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SOME LOCAL-GLOBAL NON-VANISHING RESULTS FOR THETA LIFTS FROM ORTHOGONAL GROUPS

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Abstract. We, first, improve a theorem of B. Roberts which characterizes non-vanishing of a global theta lift from $\mathrm{O}(X)$ to $\mathrm{Sp}(n)$ in terms of non-vanishing of local theta lifts. In particular, we will remove all the Archimedean conditions imposed upon his theorem. Secondly, following Roberts, we will apply our theorem to theta lifting of low rank similitude groups. Namely we characterize the non-vanishing condition of a global theta lift from $\mathrm{GO}(4)$ to $\mathrm{GSp}(2)$ in our improved setting. Also we consider non-vanishing conditions of a global theta lift from $\mathrm{GO}(4)$ to $\mathrm{GSp}(1)$ and explicitly compute the lift when it exists.

1. Introduction

Let X be a symmetric bilinear space over a number field of an even dimension m and let σ be an irreducible cuspidal automorphic representation of $\mathrm{O}(X, \mathbb{A}_F)$. Assume V_σ is a space of automorphic forms giving rise to σ . Then we consider the theta lift $\Theta_n(V_\sigma)$ of V_σ to $\mathrm{Sp}(n, \mathbb{A}_F)$ for $n = \frac{m}{2}$. One of the major questions in the theory of theta correspondence is to show that the lift $\Theta_n(V_\sigma)$ does not vanish. B. Roberts characterizes the global non-vanishing of the theta lift in terms of the local counterpart $\theta_n(\sigma_v)$. To be more precise, in [R4], Roberts proves that for $\frac{m}{2} \leq n$ the theta lift does not vanish if and only if the local theta lift of each local component σ_v does not vanish under various technical assumptions. In particular he assumes that the signature of $\mathrm{O}(X)$ at each real place is of the form $(2p, 2q)$. In this paper, first we will completely remove all the Archimedean assumptions imposed upon his theorem, although we restrict to the case $\frac{m}{2} = n$. To be precise, we prove

Theorem 1.1. *Let F be any number field, and $\sigma \cong \otimes\sigma_v$ an irreducible cuspidal automorphic representation of $\mathrm{O}(X, \mathbb{A}_F)$ with $\dim X = m$ even. Also let S_f be the finite set of finite places v such that either σ_v is ramified, $v|2$, or v is ramified in the quadratic extension $F(\sqrt{d})$ of F, where d is the discriminant of X. Assume:*

1. *The (incomplete) standard Langlands L-function $L^S(s, \sigma)$ of σ does not vanish at $s \in \{1, 2, \ldots, \frac{m}{2}\}$. (A pole is permitted.)*
2. *σ_v is tempered for all $v \in S_f$.*
3. *The local theta lift $\theta_{\frac{m}{2}}(\sigma_v)$ to $\mathrm{Sp}(\frac{m}{2}, F_v)$ exists for all places v.*

Then the global theta lift $\Theta_{\frac{m}{2}}(V_\sigma)$ to $\mathrm{Sp}(\frac{m}{2}, \mathbb{A}_F)$ does not vanish.

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Here the temperedness assumption for $v|2$ is due to the lack of a Howe duality principle for an even residual characteristic, and is thus quite crucial. The other two assumptions are due to the lack of the corresponding result of [R2] for the non-tempered case. In [R4], B. Roberts assumes that π_v is tempered for all non-Archimedean places, but here we replace the temperedness condition by the L-function condition. (This is not an improvement of his theorem, but just another way of stating the theorem, although this makes a slight difference when we apply it to the similitude case.) Also in [R4], Roberts *did not* assume π_v is tempered for Archimedean v, but in [R5, p.301] he himself pointed out that this is a mistake and it must be assumed to be tempered. But in this paper, we will show that, after all, π_v does not have to be tempered for Archimedean v.

Next we apply this theorem, as Roberts did, to theta lifting for groups of similitudes. Then we prove the following, which is an improvement of one of the main theorems of [R5].

Theorem 1.2. *Let X be a symmetric bilinear space of $\dim X = 4$ over F and σ an irreducible cuspidal automorphic representation of $\mathrm{GO}(X, \mathbb{A}_F)$. Assume that σ_v is tempered for all $v \in S_f$, where S_f is defined in the same way as in Theorem 1.1. Then the global theta lift $\Theta_2(V_\sigma)$ to $\mathrm{GSp}(2, \mathbb{A}_F)$ does not vanish if and only if the local theta lift $\theta_2(\sigma_v)$ to $\mathrm{GSp}(2, F_v)$ does not vanish for all places v.*

Notice that the group $\mathrm{GO}(X)$ is disconnected and written as $\mathrm{GO}(X) \cong \mathrm{GSO}(X) \rtimes \{1, t\}$ for some $t \in \mathrm{GO}(X)$ with $t^2 = 1$ which acts on $\mathrm{GSO}(X)$ by conjugation. Each irreducible cuspidal automorphic representation σ of $\mathrm{GO}(X, \mathbb{A}_F)$ is "extended from" an irreducible cuspidal automorphic representation π of the identity component $\mathrm{GSO}(X, \mathbb{A}_F)$ in the sense explained in Section 5. Let d be the discriminant of X. Roberts in [R5] has shown that for the purpose of similitude theta lifting, we may assume:

(1) If $d = 1$, then an irreducible cuspidal automorphic representation π of $\mathrm{GSO}(X, \mathbb{A})$ with central character χ is identified with an irreducible cuspidal automorphic representation $\tau_1 \otimes \tau_2$ of $D^\times(\mathbb{A}_F) \times D^\times(\mathbb{A}_F)$, where D is a quaternion algebra over F, and the central characters of τ_1 and τ_2 are both χ. In this case, we write $\pi = \pi(\tau_1, \tau_2)$.
(2) If $d \neq 1$, then an irreducible cuspidal automorphic representation τ of $\mathrm{GSO}(X, \mathbb{A})$ with central character χ is identified with an irreducible cuspidal automorphic representation τ of ${B_E}^\times(\mathbb{A}_F)$, where B is a quaternion algebra over the quadratic extension $E = F(\sqrt{d})$ of F, and the central character of τ is of the form $\chi \circ N_F^E$. In this case, we write $\pi = \pi(\tau, \chi)$.

Note that for $\tau_1 \otimes \tau_2$ and τ, there are Jacquet-Langlands lifts $\tau_1^{\mathrm{JL}} \otimes \tau_2^{\mathrm{JL}}$ and τ^{JL} to $\mathrm{GL}(2, \mathbb{A}_F) \times \mathrm{GL}(2, \mathbb{A}_F)$ and $\mathrm{GL}(2, \mathbb{A}_E)$, respectively. Also for each π we can consider the conjugate π^c of π, whose space V_{π^c} of representations is of the form $\{f \circ c : f \in V_\pi\}$ where $f \circ c(g) = f(tgt)$. If $\pi = \pi(\tau_1, \tau_2)$, then $\pi^c = \pi(\tau_2, \tau_1)$, and if $\pi = \pi(\tau, \chi)$, then $\pi^c = \pi(\tau^c, \chi)$ where τ^c is the Galois conjugate of τ. We will prove

Theorem 1.3. *Assume that the global theta lift $\Theta_2(V_\sigma)$ to $\mathrm{GSp}(2, \mathbb{A}_F)$ does not vanish. Then*

(1) *If $d = 1$ and σ is extended from $\pi = \pi(\tau_1, \tau_2)$, then the theta lift $\Theta_1(V_\sigma)$ to $\mathrm{GSp}(1, \mathbb{A}_F)$ does not vanish if and only if $\tau_1 = \tau_2$. Moreover, if this is the*

case, $\Theta_1(V_\sigma)$ is the space of an irreducible cuspidal representation Π such that $\Pi^\vee = \tau_1^{\mathrm{JL}} = \tau_2^{\mathrm{JL}}$.

(2) *If $d \neq 1$ and σ is extended from $\pi = \pi(\tau, \chi)$, then the theta lift $\Theta_1(V_\sigma)$ to $\mathrm{GSp}(1, \mathbb{A}_F)$ does not vanish if and only if τ^{JL} is the base change lift of an irreducible cuspidal automorphic representation τ_0 of $\mathrm{GL}(2, \mathbb{A}_F)$ whose central character is $\chi_{E/F}\chi$, where $\chi_{E/F}$ is the quadratic character for E/F. Moreover, if this is the case, $\Theta_1(V_\sigma)$ is the space of an irreducible cuspidal representation Π such that $\Pi^\vee = \tau_0$.*

In light of those two theorems, one interesting thing to investigate is, of course, when a given π can be extended to σ so that $\Theta_2(V_\sigma) \neq 0$. For certain cases, it can be shown that the answer is "always". Namely,

Theorem 1.4.

(1) *Assume π is generic. Then π can be extended to σ so that $\Theta_2(V_\sigma) \neq 0$ (without any temperedness assumption).*

(2) *Assume π is such that $\pi^c \ncong \pi$ (but not necessarily generic). If π satisfies the temperedness assumption as in Theorem 1.2, then π can be extended to σ so that $\Theta_2(V_\sigma) \neq 0$.*

We make no claim to the originality for this theorem. The first part easily follows from a theorem of Howe and Piatetski-Shapiro [HPS], although they do not explicitly state it in this way. The second part follows from Theorem 1.2 together with a theorem of Roberts in [R5], and he states its tempered version by using his notion of "global L-packets".

Finally, to give arithmetic applications of our lifts, it should be noted that if $F = \mathbb{Q}$, $d = 1$, and X is anisotropic, the lift from $\mathrm{GO}(X)$ to $\mathrm{GSp}(2)$ is known as the Yoshida lift. (See, say, [BS].) Also if $F = \mathbb{Q}$, $d > 0$, and X is anisotropic, then our lifting from $\mathrm{GO}(X)$ to $\mathrm{GSp}(2)$ together with the Jacquet-Langlands lift gives a way of constructing a holomorphic Siegel modular form from a holomorphic Hilbert modular form.

This paper is organized as follows. We first set up our notation in Section 2. In Section 3, we give a proof of Theorem 1.1. In Section 4, we discuss basics of both global and local theta lifting of groups of similitudes to the extent necessary for our purpose. In Section 5 we review the classification of the groups $\mathrm{GSO}(X)$ and $\mathrm{GO}(X)$ when $\dim X = 4$ and their representations. In Section 6, we will explicitly compute the unramified local theta lift from $\mathrm{GO}(4)$ to $\mathrm{GSp}(1)$. Then finally, in Section 7, we will give the proofs of Theorems 1.2 and 1.3. In Appendix A, we give the proof of Theorem 1.4.

2. Notation and preliminaries

In this paper, F is a local or global field of char $F = 0$. If E is a quadratic extension of F, then we denote by N_F^E (or simply by N) the norm map, and by $\chi_{E/F}$ the quadratic character obtained by local or global class field theory.

We work with smooth representations instead of K-finite ones. Namely if G is a reductive group over a global field F, then by a (cuspidal) automorphic form we mean a smooth (cuspidal) automorphic form on $G(\mathbb{A}_F)$ in the sense of [Co, Definition 2.3].

If π is a representation of a group, a Harish-Chandra module, or an automorphic representation, then by V_π we mean a space which realizes π. If π is a representation of a real Lie group, then we will denote the space of smooth vectors in V_π by V_π^∞ and the space of K-finite vectors by ${}^K V_\pi$. If π is an admissible representation of a real Lie group, then we denote the underlying Harish-Chandra module by π^{H}, and thus we have $V_{\pi^{\mathrm{H}}} = {}^K V_\pi$. If F is a local field, we denote by $\mathrm{Irr}(G(F))$ the collection of equivalence classes of irreducible smooth admissible representations of $G(F)$. Also we denote by $\mathrm{Irr}(G(F),\chi)$ the collection of equivalence classes of irreducible smooth admissible representations of $G(F)$ whose central character is χ. For each $\pi \in \mathrm{Irr}(G(F))$, we denote the contragredient by $\pi^\vee$. If π is a Harish-Chandra module, we denote by π^{CW} the Casselman-Wallach canonical completion of π. Thus ${}^K V_{\pi^{\mathrm{CW}}} = V_\pi$. (For the Casselman-Wallach canonical completion, see [C].)

For a finite dimensional vector space X over a global field F, we denote $X \otimes_F F_v$ by $X(F_v)$ for each place v and $X \otimes_F \mathbb{A}_F$ by $X(\mathbb{A}_F)$. For a natural number n, $\mathcal{S}(X(F_v)^n)$ denotes the space of Schwartz-Bruhat functions. We set $\mathcal{S}(X(\mathbb{A}_F)^n) := \otimes' \mathcal{S}(X(F_v)^n)$, where $\otimes' \mathcal{S}(X(F_v)^n)$ is the restricted tensor product over all places with respect to the characteristic function of $\mathcal{O}_v x_1 + \cdots + \mathcal{O}_v x_m$ for v finite, where $\mathcal{O}_v$ is the ring of integers of F_v and $x_1, \ldots, x_m$ is a fixed basis of $X(F_v)$.

The group $\mathrm{GSp}(n)$ is the algebraic group of symplectic similitudes of rank n over a field F. We realize this group as the group of $2n \times 2n$ matrices given by $\mathrm{GSp}(n) = \{g \in \mathrm{GL}(2n) : {}^t g J g = \nu(g) J\}$ with $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$, where I_n is the $n \times n$ identity matrix, and $\nu : \mathrm{GSp}(n) \to \mathbb{G}_m$ is the multiplier character. Then the kernel of ν is denoted by $\mathrm{Sp}(n)$. When we need to make clear that we are working with F rational points, we write $\mathrm{Sp}(n,F)$ or $\mathrm{GSp}(n,F)$, but when there is no danger of confusion, we simply write $\mathrm{Sp}(n)$ or $\mathrm{GSp}(n)$.

Let X be an even dimensional symmetric bilinear space defined over a field F of even dimension m equipped with a symmetric bilinear form. Then we denote by $\mathrm{GO}(X)$ the group of symmetric similitudes and by $\mathrm{GSO}(X)$ its identity component. If X is defined over a local or global field F of char $F = 0$, then we denote by $\operatorname{disc} X \in F^\times / F^{\times^2}$ the discriminant of X when X is viewed as a quadratic form. We let $\chi_X : F^\times \to \{\pm 1\}$ be the quadratic character of X, namely $\chi_X(a) = (a, (-1)^{\frac{m(m-1)}{2}} \operatorname{disc} X)_F$ for $a \in F^\times$, where $(\ ,\ )_F$ is the Hilbert symbol of F.

We denote the (local or global) Weil representation for $\mathrm{O}(X) \times \mathrm{Sp}(n)$ by $\omega_{n,X}$ or simply by ω when X and n are clear from the context.

If F is an Archimedean local field, then the Weil representation $\omega_{n,X}$ on $\mathcal{S}(X(F)^n)$ is a smooth Fréchet representation of the group $\mathrm{Sp}(n) \times \mathrm{O}(X)$ of moderate growth in the sense of [C]. We say that $\sigma \in \mathrm{Irr}(\mathrm{O}(X))$ and $\Pi \in \mathrm{Irr}(\mathrm{Sp}(n))$ correspond, or σ corresponds to Π, if there is a non-zero homomorphism of Harish-Chandra modules from $(\omega_{n,X})^{\mathrm{H}}$ to $(\Pi \otimes \sigma)^{\mathrm{H}} = \Pi^{\mathrm{H}} \otimes \sigma^{\mathrm{H}}$, *i.e.* $\mathrm{Hom}((\omega_{n,X})^{\mathrm{H}}, (\Pi \otimes \sigma)^{\mathrm{H}}) \neq 0$, where Hom means the set of homomorphisms of Harish-Chandra modules. It is known that the relation $\mathrm{Hom}((\omega_{X,n})^{\mathrm{H}}, (\Pi \otimes \sigma)^{\mathrm{H}}) \neq 0$ defines a graph of bijection between subsets of $\mathrm{Irr}(\mathrm{Sp}(n))$ and $\mathrm{Irr}(\mathrm{O}(X))$ up to infinitesimal equivalence (the Howe duality principle), and in particular if σ corresponds to Π, then such Π is unique up to infinitesimal equivalence, namely Π^{H} is unique, although Π might not be unique. In this case we write $(\Pi^{\mathrm{H}})^{\mathrm{CW}} = \theta_n(\sigma)$, and we call it the local theta lift of σ.

Next assume F is non-Archimedean. We say that $\sigma \in \mathrm{Irr}(\mathrm{O}(X))$ and $\Pi \in \mathrm{Irr}(\mathrm{Sp}(n))$ correspond, or σ corresponds to Π, if there is a non-zero $\mathrm{Sp}(n) \times \mathrm{O}(X)$

homomorphism from $\omega_{n,X}$ to $\Pi \otimes \sigma$, *i.e.* $\mathrm{Hom}_{\mathrm{Sp}(n)\times \mathrm{O}(X)}(\omega_{n,X}, \Pi \otimes \sigma) \neq 0$. If the residue characteristic of F is odd, it is known that the relation $\mathrm{Hom}_{\mathrm{Sp}(n)\times \mathrm{O}(X)}(\omega_{n,X}, \Pi \otimes \sigma) \neq 0$ defines a graph of bijection between subsets of $\mathrm{Irr}(\mathrm{Sp}(n))$ and $\mathrm{Irr}(\mathrm{O}(X))$ (the Howe duality principle), and in particular if σ corresponds to Π, then such a Π is unique. In this case we write $\Pi = \theta_n(\sigma)$ and call it the local theta lift of σ. If σ does not correspond to any $\Pi \in \mathrm{Irr}(\mathrm{Sp}(n))$, then we say that the theta lift of σ vanishes and write $\theta_n(\sigma) = 0$. If the residue characteristic of F is even, then in general it is not known if the same holds. However, in [R2], Roberts has shown, among other things, that if σ is tempered and corresponds to $\Pi \in \mathrm{Irr}(\mathrm{Sp}(n))$ for $n = \frac{m}{2}$, then Π is unique regardless of the residue characteristic of F. So in this case, we denote Π by $\theta_n(\sigma)$ even if the residue characteristic of F is even.

For an irreducible cuspidal automorphic representation σ of $\mathrm{O}(X, \mathbb{A}_F)$, we denote by $\Theta_n(V_\sigma)$ the space of theta lifts of σ to $\mathrm{Sp}(n, \mathbb{A}_F)$, *i.e.* the space generated by the forms of the form $\theta(f, \phi)$ for $f \in V_\sigma$ and $\phi \in \mathcal{S}(X(\mathbb{A}_F)^n)$ which are defined by

$$\theta(f;\varphi)(g) = \int_{\mathrm{O}(X,F)\backslash \mathrm{O}(X,\mathbb{A}_F)} \theta(g,h;\varphi) f(h)\, dh$$

for each $g \in \mathrm{Sp}(n, \mathbb{A}_F)$, where $\theta(g, h; \varphi)$ is the theta kernel defined by $\theta(g, h; \varphi) = \sum_{x\in X(F)^n} \omega(g, h)\varphi(x)$.

3. Theta lifting for isometry groups

In this section, we will give a proof of Theorem 1.1, which is essentially the ingenious argument of [R4], which has its origin in [BS]. What needs to be done to improve the theorem of Roberts is to prove the following key technical lemma in our improved setting.

Lemma 3.1. *Let $\sigma \cong \otimes\sigma_v$ satisfy (2) and (3) of Theorem* 1.1*. Also let S_f be as in Theorem* 1.1 *and S_∞ the set of infinite places. Set $S = S_f \cup S_\infty$. For each $k > n = \frac{m}{2}$ and for $v \in S$, the local zeta integral of the local theta lift $\theta_k(\sigma_v)$ of σ_v to $\mathrm{Sp}(k, F_v)$ can be chosen so that it has a pole at $s = k - \frac{m}{2}$. Namely, there exist a matrix coefficient f_v of $\theta_k(\sigma_v)$ and a standard K-finite χ_{V_v}-section Φ_v for $\mathrm{Sp}(k, F_v)$ such that the local zeta integral $Z(s - \frac{1}{2}, f_v, \Phi_v)$ has a pole at $s = k - \frac{m}{2}$. (See* [R4] *for the notation.)*

Proof. The proof of this lemma is given by Roberts in [R4] under the assumption that F is totally real and the signature of X at the real place is of the form $(2p, 2q)$. To remove those Archimedean conditions, we need to prove various technical lemmas, which we will prove in the next subsection. Once those technical lemmas are proven, this lemma can be shown simply by tracing the proof given by Roberts. $\square$

Once this lemma is obtained, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Apply the above lemma to the proof of the main theorem of [R4, pp.146-148]. (For the simplicity of a pole of the zeta integral, use Propositions 1.6 and 1.7 of [I], which do not assume that F is totally real unlike [KR2].) $\square$

3.1. **Some technical lemmas on zeta integrals.** What forced Roberts to impose the conditions on the infinite places in his theorem in [R4] is the unavailability of Lemma 3.1 for χ_{X_v} non-trivial and v real, and for v complex. In this subsection, we give proofs of several technical lemmas that allow us to prove Lemma 3.1 in full generality for the Archimedean case. So we assume $F = \mathbb{R}$ or $\mathbb{C}$. There are basically two technical ingredients we need. The first one is the theory of the zeta integral for the symplectic group at the Archimedean place developed in [KR1]. There, it is assumed that $F = \mathbb{R}$ and the character for the zeta integral (which corresponds to our χ_{X_v}) is trivial. The results in [KR1] are used in two places in Roberts' argument in crucial ways. (See Lemma 7.5 and Theorem 7.8 of [R4].) So we need to extend the results of [KR1] to full generality. Although, as mentioned in [KR1], there is no doubt that all the arguments there work for $F = \mathbb{C}$, it seems that the extension to the case with the non-trivial character (*i.e.* the sign character) is not completely immediate. Thus first we prove

Proposition 3.2. *All the results in* [KR1] *hold, even with the presence of the sign character.*

Proof. In this proof, all the notation as well as the numberings of propositions, lemmas, etc. are as in [KR1]. First, as in the trivial character case, the line bundle $E_{s+\rho} = P\backslash(G \times \mathbb{C})$ (see (3.3.1) on p.105) where $p \cdot (g, v) = (pg, \chi(a(g))|a(g)|^{s+\rho}v)$ is trivialized by the nowhere-vanishing section $g \mapsto (g, \chi(a(g))|a(g)|^{s+\rho})$. So each section $\Phi(g, s)$ is identified with a smooth function on Ω via

$$\Phi(g, s) \mapsto \chi(a(g))|a(g)|^{-s-\rho}\Phi(g, s).$$

For each Φ, let us write $\widetilde{\Phi}$ for the corresponding smooth function in $C^\infty(\Omega)$, *i.e.*

$$\widetilde{\Phi}(g, s) = \chi(a(g))|a(g)|^{s+\rho}\Phi(g, s).$$

Then as on p.98, the restriction of $\widetilde{\Phi}|_{K_G}$ to K_G is left invariant under $P \cap K_G$ and so we may view $\widetilde{\Phi}|_{K_G}$ as a smooth function on $\Omega \cong (P \cap K_G)\backslash K_G$.

Now we first have to prove Proposition 3.1.1, which characterizes convergence of the integral in terms of the order of non-vanishing on the negligible set. Clearly all the computations up to Lemma 3.1.3 (p.101) remain to be true for our case. What needs to be modified is the equation (3.1.14) on p.101. First notice that $|\Phi(k, s)| = |\widetilde{\Phi}(k, s)|$ for all $k \in K_G$. Thus the equation (3.1.14) becomes

$$\int_K \int_K \int_{A^+} |\widetilde{\Phi}(k(a)i(k_1, k_2), s)|\mu(a)^{-\sigma-\rho_n}\delta(a)da\ dk_1\ dk_2.$$

The equation (3.1.16) must be modified as

$$f(u) = \widetilde{\Phi}(\phi(u)i(k_1, k_2), s)$$

as a function on $[0, 1]^n$. Then everything else works identically with the trivial character case. This proves Proposition 3.1.1 and Corollary 3.1.5.

Now that both Proposition 3.1.1 and Corollary 3.1.5 have been proven, Theorem 3.2.2. on p.104 can be proven if we prove Proposition 3.2.1. But the proof for Proposition 3.2.1 works for our case because of the trivialization of the line bundle $E_{s+\rho}$. $\square$

The second ingredient we need in order to remove the Archimedean conditions on the result of [R4] and thus to obtain Lemma 3.1 is the description of a K-finite coefficient of the local lift $\theta_k(\sigma)$ for $\sigma \in \mathrm{Irr}(\mathrm{O}(X, F))$ with $\theta_n(\sigma) \neq 0$. Roberts uses the description of a K-finite coefficient of $\theta_k(\sigma)$ in terms of its Langlands data that are ordered in the standard way. In particular he uses the fact that, under his assumptions on the signature of X, the first entry of the Langlands data of $\theta_k(\sigma)$ is the character $\chi_X|\cdot|^{k-n}$. (See the remark preceding Theorem 7.7 as well as Lemma 2.2 of [R4].)

However the real point is not that the first entry of the Langlands data is $\chi_X|\cdot|^{k-n}$ in the standard order as in [R4], but that the K-finite vectors of the representation $\theta_k(\sigma)$ are realized as the image of the usual intertwining integral on the representation induced from a representation of a parabolic subgroup whose Levi factor is $\mathrm{GL}(1) \times \mathrm{Sp}(k-1)$ in which $\mathrm{GL}(1)$ acts by the character $\chi_X|\cdot|^{k-n}$. Namely, what we need is

Lemma 3.3. *Assume that σ is an irreducible admissible representation of $\mathrm{O}(X, F)$ such that $\theta_n(\sigma)$ exists for $n = \frac{1}{2}\dim X$. Then for $k > n$, $\theta_k(\sigma)$ is infinitesimally equivalent to the Langlands quotient of (i.e. the image of the intertwining integral on) $\mathrm{Ind}\,_P^{\mathrm{Sp}(k-1)}(\chi_X|\cdot|^{k-n} \otimes \sigma_2)$, where P is a (not necessarily the standard choice of) parabolic subgroup whose Levi factor is $\mathrm{GL}(1) \times \mathrm{Sp}(k-1)$, and σ_2 is a (not necessarily tempered) representation of $\mathrm{Sp}(k-1)$. (Of course, $\chi_X|\cdot|^{k-n} \otimes \sigma_2$ is extended from $\mathrm{GL}(1) \times \mathrm{Sp}(k-1)$ to P by letting the unipotent radical act trivially.)*

This can be shown by combining the following two lemmas, the first of which is essentially due to Paul [P].

Lemma 3.4. *Assume that σ is an irreducible admissible representation of $\mathrm{O}(X, F)$ such that $\theta_n(\sigma)$ exists for $n = \frac{1}{2}\dim X$. Then for $k > n$, $\theta_k(\sigma)$ is infinitesimally equivalent to the Langlands quotient of (i.e. the image of the intertwining integral on) $\mathrm{Ind}\,_{P_{k_1\ldots k_t}}^{\mathrm{Sp}(k)}(\tau_1 \otimes \cdots \otimes \tau_t \otimes \tau)$ for some parabolic $P_{k_1\ldots k_t}$ with $k_1 = 1$ and $\tau_1 = \chi_X|\cdot|^{k-n}$, where $P_{k_1\ldots k_t}$ is a parabolic subgroup of $\mathrm{Sp}(k)$ whose Levi factor is isomorphic to $\mathrm{GL}(k_1) \times \cdots \times \mathrm{GL}(k_t) \times \mathrm{Sp}(k - (k_1 + \cdots + k_t))$.*

Proof. For $F = \mathbb{R}$, this is just (a part of) Theorem 6.2(1) of [P]. The case for $F = \mathbb{C}$ is identical to the proof of [P, Theorem 6.2] by using the Induction Principle and computation of LKT. (See [A].) □

Lemma 3.5. *Assume π is an irreducible admissible representation of $\mathrm{Sp}(k)$ which is infinitesimally equivalent to the image of the intertwining integral on $\mathrm{Ind}\,_{P_{k_1\ldots k_t}}^{\mathrm{Sp}(k)}(\tau_1 \otimes \cdots \otimes \tau_t \otimes \tau_{t+1})$, where $P_{k_1\ldots k_t}$ is a parabolic subgroup whose Levi factor is isomorphic to $\mathrm{GL}(k_1) \times \cdots \times \mathrm{GL}(k_t) \times \mathrm{Sp}(k - (k_1 + \cdots + k_t))$. Let σ_2 be an irreducible admissible representation of $\mathrm{Sp}(k - k_1)$ which is infinitesimally equivalent to the image of the intertwining integral on $\mathrm{Ind}\,_{P_{k_2\ldots k_t}}^{\mathrm{Sp}(k-k_1)}(\tau_2 \otimes \cdots \otimes \tau_t \otimes \tau_{t+1})$, where $P_{k_2\ldots k_t}$ is the obvious subgroup of $P_{k_1\ldots k_t}$. (Here the convergence of this intertwining integral can be easily shown.) Then π is infinitesimally equivalent to the image of the absolutely convergent intertwining integral $J : \mathrm{Ind}\,_{P_{k_1}}^{\mathrm{Sp}(k)}(\sigma_1 \otimes \sigma_2) \rightarrow \mathrm{Ind}\,_{\overline{P}_{k_1}}^{\mathrm{Sp}(k)}(\sigma_1 \otimes \sigma_2)$ given by*

$$J(f)(g) = \int_{\overline{U}_{k_1}} f(\bar{u}g)\ d\bar{u},$$

where P_{k_1} is the parabolic subgroup whose Levi factor is $\mathrm{GL}(k_1) \times \mathrm{Sp}(k - k_1)$ and whose unipotent radical U_{k_1} is contained in the unipotent radical of $P_{k_1\ldots k_t}$. (Here we write $\overline{P}_{k_1} = {}^tP_{k_1}$ and $\overline{U}_{k_1} = {}^tU_{k_1}$.)

Proof. The proof is straightforward and is left to the reader. □

Then we can describe a K-finite coefficient of π as follows.

Lemma 3.6. *By keeping the notation of the above lemma, let M_{k_1} be the Levi factor of P_{k_1}. Also let $H : \mathrm{Sp}(k) \times \mathrm{Sp}(k) \to \mathbb{C}$ be a function satisfying the following properties:*

1. *$H(u_1 m g_1, \bar{u_2} m g_2) = H(g_1, g_2)$ for $u_1 \in U_{k_1}, \bar{u_2} \in \bar{U}_{k_1}, m \in M_{k_1}$;*
2. *for any $g_1, g_2 \in \mathrm{Sp}(k)$ the function $m \mapsto H(mg_1, g_2)$ is a coefficient for $\sigma \otimes \delta_{P_{k_1}}^{1/2}$;*
3. *H is C^∞ and $K \times K$-finite on the right,*

where $\sigma = \sigma_1 \otimes \sigma_2$, $\delta_{P_{k_1}}$ is the module of P_{k_1}, and K is the standard maximal compact subgroup of $\mathrm{Sp}(k)$. Then the function f defined by

$$f(g) = \int_{M_{k_1} \backslash G} H(hg, h) dh = \int_{\overline{U}_{k_1} \times K} H(\bar{u} kg, k) dk d\bar{u}$$

is absolutely convergent and a K-finite coefficient of π. (Note that by a coefficient of π, we mean a finite $\mathbb{C}$ linear combination of matrix coefficients of π.)

This proposition is essentially due to Jacquet, and the $\mathrm{GL}(k)$ version is stated in (5.5) of [J1] without a proof. Since no proof is given there, we will provide a detailed proof here using $\mathrm{Sp}(k)$ as our group. (The proof also works for the non-Archimedean case.) First we need the following lemma. (For this lemma, the author would like to thank H. Jacquet, who kindly showed a variant of the proof via a private correspondence [J2].)

Lemma 3.7. *Let H be as in Lemma 3.6. Then H is of the form*

$$H(g_1, g_2) = \sum_i \langle f_i(g_1), f_i'(g_2) \rangle,$$

for some $f_i \in \mathrm{Ind}\,_{P_{k_1}}^{G}(\sigma)$ and $f_i' \in \mathrm{Ind}\,_{\overline{P}_{k_1}}^{G}(\sigma^\vee)$, both of which are K-finite.

Proof. Let us simply write $G = \mathrm{Sp}(k)$, $M = M_{k_1}$, $P = P_{k_1}$, $\overline{P} = \overline{P}_{k_1}$, $U = U_{k_1}$, and $\overline{U} = \overline{U}_{k_1}$, and also write $\tau = \sigma \otimes \delta_P^{1/2}$. Let $\mathcal{M}$ be the space of coefficients of τ. Then the group $M \times M$ acts on $\mathcal{M}$ by

$$(m_1, m_2) \cdot \varphi(m) = \varphi(m_2^{-1} m m_1), \quad \text{for } \; m \in M, \;\; (m_1, m_2) \in M \times M.$$

Notice that $V_\tau \otimes V_{\tau^\vee} \cong \mathcal{M}$ as vector spaces via $(v, w) \mapsto f_{v,w}$. (Here $f_{v,w}$ is defined by $f_{v,w}(g) = \langle \pi(g) v, w \rangle$, where $\langle , \rangle$ is the canonical pairing.) So via this isomorphism, $M \times M$ acts on $V_\tau \otimes V_{\tau^\vee}$. Moreover it is easy to see that the representation obtained by this action of $M \times M$ on $V_\tau \otimes V_{\tau^\vee}$ is isomorphic to $\tau|_M \otimes \tau^\vee|_M$ as a smooth admissible representation.

Now define a function $F : K \times K \to \mathcal{M}$ by

$$F(k_1, k_2)(m) = H(mk_1, k_2) \text{ for } (k_1, k_2) \in K \times K.$$

Then via the isomorphism $V_\tau \otimes V_{\tau^\vee} \cong \mathcal{M}$, we have

$$F \in \mathrm{Ind}\,_{(K \cap M) \times (K \cap M)}^{K \times K} (\tau_0 \otimes \tau_0^\vee),$$

where $\tau_0 = \tau|_{K \cap M}$ and $\tau_0^\vee = \tau^\vee|_{K \cap M}$. This can be seen as follows. First notice that for all $g_1, g_2 \in G$ and $m \in M$, we have $H(mg_1, g_2) = H(m^{-1} m g_1, m^{-1} g_2) =$

$H(g_1, m^{-1}g_2)$. Then if we write $H(mk_1, k_2)=\sum_i\langle\tau(m)v_i, w_i\rangle$, we have, for $m_1,\ m_2 \in M\cap K$,

$$\begin{aligned}F(m_1k_1, m_2k_2)(m) &= H(mm_1k_1, m_2k_2)\\ &= H(m_2^{-1}mm_1k_1, k_2)\\ &= \sum_i\langle\tau(m_2^{-1}mm_1)v_i, w_i\rangle\\ &= (m_1, m_2)\cdot F(k_1, k_2)(m),\end{aligned}$$

and so $F(m_1k_1, m_2k_2) = (m_1, m_2)\cdot F(k_1, k_2)$.

Now notice that

$$\mathrm{Ind}\,^{K\times K}_{(K\cap M)\times(K\cap M)}(\tau_0\otimes\tau_0^\vee)\cong \mathrm{Ind}\,^{K}_{K\cap M}\tau_0\otimes\mathrm{Ind}\,^{K}_{K\cap M}\tau_0^\vee.$$

Thus we have

$$F(k_1, k_2) = \sum_i f_i(k_1)\otimes\tilde{f}_i(k_2),$$

for some $f_i\in\mathrm{Ind}\,^{K}_{K\cap M}\tau_0$ and $f'_i\in\mathrm{Ind}\,^{K}_{K\cap M}\tau_0^\vee$. By viewing $f_i(k_1)\otimes f'_i(k_2)\in V_\tau\otimes V_{\tau^\vee}$ as an element in $\mathcal{M}$ via the isomorphism $V_\tau\otimes V_{\tau^\vee}\cong\mathcal{M}$, we have

$$F(k_1, k_2)(m) = H(mk_1, k_2) = \sum_i\langle\tau(m)f_i(k_1), f'_i(k_2)\rangle.$$

Since H is $K\times K$-finite on the right, we see that the f_i's and f'_i's are K-finite on the right. Now we can extend the domain of the f_i's and f'_i's from K to all of G by

$$\begin{aligned}f_i(u_1m_1k_1) &= \tau(m_1)f_i(k_1),\\ f'_i(\bar{u}_2m_2k_2) &= \tau^\vee(m_2)\tilde{f}_i(k_2),\end{aligned}$$

for $u_1\in U$, $\bar{u}_1\in\overline{U}$ and $m_1, m_2\in M$. Those are well defined and indeed $f_i\in\mathrm{Ind}\,^G_P(\sigma)$ and $f'_i\in\mathrm{Ind}\,^G_{\overline{P}}(\sigma^\vee)$, because $\tau=\sigma\otimes\delta_P^{1/2}$ and $\tau^\vee=\sigma^\vee\otimes\delta_{\overline{P}}^{1/2}$, and also they are K-finite. □

Now we are ready to prove Lemma 3.6.

Proof of Lemma 3.6. Let us simply write $G=\mathrm{Sp}(k), P_{k_1}=P, U_{k_1}=U, \overline{U}_{k_1}=\overline{U}$, and $M_{k_1}=M$. Let us also write $\eta=\mathrm{Ind}\,^G_P(\sigma)$ and $\eta'=\mathrm{Ind}\,^G_{\overline{P}}(\sigma^\vee)$, where $\sigma^\vee$ is extended from M to $\overline{P}$ by letting $\overline{U}$ act trivially. Let J and J' be the intertwining operators for η and η', respectively, as defined in Lemma 3.5. Then $\pi^{\mathrm{H}}\cong(\eta/\ker J)^{\mathrm{H}}$ and $(\pi^\vee)^{\mathrm{H}}\cong(\eta'/\ker J')^{\mathrm{H}}$, where $\ker J$ and $\ker J'$ are characterized by the property that, for all K-finite $f\in\ker J$ and $f'\in\ker J'$,

$$\begin{aligned}J(f)(g) &= \int_{\overline{U}}\langle f(\bar{u}g), \tilde{v}\rangle\,d\bar{u}=0\quad\text{for all}\quad\tilde{v}\in\sigma^\vee,\quad\text{and}\\ J'(f')(g) &= \int_U\langle v, f'(ug)\rangle\,du=0\quad\text{for all}\quad v\in\sigma.\end{aligned}$$

Then if we write $\bar{f}$ and $\bar{f}'$ for the images in $\eta/\ker J$ and $\eta'/\ker J'$, respectively, then the canonical pairing of π and $\pi^\vee$ is given by

$$\langle\bar{f}, \bar{f}'\rangle = \int_{M\backslash G}\langle f(h), f'(h)\rangle\,dh,$$

for f, f' K-finite. This can be proven as follows. First of all, clearly the function $g \mapsto \langle f(g), f'(g)\rangle$ is M-invariant, and so the integral makes sense. Second of all, this integral absolutely converges, because

$$\begin{aligned}\int_{M\backslash G}\langle f(h), f'(h)\rangle \, dh &= \int_{\overline{P}\backslash G}\int_{\overline{U}}\langle f(\bar{u}k), f'(\bar{u}k)\rangle \, d\bar{u}dk \\ &= \int_{K}\int_{\overline{U}}\langle f(\bar{u}k), f'(k)\rangle \, d\bar{u}dk,\end{aligned}$$

where the integral $\int_{\overline{U}}\langle f(\bar{u}k), f'(k)\rangle \, d\bar{u}$ converges absolutely by Lemma 3.5. Finally, the characterizing property of $\ker J$ and $\ker J'$ guarantees that the integral is independent of the choice of the representatives of $\bar{f}$ and $\bar{f}'$. Therefore a coefficient of π is a finite $\mathbb{C}$ linear combination of functions of the form

$$g \mapsto \langle \pi(g)\bar{f}, \bar{f}'\rangle = \int_{M\backslash G}\langle f(hg), f'(h)\rangle \, dh = \int_{K}\int_{\overline{U}}\langle f(\bar{u}kg), f'(k)\rangle \, d\bar{u}dk,$$

where $f \in \eta$ and $f' \in \eta'$ are K-finite.

Now if H is a function satisfying all three properties, then by Lemma 3.7 we have

$$H(g_1, g_2) = \sum_i \langle f_i(g_1), f'_i(g_2)\rangle$$

for some f_i's $\in \eta$ and f'_i's $\in \eta'$, all of which are K-finite. Thus the lemma follows. $\square$

Once all those lemmas are proven, Lemma 3.1 for $F = \mathbb{R}$ can be proven by exactly the same computation as in [R4].

Finally to apply Roberts' argument to the $F = \mathbb{C}$ case, we need the following.

Lemma 3.8. *Assume $F = \mathbb{C}$. The local zeta integral $Z(s-1/2, f_1 \otimes \delta_P^{-1/2}, \Phi_1)$ as in Proposition 8.7 of* [R4] *has a simple pole at $k-n$ for suitably chosen Φ_1. (See* [R4] *for the notation.)*

Proof. Let $\phi_1 \in C_c^\infty(\mathbb{C}^\times)$ have support in the ball $B = B(1, \epsilon) \subset \mathbb{C}$ of radius $\epsilon < 1$ and center 1, and $\phi_1(1) = 0$. Let Φ_1 be the section obtained from ϕ_1 as in the proof of Proposition 8.7 of [R4]. Then the proof is completely analogous to the real case as in [R4, p.188]. $\square$

This lemma is the complex analogue of Proposition 8.7 of [R4]. The rest of the proof of Lemma 3.1 is identical to the real case.

4. Theta lifting for similitude groups

In this section, we will first review the theory of both local and global theta lifting for groups of similitudes, and then discuss some relations between the two. The main references for similitude theta lifting are [R5] and [HST].

Following [HST], we extend the Weil representation for $\mathrm{Sp}(n) \times \mathrm{O}(X)$ to the group

$$R = \{(g, h) \in \mathrm{GSp}(n) \times \mathrm{GO}(X) : \nu(g)\nu(h) = 1\}.$$

(See Section 2 of [HST] for more details.) We denote this extended Weil representation, again, by $\omega_{n,X}$ or simply by ω.

First assume F is non-Archimedean. For $\sigma \in \mathrm{Irr}(\mathrm{GO}(X))$ and $\Pi \in \mathrm{Irr}(\mathrm{GSp}(n))$, we say that σ and Π correspond, or σ corresponds to Π, if there is a non-zero R homomorphism from $\omega_{n,X}$ to $\Pi \otimes \sigma$, *i.e.* $\mathrm{Hom}_R(\omega_{n,X}, \Pi \otimes \sigma) \neq 0$. Let $\mathrm{GSp}(n)^+ = \{g \in \mathrm{GSp}(n) : \nu(g) \in \nu(\mathrm{GO}(X))\}$. If the residue characteristic of F is odd, it is known that the relation $\mathrm{Hom}_R(\omega_{n,X}, \Pi \otimes \sigma) \neq 0$ defines a graph of bijection between subsets of $\mathrm{Irr}(\mathrm{GSp}(n)^+)$ and $\mathrm{Irr}(\mathrm{GO}(X))$ (the Howe duality principle). (This follows from Theorem 4.4 of [R1] together with the multiplicity one theorem of [AP, Theorem 1.4].) Unlike the isometry case, it is still unknown if the group $\mathrm{GSp}(n)^+$ can be replaced by $\mathrm{GSp}(n)$ even for the odd residual characteristic case, although it is known to be true for certain cases. (See Theorem 1.8 of [R5].) For our purposes, however, the following is enough.

Lemma 4.1. *Let X be a four dimensional quadratic form over a non-Archimedean local field F of* char $F = 0$*. First assume the residual characteristic of F is odd. Then if $\sigma \in \mathrm{Irr}(\mathrm{GO}(X))$ corresponds to $\Pi \in \mathrm{Irr}(\mathrm{GSp}(2))$, then Π is unique. Next assume the residual characteristic is even. Then the same holds as long as σ is tempered, and in this case Π is also tempered. Hence if such a Π exists, we write $\theta_2(\sigma) = \Pi$, assuming σ is tempered if the residual characteristic is even. If no such Π exists, we write $\theta_2(\sigma) = 0$.*

Proof. This is just a part of Theorem 1.8 in [R5]. $\square$

Next assume F is Archimedean. Then just as in the non-Archimedean case, the extended Weil representation $\omega_{n,X}$ on $\mathcal{S}(X(F)^n)$ is defined, which is a smooth Fréchet representation of the group R of moderate growth in the sense of [C]. In particular $(\omega^{\mathrm{H}})^{\mathrm{CW}} = \omega$. Then for $\sigma \in \mathrm{Irr}(\mathrm{GO}(X))$ and $\Pi \in \mathrm{Irr}(\mathrm{GSp}(n))$, we say that σ and Π correspond, or σ corresponds to Π, if there is a non-zero homomorphism of Harish-Chandra modules from $(\omega_{n,X})^{\mathrm{H}}$ to $((\Pi \otimes \sigma)|_R)^{\mathrm{H}}$, *i.e.* $\mathrm{Hom}((\omega_{n,X})^{\mathrm{H}}, ((\Pi \otimes \sigma)|_R)^{\mathrm{H}}) \neq 0$, where Hom means the set of homomorphisms of Harish-Chandra modules for smooth representations of R. Although, just as in the non-Archimedean case, the Howe duality for similitude groups is not known in full generality, we only need the following for our purposes.

Lemma 4.2. *Let X be a four dimensional quadratic form over $F = \mathbb{R}$ or $\mathbb{C}$. If $\sigma \in \mathrm{Irr}(\mathrm{GO}(X))$ corresponds to $\Pi \in \mathrm{Irr}(\mathrm{GSp}(2))$, then Π is unique up to infinitesimal equivalence. In this case, we write $\theta_2(\sigma) = (\Pi^{\mathrm{H}})^{\mathrm{CW}}$. If no such Π exists, we write $\theta_2(\sigma) = 0$.*

Proof. This is again essentially a part of Theorem 1.8 in [R5]. In [R5], the signature of X is assumed to be of the form (p, q) with both p and q even, but this assumption is unnecessary. Also if $F = \mathbb{C}$, this is obvious because in this case we have $\mathrm{GSp}(n)^+ = \mathrm{GSp}(n)$. $\square$

Now let us consider the global case, so assume F is a global field. Just as in the isometry case, define the theta kernel by $\theta(g, h; \varphi) = \sum_{x \in X(F)^n} \omega(g, h)\varphi(x)$ for $(g, h) \in R(\mathbb{A})$ and $\varphi \in \mathcal{S}(X(\mathbb{A}_F)^n)$. Then for each automorphic representation σ of $\mathrm{GO}(X, \mathbb{A}_F)$ and for $f \in V_\sigma$, consider the integral $\theta(f; \varphi)(g) = \int_{\mathrm{O}(X,F)\backslash \mathrm{O}(X,\mathbb{A}_F)} \theta(g, h_1 h; \varphi) f(h_1 h)\, dh_1$, where $h \in \mathrm{GO}(X, \mathbb{A}_F)$ is any element such that $\nu(g)\nu(h) = 1$. For a suitable choice of the Haar measure dh_1 as in [HK], it can be shown that this integral is absolutely convergent. Also the invariance property of the measure guarantees that this integral is independent of the choice of h. Now set $\mathrm{GSp}(n,\mathbb{A}_F)^+ = \{g \in \mathrm{GSp}(n,\mathbb{A}_F) : \nu(g) \in \nu(\mathrm{GO}(X,\mathbb{A}))\}$. Then $\theta(f;\varphi)$ is a function on $\mathrm{GSp}(n,\mathbb{A}_F)^+$ which is left $\mathrm{GSp}(n,F)^+$ invariant. We can extend this function to an automorphic form on $\mathrm{GSp}(n,\mathbb{A}_F)$ by insisting that it is left $\mathrm{GSp}(n,F)$ invariant and zero outside $\mathrm{GSp}(n,F)\,\mathrm{GSp}(n,\mathbb{A}_F)^+$. We denote this automorphic form also by $\theta(f;\varphi)$, whose central character is $\chi^{-1}\chi_V^n$, where χ is the central character of σ. Then we denote by $\Theta_n(V_\sigma)$ the space generated by the automorphic forms $\theta(f;\varphi)$ for all $f \in V_\sigma$ and all $\varphi \in \mathcal{S}(V(\mathbb{A}_F)^n)$. If $\Theta_n(V_\sigma)$ is in the space of non-zero cusp forms, then each of the irreducible constituents under right translation provides an irreducible cuspidal automorphic representation of $\mathrm{GSp}(n,\mathbb{A}_F)$. So let us write $\Theta_n(V_\sigma) = \bigoplus_i \Pi_i$, where each Π_i is an irreducible cuspidal automorphic representation of $\mathrm{GSp}(n,\mathbb{A}_F)$. If we write $\sigma \cong \otimes\sigma_v$ and $\Pi_i \cong \otimes\Pi_{i,v}$, then each $\sigma_v^\vee$ corresponds to $\Pi_{i,v}$.

Remark 4.3. We should mention a certain conventional discrepancy found in the literature. In [R1] and [HK], the extended Weil representation is defined for the group $R' = \{(g,h) \in \mathrm{GSp}(n) \times \mathrm{GO}(V) : \nu(g) = \nu(h)\}$. On the other hand, in [HST] it is defined for our group R. Let us denote the extended Weil representations of R' by ω'. By direct computation, it can be shown that ω' is obtained from ω via the isomorphism $R' \to R$ given by $(g,h) \mapsto (\nu(g)^{-1}g,h)$. Then for the local case, if $\sigma \in \mathrm{Irr}(\mathrm{GO}(X))$ corresponds to $\Pi \in \mathrm{Irr}(\mathrm{GSp}(n))$ via ω, then π corresponds to $\tilde{\Pi}$ via ω' where $\tilde{\Pi}$ is defined by $\tilde{\Pi}(g) = \chi(\nu(g))^{-1}\Pi(g)$ for χ the central character of Π.

The choice of R seems to be completely conventional, but the reader should be aware that it also affects the global theta lift. Indeed if we use R', then for the integral defining $\theta(f;\varphi)(g)$, we have to choose h to be such that $\nu(g) = \nu(h)$. (Note that the integral on p.389 of [HST] is not quite correct.) Accordingly, the central character of $\theta(f;\varphi)$ is $\chi\chi_V^n$, which is proved in [HK, Lemma 5.1.9].

To consider the non-vanishing problem for our similitude case, we first consider the restriction to the isometry case. If f is an automorphic form on $\mathrm{GO}(X,\mathbb{A}_F)$, then clearly $f|_{\mathrm{O}(X,\mathbb{A}_F)}$ is an automorphic form on $\mathrm{O}(X,\mathbb{A}_F)$. The same thing can be said of automorphic forms on $\mathrm{GSp}(n,\mathbb{A})$. If V is a space of automorphic forms on $\mathrm{GSp}(n,\mathbb{A}_F)$, then we let $V|_{\mathrm{Sp}(n)} = \{f|_{\mathrm{Sp}(n,\mathbb{A}_F)} : f \in V\}$. Then we have

Lemma 4.4. *Let σ be an automorphic representation of $\mathrm{GO}(X,\mathbb{A}_F)$. Then*

(1) *For $f \in V_\sigma$, $\theta_n(f;\varphi)|_{\mathrm{Sp}(n,\mathbb{A}_k)} = \theta_n(f|_{\mathrm{O}(X,\mathbb{A}_F)};\varphi)$, where $\theta_n(f|_{\mathrm{O}(X,\mathbb{A}_F)};\varphi)$ is the isometry theta lift of $f|_{\mathrm{O}(X,\mathbb{A}_F)}$.*
(2) *$\Theta(V_\sigma) \neq 0$ if and only if $\Theta(V_\sigma)|_{\mathrm{Sp}(n)} \neq 0$.*
(3) *$\Theta(V_\sigma)$ is in the space of cusp forms if and only if $\Theta(V_\sigma)|_{\mathrm{Sp}(n)}$ is.*

Proof. (1). This is obvious.

(2). Assume $\Theta(V_\sigma) \neq 0$. Then for some $f \in V_\sigma$, $g \in \mathrm{GSp}(n,\mathbb{A}_F)$, and $\varphi \in \mathcal{S}(X(\mathbb{A}_F)^n)$, we have $\theta_n(f;\varphi)(g) \neq 0$. By definition of $\theta_n(f;\varphi)$, we may assume $g \in \mathrm{GSp}(n,\mathbb{A}_F)^+$. Let $g_1 = g\begin{pmatrix} I_n & 0 \\ 0 & \nu(g)^{-1}I_n \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{A}_F)$. Then for $h \in \mathrm{GO}(X,\mathbb{A}_F)$

with $\nu(g)\nu(h) = 1$, we have

$$\begin{aligned}
\theta(f;\varphi)(g) &= \int_{\mathrm{O}(X,F)\backslash \mathrm{O}(X,\mathbb{A}_F)} \left(\sum_{x\in X(F)^n} \omega(g, h_1 h)\varphi(x) \right) f(h_1 h)\, dh_1 \\
&= \int_{\mathrm{O}(X,F)\backslash \mathrm{O}(X,\mathbb{A}_F)} \left(\sum_{x\in X(F)^n} |\nu(g)|^{\frac{mn}{2}} \omega(g_1, 1)\varphi(x \circ h^{-1} h_1^{-1}) \right) f(h_1 h)\, dh_1 \\
&= |\nu(g)|^{\frac{mn}{2}} \int_{\mathrm{O}(X,F)\backslash \mathrm{O}(X,\mathbb{A}_F)} \left(\sum_{x\in X(F)^n} \omega(g_1, h_1)\varphi'(x) \right) f'(h_1)\, dh_1 \\
&= |\nu(g)|^{\frac{mn}{2}} \theta_n(f'|_{\mathrm{O}(X,\mathbb{A}_F)};\varphi')(g_1),
\end{aligned}$$

where $\varphi' \in \mathcal{S}(X(\mathbb{A}_F)^n)$ is given by $\varphi'(x) = \varphi(x \circ h^{-1})$ and $f' = h \cdot f \in V_\sigma$. (For the action of $\omega(g, h_1 h)$, see, for example, p.261 of [R5] together with Remark 4.3 above.) Now $\theta_n(f'|_{\mathrm{O}(X,\mathbb{A}_F)};\varphi') = \theta_n(f';\varphi')|_{\mathrm{Sp}(n,\mathbb{A}_F)} \in \Theta(V_\sigma)|_{\mathrm{Sp}(n)}$. This proves the only if part. The converse is obvious.

(3). First notice that $\mathrm{GSp}(n) \cong \mathrm{Sp}(n) \rtimes \mathbb{G}_m$, and if $P \subset \mathrm{Sp}(n)$ is a parabolic subgroup, then $P \rtimes \mathbb{G}_m$ is a parabolic subgroup of $\mathrm{GSp}(n)$, and every parabolic subgroup of $\mathrm{GSp}(n)$ is of this form. Then if $N_P \subset P$ is the unipotent radical of P, then N_P is also the unipotent radical of $P \rtimes \mathbb{G}_m$. Now assume $\Theta(V_\sigma)|_{\mathrm{Sp}(n)}$ is in the space of cusp forms. Then for each $f \in \Theta(V_\sigma)$ and each N_P, we have $\int_{N_P(\mathbb{A}_F)} f(nh)\, dn = 0$ for all $h \in \mathrm{Sp}(n,\mathbb{A}_F)$. We have to show $\int_{N_P(\mathbb{A}_F)} f(ng)\, dn = 0$ for each $g \in \mathrm{GSp}(n,\mathbb{A}_F)$. Let $g_1 \in \mathrm{Sp}(n,\mathbb{A}_F)$ be as in (2), and $f' = (g_1^{-1}g)\cdot f$. Then $f'|_{\mathrm{Sp}(n,\mathbb{A}_F)} \in \Theta(V_\sigma)|_{\mathrm{Sp}(n)}$. So $\int_{N_P(\mathbb{A}_F)} f'(ng_1)\, dn = 0$. But $\int_{N_P(\mathbb{A}_F)} f'(ng_1)\, dn = \int_{N_P(\mathbb{A}_F)} f(ng_1 g_1^{-1} g)\, dn = \int_{N_P(\mathbb{A}_F)} f(ng)\, dn$. The converse is obvious. □

We should also mention the following, whose proof is elementary and left to the reader.

Lemma 4.5. *Let σ be an automorphic representation of $\mathrm{GO}(X,\mathbb{A}_F)$, and σ_1 an irreducible constituent of $\{f|_{\mathrm{O}(X,\mathbb{A}_F)} : f \in V_\sigma\}$ as an automorphic representation of $\mathrm{O}(X,\mathbb{A}_F)$. If we write $\sigma \cong \otimes\sigma_v$ and $\sigma_1 \cong \otimes\sigma_{1v}$, then each σ_{1v} is an irreducible constituent of the restriction $\sigma_v|_{\mathrm{O}(X,F_v)}$ of σ_v to $\mathrm{O}(X,F_v)$.*

5. The groups $\mathrm{GSO}(X)$ and $\mathrm{GO}(X)$ and their representations

In this section, we briefly review the classification of the groups $\mathrm{GSO}(X)$ and $\mathrm{GO}(X)$ and their representations when $\dim X = 4$. All the proofs are found in [R5], [HST], or the citations therein.

5.1. **The local case.** First consider the case where F is a local field. Then we have

Lemma 5.1. *For the sake of similitude theta correspondence, we may assume*

(1) *If $d = 1$, then $\mathrm{GSO}(X) \cong \mathrm{GSO}(M_2)$ or $\mathrm{GSO}(D)$, where M_2 is the space of 2×2 matrices over F with the quadratic form given by $-\det$, and D is the unique division quaternion algebra over F made into a quadratic form in the usual way. Then there is a natural bijection between $\mathrm{Irr}(\mathrm{GSO}(X),\chi)$ and the set of irreducible admissible representations $\tau_1\otimes\tau_2$ of $\mathrm{GL}(2)\times\mathrm{GL}(2)$ if $\mathrm{GSO}(X) \cong \mathrm{GSO}(M_2)$, or of $D^\times \times D^\times$ if $\mathrm{GSO}(X) \cong \mathrm{GSO}(D)$, such that*

both τ_1 and τ_2 have the same central character χ. In this case, we write $\pi = \pi(\tau_1, \tau_2)$.

(2) *If $d \neq 1$, then $\mathrm{GSO}(X) \cong \mathrm{GSO}(X_E)$, where $X_E = \{x \in \mathrm{M}_{2\times 2}(E) | \ {}^c x^t = x\}$ is the space of Hermitian matrices over $E = F(\sqrt{d})$ with the quadratic form given by $-\det$. Then there is a natural bijection between $\mathrm{Irr}(\mathrm{GSO}(X), \chi)$ and $\mathrm{Irr}(\mathrm{GL}(2, E), \chi \circ N_F^E)$. In this case, we write $\pi = \pi(\tau, \chi)$.*

Let $\pi \in \mathrm{Irr}(\mathrm{GSO}(X))$. We define π^c by taking $V_{\pi^c} = V_\pi$ and by letting $\pi^c(g)f = \pi(tgt)f$ for all $g \in \mathrm{GSO}(X)$ and $f \in V_\pi$. Also notice that $\mathrm{GO}(X) \cong \mathrm{GSO}(X) \rtimes \{1, t\}$, where we choose t to act on X as the matrix transpose if $X = M_2$ or $X = X_E$ and the quaternion conjugation if $X = D$. Then we have

- If $\pi \ncong \pi^c$, then $\mathrm{Ind}\,_{\mathrm{GSO}(X)}^{\mathrm{GO}(X)} \pi$ is irreducible, and we denote it by π^+.
- If $\pi \cong \pi^c$, then $\mathrm{Ind}\,_{\mathrm{GSO}(X)}^{\mathrm{GO}(X)} \pi$ is reducible. Indeed, it is the sum of two irreducible representations, and we write $\mathrm{Ind}\,_{\mathrm{GSO}(X)}^{\mathrm{GO}(X)} \pi \cong \pi^+ \oplus \pi^-$. Here, t acts on $\pi^\pm$ via a linear operator $\theta^\pm$ with the property that $(\theta^\pm)^2 = \mathrm{Id}$ and $\theta^\pm \circ g = tgt \circ \theta^\pm$ for all $g \in \mathrm{GSO}(X)$.

We can be more explicit about the irreducible components π^+ and π^-. First assume $d = 1$. In this case, it is easy to see that, via ρ, t acts on $\mathrm{GL}(2) \times \mathrm{GL}(2)$ or $D^\times \times D^\times$ by $t \cdot (g_1, g_2) = (g_2, g_1)$, and if $\pi = \pi(\tau_1, \tau_2)$ is such that $\pi \cong \pi^c$, then $\tau_1 \cong \tau_2$. Then we can choose $\theta^\pm$ to be such that $\theta^+(x_1 \otimes x_2) = x_2 \otimes x_1$ and $\theta^-(x_1 \otimes x_2) = -x_2 \otimes x_1$ for $x_1 \otimes x_2 \in V_{\tau_1 \otimes \tau_2}$. We choose π^+ and π^- accordingly. Note that our choice of π^+ is the preferred extension defined in [PSP, p.227].

Next assume $d \neq 1$. In this case t acts, via ρ, on $\mathrm{GL}(2, E)$ in such a way that $t \cdot g = {}^c g$. If $\pi = \pi(\tau, \chi)$ is such that $\pi \cong \pi^c$, then $\tau \cong \tau^c$. Note that τ has a unique Whittaker model; namely, it is realized as a space of functions $f : \mathrm{GL}(2, E) \to \mathbb{C}$ such that $f\left(\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)\right) = \psi_v(\mathrm{tr}\ a) f(g)$ for all $a \in E$ and $g \in \mathrm{GL}(2, E)$, where ψ_v is a fixed additive character of F. Then we define $\theta^\pm$ to be the linear operator that acts on this space of Whittaker functions by $f \mapsto \pm f \circ c$, and θ^+ is chosen to be the one that acts as $f \mapsto f \circ c$. We choose π^+ and π^- accordingly.

Remark 5.2. We should note that our choice of π^+ and π^- is different from that of Roberts in [R5], but rather we follow [HST], although, if π is spherical, it turns out that our π^+ is spherical and coincides with the notation of [R5]. Also the reader should notice that in the above discussion the fields F and E do not have to be non-Archimedean.

5.2. **The global case.** Now we consider the global case, and hence in this subsection F is a global field of char $F = 0$ and the groups $\mathrm{GSO}(X)$, $\mathrm{GO}(X)$, etc. denote algebraic groups over F. If $d \neq 1$ and $E = F(\sqrt{d})$, then let c be the non-trivial element in $\mathrm{Gal}(E/F)$. For each quaternion algebra D over F, let $B_{D,E} = D \otimes E$. Then for each $g \in B_{D,E}$, we define ${}^c g^*$ by linearly extending the operation ${}^c(x \otimes a)^* = x^* \otimes {}^c a$ where ^{c} is the Galois conjugation and * is the quaternion conjugation. Then the space $X_{D,E} = \{g \in B_{D,E} : {}^c g^* = g\}$ can be made into a four dimensional quadratic space over F via the reduced norm of the quaternion $B_{D,E}$. Similarly to the local case, we have

Lemma 5.3. *For the sake of similitude theta correspondence, we may assume*

(1) *If $d = 1$, then $\mathrm{GO}(X)$ is isomorphic to $\mathrm{GO}(D)$ for some (possibly split) quaternion algebra over F. Then there is a natural bijective correspondence*

between an irreducible cuspidal automorphic representation π *of* $\mathrm{GSO}(X,\mathbb{A}_F) \cong \mathrm{GSO}(D,\mathbb{A}_F)$ *whose central character is* χ *and an irreducible cuspidal automorphic representation* $\tau_1 \otimes \tau_2$ *of* $D^\times(\mathbb{A}_F) \times D^\times(\mathbb{A}_F)$ *such that both* τ_1 *and* τ_2 *have the central character* χ*. In this case, we write* $\pi = \pi(\tau_1,\tau_2)$*.*

(2) *If* $d \neq 1$*, then there exists a quaternion algebra* D *over* F *such that* $\mathrm{GO}(X) \cong \mathrm{GO}(X_{D,E})$*. Then there is a natural bijective correspondence between an irreducible cuspidal automorphic representation* π *of* $\mathrm{GSO}(X,\mathbb{A}_F) \cong \mathrm{GSO}(X_{D,E},\mathbb{A}_F)$ *whose central character is* χ *and an irreducible cuspidal automorphic representation* τ *of* $B^\times_{D,E}(\mathbb{A}_E)$ *whose central character is of the form* $\chi \circ N^E_F$*. In this case, we write* $\pi = \pi(\tau,\chi)$*.*

Finally, we consider the relation between irreducible cuspidal automorphic representations of the two groups $\mathrm{GSO}(X,\mathbb{A}_F)$ and $\mathrm{GO}(X,\mathbb{A}_F)$. First define π^c by taking $V_{\pi^c} = \{f \circ c : f \in V_\pi\}$, where $c : \mathrm{GSO}(X,\mathbb{A}_F) \to \mathrm{GSO}(X,\mathbb{A}_F)$ is the isomorphism given by conjugation $g \mapsto tgt$. Then clearly π^c is an irreducible cuspidal automorphic representation of $\mathrm{GSO}(X,\mathbb{A}_F)$. (Note that as an admissible representation, π^c is isomorphic to the representation π' with $V_{\pi'} = V_{\pi^c}$ and the action defined by $\pi'(g)f = \pi(tgt)f$, and so if we write $\pi \cong \otimes\pi_v$, then $\pi^c \cong \otimes\pi^c_v$.) By the multiplicity one theorem, $\pi \cong \pi^c$ implies $V_\pi = V_{\pi^c}$ and in this case $f \circ c \in V_\pi$. Also let σ be an irreducible cuspidal automorphic representation of $\mathrm{GO}(X,\mathbb{A}_F)$. Define $V^\circ_\sigma = \{f|_{\mathrm{GSO}(X,\mathbb{A}_F)} : f \in V_\sigma\}$. Then either $V^\circ_\sigma = V_\pi$ for some irreducible cuspidal automorphic representation π of $\mathrm{GSO}(X,\mathbb{A}_F)$ such that $\pi = \pi^c$, or $V^\circ_\sigma = V_\pi \oplus V_{\pi^c}$ for some irreducible cuspidal automorphic representation π of $\mathrm{GSO}(X,\mathbb{A}_F)$ such that $\pi \neq \pi^c$. (See [HST, pp.381–382].) Then we have

Proposition 5.4. *Define* $\widehat{\pi}$ *to be the sum of all the irreducible cuspidal automorphic representations of* $\mathrm{GO}(X,\mathbb{A}_F)$ *lying above* π*, i.e.* $\widehat{\pi} = \bigoplus_i \sigma_i$ *where* σ_i *runs over all the irreducible cuspidal automorphic representations of* $\mathrm{GO}(X,\mathbb{A}_F)$ *such that* $V^\circ_{\sigma_i} = V_\pi$ *if* $\pi = \pi^c$*, or* $V^\circ_{\sigma_i} = V_\pi \oplus V_{\pi^c}$ *otherwise.*

Then

$$\widehat{\pi} \cong \bigoplus_\delta \bigotimes_v \pi_v^{\delta(v)},$$

where δ *runs over all the maps from the set of all places of* F *to* $\{\pm\}$ *with the property that* $\delta(v) = +$ *for almost all places of* F*, and* $\delta(v) = +$ *if* $\pi_v \ncong \pi^c_v$*, and further if* $\pi \cong \pi^c$*, then* $\prod_v \delta(v) = +$*. Moreover each* $\bigotimes \pi_v^{\delta(v)}$ *is (isomorphic to) an irreducible cuspidal automorphic representation of* $\mathrm{GO}(X,\mathbb{A}_F)$*.*

Proof. When π is generic, the proof is given in [HST, pp.382-383]. Also one can see that the proof there works even if π is not generic as long as π satisfies $\pi \ncong \pi^c$. So we assume that π is not generic and $\pi \cong \pi^c$. The basic idea of the proof is in [PSP, pp.227-228]. Note that our choice of π^+ is the preferred extension defined in [PSP, p.227]. Then following [PSP], define a linear functional $e_\pi : V_\pi \to \mathbb{C}$ given by evaluation at 1, *i.e.* $e_\pi(f) = f(1)$. Now notice that we have an isomorphism $V_\pi \cong V_{\pi^\delta}$ of vector spaces, because $\pi \cong \bigotimes \pi_v$ and for each place v, $V_{\pi_v} \cong V_{\pi_v^{\delta(v)}}$. Then via this isomorphism we define a linear functional $e_{\pi^\delta} : \pi^\delta \to \mathbb{C}$. Note that there is a natural action of $\mathrm{GO}(X,\mathbb{A}_F)$ on e_{π^δ} via $g \cdot e_{\pi^\delta}(f) = e_{\pi^\delta}(g^{-1} \cdot f)$. Then one can see that π^δ is automorphic if and only if $g \cdot e_{\pi^\delta} = e_{\pi^\delta}$ for all $g \in \mathrm{GO}(X,F)$. But clearly for all $g \in \mathrm{GSO}(X,F)$, $g \cdot e_{\pi^\delta} = e_{\pi^\delta}$. Hence it suffices to show that

$t \cdot e_{\pi^\delta} = e_{\pi^\delta}$. (Note that $t = \bigotimes_v t_v$ is coherent in the sense of [PSP, p.228]. Also note that in [PSP] our t_v is denoted by σ_v.) Then it can be easily seen that $t_v \cdot e_{\pi^\delta} = \delta(v) e_{\pi^\delta}$. Then $t = \bigotimes_v t_v$ acts as multiplication by $\prod_v \delta(v)$. Thus e_{π^δ} is invariant under the action of t if and only if $\prod_v \delta(v) = 1$. This completes the proof. $\square$

This proposition tells us that if π is an irreducible cuspidal automorphic representation of $\mathrm{GSO}(X, \mathbb{A}_F)$ and δ is a map from the set of all places of F to $\{\pm\}$ having the property described in the above proposition, then there is an irreducible cuspidal automorphic representation $\sigma = (\pi, \delta)$ of $\mathrm{GO}(X, \mathbb{A}_F)$ lying above π such that $\sigma \cong \bigotimes \pi_v^{\delta(v)}$. We call such a map δ an "extension index" of π, and (π, δ) the extension of π with an extension index δ.

6. Local parameters of unramified theta lifts

After some preliminaries, we will explicitly compute the local parameters of the unramified theta lifts from GO(4) to GSp(1) (= GL(2)).

In this section, the groups $\mathrm{GO}(X, F_v), \mathrm{GSp}(n, F_v)$, etc. are all denoted simply by $\mathrm{GO}(X)$, $\mathrm{GSp}(n)$, etc., and we simply write F for F_v. Moreover we assume that v is finite. Also "Ind" always means unnormalized induction, and whenever we use normalized induction, we use the notation "n-Ind".

6.1. **Preliminaries.** For our computation of the local parameters, we need the Jacquet module of the Weil representation, which is done in [HST], which, in turn, comes from [Kd1]. We will repeat the essential point. For this, let us decompose X as $X = Y_r \oplus W \oplus Y_r^*$, where Y_r is a totally isotropic space and Y_r^* is its complement so that $Y_r \oplus Y_r^*$ is r copies of the hyperbolic plane. We denote the standard basis of Y_r by $\{f_1, f_2, \ldots, f_r\}$, and write $l = \dim W$ so that $m = 2r + l$. Now let Q_r be the parabolic subgroup of $\mathrm{GO}(X)$ preserving the flag $\langle f_1 \rangle \subset \langle f_1, f_2 \rangle \subset \cdots \subset \langle f_1, f_2, \ldots, f_r \rangle$ so that its Levi factor is isomorphic to $\mathrm{GO}(W) \times \mathbb{G}_m^r$. Let Q be the parabolic subgroup preserving the flag $\langle f_1, f_2, \ldots, f_r \rangle$, so that its Levi factor is isomorphic to $\mathrm{GO}(W) \times \mathrm{GL}(r)$. Further let $S_Q = R \cap (\mathrm{GSp}(n) \times Q)$ be the parabolic subgroup of R whose Levi factor M_Q is isomorphic to $R_{n,W} \times \mathrm{GL}(r)$, where $R_{n,W}$ is defined in the same way as R, but with respect to $\mathrm{GSp}(n)$ and $\mathrm{GO}(W)$. We denote by N_Q its unipotent radical. Also let S_{Q_r} be the parabolic subgroup of M_Q whose Levi factor M_{Q_r} is isomorphic to $R_{n,W} \times \mathbb{G}_m^r$, *i.e.* $S_{Q_r} = M_Q \cap (\mathrm{GSp}(n) \times Q_r)$. We denote by N_{Q_r} its unipotent radical. Now let P_i be the standard parabolic subgroup of $\mathrm{GSp}(n)$ whose Levi factor is isomorphic to $\mathbb{G}_m^i \times \mathrm{GSp}(n-i)$. Then we define S_{P_i,Q_r} to be the parabolic subgroup of M_{Q_r} whose Levi factor M_{P_i,Q_r} is isomorphic to $\mathbb{G}_m^i \times R_{n-i,W} \times \mathbb{G}_m^r$, *i.e.* $S_{P_i,Q_r} = M_{Q_r} \cap (P_i \times Q_r)$. We write a typical element in M_{P_i,Q_r} by $(\alpha_1, \cdots, \alpha_i, (g,h), \beta_1, \ldots, \beta_r)$. Notice we have the inclusions $S_{P_i,Q_r} \subset M_{Q_r} \subset S_{Q_r} \subset M_Q$. Also we can set $P_0 = \mathrm{GSp}(n)$ and so $S_{P_0,Q_r} = M_{P_0,Q_r} = M_{Q_r}$. Now the unnormalized Jacquet module of $\omega_{n,X}$ is computed as follows, which is nothing but Lemma 4 of [HST] with the notation adjusted to ours.

Proposition 6.1. *The unnormalized Jacquet module $J = J(\omega_{n,X})_{N_Q}$ of $\omega_{n,X}$ with respect to N_Q has a filtration*

$$0 = J^{(s+1)} \subset J^{(s)} \subset \cdots \subset J^{(1)} \subset J^{(0)} = J$$

of M_Q-modules, where $s = \min\{n, r\}$. Let $I^{(i)} = J^{(i)}/J^{(i+1)}$. Then the unnormalized Jacquet module $I^{(i)}_{N_{Q_r}}$ of $I^{(i)}$ with respect to N_{Q_r}, which is an M_{Q_r}-module, is *given by*

$$I^{(i)}_{N_{Q_r}} = \mathrm{Ind}\,^{M_{Q_r}}_{S_{P_i,Q_r}} \sigma_{i,r},$$

where $\sigma_{i,r}$ is given by the representation of M_{P_i,Q_r}, which is of the form

$$\begin{aligned}(\alpha_1, \cdots, \alpha_i, (g,h), \beta_1, \dots, \beta_r) \mapsto & |\nu(g)|^{nr/2-ni-li/4}|\alpha|^{n+l/2}(\alpha, (-1)^{l/2}D_W)|\beta|^n \\ & \cdot \prod_{j=1}^{i} \mu_{r-i+j}(\alpha^{-1}_{i-j+1}\beta_{r-i+j}\nu(g))\omega_{n-i,W}(g,h)\end{aligned}$$

for some characters μ_{r-i+j}, where $\alpha = \alpha_1 \cdots \alpha_i$, $\beta = \beta_1 \cdots \beta_{r-i}$, $D_W = \mathrm{disc}\,W$ and $(,)$ is the Hilbert symbol.

Remark 6.2. In the above notation, if one of $n-i$ and W is zero, $\omega_{n-i,W}$ is taken to be the trivial representation. If $n-i$ is zero, we write a typical element in M_{P_i,Q_r} by $(\alpha_1, \cdots, \alpha_i, (h), \beta_1, \dots, \beta_r)$ where $h \in \mathrm{GO}(W)$, and we have to replace $\nu(g)$ by $\nu(h)^{-1}$ in the above formula. If W is zero, we write a typical element in M_{P_i,Q_r} by $(\alpha_1, \cdots, \alpha_i, (g), \beta_1, \dots, \beta_r)$, where $g \in \mathrm{GSp}(n-i)$, and g acts as in the above formula. If both $n-i$ and W are zero, we have $M_{P_i,Q_r} \cong \mathbb{G}_m^i \times \mathbb{G}_m \times \mathbb{G}_m^r$ and write a typical element by $(\alpha_1, \cdots, \alpha_i, (\lambda), \beta_1, \dots, \beta_r)$ if, for the natural projection $\iota : M_{P_i,Q_r} \to P_i$, we have $\nu(\iota(\alpha_1, \cdots, \alpha_i, (\lambda), \beta_1, \dots, \beta_r)) = \lambda$, and we have to replace $\nu(g)$ by λ.

Remark 6.3. Although this is a small point, the reader should notice that the choice of the parabolic $R_{P_i,Q}$ in [HST] is not quite correct and should be replaced by our S_{P_i,Q_r}. Also in [HST] there is a misprint for the index of α inside ν_{r-i+j}.

The following lemma will be necessary later.

Lemma 6.4. *Keeping the above notation, let μ and δ be admissible representations of Q_r and P_i, respectively. Then the natural map* $\mathrm{Ind}\,^{\mathrm{GSp}(n)\times\mathrm{GO}(V)}_{P_i\times Q_r}(\delta\otimes\mu) \to \mathrm{Ind}\,^{R}_{R\cap(P_i\times Q_r)}(\delta\otimes\mu)$ *is an injective R-homomorphism.*

Proof. Let $F \in \mathrm{Ind}\,^{\mathrm{GSp}(n)\times\mathrm{GO}(V)}_{P_i\times Q_r}(\delta\otimes\mu)$ and let $\bar{F}$ be its image under the natural map, namely $\bar{F} = F|_R$. Assume $\bar{F} = 0$. Then for each $(g,h) \in \mathrm{GSp}(n) \times \mathrm{GO}(V)$, let us define

$$u = u(g,h) = \begin{pmatrix} I_n & O \\ O & \nu(g)\nu(h)I_n \end{pmatrix} \in \mathrm{GSp}(n),$$

where I_n is the $n \times n$ identity matrix. Then $(u,1) \in P_i \times Q_r$, $\nu(u) = \nu(g)\nu(h)$, and $(u^{-1}g,h) \in R$. So we have

$$F(g,h) = F(uu^{-1}g,h) = (\delta\otimes\mu)(u,1)F(u^{-1}g,h) = (\delta\otimes\mu)(u,1)\bar{F}(u^{-1}g,h) = 0.$$

Thus the map is injective. □

6.2. **Computation of local parameters.** We will compute the local parameters of unramified theta lifts from $\mathrm{GO}(X)$ to $\mathrm{GSp}(1)$ $(= \mathrm{GL}(2))$. First assume $d \neq 1$. First notice the following.

Remark 6.5. Assume the extension E/F is unramified. Then it is easy to see that any unramified character η on $E^\times$ is Galois invariant and written as $\eta = \chi \circ N$ for some unramified character χ on $F^\times$. Accordingly if $\tau = \text{n-Ind}\,(\eta, \eta')$ is unramified, then $\eta = \chi_1 \circ N$ and $\eta' = \chi_2 \circ N$ for unramified characters χ_1 and χ_2 on $F^\times$.

Then we have

Proposition 6.6. *Assume the extension E/F is unramified, and $\sigma \in \mathrm{Irr}(\mathrm{GO}(X))$ is unramified; i.e., $\sigma = \pi^+$ for some unramified $\pi = \pi(\tau, \chi)$. (So by Remark 6.5, $\pi = \pi^c$.) If σ corresponds to $\Pi \in \mathrm{Irr}(\mathrm{GSp}(1))$ and Π is unramified, then τ is the base change lift of Π. In particular such a Π is unique. Moreover the central character of Π is $\chi_{E/F}\chi$.*

Proof. Assume σ corresponds to an unramified Π. Since τ is unramified, we can write $\tau = \text{n-Ind}\,_P^{\mathrm{GL}(2,E)}(\eta \otimes \eta')$ for unramified characters η and η' on $E^\times$, or by using unnormalized induction, $\tau = \mathrm{Ind}\,_P^{\mathrm{GL}(2,E)}(\tilde{\eta} \otimes \tilde{\eta}')$, where $\tilde{\eta} = |\cdot|_E^{1/2}\eta$ and $\tilde{\eta}' = |\cdot|_E^{-1/2}\eta'$. (Here we note that $|\cdot|_E = |\cdot| \circ N$.) By Remark 6.5, $\eta = \chi_1 \circ N$ and $\eta' = \chi_2 \circ N$ for unramified characters χ_1 and χ_2 on $F^\times$ with $\chi_1\chi_2 = \chi$. Then $\pi = \pi(\chi, \tau) = \mathrm{Ind}\,_Q^{\mathrm{GSO}(X)}(\mu)$, where Q is the parabolic preserving the flag

$$\langle \begin{pmatrix} 0 & \sqrt{d} \\ 0 & 0 \end{pmatrix} \rangle,$$

and μ is defined by

$$\mu(\begin{pmatrix} \beta_1 & * & * & * \\ 0 & a & b\,d & * \\ 0 & b & a & * \\ 0 & 0 & 0 & \lambda{\beta_1}^{-1} \end{pmatrix}) = (\tilde{\eta}'/\chi)(\lambda{\beta_1}^{-1})\tilde{\eta}(a + b\sqrt{d}) = (|\cdot|\chi_1/\chi_2)(\beta_1)(|\cdot|^{-1/2}\chi_2)(\lambda),$$

where the matrix representation of each element of $\mathrm{GO}(X)$ is with respect to the ordered basis

$$\begin{pmatrix} 0 & \sqrt{d} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \sqrt{d} & 0 \\ 0 & -\sqrt{d} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ (2/d)\sqrt{d} & 0 \end{pmatrix}.$$

(Here we view $\tilde{\eta}'$ as a character on $F^\times$ via $F^\times \hookrightarrow E^\times$. See also pp.297-298 in [R5].) The second equality can be seen as follows. First notice that the middle block

$$h = \begin{pmatrix} a & b\,d \\ b & a \end{pmatrix}$$

is identified with $E^\times$ by $h \leftrightarrow a+b\sqrt{d}$. In particular the Levi factor of Q is isomorphic to $\mathrm{GO}(W) \times \mathbb{G}_m$, where W is the quadratic space E equipped with the quadratic form $-N$. Thus $\lambda = N(h)$ and so $\tilde{\eta}(a + b\sqrt{d}) = (|\cdot|_E^{1/2}\eta)(h) = (|\cdot|^{1/2}\chi_1)(N(h)) = (|\cdot|^{1/2}\chi_1)(\lambda)$.

Now assume Π corresponds to σ, so there is a non-zero R-homomorphism $\omega_{1,X} \longrightarrow \Pi \otimes \sigma$. Notice that there is an injective $\mathrm{GO}(X)$-homomorphism $\sigma \hookrightarrow \mathrm{Ind}\,_{\mathrm{GSO}(X)}^{\mathrm{GO}(X)}\sigma$ ($\cong \mathrm{Ind}\,_Q^{\mathrm{GO}(X)}\mu$). Then by composing with this injection, we have a non-zero R-homomorphism

$$\omega_{1,X} \longrightarrow \Pi \otimes \mathrm{Ind}\,_Q^{\mathrm{GSO}(X)}\mu.$$

By Lemma 6.4, we have a non-zero R-homomorphism

$$\omega_{1,X} \longrightarrow \mathrm{Ind}\,_{R \cap (\mathrm{GSp}(1) \times Q)}^{R}\Pi \otimes \mu.$$

Then if we take the Jacquet module of $\omega_{1,X}$ with respect to N_Q, the Frobenius reciprocity together with Proposition 6.1 gives M_Q-homomorphisms

$$0 \subset J^{(1)} \subset J^{(0)} \xrightarrow{\ \varphi\ } \Pi \otimes \mu.$$

(Here Π is actually $\Pi|_{\mathrm{GSp}(1)^+}$, where $\mathrm{GSp}(1)^+ = \{g \in \mathrm{GSp}(1) : \nu(g) \in \nu(\mathrm{GO}(X))\}$.) It is easy to see that $\ker\varphi \not\supseteq J^{(1)}$. (Indeed, if $\ker\varphi \supseteq J^{(1)}$, by applying Proposition 6.1, we will get a contradiction.)

So by restricting φ to $J^{(1)}$, Proposition 6.1 together with $M_{Q_1} = M_Q$ gives a non-zero M_Q-homomorphism

$$J^{(1)}(=I^{(1)}) = \mathrm{Ind}\,_{S_{P_1,Q}}^{M_Q}\sigma_{1,1} \xrightarrow{\varphi'} \Pi\otimes\mu.$$

By considering the action of the elements of the form $(1,(1),\beta_1)$ on $\mathrm{Ind}\,_{S_{P_1,Q}}^{M_Q}\sigma_{1,1}$ and $\Pi\otimes\mu$, we see that

$$\mu_1 = \chi/\tilde{\eta}' = |\cdot|\chi_1/\chi_2,$$

where μ_1 is as in Proposition 6.1.

Assume $\Pi = \mathrm{Ind}\,_{P_1}^{\mathrm{GSp}(1)}\delta$ for the character δ on P_1 given by

$$\delta(\begin{pmatrix}\alpha_1 & * \\ 0 & \lambda\alpha_1^{-1}\end{pmatrix}) = \delta_1(\alpha_1)\delta_2(\lambda).$$

Then $\Pi\otimes\mu \cong \mathrm{Ind}\,_{P_1\times Q}^{\mathrm{GSp}(1)\times Q}(\delta\otimes\mu)$. By Lemma 6.4, φ' gives rise to a non-zero M_Q-homomorphism

$$\mathrm{Ind}\,_{S_{P_1,Q}}^{M_Q}\sigma_{1,1} \xrightarrow{\varphi'} \mathrm{Ind}\,_{S_{P_1,Q}}^{M_Q}(\delta\otimes\mu),$$

which we also call φ'. Let us view φ' as an R-homomorphism by restricting to the elements of the form $((g,h),1)$, and so we have a non-zero R-homomorphism

$$\mathrm{Ind}\,_{R_{1,W}}^{R}\sigma_{1,1} \xrightarrow{\varphi'} \mathrm{Ind}\,_{R_{1,W}}^{R}(\delta\otimes\mu),$$

where $\sigma_{1,1}$ and $\delta\otimes\mu$ are restricted to $R_{1,W}$. This φ' can be made into a non-zero $\mathrm{GSp}(1)$-homomorphism φ'' via the diagram

$$\begin{tikzcd}
\mathrm{Ind}\,_{R_{1,W}}^{R}\sigma_{1,1} \arrow[r, hook, "\tilde{\Phi}"] \arrow[d, "\varphi'"] & \mathrm{Ind}\,_{P_1^+}^{\mathrm{GSp}(1)^+}\widehat{\sigma_{1,1}} \arrow[r, hook] \arrow[d] & \mathrm{Ind}\,_{P_1}^{\mathrm{GSp}(1)}\widehat{\sigma_{1,1}} \arrow[d, "\varphi''"] \\
\mathrm{Ind}\,_{R_{1,W}}^{R}(\delta\otimes\mu) \arrow[r, hook] & \mathrm{Ind}\,_{P_1^+}^{\mathrm{GSp}(1)^+}(\widehat{\delta\otimes\mu}) \arrow[r, hook] & \mathrm{Ind}\,_{P_1}^{\mathrm{GSp}(1)}(\widehat{\delta\otimes\mu}),
\end{tikzcd}$$

where all the horizontal arrows are injective and the vertical arrows are non-zero intertwining operators for the corresponding groups. This diagram is given as follows. First, define $\iota : R \to \mathrm{GSp}(1)^+$ by $(g,h)\mapsto g$. Also let $\widehat{\sigma_{1,1}}$ be the character on $P_1^+(=\mathrm{GSp}(1)\cap P_1)$ defined by

$$\begin{pmatrix}\alpha_1 & * \\ 0 & \lambda\alpha_1^{-1}\end{pmatrix} \mapsto \sigma_{1,1}(\alpha_1,(h),1) = |\nu(h)||\alpha_1|^2(\alpha_1,-D_W)\mu(\alpha_1^{-1}\nu(h)^{-1}),$$

where $h \in \mathrm{GO}(W)$ is such that $\nu(h)=\lambda^{-1}$. This is well defined. (Also note that $(\alpha_1,-D_W) = (\alpha_1,d)$ is the quadratic character for the quadratic extension E/F and we put $(\alpha_1,d) = \chi_{E/F}(\alpha_1)$.) This gives us a natural map $\Phi : \sigma_{1,1} \to \widehat{\sigma_{1,1}}$ that respects the actions of $R_{1,W}$ and P_1^+ via ι. Then Φ can be extended to a non-zero map

$$\tilde{\Phi} : \mathrm{Ind}\,_{R_{1,W}}^{R}\sigma_{1,1} \longrightarrow \mathrm{Ind}\,_{P_1}^{R}\widehat{\sigma_{1,1}}$$

defined by $\tilde{\Phi}(F)(g) = \Phi(F((g,h),1))$, where $h \in \mathrm{GO}(W)$ is such that $\iota(g,h) = g$. This is well-defined, injective, and respects the actions of R and $\mathrm{GSp}(1)^+$ via ι. We can similarly define a non-zero injective map

$$\mathrm{Ind}\,{}_{R_{1,W}}^{R}(\delta \otimes \mu) \longrightarrow \mathrm{Ind}\,{}_{P_1^+}^{\mathrm{GSp}(1)^+}(\widehat{\delta \otimes \mu}).$$

Here $\widehat{\delta \otimes \mu}(\left(\begin{smallmatrix} \alpha_1 & * \\ 0 & \lambda\alpha_1{}^{-1}\end{smallmatrix}\right)) = \delta_1(\alpha_1)\delta_2(\lambda)(|\cdot|^{-1/2}\chi_2)(\lambda^{-1})$. To see the left square of the diagram, notice that, in general, if γ is a character on P_1, there is an injective map $\mathrm{Ind}\,{}_{P_1{}^+}^{\mathrm{GSp}(1)^+}\gamma \to \mathrm{Ind}\,{}_{P_1}^{\mathrm{GSp}(1)}\gamma$ given by $F \mapsto \tilde{F}$ where

$$\tilde{F}(g) = \gamma(\left(\begin{smallmatrix} 1 & 0 \\ 0 & \nu(g)^{-1}\end{smallmatrix}\right))F((\left(\begin{smallmatrix} 1 & 0 \\ 0 & \nu(g)\end{smallmatrix}\right)g).$$

This map respects the actions of $\mathrm{GSp}(1)^+$ and $\mathrm{GSp}(1)$ via the inclusion $\mathrm{GSp}(1)^+ \hookrightarrow \mathrm{GSp}(1)$. Since both $\widehat{\sigma_{1,1}}$ and $\widehat{\delta \otimes \mu}$ can be viewed as characters on P_1 in the obvious way, we have the injective maps as in the above diagram.

Now let us switch to the notation $P = \left(\begin{smallmatrix} a & * \\ 0 & d\end{smallmatrix}\right)$ for the parabolic. Then

$$\widehat{\sigma_{1,1}}(\begin{pmatrix} a & * \\ 0 & d\end{pmatrix}) = (|\cdot|\chi_{E/F})(a)(\chi_1/\chi_2)(d)$$

and

$$\widehat{\delta \otimes \mu}(\begin{pmatrix} a & * \\ 0 & d\end{pmatrix}) = (\delta_1\delta_2|\cdot|^{1/2}/\chi_2)(a)(|\cdot|^{1/2}\delta_2/\chi_2)(d).$$

Note that, by twisting by $\chi_2/|\cdot|^{1/2}$, φ'' gives rise to a non-zero $\mathrm{GL}(2)$-homomorphism

$$(\chi_2/|\cdot|^{1/2}) \otimes \mathrm{Ind}\,{}_P^{\mathrm{GL}(2)}(\widehat{\sigma_{1,1}}) \longrightarrow (\chi_2/|\cdot|^{1/2}) \otimes \mathrm{Ind}\,{}_P^{\mathrm{GL}(2)}(\widehat{\delta \otimes \mu}),$$

which can be rewritten as

$$\text{n-Ind}\,{}_P^{\mathrm{GL}(2)}(\chi_{E/F}\chi_2 \otimes \chi_1) \longrightarrow \Pi.$$

Therefore $\Pi \cong \text{n-Ind}\,{}_P^{\mathrm{GL}(2)}(\chi_{E/F}\chi_2 \otimes \chi_1)$, and so τ is the base change lift of Π whose central character is $\chi_{E/F}\chi_1\chi_2 = \chi_{E/F}\chi$. ☐

The case $d = 1$ is well known. Namely,

Proposition 6.7. *Assume $d = 1$ and $\sigma \in \mathrm{Irr}(\mathrm{GO}(X))$ is unramified,* i.e. $\sigma = \pi^+$ *for some unramified $\pi = \pi(\tau_1, \tau_2)$ with τ_i's unramified representations of* $\mathrm{GL}(2, F)$*. If σ corresponds to $\Pi \in \mathrm{Irr}(\mathrm{GSp}(1))$, then $\tau_1 \cong \tau_2 \cong \Pi$. In particular, such a Π is unique.*

This proposition is known to be true even for the ramified case. But it might be a good exercise to prove the unramified case in the way we did for the $d \neq 1$ case.

7. Proof of Theorems 1.2 and 1.3

Now we are ready to prove our main theorems. First we give a proof of Theorem 1.2, the major arguments of which are essentially given in [R5].

Proof of Theorem 1.2. Let σ be a cuspidal automorphic representation of $\mathrm{GO}(X, \mathbb{A}_F)$ satisfying the hypotheses of Theorem 1.2. Also assume that σ is extended from an irreducible cuspidal automorphic representation π of $\mathrm{GSO}(X, \mathbb{A}_F)$, *i.e.* $\sigma = (\pi, \delta)$ for some δ as discussed at the end of Section 5. Also let σ_1 be an irreducible constituent of the space $\{f|_{\mathrm{O}(X,\mathbb{A}_F)} : f \in V_\sigma\}$ viewed as an automorphic representation of $\mathrm{O}(X, \mathbb{A}_F)$.

First let us show the if part. So assume $\sigma \cong \bigotimes \sigma_v$ has the property that σ_v has a local theta lift to $\mathrm{GSp}(2, F_v)$ for all v. Then as discussed in [R1, Lemma 4.2], $\sigma_1 \cong \bigotimes \sigma_{1v}$ has the property that σ_{1v} has a local theta lift to $\mathrm{Sp}(2, F_v)$. Also note that

$$L^S(s,\sigma_1) = \begin{cases} L^S(s,\sigma_1) = L^S(s,\tau_1^{\mathrm{JL}} \times \tau_2^{\mathrm{JL}\vee}) \text{ if } d=1 \text{ and } \pi = \pi(\tau_1,\tau_2), \\ L^S(s,\tau^{\mathrm{JL}},\chi^{-1},\mathrm{Asai}) \text{ if } d \neq 1 \text{ and } \pi = \pi(\tau,\chi), \end{cases}$$

where ${}^{\mathrm{JL}}$ indicates the Jacquet-Langlands lift, $L^S(s,\tau_1^{\mathrm{JL}} \times \tau_2^{\mathrm{JL}})$ is the (incomplete) Rankin-Selberg L-function, and $L^S(s,\tau^{\mathrm{JL}},\chi^{-1},\mathrm{Asai})$ is the (incomplete) Asai L-function. (See Lemma 8.1 of [R5] for the proof.) It is well known that the Rankin-Selberg L-function does not vanish at $s = 1, 2$. The Asai L-function is also non-vanishing at $s = 1, 2$. This can be shown as follows. First, the non-vanishing at $s = 1$ is given in, say, [F1], which is also discussed in [R5, p.302]. The non-vanishing at $s = 2$ follows from absolute convergence of the Euler product for $\Re(s) > 1 + \frac{2}{9}$. The proof for this is elementary. Namely, by the explicit description of the unramified factor of the Euler product of the Asai L-function found in, say, [F2, p.200], and the known estimate of the bound of the Langlands parameters given by [KS], one sees that each unramified Euler factor of the Asai L-function is the product of four factors of the form

$$(1 - a_v q_v^{-s})^{-1} \quad \text{with} \quad |a_v| < q_v^{2/9}.$$

Then the convergence of this product can be easily proven by using Lemma 2 on p.187 and the first paragraph of p.188 of [M]. Thus by Theorem 1.2 and Lemma 4.4, we see that the global theta lift of σ to $\mathrm{GSp}(2,\mathbb{A}_F)$ does not vanish.

Conversely, assume the global theta lift $\Theta_2(V_\sigma)$ to $\mathrm{GSp}(2,\mathbb{A}_F)$ does not vanish. If $\Theta_2(V_\sigma)$ is not contained in the space of cusp forms, then by the persistence principle the global theta lift $\Theta_1(V_\sigma)$ to $\mathrm{GSp}(1,\mathbb{A}_F)$ is non-zero and is contained in the space of cusp forms. Now let (Π, V_Π) be an irreducible constituent of the lift $\Theta_1(V_\sigma)$. Then each σ_v and $\Pi_v^\vee$ correspond under the local theta correspondence for the similitude in the sense defined at the beginning of Section 4. Hence by the persistence principle, for each v, $\theta_2(\sigma_v) \neq 0$. If $\Theta_2(V_\sigma)$ is contained in the space of cusp forms, similarly one sees that $\theta_2(\sigma_v) \neq 0$ for each v. □

Next we prove Theorem 1.3. For this, we need

Proposition 7.1. *Let E/F be a quadratic extension of global fields, and τ an irreducible cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A}_E)$ whose central character is $\chi \circ N_F^E$ for a Hecke character χ on $\mathbb{A}_F^\times$. Then the incomplete Asai L-function $L^S(s,\tau,\chi^{-1},Asai)$ has a pole at $s = 1$ if and only if τ is the base change lift of a cuspidal automorphic representation τ_0 of $\mathrm{GL}(2,\mathbb{A}_F)$ whose central character is $\chi_{E/F}\chi$.*

Proof. The case where χ is trivial is essentially done on p.311 of [F1]. (Also see [FZ] for the Archimedean assumption imposed on [F1].) We can extend his method to the case where χ is non-trivial by making the following small modifications to [F1], though we restrict to the case $G = \mathrm{GL}(2)$. (Indeed, if $G \neq \mathrm{GL}(2)$, our argument would not work.) So in the following, all the notation is as in [F1].

First for each $\phi \in V_\tau$, let us define

$$I(s,\Phi,\phi,\chi) = \int_{Z(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)} E(g,\Phi,s)\phi(g)\chi^{-1}(\det g)dg,$$

where Φ is a Schwartz-Bruhat function on $\mathbb{A}_F^2$ and $E(g,\Phi,s)$ is the Eisenstein series as in [F1]. (Compare this with the integral in [F1, p.302].) Then as in [F1], we see that $I(s,\Phi,\phi,\chi)$ has a pole at $s=1$ if and only if $\hat{\Phi}(0)\neq 0$ and $\int \phi(g)\chi^{-1}(\det g)dg \neq 0$.

Next consider the integral

$$\Psi(s,\Phi,W,\chi)=\int_{N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} W(g)\Phi(\epsilon g)|g|^s\chi^{-1}(\det g)dg,$$

where W is the Whittaker vector as defined in [F1, p.302]. Then by the same computation as in [F1, p.303], we have

$$I(s,\Phi,\phi,\chi)=\Psi(s,\Phi,W,\chi),$$

and

$$\Psi(s,\Phi,W,\chi)=A(s,\Phi_S,W_S,\chi_S)L^S(s,\tau,\chi^{-1},\mathrm{Asai})$$

for some function $A(s,\Phi_S,W_S,\chi_S)$ with $\hat{\Phi}(0)\neq 0$. (See [F1, p.310].) Therefore the (incomplete) Asai L-function $L^S(s,\tau,\chi^{-1},\mathrm{Asai})$ has a pole at $s=1$ if and only if $\Psi(s,\Phi,W,\chi)$ has a pole at $s=1$, *i.e.* if and only if $\int\phi(g)\chi^{-1}(\det g)dg\neq 0$. The rest of the proof is basically the same as [F1], and is left to the reader. □

Now we have

Proof Theorem 1.3. Assume σ is an irreducible cuspidal automorphic representation of $\mathrm{GO}(X,\mathbb{A}_F)$ with the global theta lift $\Theta_2(V_\sigma)$ non-vanishing. (We write $\sigma=(\pi,\delta)$ as in Section 5.) Also let σ_1 be as in the proof of Theorem 1.2. Then by Lemma 4.4 the global theta lift $\Theta_2(V_{\sigma_1})$ to $\mathrm{Sp}(2,\mathbb{A}_F)$ does not vanish. Now assume $\Theta_1(V_{\sigma_1})=0$ so that $\Theta_2(V_{\sigma_1})$ provides an irreducible cuspidal automorphic representation of $\mathrm{Sp}(2,\mathbb{A}_F)$, which we denote by Π. Then by functoriality of the unramified theta correspondence, we have $L^S(\Pi,s,\chi_X)=\zeta_F^S(s)L^S(\sigma_1,s)$. It is well known that $\zeta_F^S(s)$ has a simple pole at $s=1$. So if $L^S(\sigma_1,s)$ has a simple pole at $s=1$, then $L^S(\Pi,s,\chi_X)$ would have a double pole, which contradicts the fact that the poles of the standard (incomplete) Langlands L-function of $\mathrm{Sp}(n)$ are at most simple. (See the remark after Proposition 1.7 of [I], and also Theorem 7.2.5 of [KR2].) Therefore if $\Theta_1(V_{\sigma_1})=0$, then $L^S(\sigma_1,s)$ cannot have a pole at $s=1$. Hence if $d=1$ and so $\pi=\pi(\tau_1,\tau_2)$, then $L^S(s,\tau_1^{\mathrm{JL}}\times\tau_2^{\mathrm{JL}^\vee})$ has a pole at $s=1$, *i.e.* $\tau_1=\tau_2$. Also if $d\neq 1$ and so $\pi=\pi(\tau,\chi)$, then $L^S(s,\tau^{\mathrm{JL}},\chi^{-1},\mathrm{Asai})$ has a pole at $s=1$, and thus by Proposition 7.1 we see that τ^{JL} is the base change lift of a cuspidal automorphic representation τ_0 of $\mathrm{GL}(2,\mathbb{A}_F)$ whose central character is $\chi_{E/F}\chi$.

Conversely, assume $\Theta_1(V_\sigma)\neq 0$, and so it gives rise to a cuspidal automorphic representation Π of $\mathrm{GL}(2,\mathbb{A}_F)$. First assume $d=1$ and so $\pi=\pi(\tau_1,\tau_2)$. Then note that for almost all v, we have $\pi_v=\pi(\tau_{1v},\tau_{2v})$, where τ_{1v} and τ_{2v} are spherical representations of $\mathrm{GL}(2,F_v)$. Then by Proposition 6.7 we have $\tau_{1v}\cong\tau_{2v}$. So by the strong multiplicity one theorem, $\tau_1\cong\tau_2$. Also in this case we know that $(\Pi_v)^\vee\cong\tau_{1v}\cong\tau_{2v}$ for almost all v. Thus we have $\Pi^\vee\cong\tau_1\cong\tau_2$. Moreover since $\Theta_1(V_\sigma)$ is closed under the right action of $\mathrm{GL}(2,\mathbb{A}_F)$, the multiplicity one theorem gives us $\Theta_1(V_\sigma)=V_\Pi=V_{\pi_1^\vee}=V_{\pi_2^\vee}$. We can similarly prove the $d\neq 1$ case by using strong multiplicity one and Propositions 6.7 and 6.6. This completes the proof. □

7.1. **A remark for the $d \neq 1$ case.** Our theorem for the $d \neq 1$ case reveals the following interesting phenomenon. Let τ be an irreducible cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_E)$ that is the base change lift of an irreducible cuspidal automorphic representation τ_0 of $\mathrm{GL}(2, \mathbb{A}_F)$. (Note that such a τ_0 is unique.) Assume the central character of τ_0 is χ, and so the central character of τ is $\chi \circ N_F^E$. Then we can identify τ with an irreducible cuspidal automorphic representation $\pi = \pi(\tau, \chi)$ of $\mathrm{GSO}(X, \mathbb{A}_F)$ whose central character is χ. Since τ is generic, π can always be extended to an irreducible cuspidal automorphic representation $\sigma = (\pi, \delta)$ of $\mathrm{GO}(X, \mathbb{A}_F)$ such that $\Theta_2(V_\sigma) \neq 0$. (See Appendix A.) In this case by Theorem 1.3 we have $\Theta_1(V_\sigma) = 0$. Thus $\Theta_2(V_\sigma)$ is cuspidal.

However since $\chi \circ N = \chi_{E/F}\chi \circ N_F^E$, we can also identify τ with an irreducible cuspidal automorphic representation $\pi' = \pi'(\tau, \chi_{E/F}\chi)$ of $\mathrm{GSO}(X, \mathbb{A}_F)$ whose central character is $\chi_{E/F}\chi$. Again we can extend π' to $\sigma' = (\pi', \delta')$ such that $\Theta_2(V_{\sigma'}) \neq 0$. Then this time, by Theorem 1.3 we have $\Theta_1(V_{\sigma'}) = V_{\tau_0^\vee} \neq 0$, and $\Theta_2(V_{\sigma'})$ is non-zero but non-cuspidal. This observation can be summarized as follows:

$$
\begin{tikzcd}[column sep=small]
 & & \pi = \pi(\chi, \tau) \arrow[r] & \sigma = (\pi, \delta) \arrow[r, "\substack{\text{theta}\\ \text{lift}}"] & \begin{cases} \Theta_2(V_\sigma) \neq 0 \\ \Theta_1(V_\sigma) = 0 \end{cases} \\
\tau_0 \arrow[r, "\substack{\text{base}\\ \text{change}}"] & \tau \arrow[ur] \arrow[dr] & & & \\
 & & \pi' = \pi'(\chi_{E/F}\chi, \tau) \arrow[r] & \sigma' = (\pi', \delta') \arrow[r, "\substack{\text{theta}\\ \text{lift}}"] & \begin{cases} \Theta_2(V_{\sigma'}) \neq 0 \\ \Theta_1(V_{\sigma'}) = V_{\tau_0^\vee}. \end{cases}
\end{tikzcd}
$$

(Of course this diagram makes sense only with the assumption that τ is cuspidal. It is known that τ is cuspidal if and only if τ_0 is not the automorphic induction of a Hecke character of E.)

Appendix A. Proof of Theorem 1.4

In this appendix, we will prove Theorem 1.4. Let us start with the following theorem due to [HPS].

Proposition A.1. *Assume σ is a generic irreducible cuspidal automorphic representation of $\mathrm{O}(X, \mathbb{A}_F)$ or $\mathrm{GO}(X, \mathbb{A}_F)$. Also let $T = \mathrm{O}(X)/\mathrm{SO}(X) \cong \mathrm{GO}(X)/\mathrm{GSO}(X)$. Then there is a unitary character μ of $T(F)\backslash T(\mathbb{A}_F)$ such that the space $\mu \otimes V_\sigma = \{\mu f : f \in V_\sigma\}$ has a non-zero theta lift $\Theta_2(\mu \otimes V_\sigma)$ to $\mathrm{Sp}(2, \mathbb{A}_F)$ or $\mathrm{GSp}(2, \mathbb{A}_F)$, respectively.*

Proof. The isometry case is essentially Theorem 8.1 of [HPS], which follows from Theorem 5.7 of the same paper. However, we should mention that in [HPS] they state that μ is on $T(\mathbb{A}_F)$, but from the proof of Theorem 5.7 it is easy to see that μ is actually on $T(F)\backslash T(\mathbb{A})$. Indeed if μ is not trivial on $T(F)$, μf cannot be automorphic. The similitude case follows immediately from the isometry case and Lemma 4.4. □

We should also mention the following, which is due to Roberts [R3, R5].

Proposition A.2. *Let π_v be an irreducible admissible representation of $\mathrm{GSO}(X, F_v)$. Then at least one of π_v^+ and π_v^- has non-zero theta lift to $\mathrm{GSp}(2, F_v)$. Moreover, if π_v is spherical, then π_v^+ always has non-zero theta lift.*

Proof. The first part is one of the main theorems of [R3]. (Note that the proof works even if $v | \infty$.) The second part is proven in [R5, Proposition 4.3]. □

Finally we can prove Theorem 1.4.

Proof of Theorem 1.4. (1). First consider any extension $\sigma = (\pi, \delta)$ of π. Then by Proposition A.1, there exists a unitary character μ so that $\mu \otimes V_\sigma$ has non-zero theta lift. Now for each $f \in V_\sigma$, clearly $f|_{\mathrm{GSO}(X,\mathbb{A}_F)} = (\mu f)|_{\mathrm{GSO}(X,\mathbb{A}_F)}$, which means that $V_\sigma^\circ = (\mu \otimes V_\sigma)^\circ$, *i.e.* both V_σ and $\mu \otimes V_\sigma$ lie above V_π. Hence there is an extension index δ' of π so that $\mu \otimes V_\sigma = V_{\sigma'}$ for $\sigma' = (\pi, \delta')$, which we re-choose to be σ.

(2). This immediately follows from Proposition 5.4, Theorem 1.2, and Proposition A.2. $\square$

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