the proof of Theorem B.

From Theorem B we immediately obtain a classification of all cuspidal Levi subsuperalgebras.

**Corollary 6.1.** *Any simple cuspidal Levi subsuperalgebra of an affine Lie superalgebra is isomorphic to one of the following:*

- (1) Simple Lie subalgebra of type  $A$  or  $C$ ;
- (2)  $\mathfrak{osp}(m, 2n)$ ,  $m = 1, 3, 4, 5, 6$ ;
- (3)  $D(2, 1; a)$ .

Let  $\mathfrak{A}$  be a Lie (super)algebra with a fixed Cartan subalgebra and the corresponding root decomposition. Given a module  $M$  over  $\mathfrak{A}$  we denote by  $injM$  the set of all roots  $\alpha$  for which there exists a root element in  $\mathfrak{A}_{\alpha}$  that acts injectively on  $M$ .

**Corollary 6.2.** *Let  $V \simeq L_{\mathcal{P}}(N)$  be an irreducible module in  $K(\mathfrak{G})$ , where  $\mathcal{P}$  is a parabolic subset of  $\Delta$  and  $N$  is a cuspidal  $\mathfrak{G}_{\mathcal{P}}^0$ -module. Then either  $\mathcal{P}^0 = \emptyset$  or  $injN$  contains an even root of  $\mathfrak{G}$ .*

*Proof.* Suppose  $\mathcal{P}^0 \neq \emptyset$ . If  $\mathcal{P}^0$  contains a root  $\alpha \in \pi_0$ , then  $\alpha \in injV$  by definition. Let  $\mathcal{P}^0 \cap \pi_0 = \emptyset$ . Then  $\mathcal{P}^0$  contains an odd root  $\beta$  and  $2\beta \in \Delta_0$ . If  $2\beta \notin injN$ , then  $N$  contains a nonzero element  $v$  such that  $\mathfrak{G}_{2\beta}v = 0$ . Then  $v$  is a  $\mathcal{P}$ -primitive element with  $\mathcal{P}^+ \subsetneq \mathcal{P}^+$ , which is a contradiction.  $\square$

*Remark 6.3.* (1) Combining Theorem B with Mathieu’s classification of cuspidal modules over simple Lie algebras of types  $A$  and  $C$  we obtain a complete classification of irreducible modules in  $K(\mathfrak{G})$  when  $\mathfrak{G}$  is of type  $A(m, n)$  and  $C(n)$ . Note that our reduction in these cases is fully independent of the results of [5].

- (2) If  $\mathfrak{G}$  is not of types  $A(m, n)$  and  $C(n)$ , then our reduction relies on the classification of cuspidal Levi subalgebras in [5]. In this case the classification of irreducible modules in  $K(\mathfrak{G})$  is reduced to a finite indeterminacy by [5], Proposition 6.3.
- (3) In cases  $C(n)$  and  $G(3)$  the irreducible modules in  $K(\mathfrak{G})$  are parabolically induced but not always isomorphic to modules of type  $L^S(N)$ .
- (4) Classification of noncuspidal irreducible modules in  $K(\mathfrak{G})$  for  $D(2, 1; a)$  is based on the classification of cuspidal  $\mathfrak{sl}(2)$ -modules and is, therefore, complete.
- (5) Classification of all irreducible modules over  $\mathfrak{osp}(1, 2)$  was obtained in [1].
- (6) Classification of irreducible modules in  $K(\mathfrak{G})$  will be complete after the classification of cuspidal modules over  $\mathfrak{osp}(m, 2n)$ ,  $m = 1, 3, 4, 5, 6$ , and  $D(2, 1; a)$ .
- (7) Cuspidal Lie subalgebras described in Corollary 6.1 coincide with cuspidal superalgebras in the finite-dimensional setting which were classified in [6].

## 7. INTEGRABLE MODULES OVER AFFINE KAC-MOODY ALGEBRAS

Let  $\mathfrak{G}$  be an affine Lie algebra,  $\Delta$  its root system,  $\mathfrak{g}$  the underlying finite-dimensional simple Lie subalgebra of  $\mathfrak{G}$ , and  $\dot{\Delta}$  the root system of  $\mathfrak{g}$ . A weight  $\mathfrak{G}$ -module is *integrable* if all  $\mathfrak{G}_\alpha$ ,  $\alpha \in \Delta^{\text{re}}$ , are locally finite on  $V$ . We will now recover the classification of irreducible integrable modules in  $K(\mathfrak{G})$  which is due to Chari.

**Proposition 7.1.** (i) *If  $\mathfrak{g}$  is locally finite on an irreducible module  $V$  in  $K(\mathfrak{G})$ , then  $V$  is a highest weight module.*

(ii) ([3]) *Any irreducible integrable module in  $K(\mathfrak{G})$  is a highest weight module.*

*Proof.* Let  $V$  be an irreducible module in  $K(\mathfrak{G})$  on which  $\mathfrak{g}$  is locally finite. By Theorem 4.4(ii),  $V \simeq L_{\mathfrak{G}}^S(N)$  for some basis  $\pi$  of  $\Delta$ , a proper subset  $S \subset \pi$  and an irreducible cuspidal  $\mathfrak{G}(S)$ -module  $N$ . Suppose  $S \neq \emptyset$ ; hence  $V$  is not a highest weight module. Since  $\mathfrak{g}$  is locally finite then  $S \cap \dot{\pi} = \emptyset$ , where  $\dot{\pi}$  is a basis of  $\dot{\Delta}$ . Suppose that  $S$  contains a root  $\alpha - m\delta$  where  $\alpha \in \dot{\Delta}$ ,  $m \in \mathbb{Z}$ . Let  $\mathcal{B}(\alpha - m\delta) \simeq \mathfrak{sl}(2)$  be the subalgebra of  $\mathfrak{G}$  generated by  $\mathfrak{G}_{\pm(\alpha - m\delta)}$ . Fix a nonzero weight vector  $v \in N$ . Then  $U(\mathcal{B}(\alpha - m\delta))v \subset N$  is a torsion free  $\mathcal{B}(\alpha - m\delta)$ -module. Without loss of generality we can assume that  $\mathfrak{G}_\alpha v = 0$  and  $\mathfrak{G}_{k\delta} v = 0$  for all  $k > 0$ . Under such an assumption  $m$  must be positive. Now consider a subalgebra  $\mathcal{B}(\alpha) \simeq \mathfrak{sl}(2)$  of  $\mathfrak{G}$  generated by  $\mathfrak{G}_{\pm\alpha}$ . Let  $V_1 = U(\mathcal{B}(\alpha))v$ . Then  $V_1$  is a finite-dimensional  $\mathfrak{sl}(2)$ -module. Assume  $\dim V_1 = r$ . Choose a basis element  $X_{\alpha+k\delta}$  in each space  $\mathfrak{G}_{\pm(\alpha+k\delta)}$ ,  $k \in \mathbb{Z}$ , and consider  $w_1 = X_{-\alpha+m\delta}v$ . Let  $V_2 = U(\mathcal{B}(\alpha))w_1$ . Note that  $\mathfrak{G}_{k\delta}w_1 = \mathfrak{G}_\alpha w_1 = 0$  for all  $k > 0$  due to the induced structure of  $V$ . Assume now that  $\mathfrak{G}$  is not of type  $A_{2n}^{(2)}$ . Then  $[X_{-\alpha}, X_{-\alpha+m\delta}] = 0$  and hence  $X_{-\alpha}^r w_1 = 0$ . Therefore,  $V_2$  is a finite-dimensional module of dimension  $\leq r$ . Applying the same arguments to nonzero elements  $w_i = X_{-\alpha+m\delta}^i v$ ,  $i = 1, \dots, r+1$ , we construct  $r+1$  nonzero modules of dimension  $\leq r$  over  $\mathfrak{sl}(2)$ . Since all these module have different highest weights as  $\mathfrak{sl}(2)$ -modules, we obtain a contradiction.

Now consider the case when  $\mathfrak{G}$  is of type  $A_{2n}^{(2)}$ . In this case we have

$$[X_{-\alpha}, [X_{-\alpha}, X_{-\alpha+m\delta}]] = 0.$$

Hence  $X_{-\alpha}^{r+1}w_1 = 0$ . Therefore,  $V_2$  is a finite-dimensional module of dimension  $\leq r + 1$ . But the difference of highest weights of  $V_1$  and  $V_2$  is  $\alpha$ , thus  $V_2$  cannot have dimension  $r + 1$ . It has dimension  $\leq r$ , and the same arguments as above lead to a contradiction. Thus (i) follows. Statement (ii) is an immediate corollary of (i).  $\square$

8. WEAKLY INTEGRABLE MODULES IN  $K(\mathfrak{G})$

Let  $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1$  be a nontwisted affine Lie superalgebra and  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$  the underlying basic classical Lie superalgebra. Also let  $\mathfrak{g}_0 = \sum_{j=0}^s \mathfrak{g}_{0j}$ , where  $\mathfrak{g}_{00}$  is abelian and  $\mathfrak{g}_{0i}$ ,  $i = 1, \dots, s$ , are simple Lie algebras ( $s \leq 3$ ). Following [14] we define a *weakly integrable*  $\mathfrak{G}$ -module  $V$  as a module which is integrable over the affinization  $\mathfrak{G}_{0j}$  of  $\mathfrak{g}_{0j}$  for some  $j \geq 1$ , and on which  $\mathfrak{g}_0$  is locally finite. Of course, for  $C(n)$  a weak integrability coincides with the usual integrability since  $s = 1$  in this case. Theorem C states that the only weakly integrable irreducible modules in  $K(\mathfrak{G})$  are highest weight modules which is a generalization of Proposition 7.1 for affine Lie superalgebras.

**Theorem 8.1.** *Let  $V$  be an irreducible module in  $K(\mathfrak{G})$  such that  $\mathfrak{g}_0$  is locally finite on  $V$ . Then  $V$  is a highest weight module.*

*Proof.* By Theorem A there exists a parabolic subset  $\mathcal{P} \subset \Delta$  and an irreducible cuspidal  $\mathfrak{G}_{\mathcal{P}}^0$ -module  $N$  such that  $V \simeq L_{\mathcal{P}}(N)$ . Since  $\mathfrak{g}_0$  is locally finite on  $V$ , then  $\mathfrak{G}_{\mathcal{P}}^0 \neq \mathfrak{G}$ . Suppose first that  $\mathcal{P}^0 \neq \emptyset$ . Since  $N$  is cuspidal, then by Corollary 6.2 there exists an even root  $\alpha$  such that  $N$  is cuspidal over a subalgebra  $\mathcal{B}(\alpha) \simeq \mathfrak{sl}(2)$ , generated by  $\mathfrak{G}_{\pm\alpha}$ . If  $\alpha \in \dot{\Delta}_0$ , then  $\mathfrak{g}_0$  is not locally finite on  $N$ , and hence on  $V$ , which is a contradiction. Suppose now that  $\alpha = \phi - m\delta$  for some  $\phi \in \dot{\Delta}_0$  and  $m \in \mathbb{Z}$ . Without loss of generality we may assume that  $\phi \in \mathcal{P}^+$  (changing  $\alpha$  to  $-\alpha$  if necessary) and  $\delta \in \mathcal{P}^+$ . It implies  $m > 0$ . As in the proof of Proposition 7.1 we can consider a subalgebra  $\mathcal{B}(\phi - m\delta) \simeq \mathfrak{sl}(2)$  generated by  $\mathfrak{G}_{\pm(\phi - m\delta)}$ . Let  $M = U(\mathcal{B}(\phi - m\delta))v$  for some nonzero weight element  $v \in N$  and  $\tilde{M} = U(\mathfrak{g}_0)M \subset V$ . Then  $\tilde{M}$  is a  $\mathfrak{G}(\phi)$ -module, where  $\mathfrak{G}(\phi) \simeq \mathfrak{sl}(2)$  is generated by  $\mathfrak{G}_{\pm\phi}$ . Applying the same argument as in the proof of Proposition 7.1 we conclude that the action of  $\mathfrak{G}_{-\phi} = \mathfrak{g}_{-\phi}$  is not locally finite on  $\tilde{M}$ , which is a contradiction. Therefore we can assume that  $\mathcal{P}^0 = \emptyset$ . Let  $v$  be a nonzero  $\mathcal{P}$ -primitive weight element of  $V$ . Fix  $j$  and let  $\mathfrak{G}_{0j}$  be the affinization of  $\mathfrak{g}_{0j}$ . Let  $\Delta_{0j}$  be the root system of  $\mathfrak{G}_{0j}$  and  $\Delta_{0j}^+ = \Delta_{0j} \cap \Delta_+$ . Consider a  $\mathfrak{G}_{0j} + \mathcal{H}$ -submodule  $V' = U(\mathfrak{G}_{0j})v$  of  $V$ . Since  $v$  is a  $\mathcal{P}$ -primitive element, then  $V'$  is a homomorphic image of the Verma type module  $\text{ind}(\mathcal{H} \oplus (\mathfrak{G}_{0j})_{\mathcal{P}}^+, \mathfrak{G}_{0j}; \mathbb{C})$  with a nonzero central charge, where  $(\mathfrak{G}_{0j})_{\mathcal{P}}^+ = \mathfrak{G}_{0j} \cap \mathfrak{G}_{\mathcal{P}}^+$ . Suppose that  $\delta \in \mathcal{P}$  and  $w\mathcal{P} \cap \Delta_{0j}^+ \neq \Delta_{0j}^+$  for any  $w$  in the Weyl group of  $\Delta_{0j}$ . Then  $V'$  has some infinite-dimensional weight spaces by [10], Theorem 5.14. But this is a contradiction since  $V'$  is an object of the category  $K(\mathfrak{G})$ . Therefore, there exists an element  $w$  of the Weyl group of  $\mathfrak{G}$  such that  $w\mathcal{P} \cap \Delta_{0j}^+ = \Delta_{0j}^+$  for all  $j$ . It follows immediately that  $\Sigma_{\beta} \subset \Omega_v$  for all  $\beta \in \dot{\Delta}_0^+(w^{-1}\pi)$ . Applying Lemma 5.3(iii), we obtain that  $V$  is a highest weight module. This completes the proof.  $\square$

Theorem C follows immediately from Theorem 8.1.

*Remark 8.2.* 1. Together with the Kac-Wakimoto classification of integrable irreducible highest weight modules, Theorem C gives a complete classification of weakly integrable irreducible modules in  $K(\mathfrak{G})$ .

2. Using [5], Theorem 6.1, one can show that Theorem 8.1 also holds for basic classical Lie superalgebras. Therefore any irreducible integrable  $\mathfrak{g}$ -module with finite-dimensional weight spaces is of highest weight.
3. If we relax the requirement of local finiteness of  $\mathfrak{g}_0$  in the definition of weak integrability, then we get nonhighest weight “integrable” modules in  $K(\mathfrak{G})$ . Such modules are partially integrable in the sense of Dimitrov and Penkov ([7]).

## 9. CONJECTURES

In conclusion we formulate several conjectures for affine Lie algebras.

Let  $\mathcal{B}$  be an affine Lie algebra with a Cartan subalgebra  $H$ . Let  $\mathfrak{k}(\mathcal{B})$  be the category of all weight  $\mathcal{B}$ -modules,  $\tilde{K}(\mathcal{B})$  the full subcategory of  $\mathfrak{k}(\mathcal{B})$  consisting of modules with finite-dimensional weight subspaces and  $K_0(\mathcal{B})$  the full subcategory of  $\tilde{K}(\mathcal{B})$  of modules of zero level.

A  $\mathcal{B}$ -module  $V \in \mathfrak{k}(\mathcal{B})$  is *dense* if the set of weights is a coset of  $H^*/Q$ . The following conjecture was formulated in [10]:

**Conjecture 1.** *An irreducible module  $V \in \mathfrak{k}(\mathcal{B})$  is dense if and only if it is cuspidal.*

This conjecture would reduce the classification problem to the classification of irreducible weight modules with infinite-dimensional weight spaces over the Heisenberg subalgebra of  $\mathcal{B}$  and irreducible dense modules over affine Lie subalgebras of  $\mathcal{B}$  (cf. [10]). Irreducible modules with infinite-dimensional weight spaces over the Heisenberg algebra necessarily have a nonzero level. Some examples were constructed in [2]. Families of dense  $\mathcal{B}$ -modules with infinite-dimensional weight spaces were constructed in [4]. These classification problems are still open.

Theorem 4.4(ii) confirms Conjecture 1 for the category  $K(\mathcal{B})$  and describes irreducible modules. On the other hand all irreducible modules in  $K_0(\mathcal{B})$  which are not irreducible over  $[\mathcal{B}, \mathcal{B}]$  were classified in [16]. Such modules are called *Loop* modules. We believe that the following holds:

**Conjecture 2.** *An irreducible module  $V \in K_0(\mathcal{B})$  is either a Loop module or is parabolically induced from a standard parabolic subalgebra of  $\mathcal{B}$  (cf. Section 4).*

Conjecture 2 together with Theorem 4.4(ii) would complete a classification of irreducible modules in  $\tilde{K}(\mathcal{B})$ .

It was shown in [9] that Conjecture 1 holds for modules over  $\mathfrak{sl}(2)$  with at least one finite-dimensional weight space. This suggests the following weaker version of Conjecture 1:

**Conjecture 3.** *An irreducible module  $V \in \mathfrak{k}(\mathcal{B})$  with at least one finite-dimensional weight space is dense if and only if it is cuspidal.*

Irreducible integrable modules in  $\mathfrak{k}(\mathcal{B})$  are quotients of a tensor product of a standard highest weight and a dual to standard modules by [4].

**Conjecture 4.** *An irreducible integrable module  $V \in \mathfrak{k}(\mathcal{B})$  of a nonzero level is of highest weight.*

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