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ADDITIVITY OF SPIN^c-QUANTIZATION UNDER CUTTING

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ABSTRACT. A G-equivariant spin^c-structure on a manifold gives rise to a virtual representation of the group G, called *the spin^c-quantization* of the manifold. We present a cutting construction for S^1-equivariant spin^c-manifolds and show that the quantization of the original manifold is isomorphic to the direct sum of the quantizations of the cut spaces. Our proof uses Kostant-type formulas, which express the quantization in terms of local data around the fixed point set of the S^1-action.

1. INTRODUCTION

In this paper we discuss S^1-equivariant spin^c-structures on compact oriented Riemannian S^1-manifolds and the Dirac operator associated to those structures. The index of the Dirac operator is a virtual representation of S^1 and is called *the spin^c-quantization* of the spin^c-manifold.

Also, we describe a cutting construction for spin^c-structures. Cutting was first developed by E. Lerman for symplectic manifolds (see [4]) and then extended to manifolds that posses other structures. In particular, our recipe is closely related to the one described in [6].

The goal of this paper is to point out a relation between spin^c-quantization and cutting. We claim that the quantization of our original manifold is isomorphic (as a virtual representation) to the direct sum of the quantizations of the cut spaces. We refer to this property as 'additivity under cutting'.

In [5], Guillemin, Sternberg and Weitsman define *signature quantization* and show that it satisfies 'additivity under cutting'. In fact, this observation motivated the present paper.

It is important to mention that this property does not hold for the most common 'almost-complex quantization'. In this case, we start with an almost complex compact manifold and a Hermitian line bundle with Hermitian connection, and we construct the Dolbeaut-Dirac operator associated to this data. Its index is a virtual vector space, and in the presence of an S^1-action on the manifold and the line bundle we get a virtual representation of S^1, called the Dolbeau-Dirac quantization of the manifold (see [2] or [12]). This is a special case of our spin^c-quantization, since an almost complex structure and a complex line bundle determine a spin^c-structure, which gives rise to the same Dirac operator (See Lemma 2.7 and Remark 2.9 in [6], and Appendix D in [3]). However, in the almost complex case, the cutting is done along the zero level set of the moment map determined by the line bundle and the connection. This results in additivity for all weights except zero. More precisely, if

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$N_{\pm}(\mu)$ denotes the multiplicity of the weight μ in the almost complex quantization of the cut spaces and $N(\mu)$ is the weight of μ in the quantization of the original manifold, we have (see p. 258 in [12])

$$N(\mu) = N_{+}(\mu) + N_{-}(\mu), \qquad \mu \neq 0,$$

but

$$N(0) = N_{+}(0) = N_{-}(0),$$

and therefore there is no additivity in general.

On the other hand, if spin^c-cutting is done for a spin^c-manifold M (in particular, the spin^c-structure can come from an almost complex structure), then the additivity will hold for any weight. Roughly speaking, this happens because the spin^c-cutting is done at the level set $1/2$ of the 'moment map', which is not a weight (i.e., an integer) for the group S^1.

A review of the necessary standard material is presented in the Appendix in Section 8. Important terminology and notation is introduced in this Appendix, and the reader is encouraged to quickly go over it and refer to it during the reading of the paper, when needed. In Section 2 we describe in detail the cutting process. In Sections 3 and 4 we develop Kostant-type formulas for spin^c-quantizations in terms of local data around connected components of the fixed point set, and finally in Section 5 we prove the additivity result. In Section 6, we give a detailed example that illustrates the additivity property of spin^c-quantization. In particular, we classify and cut all the S^1-equivariant spin^c-structures on the two-sphere. In Section 7, we comment about the relation of our work to the original symplectic cutting construction.

Throughout this paper, all spaces are assumed to be smooth manifolds and all maps and actions are assumed to be smooth. The principal action in a principal bundle will be always a right action. A real vector bundle E, equipped with a fiberwise inner product, will be called a *Riemannian vector bundle*. If the fibers are also oriented, then its bundle of oriented orthonormal frames will be denoted by $SOF(E)$. For an oriented Riemannian manifold M, we will simply write $SOF(M)$ instead of $SOF(TM)$.

2. SPIN^c-CUTTING

In [4] Lerman describes the symplectic cutting construction for symplectic manifolds equipped with a Hamiltonian G-action. In [6] this construction is generalized to manifolds with other structures, including spin^c-manifolds. However, the cutting of a spin^c-structure is incomplete in [6], since it only produces a spin^c-principal bundle on the cut spaces $P_{cut} \to M_{cut}$, without constructing a map $P_{cut} \to SOF(M_{cut})$.

In this section, we describe the construction from section 6 in [6] and fill the necessary gaps.

From now on we will work with G-equivariant spin^c-structures (see Section 8 for the definitions). This includes the nonequivariant case when G is taken to be the trivial group $\{e\}$.

2.1. The product of two spin^c-structures.

Note that the group $SO(m)\times SO(n)$ naturally embeds in $SO(n+m)$ as block matrices, and therefore it acts on $SO(n+m)$ from the left by left multiplication.

The proof of the following claim is straightforward.

Claim 2.1. Let M and N be two oriented Riemannian manifolds of respective dimensions m and n. Then the map

$$(SOF(M) \times SOF(N)) \times_{SO(m)\times SO(n)} SO(n+m) \to SOF(M \times N),$$
$$[(f,g),K] \mapsto (f,g) \circ K$$

is an isomorphism of principal $SO(n+m)$-bundles.

Here, $f : \mathbb{R}^m \xrightarrow{\sim} T_aM$ and $g : \mathbb{R}^n \xrightarrow{\sim} T_bN$ are frames, and $K : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ is in $SO(m+n)$.

The above claim suggests a way to define the product of two spin^c-manifolds (see also Lemma 6.10 from [6]). There is a natural group homomorphism $j : Spin(m) \times Spin(n) \to Spin(m+n)$, which is induced from the embeddings

$$\mathbb{R}^m \hookrightarrow \mathbb{R}^m \times \{0\} \subset \mathbb{R}^{m+n} \qquad \text{and} \qquad \mathbb{R}^n \hookrightarrow \{0\} \times \mathbb{R}^n \subset \mathbb{R}^{m+n} .$$

This gives rise to a homomorphism

$$j^c : Spin^c(m) \times Spin^c(n) \to Spin^c(m+n), \qquad ([A,a],[N,b]) \mapsto [j(A,B),ab] ,$$

and therefore $Spin^c(m) \times Spin^c(n)$ acts from the left on $Spin^c(m+n)$ via j^c.

If a group G acts on two manifolds M and N, then it clearly acts on $M \times N$ by $g \cdot (m,n) = (g \cdot m, g \cdot n)$, and the above claim generalizes to this case as well.

Definition 2.1. Let G be a Lie group that acts on two oriented Riemannian manifolds M,N by orientation preserving isometries. Let $P_M \to SOF(M)$ and $P_N \to SOF(N)$ be G-equivariant spin^c-structures on M and N. Then

$$P = (P_M \times P_N) \times_{Spin^c(m)\times Spin^c(n)} Spin^c(m+n) \to SOF(M \times N)$$

is a G-equivariant spin^c-structure on $M \times N$, called *the product* of the two given spin^c-structures.

Remark 2.1. In the above setting, if L_M and L_N are the determinant line bundles of the spin^c-structures on M and N, respectively, then the determinant line bundle of $P \to SOF(M \times N)$ is $L_M \boxtimes L_N$ (exterior tensor product). See Lemma 6.10 from [6] for details.

2.2. **Restriction of a spin^c-structure.** In general, it is not clear how to restrict a spin^c-structure from a Riemannian oriented manifold to a submanifold. However, for our purposes, it suffices to work with co-oriented submanifolds of co-dimension 1.

The proof of the following claim is straightforward.

Claim 2.2. Assume that the following data is given:

(1) M an oriented Riemannian manifold of dimension m.
(2) G a Lie group that acts on M by orientation preserving isometries.
(3) $Z \subset M$ a G-invariant co-oriented submanifold of co-dimension 1.
(4) $P \to SOF(M)$ a G-equivariant spin^c-structure on M.

Define an injective map

$$i : SOF(Z) \to SOF(M), \qquad i(f)(a_1,\dots,a_m) = f(a_1,\dots,a_{m-1}) + a_m \cdot v_p,$$

where $f : \mathbb{R}^{m-1} \xrightarrow{\sim} T_pZ$ is a frame in $SOF(Z)$ and $v \in \Gamma(TM|_Z)$ is the vector field of positive unit vectors, orthogonal to TZ.

Then the pullback $P' = i^*(P) \to SOF(Z)$ is a G-equivariant spin^c-structure on Z, called *the restriction of P to Z*.

Remark 2.2.

(1) This is the relevant commutative diagram for the claim:

$$\begin{array}{ccc} P' = i^*(P) & \longrightarrow & P \\ \big\downarrow & & \big\downarrow \\ SOF(Z) & \xrightarrow{\ i\ } & SOF(M) \\ \big\downarrow & & \big\downarrow \\ Z & \longrightarrow & M \end{array}$$

(2) The principal action of $Spin^c(m-1)$ on P' is obtained using the natural inclusion $Spin^c(m-1) \hookrightarrow Spin^c(m)$.

(3) The determinant line bundle of P' is the restriction to Z of the determinant line bundle of P.

2.3. **Quotients of spin^c-structures.** We now discuss the process of taking quotients of a spin^c-structure with respect to a group action. Since the basic cutting construction involves an S^1-action, we will only deal with circle actions.

Assume that the following data is given:

(1) An oriented Riemannian manifold Z of dimension n.
(2) A free action $S^1 \circlearrowleft Z$ by isometries.
(3) $P \to SOF(Z)$ an S^1-equivariant spin^c-structure on Z.

Denote by $\frac{\partial}{\partial\theta} \in Lie(S^1)$ an infinitesimal generator, by $\left(\frac{\partial}{\partial\theta}\right)_Z \in \chi(Z)$ the corresponding vector field, and by $\pi : Z \to Z/S^1$ the quotient map. Also let $V = \pi^*\left(T\left(Z/S^1\right)\right)$. This is an S^1-equivariant vector bundle over Z.

$$\begin{array}{ccc} V = \pi^*\left(T\left(Z/S^1\right)\right) & \longrightarrow & T\left(Z/S^1\right) \\ \big\downarrow & & \big\downarrow \\ Z & \xrightarrow{\ \pi\ } & Z/S^1 \end{array}$$

We have the following simple fact.

Lemma 2.1. *The map from the subbundle* $\left(\left(\frac{\partial}{\partial\theta}\right)_Z\right)^\perp$ *of* TZ *to* V*, given by*

$$v \in T_pZ \longmapsto (p, \pi_* v) \in V_p,$$

is an isomorphism of S^1*-equivariant vector bundles over* Z*.*

Remark 2.3. Using this lemma, we identify V with a subbundle of TZ and endow V with the induced Riemannian metric and orientation. This identification endows Z/S^1 with an orientation and a Riemannian metric, and hence it makes sense to speak of $SOF(Z/S^1)$.

Now define a map $\eta\colon SOF(V) \to SOF(Z)$ in the following way. If $f : \mathbb{R}^{n-1} \xrightarrow{\simeq} V_p$ is a frame, then $\eta(f)\colon \mathbb{R}^n \to T_pZ$ will be given by $\eta(f)e_i = f(e_i)$ for $i = 1, \ldots, n-1$ and $\eta(f)e_n$ is a unit vector in the direction of $\left(\frac{\partial}{\partial\theta}\right)_{Z,p}$.

The following lemmas are used to get a spin^c-structure on Z/S^1. Their proofs are straightforward and left to the reader.

Lemma 2.2. *The pullback* $\eta^*(P) \subset SOF(V) \times P$ *is an* S^1*-equivariant spin*^{c}*-structure on* $SOF(V)$*.*

(The S^1-action on $\eta^*(P)$ is induced from the S^1-actions on $SOF(V)$ and P, and the right action of $Spin^c(n-1)$ is induced by the natural inclusion $Spin^c(n-1) \subset Spin^c(n)$).

$$\begin{array}{ccc} \eta^*(P) & \longrightarrow & P \\ \downarrow & & \downarrow \\ SOF(V) & \xrightarrow{\eta} & SOF(Z) \\ & \searrow \quad \swarrow & \\ & Z & \end{array}$$

Lemma 2.3. *Consider the S^1-equivariant spin^c-structure $\eta^*(P) \to SOF(V) \to Z$. The quotient of each of the three components by the left S^1 action gives rise to a spin^c-structure on Z/S^1, called* the quotient of the given spin^c-structure.

$$\begin{array}{c} \overline{P} := \eta^*(P)/S^1 \\ \Big\downarrow \\ SOF(Z/S^1) = SOF(V)/S^1 \\ \Big\downarrow \\ Z/S^1 \end{array}$$

Remark 2.4. If L is the determinant line bundle of the given spin^c-structure on Z, then the determinant line bundle of $\overline{P}$ is L/S^1.

2.4. **spin^c-cutting.** We are now in the position of describing the process of cutting a given S^1-equivariant spin^c-structure on a manifold. Assume that the following data is given:

(1) An oriented Riemannian manifold M of dimension m.
(2) An action of S^1 on M by isometries.
(3) A co-oriented submanifold $Z \subset M$ of co-dimension 1 that is S^1-invariant. We also demand that S^1 acts freely on Z and that $M \setminus Z$ is a disjoint union of two open pieces M_+, M_-, such that positive (resp. negative) normal vectors point into M_+ (resp. M_-). Such submanifolds are called *reducible splitting hypersurfaces* (see Definitions 3.1 and 3.2 in [6]).
(4) $P \to SOF(M)$ an S^1-equivariant spin^c-structure on M.

We will use the following fact.

Claim 2.3. There is an invariant (smooth) function $\Phi : M \to \mathbb{R}$, such that $\Phi^{-1}(0) = Z$, $\Phi^{-1}(0,\infty) = M_+$, $\Phi^{-1}(-\infty, 0) = M_-$ and 0 is a regular value of Φ.

To prove this claim, first define Φ locally on a chart, use a partition of unity to get a globally well defined function on the whole manifold, and then average with respect to the group action to get S^1-invariance.

This function Φ plays the role of a 'moment map' for the S^1 action. To define *the cut space* M^+_{cut}, first introduce an S^1-action on $M \times \mathbb{C}$:

$$a \cdot (m, z) = (a \cdot m, a^{-1} z)$$

and then let $M^+_{cut} = \left\{(m, z) | \Phi(m) = |z|^2\right\}/S^1$. The cut space M^-_{cut} is defined similarly, using the diagonal action on $M \times \mathbb{C}$:

$$a \cdot (m, z) = (a \cdot m, a \cdot z)$$

and by setting $M^-_{cut} = \left\{(m, z) | \Phi(m) = -|z|^2\right\} / S^1$.

Remark 2.5. The orientation and the Riemannian metric on M (and on $\mathbb{C}$) descend to the cut spaces $M^{\pm}_{cut}$ as follows. $M \times \mathbb{C}$ is naturally an oriented Riemannian manifold. Consider the map

$$\widetilde{\Phi} : M \times \mathbb{C} \to \mathbb{R}, \qquad \widetilde{\Phi}(m, z) = \Phi(m) - |z|^2.$$

Zero is a regular value of $\widetilde{\Phi}$, and therefore $\widetilde{Z} = \widetilde{\Phi}^{-1}(0)$ is a manifold. It inherits a metric and is co-oriented (hence oriented). Since S^1 acts freely on $\widetilde{Z}$, the quotient $M^+_{cut} = \widetilde{Z}/S^1$ is an oriented Riemannian manifold (see Remark 2.3).

A similar procedure, using $\widetilde{\Phi}(m, z) = \Phi(m) + |z|^2$, is carried out in order to get an orientation and a metric on M^-_{cut}.

We also have an S^1 action of the cut spaces (see Remark 2.7).

The purpose of this subsection is to describe how to get spin^c-structures on $M^{\pm}_{cut}$ from the given spin^c-structure on M. We start by constructing a spin^c-structure on M^+_{cut}.

Step 1. Consider $\mathbb{C}$ with its natural structure as an oriented Riemannian manifold, and let

$$P_{\mathbb{C}} = \mathbb{C} \times Spin^c(2) \longrightarrow SOF(\mathbb{C}) = \mathbb{C} \times SO(2) \longrightarrow \mathbb{C}$$

be the trivial spin^c-structure on $\mathbb{C}$. Turn it into an S^1-equivariant spin^c-structure by letting S^1 act on $P_{\mathbb{C}}$:

$$e^{i\theta} \cdot (z, [a, b]) = (e^{-i\theta} z, [x_{-\theta/2} \cdot a, e^{i\theta/2} \cdot b]), \qquad z \in \mathbb{C}\ ,\ [a, b] \in Spin^c(2),$$

where $x_\theta = \cos\theta + \sin\theta \cdot e_1 e_2 \in Spin(2)$.

Remark 2.6. Note that although θ is defined up to 2π, and therefore each of the elements $x_{-\theta/2} \cdot a$ and $e^{i\theta/2} \cdot b$ are defined only up to π, the element $[x_{-\theta/2} \cdot a, e^{i\theta/2} \cdot b] \in Spin^c(2)$ is well defined (i.e., up to 2π).

Here is a diagram for this structure:

$$\begin{array}{ccccc}
S^1 \times P_{\mathbb{C}} & \longrightarrow & P_{\mathbb{C}} & \longleftarrow & P_{\mathbb{C}} \times Spin^c(2) \\
\downarrow & & \downarrow & & \downarrow \\
S^1 \times SOF(\mathbb{C}) & \longrightarrow & SOF(\mathbb{C}) & \longleftarrow & SOF(\mathbb{C}) \times SO(2) \\
\downarrow & & \downarrow & & \\
S^1 \times \mathbb{C} & \longrightarrow & \mathbb{C} & &
\end{array}$$

Step 2. Taking the product of the spin^c-structures P (on M) and $P_{\mathbb{C}}$ (on $\mathbb{C}$), we get an (S^1 equivariant) spin^c-structure $P_{M\times\mathbb{C}}$ on $M \times \mathbb{C}$ (see §2.1).

Step 3. It is easy to check that

$$\widetilde{Z} = \{(m, z) | \Phi(m) = |z|^2\} \subset M \times \mathbb{C}$$

is an S^1-invariant co-oriented submanifold of co-dimension one, and therefore we can restrict $P_{M\times\mathbb{C}}$ and get an S^1-equivariant spin^c-structure $P_{\widetilde{Z}}$ on $\widetilde{Z}$ (see §2.2).

Step 4. Since $P_{\widetilde{Z}} \to SOF(\widetilde{Z}) \to \widetilde{Z}$ is an S^1-equivariant spin^c-structure, we can take the quotient by the S^1-action to get a spin^c-structure P^+_{cut} on $M^+_{cut} = \widetilde{Z}/S^1$ (see §2.3).

Remark 2.7. The spin^c-structure P^+_{cut} can be turned into an S^1-equivariant one. This is done by observing that we actually have two S^1 actions on $M \times \mathbb{C}$: the anti-diagonal action $a \cdot (m, z) = (a \cdot m, a^{-1} \cdot z)$ and the *M-action* $a \cdot (m, z) = (a \cdot m, z)$. These actions commute with each other, and the M-action naturally decends to the cut space M^+_{cut} and lifts to the spin^c-structure P^+_{cut}.

Let us now briefly describe the analogous construction for M^-_{cut}.

Step 1. Define $P_{\mathbb{C}}$ as before, but with the action

$$e^{i\theta} \cdot (z, [a, b]) = (e^{i\theta} z, [x_{\theta/2} \cdot a, e^{i\theta/2} \cdot b]).$$

Step 2. Define the spin^c-structure $P_{M\times\mathbb{C}}$ on $M \times \mathbb{C}$ as before.

Step 3. As before, replacing $\widetilde{Z}$ with $\{(m, z)|\Phi(m) = -|z|^2\} \subset M \times \mathbb{C}$.

Step 4. Repeat as before to get a spin^c-structure P^-_{cut} on M^-_{cut}.

Remark 2.8. In Step 1 we defined a spin^c-structure on $\mathbb{C}$. The corresponding determinant line bundle is the trivial line bundle $\mathbb{L}_{\mathbb{C}} = \mathbb{C}\times\mathbb{C}$ over $\mathbb{C}$ (with projection $(z, b) \mapsto z$). The S^1 action on $\mathbb{L}_{\mathbb{C}}$ is given by

$$a \cdot (z, b) = \begin{cases} (a^{-1} \cdot z, a \cdot b) & \text{for } P^+_{cut}, \\ (a \cdot z, a \cdot b) & \text{for } P^-_{cut}. \end{cases}$$

If $\mathbb{L}$ is the determinant line bundle of the given spin^c-structure on M, then the determinant line bundle on $M^{\pm}_{cut}$ is given by

$$\mathbb{L}^{\pm}_{cut} = \left[(\mathbb{L} \boxtimes \mathbb{L}_{\mathbb{C}}) \,|_{\widetilde{Z}} \right] / S^1,$$

where we divide by the diagonal action of S^1 on $\mathbb{L} \times \mathbb{L}_{\mathbb{C}}$. This is an S^1-equivariant complex line bundle (with respect to the M-action).

3. The generalized Kostant formula for isolated fixed points

Assume that the following data is given:

1. An oriented compact Riemannian manifold M of dimension $2m$.
2. $T = \mathbb{T}^n$ an n-dimensional torus that acts on M by isometries.
3. $P \to SOF(M)$ a T-equivariant spin^c-structure, with determinant line bundle $\mathbb{L}$.
4. A $U(1)$-invariant connection on $P_1 = P/Spin(2m)$.

This data determines a complex virtual representation $Q(M) = ker(D^+) - coker(D^+)$ of T (see §8.5 for details). Denote by $\chi \colon T \to \mathbb{C}$ its character.

Lemma 3.1. *Let $x \in M^T$ be a fixed point, and choose a T-invariant complex structure $J : T_xM \to T_xM$. Denote by $\alpha_1, \ldots, \alpha_m \in \mathfrak{t}^* = Lie(T)^*$ the weights of the action $T \circlearrowright T_xM$ and by μ the weight of $T \circlearrowright \mathbb{L}_x$. Then $\frac{1}{2}\left(\mu - \sum_{j=1}^{m} \alpha_j\right)$ is in the weight lattice of T.*

Proof. Decompose $T_xM = L_1 \oplus \cdots \oplus L_m$, where each L_j is a 1-dimensional T-invariant complex subspace of T_xM, on which T acts with weight α_j. Fix a point $p \in P_x$.

For each $z \in T$, there is a unique element $[A_z, w_z] \in Spin^c(2m)$ such that $z \cdot p = p \cdot [A_z, w_z]$. This gives a homomorphism

$$\eta\colon T \to Spin^c(2m), \qquad z \mapsto [A_z, w_z]$$

(note that A_z and w_z are defined only up to sign, but the element $[A_z, w_z]$ is well defined).

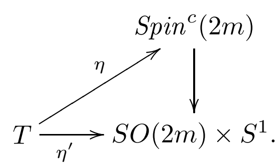
Choose a basis $\{e_j\} \subset T_xM$ (over $\mathbb{C}$) with $e_j \in L_j$ for all $1 \leq j \leq m$. With respect to this basis, each element $z \in T$ acts on T_xM through the matrix

$$A'_z = \begin{pmatrix} z^{\alpha_1} & & & 0 \\ & z^{\alpha_2} & & \\ & & \ddots & \\ 0 & & & z^{\alpha_m} \end{pmatrix} \in U(m) \subset SO(2m)\ .$$

This enables us to define another homomorphism,

$$\eta'\colon T \to SO(2m) \times S^1, \qquad z \mapsto (A'_z, z^{\mu})\ .$$

It is not hard to see that the relation $z \cdot p = p \cdot [A_z, w_z]$ (for all $z \in T$) will imply the commutativity of the following diagram:



(The vertical map is the double cover taking $[A, z] \in Spin^c(2m)$ to $(\lambda(A), z^2)$.) For any $z = e^{i\theta} \in T$ we have

$$\lambda(A_z) = A'_z \quad \Rightarrow \quad A_z = \prod_{j=1}^{m} \left[\cos\left(\frac{\theta \cdot \alpha_j}{2}\right) + \sin\left(\frac{\theta \cdot \alpha_j}{2}\right) e_j J(e_j)\right] \in Spin(2m)$$

(where the spin group is thought of as sitting inside the Clifford algebra) and

$$w_z^2 = z^{\mu} \qquad \Rightarrow \qquad w_z = z^{\mu/2}\ .$$

Note that

$$T_{Spin^c(2m)} = \left\{ \left[\prod_{j=1}^{m} (\cos t_j + \sin t_j \cdot e_j J(e_j))\,, u\right] : t_j \in \mathbb{R}\ ,\ u \in S^1 \right\} \subset Spin^c(2m)$$

is a maximal torus and that in fact η is a map from T to $T_{Spin^c(2m)}$.

Now define another map

$$\psi\colon T_{Spin^c(2m)} \to S^1, \qquad \left[\prod_{j=1}^{m} (\cos t_j + \sin t_j \cdot e_j J(e_j), u)\right] \mapsto u \cdot e^{-i\sum_j t_j}.$$

By composing η and ψ we get a well defined map $\psi \circ \eta \colon T \to S^1$ which is given by

$$e^{i\theta} \mapsto \left(e^{i\theta}\right)^{\frac{1}{2}\left(\mu - \sum_j \alpha_j\right)},$$

and therefore $\frac{1}{2}\left(\mu - \sum_j \alpha_j\right)$ must be a weight of T. □

Remark 3.1. The idea in the above proof is simple. To show that $\beta = \frac{1}{2}\left(\mu - \sum_j \alpha_j\right)$ is a weight, we want to construct a 1-dimensional complex representation of T with weight β. The map η is a natural homomorphism $T \to Spin^c(2m)$. The map ψ is nothing but the action of a maximal torus of $Spin^c(2m)$ on the lowest weight space of the spin representation Δ^+_{2m} (see Proposition 8.3, and Lemma 12.12 in [7]). Finally, $\psi \circ \eta \colon T \to S^1$ is the required representation.

The following is Proposition 11.3 from [7].

Proposition 3.1. *Assume that the fixed points M^T of the action on M are isolated. For each $p \in M^T$, choose a complex structure on T_pM, and denote by*

(1) $\alpha_{1,p}, \dots, \alpha_{m,p} \in \mathfrak{t}^*$ *the weights of the action of T on T_pM.*
(2) μ_p *the weight of the action of T on $\mathbb{L}_p$.*
(3) $(-1)^p$ *will be $+1$ if the orientation coming from the choice of the complex structure on T_pM coincides with the orientation of M, and -1 otherwise.*

Then the character $\chi \colon T \to \mathbb{C}$ of $Q(M)$ is given by

$$\chi(\lambda) = \sum_{p \in M^G} \nu_p(\lambda), \qquad \nu_p(\lambda) = (-1)^p \cdot \lambda^{\mu_p/2} \prod_{j=1}^{m} \frac{\lambda^{-\alpha_{j,p}/2} - \lambda^{\alpha_{j,p}/2}}{(1 - \lambda^{\alpha_{j,p}})(1 - \lambda^{-\alpha_{j,p}})},$$

where $\lambda^\beta : T \to S^1$ is the representation that corresponds to the weight $\beta \in \mathfrak{t}^*$.

Remark 3.2. (1) Although $\pm\alpha_{j,p}/2$ may not be in the weight lattice of T, the expression $\nu_p(\lambda)$ can be equivalently written as

$$(-1)^p \cdot \lambda^{\left(\mu_p - \sum_j \alpha_{j,p}\right)/2} \prod_{j=1}^{m} \frac{1 - \lambda^{\alpha_{j,p}}}{(1 - \lambda^{\alpha_{j,p}})(1 - \lambda^{-\alpha_{j,p}})} .$$

By Lemma 3.1, $\left(\mu_p - \sum_j \alpha_{j,p}\right)/2$ is a weight, so $\nu_p(\lambda)$ is well defined.

(2) Since the fixed points of the action $T \circlearrowleft M$ are isolated, all the $\alpha_{j,p}$'s are nonzero. This follows easily from theorem B.26 in [2].

Now we present the generalized Kostant formula for spin^c-quantization.

Assume that the fixed points of $T \circlearrowleft M$ are isolated, choose a complex structure on T_pM for each $p \in M^G$, and use the notation of Proposition 3.1. By the above remark, we can find a *polarizing vector* $\xi \in \mathfrak{t}$ such that $\alpha_{j,p}(\xi) \neq 0$ for all j, p. We can choose our complex structures on T_pM such that $\alpha_{j,p}(\xi) \in i\mathbb{R}^+$ for all j, p.

For each weight $\beta \in \mathfrak{t}^*$ denote by $\#(\beta, Q(M))$ the multiplicity of this weight in $Q(M)$. Also, for $p \in M^T$ define the partition function $\overline{N}_p : \mathfrak{t}^* \to \mathbb{Z}^+$ by setting

$$\overline{N}_p(\beta) = \left| \left\{ (k_1, \dots, k_m) \in \left(\mathbb{Z} + \frac{1}{2}\right)^m \ : \ \beta + \sum_{j=1}^{m} k_j \alpha_{j,p} = 0, \qquad k_j > 0 \right\} \right| .$$

The right hand side is always finite since our weights are polarized.

Theorem 3.1 (Kostant formula)**.** *For any weight $\beta \in \mathfrak{t}^*$ of T, we have*

$$\#(\beta, Q(M)) = \sum_{p \in M^G} (-1)^p \cdot \overline{N}_p \left(\beta - \frac{1}{2}\mu_p \right).$$

Proof. For $p \in M^T$ and $\lambda \in T$, set $\alpha_j = \alpha_{j,p}$ and $\mu = \mu_p$. From Proposition 3.1 we then get

$$\nu_p(\lambda) = (-1)^p \cdot \lambda^{\mu/2} \prod_{j=1}^{m} \frac{\lambda^{-\alpha_j/2}(1-\lambda^{\alpha_j})}{(1-\lambda^{\alpha_j})(1-\lambda^{-\alpha_j})} = (-1)^p \cdot \lambda^{\frac{1}{2}(\mu - \sum_j \alpha_j)} \prod_{j=1}^{m} \frac{1}{1-\lambda^{-\alpha_j}}.$$

Note that we have

$$\prod_{j=1}^{m} \frac{1}{1-\lambda^{-\alpha_j}} = \sum_{\beta} N_p(\beta) \cdot \lambda^{\beta},$$

where the sum is taken over all weights $\beta \in \mathfrak{t}^*$ in the weight lattice ℓ^* of T and $N_p(\beta)$ is the number of *nonnegative* integer solutions $(k_1, \ldots, k_m) \in (\mathbb{Z}_+)^m$ to

$$\beta + \sum_{j=1}^{m} k_j \alpha_j = 0$$

(see formula 5 in [5]). Hence,

$$\nu_p(\lambda) = (-1)^p \cdot \sum_{\beta \in \ell^*} N_p(\beta) \cdot \lambda^{\beta + \frac{1}{2}(\mu - \sum_j \alpha_j)}.$$

By Lemma 3.1, $\frac{1}{2}(\mu - \sum_j \alpha_j) \in \ell^*$ (i.e., it is a weight), so by change of variable $\beta \mapsto \beta - \frac{1}{2}(\mu - \sum_j \alpha_j)$ we get

$$\nu_p(\lambda) = (-1)^p \cdot \sum_{\beta \in \ell^*} N_p \left(\beta - \frac{1}{2}\mu + \frac{1}{2} \sum_j \alpha_j \right) \cdot \lambda^{\beta}.$$

By definition, $N_p \left(\beta - \frac{1}{2}\mu + \frac{1}{2} \sum_j \alpha_j \right)$ is the number of nonnegative integer solutions for the equation

$$\beta - \frac{1}{2}\mu + \frac{1}{2} \sum_j \alpha_j + \sum_j k_j \alpha_j = 0$$

or, equivalently, for

$$\beta - \frac{1}{2}\mu + \sum_j \left(k_j + \frac{1}{2} \right) \alpha_j = 0.$$

Using the definition of $\overline{N}_p$ (see above) we conclude that

$$N_p \left(\beta - \frac{1}{2}\mu + \frac{1}{2} \sum_j \alpha_j \right) = \overline{N}_p \left(\beta - \frac{1}{2}\mu \right)$$

and then

$$\nu_p(\lambda) = (-1)^p \cdot \sum_{\beta \in \ell^*} \overline{N}_p \left(\beta - \frac{1}{2}\mu \right) \lambda^{\beta}.$$

This means that the formula to the character can be written as

$$\chi(\lambda) = \sum_{\beta \in \ell^*} \left[\sum_{p \in M^G} (-1)^p \cdot \overline{N}_p \left(\beta - \frac{1}{2}\mu \right) \right] \lambda^\beta$$

and the multiplicity of β in $Q(M)$ is given by

$$\#(\beta, Q(M)) = \sum_{p \in M^G} (-1)^p \cdot \overline{N}_p \left(\beta - \frac{1}{2}\mu \right),$$

as desired. □

4. The generalized Kostant formula for nonisolated fixed points

4.1. **Equivariant characteristic classes.** Let an *abelian* Lie group G (with Lie algebra $\mathfrak{g}$) act *trivially* on a smooth manifold X. We now define the equivariant cohomology (with generalized coefficients) and equivariant characteristic classes for this special case. For the more general case, see [9] or Appendix C in [2].

Definition 4.1. A real-valued function α is called an *almost everywhere analytic function* (a.e.a.) if

(1) its domain is of the form $\mathfrak{g} \setminus P$, and $P \subset \mathfrak{g}$ is a closed set of measure zero,
(2) it is analytic on $\mathfrak{g} \setminus P$.

Denote by $C^\#(\mathfrak{g})$ the space of all equivalence classes of a.e.a. functions on $\mathfrak{g}$ (two such functions are equivalent if they coincide outside a set of measure zero).

Let $\mathcal{A}_G^\#(X) = C^\#(\mathfrak{g}) \otimes \Omega^\bullet(X;\mathbb{C})$ be the space of all a.e.a. functions $\mathfrak{g} \to \Omega^\bullet(X;\mathbb{C})$, where $\Omega^\bullet(X;\mathbb{C})$ is the (ordinary) de Rham complex of X with complex coefficients.

Define a differential (recall that G is abelian and the action is trivial)

$$d_{\mathfrak{g}} : \mathcal{A}_G^\#(X) \to \mathcal{A}_G^\#(X), \qquad (d_{\mathfrak{g}}\alpha)(u) = d(\alpha(u)),$$

and the *G-equivariant (de Rham) cohomology* of X:

$$H_G^\#(X) = \frac{Ker(d_{\mathfrak{g}})}{Im(d_{\mathfrak{g}})} .$$

Note that $H_G^\#(X)$ is isomorphic to the space $C^\#(\mathfrak{g}) \otimes H^\bullet(X;\mathbb{C})$ of a.e.a. functions $\mathfrak{g} \to H^\bullet(X;\mathbb{C})$. Equivariant characteristic classes will be elements of the ring $H_G^\#(X)$.

If X is compact and oriented, then equivariant cohomology classes can be integrated over X. For any class $[\alpha] \in H_G^\#(X)$ and u in the domain of α, let

$$\left(\int_X [\alpha] \right)(u) = \int_X (\alpha(u)),$$

and thus $\int_X [\alpha]$ is an element of $C^\#(\mathfrak{g}) \otimes \mathbb{C}$.

Assume now that both X and G are connected, and let $\pi \colon L \to X$ be a complex line bundle over X. Assume that G acts on the fibers of the bundle with weight $\mu \in \mathfrak{g}^*$, i.e., $\exp(u) \cdot y = e^{i\mu(u)} \cdot y$ for all $u \in \mathfrak{g}$ and $y \in L$ (so the action on the base space is still trivial). Denote by $c_1(L) = [\omega] \in H^2(X)$ the (ordinary) first Chern class of the line bundle. Here $\omega \in \Omega^2(X)$ is a real two-form. Then the *first equivariant Chern class* of the equivariant line bundle $L \to X$ is defined to be $[\omega + \mu] \in H_G^\#(X)$. We will denote this class by $\tilde{c_1}(L)$.

Now assume that $E \to X$ is a G-equivariant complex vector bundle of complex rank k (where G acts trivially on X) that splits as a sum of k equivariant complex line bundles $E = L_1 \oplus \cdots \oplus L_k$ (one can avoid this assumption by using *the (equivariant) splitting principle*). Let $\tilde{c_1}(L_1) = [\omega_1 + \mu_1], \ldots, \tilde{c_1}(L_k) = [\omega_k + \mu_k]$ be the equivariant first Chern classes of these line bundles, and define *the equivariant Euler class* of E by

$$\tilde{Eu}(E) = \prod_{j=1}^{k} \tilde{c_1}(L_j) = \left[\prod_{j=1}^{k} (\omega_j + \mu_j)\right] \in H_G^{\#}(X).$$

We will also need the equivariant A-roof class, which we will denote by $\tilde{A}(E)$. To define this class, consider the following meromorphic function:

$$f(z) = \frac{z}{e^{z/2} - e^{-z/2}} = \frac{z/2}{\sinh(z/2)}, \qquad f(0) = 1 .$$

Its domain is $D = \mathbb{C} \setminus \{\pm 2\pi i, \pm 4\pi i, \ldots\}$. Define, for each $1 \le j \le k$,

$$f(\tilde{c_1}(L_j))(u) = f(c_1(L_j) + \mu_j(u)) = \sum_{n=1}^{\infty} \frac{f^{(n)}(\mu_j(u))}{n!} \cdot (c_1(L_j))^n$$

whenever $\mu_j(u) \in D$ for all $1 \le j \le k$, and also

$$\tilde{A}(E) = \prod_{j=1}^{k} f(\tilde{c_1}(L_j)).$$

Also note that the quotient

$$\frac{\tilde{A}(E)}{\tilde{Eu}(E)}$$

can be defined using the same procedure, replacing $f(z)$ with $\frac{1}{2\sinh(z/2)}$. If all the μ_j's are nonzero, then

$$\frac{\tilde{A}(E)}{\tilde{Eu}(E)} \in H_G^{\#}(X).$$

4.2. **The Kostant formula.** Assume that the following data is given:

(1) An oriented compact Riemannian manifold M of dimension $2m$.
(2) A circle action $S^1 \circlearrowright M$ by isometries.
(3) An S^1-equivariant spin^c-structure $P \to SOF(M)$, with determinant line bundle $\mathbb{L}$.
(4) A $U(1)$-invariant connection on $P_1 = P/Spin(2m) \to M$.

In this section we present a formula for the character $\chi \colon S^1 \to \mathbb{C}$ of the virtual representation $Q(M)$ determined by the above data (see §8.5). We *do not* assume, however, that the fixed points are isolated.

We use the following conventions and notation:

- M^{S^1} is the fixed points set.
- For each connected component $F \subset M^{S^1}$, let NF denote the normal bundle to $TF \subset TM$. The bundles NF and TF are S^1-equivariant real vector bundles of even rank, with trivial fixed subspace, and therefore are equivariantly isomorphic to complex vector bundles. Choose an equivariant complex structure on the fibers of TF and NF, and denote the rank of NF as a complex vector bundle by $m(F)$.

- The complex structures on NF and TF induce an orientation on those bundles. Let $(-1)^F$ be $+1$ if the orientation of F followed by that of NF is the given orientation on M, and -1 otherwise.

With respect to the above data, choices and notation, we have

Proposition 4.1. *For all $u \in \mathfrak{g} = Lie(S^1)$ such that the right hand side is defined,*

$$\chi(exp(u)) = \sum_{F \subset M^{S^1}} (-1)^F \cdot (-1)^{m(F)} \cdot \int_F e^{\frac{1}{2}\tilde{c_1}(\mathbb{L}|_F)} \cdot \tilde{A}(TF) \cdot \frac{\tilde{A}(NF)}{\tilde{Eu}(NF)},$$

where the sum is taken over the connected components of M^{S^1}.

This formula is derived from the Atiyah-Segal-Singer index theorem (see [10]). For some details, see p. 547 in [6].

Assume that the normal bundle splits as a direct sum of (equivariant) complex line bundles

$$NF = L_1^F \oplus \cdots \oplus L_{m(F)}^F \quad .$$

For each fixed component $F \subset M^{S^1}$, denote by $\{\alpha_{j,F}\}$ the weights of the action of S^1 on $\{L_j^F\}$. As in the previous section, all the $\alpha_{j,F}$'s are nonzero, and we can polarize them, i.e., we can choose our complex structure on NF in such a way that $\alpha_{j,F}(\xi) > 0$ for some fixed $\xi \in \mathfrak{g}$ and for all j's and F's. Also denote by μ_F the weight of the action of S^1 on $\mathbb{L}|_F$.

For each $\beta \in \mathfrak{g}^* = Lie(S^1)^*$, define the following set (which is finite, since our weights are polarized):

$$\mathcal{S}_\beta = \left\{ (k_1, \ldots, k_{m(F)}) \in \left(\mathbb{Z} + \frac{1}{2}\right)^{m(F)} \quad : \quad \beta + \sum_{j=1}^{m(F)} k_j \alpha_{j,F} = 0, \qquad k_j > 0 \right\}$$

and for each tuple $k = (k_1, \ldots, k_{m(F)})$, let

$$\overline{p}_{k,F} = (-1)^{m(F)} \int_F e^{\frac{1}{2}\left(c_1(\mathbb{L}|_F) - \sum_j c_1(L_j^F)\right)} \cdot \tilde{A}(TF) \cdot e^{-\sum_j k_j c_1(L_j^F)} \quad .$$

Now define

$$\overline{N}_F(\beta) = \sum_{k \in \mathcal{S}_\beta} \overline{p}_{k,F} \quad .$$

With this notation, the Kostant formula in this case of nonisolated fixed points becomes identical to the formula for isolated fixed points (from §3).

Theorem 4.1. *For each weight $\beta \in \mathfrak{g}^* = Lie(S^1)^*$, the multiplicity of β in $Q(M)$ is given by*

$$\#(\beta, Q(M)) = \sum_{F \subset M^{S^1}} (-1)^F \cdot \overline{N}_F\left(\beta - \frac{1}{2}\mu_F\right) \ ,$$

where the sum is taken over the connected components of M^{S^1}.

Proof. For a fixed connected component $F \subset M^{S^1}$, omit the F in $\alpha_{j,F}$, μ_F and L_j^F, and compute

$$\int_F e^{\frac{1}{2}\tilde{c}_1(\mathbb{L}|_F)} \cdot \tilde{A}(TF) \cdot \frac{\tilde{A}(NF)}{\tilde{Eu}(NF)}$$

$$= \int_F e^{\frac{1}{2}c_1(\mathbb{L}|_F)+\frac{1}{2}\mu} \cdot \tilde{A}(TF) \cdot \prod_{j=1}^{m(F)} \frac{1}{e^{[c_1(L_j)+\alpha_j]/2} - e^{-[c_1(L_j)+\alpha_j]/2}}$$

$$= e^{\frac{1}{2}\mu} \cdot \int_F e^{\frac{1}{2}c_1(\mathbb{L}|_F)} \cdot \tilde{A}(TF) \cdot \prod_{j=1}^{m(F)} \frac{e^{-[c_1(L_j)+\alpha_j]/2}}{1 - e^{-[c_1(L_j)+\alpha_j]}}$$

$$= e^{\left[\mu - \sum_j \alpha_j\right]/2} \cdot \int_F e^{\left[c_1(\mathbb{L}|_F) - \sum_j c_1(L_j)\right]/2} \cdot \tilde{A}(TF) \cdot \prod_{j=1}^{m(F)} \frac{1}{1 - e^{-[c_1(L_j)+\alpha_j]}}.$$

Using the geometric series

$$\frac{1}{1-z} = \sum_{l=0}^{\infty} z^l$$

and the notation $z = \exp(u)$ we get, for each j, and for each $u \in \mathfrak{g}$ such that the series converges,

$$\frac{1}{1 - e^{-[c_1(L_j)+\alpha_j(u)]}} = \sum_{l=0}^{\infty} e^{-l\cdot[c_1(L_j)+\alpha_j(u)]} = \sum_{l=0}^{\infty} e^{-l\cdot c_1(L_j)} z^{-l\cdot\alpha_j}$$

(where $z^{-l\cdot\alpha_j}$ is the representation of S^1 that corresponds to the weight $-l \cdot \alpha_j \in \ell^* \subset \mathfrak{g}^*$) and thus

$$\prod_{j=1}^{m(F)} \frac{1}{1 - e^{-[c_1(L_j)+\alpha_j(u)]}} = \sum_{l\in\ell^*} \left[\sum_{l+\sum_j k_j\alpha_j = 0} e^{-\sum_j k_j c_1(L_j)} \right] z^l.$$

The formula that we get for the character is

$$\chi(z) = \sum_{F\subset M^{S^1}} (-1)^F \cdot (-1)^{m(F)} \cdot e^{\frac{1}{2}\left[\mu_F(u) - \sum_j \alpha_{j,F}(u)\right]}$$

$$\cdot \int_F e^{\frac{1}{2}\left[c_1(\mathbb{L}|_F) - \sum_j c_1(L_j^F)\right]} \cdot \tilde{A}(TF) \cdot \sum_{l\in\ell^*} \left[\sum_{l+\sum_j k_j\alpha_{j,F}=0} e^{-\sum_j k_j c_1(L_{j,F})} \right] z^l$$

$$= \sum_{l\in\ell^*} \sum_{F\subset M^{S^1}} \sum_{l+\sum_j k_j\alpha_{j,F}=0} (-1)^F \cdot \overline{p}_{k,F} \cdot z^{l+\frac{1}{2}\left(\mu_F - \sum_j \alpha_{j,F}\right)}.$$

Lemma 3.1 implies that $\frac{1}{2}\left(\mu_F - \sum_j \alpha_{j,F}\right)$ is a weight of S^1 (so the previous formula is well defined); hence we can make a change of variables $\beta = l + \frac{1}{2}\mu_F - \frac{1}{2}\sum_j \alpha_{j,F}$

and get

$$\chi(z) = \sum_{\beta\in\ell^*}\left[\sum_{F\subset M^{S^1}}\sum_{k\in\mathcal{S}_{\beta-\frac{1}{2}\mu_F}}(-1)^F\cdot\overline{p}_{k,F}\right]z^\beta$$
$$= \sum_{\beta\in\ell^*}\left[\sum_{F\subset M^{S^1}}(-1)^F\cdot\overline{N}_F\left(\beta-\frac{1}{2}\mu_F\right)\right]z^\beta.$$

From this we conclude that the multiplicity of $\beta\in\ell^*\subset\mathfrak{g}^*$ in $Q(M)$ is given by

$$\#(\beta, Q(M)) = \sum_{F\subset M^{S^1}}(-1)^F\cdot\overline{N}_F\left(\beta-\frac{1}{2}\mu_F\right),$$

as desired (the sum is taken over the connected components of the fixed point set M^{S^1}). □

4.3. **The case** $m(F) = 1$**.** To prove the additivity of spin^c-quantization under cutting, we will need the terms of the Kostant formula for nonisolated fixed points in the special case where $m(F) = 1$; i.e., when the normal bundle to the fixed components has complex dimension 1. Therefore, assume that we are given the same data as in §4.2, and also that

- Each fixed component $F\subset M^{S^1}$ is of real codimension 2 in M, i.e., the normal bundle $NF = TM/TF$ is of real dimension 2.

For a fixed component F, we adopt all the notation from §4.2. Since $m(F)$ is assumed to be 1, we have

$$NF = L_1^F$$

and only one weight,

$$\alpha_{1,F} = \alpha_F.$$

For each $\beta\in\mathfrak{g}^*$, the corresponding set $\mathcal{S}_\beta$ becomes

$$\mathcal{S}_\beta = \left\{k\in\mathbb{Z}+\frac{1}{2}\quad:\quad\beta+k\cdot\alpha_F=0,\qquad k>0\right\}$$

which is either empty or contains only one element. The expression for $\overline{p}_{k,F}$ also simplifies to

$$\overline{p}_{k,F} = -\int_F e^{[c_1(\mathbb{L}|_F)-c_1(NF)]/2}\cdot\tilde{A}(TF)\cdot e^{-k\cdot c_1(NF)},$$

and this implies that

$$\overline{N}_F\left(\beta-\frac{1}{2}\mu_F\right) = \begin{cases}0 & \text{if } \mathcal{S}_{\beta-\frac{1}{2}\mu_F}=\phi,\\ \overline{p}_{k,F} & \text{if } \mathcal{S}_{\beta-\frac{1}{2}\mu_F}=\{k\}.\end{cases}$$

5. Additivity under cutting

In this section we prove our main result, namely, the additivity of spin^c-quantization under the cutting construction described in §2.4.

Our setting is as follows:

(1) A compact oriented connected Riemannian manifold M of dimension $2m$.
(2) An action of S^1 on M by isometries.

(3) An S^1-equivariant spin^c-structure $P \to SOF(M) \to M$.
(4) A co-oriented splitting hypersurface $Z \subset M$ on which S^1 acts freely.

After choosing a $U(1)$-invariant connection on $P_1 = P/Spin(2m)$, we can construct a Dirac operator D^+, whose index $Q(M)$ is independent of the connection. We call $Q(M)$ *the spin^c-quantization of* M (see §8.5).

We can now perform the cutting construction from §2.4 to obtain two other manifolds $M^{\pm}_{cut}$ (the *cut spaces*). Those cut spaces are also compact oriented Riemannian manifolds of dimension $2m$, endowed with a circle action and with S^1-equivariant spin^c-structures $P^{\pm}_{cut}$. Thus, we can quantize them (after choosing a suitable connection) and obtain two virtual representations $Q\left(M^{\pm}_{cut}\right)$.

Theorem 5.1. *As virtual representations of S^1, we have*

$$Q(M) = Q\left(M^{+}_{cut}\right) \oplus Q\left(M^{-}_{cut}\right).$$

We will need a few preliminary lemmas for the proof of the theorem. Those are similar to Proposition 6.1 from [6], where a few gaps were found.

5.1. First lemma - the normal bundle.

Recall the construction of $M^{\pm}_{cut}$ from §2.4.

- Choose an S^1-invariant smooth function $\phi\colon M \to \mathbb{R}$ such that $\phi^{-1}(0) = Z$, $\phi^{-1}(0,\infty) = M_+$, $\phi^{-1}(-\infty,0) = M_-$, and 0 is a regular value of ϕ.
- Define $\tilde{Z}^{\pm} = \left\{(m,z) \mid \phi(m) = \pm|z|^2\right\} \subset M \times \mathbb{C}$, and let S^1 act on $\tilde{Z}^{\pm}$ by $a \cdot (m,z) = (a \cdot m, a^{\mp 1} \cdot z)$.
- Finally, define $M^{\pm}_{cut} = \tilde{Z}^{\pm}/S^1$.

Remark 5.1. Note that we have S^1-equivariant embeddings

$$Z \to \tilde{Z}^{\pm}\ ,\ m \mapsto (m,0) \qquad \text{and} \qquad Z/S^1 \to M^{\pm}_{cut}\ ,\ [m] \mapsto [m,0]$$

and therefore we can think of Z and Z/S^1 as submanifolds of $\tilde{Z}^{\pm}$ and $M^{\pm}_{cut}$, respectively.

Lemma 5.1.

(1) *The maps*

$$\eta\colon T(\tilde{Z}^{\pm})|_Z \to Z \times \mathbb{C}, \qquad \eta\colon (v,w) \in T_{(m,0)}\tilde{Z}^{\pm} \mapsto (m,w)$$

give rise to short exact sequences

$$0 \longrightarrow TZ \longrightarrow T\tilde{Z}^{\pm}|_Z \xrightarrow{\ \eta\ } Z \times \mathbb{C} \longrightarrow 0$$

of S^1-equivariant vector bundles (with respect to both the diagonal (anti-diagonal) action and the M-action) over Z. The action on $Z \times \mathbb{C}$ is taken to be

$$a \cdot (m,z) = (a \cdot m, a^{\mp 1} \cdot z)\ .$$

(2) *The short exact sequences above descend to the short exact sequences*

$$0 \longrightarrow T(Z/S^1) \longrightarrow T(M^{\pm}_{cut})|_{Z/S^1} \longrightarrow Z \times_{S^1} \mathbb{C} \longrightarrow 0$$

of equivariant vector bundles over Z/S^1. The S^1 action on $Z \times_{S^1} \mathbb{C}$ is induced from the action on Z.

Proof.

(1) The S^1-equivariant embedding $Z \to \widetilde{Z}^{\pm}$ gives rise to an injective map $TZ \to T\widetilde{Z}^{\pm}$, which is an S^1-equivariant map of vector bundles over Z. The map η is onto, since for any $(m,w) \in Z \times \mathbb{C}$ we have $\eta(0,w) = (m,w)$, and it is equivariant since for $(v,w) \in T_{(m,0)}\widetilde{Z}^{\pm},\ m \in Z$, we have

$$\eta(a \cdot (v,w)) = \eta(a \cdot v, a^{\mp} \cdot w) = (a \cdot m, a^{\mp} \cdot w) = a \cdot (m,z)$$

(and similarly for the M-action).

To prove $ker(\eta) = TZ$, note that the definitions of ϕ and $\widetilde{Z}$ imply that

$$T\widetilde{Z}^{\pm} = \left\{(v,w) \in T_{(m,z)}M \times \mathbb{C} \ : \ d\phi_m(v) = z \cdot \overline{w} + \overline{z} \cdot w\right\},$$
$$TZ = \{v \in T_m M \ : \ d\phi_m(v) = 0\},$$

so $(v,w) \in T_{(m,0)}\widetilde{Z}^{\pm}$ satisfies $\eta(v,w) = (m,0)$ if and only if

$$w = 0 \ \Leftrightarrow\ d\phi_m(v) = 0 \ \Leftrightarrow\ v = (v,0) \in T_m Z \subset T_{(m,0)}\widetilde{Z}^{\pm},$$

and hence $ker(\eta) = TZ$ and the sequence is exact.

(2) is a direct consequence of (1). □

Let $N^{\pm} \to Z$ be the normal bundle to Z in $\tilde{Z}^{\pm}$, and $\overline{N}^{\pm} \to Z/S^1$ be the normal bundle to Z/S^1 in $M^{\pm}_{cut}$. The above lemma implies:

Corollary 5.1. *The short exact sequences of Lemma* 5.1 *induce isomorphisms*

$$N^{\pm} \xrightarrow{\ \simeq\ } Z \times \mathbb{C}, \qquad \overline{N}^{\pm} \xrightarrow{\ \simeq\ } Z \times_{S^1} \mathbb{C}$$

of equivariant vector bundle, and hence an orientation on the fibers of the bundles $\overline{N}^{\pm}$ *(coming from the complex orientation on* $\mathbb{C}$*).*

Remark 5.2. Note that the map

$$\overline{N}^{+} = Z \times_{S^1} \mathbb{C} \longrightarrow \overline{N}^{-} = Z \times_{S^1} \mathbb{C}, \qquad [z,a] \mapsto [z,\overline{a}]$$

is an S^1-equivariant orientation-reversing bundle isomorphism.

Claim 5.1. The natural orientation on $Z/S^1 \subset M^{\pm}_{cut}$, coming from the reduction process, followed by the orientation of $\overline{N}^{\pm}$, gives the orientation on $M^{\pm}_{cut}$.

Proof. Fix $x \in Z$. Choose an oriented orthonormal basis for $T_x M$ of the form

$$v_1, \ldots, v_{2m-2}, v_\theta, v_N,$$

where $v_\theta = c \cdot \left(\frac{\partial}{\partial\theta}\right)_{M,x}$ is a positive multiple of the generating vector field at $x \in Z$ ($c > 0$ is chosen such that v_θ has length 1), $\{v_1, \ldots, v_{2m-2}, v_\theta\}$ are an oriented orthonormal basis for $T_x Z$, and v_N is a positively oriented normal vector to Z.

By the definition of the metric and orientation on the reduced space, the push-forward of $v_1, \ldots, v_{2m-2}$ by the quotient map $Z \to Z/S^1$ is an oriented orthonormal basis for $T_{[x]}\left(Z/S^1\right)$.

Now the vectors

$$v_1, \ldots, v_{2m-2}, 1, i, v_\theta, v_N \in T_{(m,0)}M \times \mathbb{C}$$

are an oriented orthonormal basis, where $1, i \in \mathbb{C}$. Note that $\left(\frac{\partial}{\partial\theta}\right)_M = \left(\frac{\partial}{\partial\theta}\right)_{M \times \mathbb{C}}$ on $Z \cong Z \times \{0\} \subset M \times \mathbb{C}$, and that the normal to Z in M can be identified with the normal to $\tilde{Z}^{\pm}$ in $M \times \mathbb{C}$, when restricted to $Z \subset \tilde{Z}^{\pm}$. Hence, the push forward of $v_1, \dots, v_{2m-2}, 1, i$ by the quotient map $\tilde{Z}^{\pm} \to M^{\pm}_{cut}$ is an orthonormal basis for $T_{[m,0]} M^{\pm}_{cut}$.

Since $1, i$ descend to an oriented orthonormal basis for $(\overline{N}^{\pm})_x$, when identified with $\mathbb{C}$ using Corollary 5.1, the claim follows. $\square$

5.2. **Second lemma - the determinant line bundle.** We would like to relate the determinant line bundles of $P^{\pm}_{cut}$ (over $M^{\pm}_{cut}$), which will be denoted by $\mathbb{L}^{\pm}_{cut}$, to the determinant line bundle $\mathbb{L}$ of the spin^c-structure P on M. Denote $\mathbb{L}_{red} = (\mathbb{L}|_Z)/S^1$. This is a line bundle over $Z/S^1 \subset M^{\pm}_{cut}$.

Then we have:

Lemma 5.2. *The restriction of* $\mathbb{L}^{\pm}_{cut}$ *to* Z/S^1 *is isomorphic, as an* S^1*-equivariant complex line bundle, to* $\mathbb{L}_{red} \otimes \overline{N}^{\,-}$.

Remark 5.3. This is not a typo. Both $\mathbb{L}^{+}_{cut}$ and $\mathbb{L}^{-}_{cut}$ are isomorphic to $\mathbb{L}_{red} \otimes \overline{N}^{\,-}$.

Proof. Recall that the determinant line bundle over the cut spaces is given by

$$\mathbb{L}^{\pm}_{cut} = \left[(\mathbb{L} \boxtimes \mathbb{L}_{\mathbb{C}}{}^{\pm})|_{\tilde{Z}^{\pm}} \right] / S^1,$$

where $\mathbb{L}_{\mathbb{C}}{}^{\pm}$ is the determinant line bundle of the spin^c-structure on $\mathbb{C}$, defined in the process of constructing $P^{\pm}_{cut}$, and we divide by the diagonal action of S^1 on $\mathbb{L} \times \mathbb{L}_{\mathbb{C}}{}^{\pm}$.

Therefore we have

$$\mathbb{L}^{\pm}_{cut}|_{Z/S^1} = \left[(\mathbb{L} \boxtimes \mathbb{L}_{\mathbb{C}}{}^{\pm})|_Z \right] / S^1 = \left[\mathbb{L}|_Z \boxtimes \mathbb{L}_{\mathbb{C}}{}^{\pm}|_{\{0\}} \right] / S^1.$$

Since the S^1 action on the vector space $\mathbb{L}_{\mathbb{C}}{}^{\pm}|_{\{0\}}$ has weight $+1$ (see Remark 2.8) we end up with

$$\mathbb{L}^{\pm}_{cut}|_{Z/S^1} = \mathbb{L}_{red} \otimes (N^{-}/S^1) = \mathbb{L}_{red} \otimes \overline{N}^{\,-},$$

as desired. $\square$

Corollary 5.2. *If* $F \subset Z/S^1 \subset M^{\pm}_{cut}$ *is a connected component, then* S^1 *acts on the fibers of* $\left(\mathbb{L}^{\pm}_{cut}\right)|_F$ *with weight* $+1$.

Proof. The previous lemma implies that

$$\mathbb{L}^{\pm}_{cut}|_F = \mathbb{L}_{red}|_F \otimes \overline{N}^{\,-}|_F.$$

The action of S^1 on $\mathbb{L}_{red}$ is trivial. Using the isomorphism $\overline{N}^{\,-} \simeq Z \times_{S^1} \mathbb{C}$ from Corollary 5.1, we see that the action of S^1 on the fibers of $\overline{N}^{\,-}|_F$ will have weight $+1$. $\square$

5.3. **Third lemma - the spaces $M_{\pm}$.** Recall that $M \setminus Z = M_+ \amalg M_-$ (disjoint union), where $M_{\pm} \subset M$ are open submanifolds. We have the embeddings

$$i_{\pm} \colon M_{\pm} \to M^{\pm}_{cut} \qquad m \mapsto [m, \sqrt{\pm\phi(m)}]$$

which are equivariant and preserve the orientation (see Proposition 6.1 in [6]). Also recall that, as sets, we have $M^{\pm}_{cut} = Z/S^1 \amalg M_{\pm}$.

It is important to note that the embeddings $M_{\pm} \to M^{\pm}_{cut}$ do not preserve the metric. This, however, will not effect our calculations.

Lemma 5.3. *The restriction of* $\mathbb{L}$ *to* $M_{\pm}$ *is isomorphic to the restriction of* $L^{\pm}_{cut}$ *to* $M_{\pm}$*. In other words,*

$$\mathbb{L}|_{M_{\pm}} \simeq \left(\mathbb{L}^{\pm}_{cut}\right)|_{M_{\pm}}.$$

Proof. Let
$$\widetilde{M}_\pm = \left\{(m, \sqrt{\pm\phi(m)}) : m \in M_\pm\right\} \subset \widetilde{Z}^\pm ,$$
and let
$$pr_1 \colon M \times \mathbb{C} \to M, \qquad pr_2 \colon M \times \mathbb{C} \to \mathbb{C}$$
be the projections. Then
$$\mathbb{L}^\pm_{cut} = \left[(pr_1^*(\mathbb{L}) \otimes pr_2^*(\mathbb{L}_\mathbb{C}))\,|_{\widetilde{Z}^\pm}\right] / S^1,$$
and when restricting to $M_\pm$, we get
$$\mathbb{L}^\pm_{cut}|_{M_\pm} = pr_1^*(\mathbb{L})|_{\widetilde{M}_\pm} \otimes pr_2^*(\mathbb{L}_\mathbb{C})|_{\widetilde{M}_\pm} = \mathbb{L}|_{M_\pm} \otimes pr_2^*(\mathbb{L}_\mathbb{C})|_{\widetilde{M}_\pm},$$
since $M_\pm \simeq \widetilde{M}_\pm$. The term $pr_2^*(\mathbb{L}_\mathbb{C})|_{\widetilde{M}_\pm}$ is a trivial equivariant complex line bundle, so we conclude that
$$\mathbb{L}^\pm_{cut}|_{M_\pm} = \mathbb{L}|_{M_\pm} \otimes \mathbb{C} = \mathbb{L}|_{M_\pm},$$
as needed. □

5.4. **The proof of additivity under cutting.** Using all the preliminary lemmas, we can now prove our main theorem.

Proof of Theorem 5.1. Write $M \setminus Z = M_+ \sqcup M_-$. Because the action $S^1 \circlearrowleft Z$ is free, the submanifold $Z \subset M$ is a reducible splitting hypersurface (see §2.4). Every connected component $F \subset M^{S^1}$ of the fixed point set must be a subset of either M_+ or M_- .

Also recall that $M^\pm_{cut} = M_\pm \sqcup Z/S^1$ and the action of S^1 on Z/S^1 is trivial (and hence Z/S^1 is a subset of the fixed point set under the action $S \circlearrowleft M^\pm_{cut}$).

Using the Kostant formula (Theorem 4.1) we get, for any weight $\beta \in Lie(S^1)^*$,
$$\begin{aligned}\#(\beta, Q(M)) &= \sum_{F \subset M^{S^1}} (-1)^F \cdot \overline{N}_F\left(\beta - \frac{1}{2}\mu_F\right)\\ &= \sum_{F \subset (M_+)^{S^1}} (-1)^F \cdot \overline{N}_F\left(\beta - \frac{1}{2}\mu_F\right) + \sum_{F \subset (M_-)^{S^1}} (-1)^F \cdot \overline{N}_F\left(\beta - \frac{1}{2}\mu_F\right),\end{aligned}$$
where the sum is taken over the connected components of the fixed point sets. For the cut spaces we have the following equalities:
$$\begin{aligned}\#(\beta, Q(M^\pm_{cut})) &= \sum_{F \subset \left(M^\pm_{cut}\right)^{S^1}} (-1)^F \cdot \overline{N}_F\left(\beta - \frac{1}{2}\mu_F\right)\\ &= \sum_{F \subset (M_\pm)^{S^1}} (-1)^F \cdot \overline{N}_F\left(\beta - \frac{1}{2}\mu_F\right) + \sum_{F \subset Z/S^1} (-1)^F \cdot \overline{N}_F\left(\beta - \frac{1}{2}\mu_F\right) .\end{aligned}$$

In order to prove additivity, we need to show that
$$\sum_{F \subset Z/S^1 \subset M^+_{cut}} (-1)^F \cdot \overline{N}_F\left(\beta - \frac{1}{2}\mu_F\right) + \sum_{F \subset Z/S^1 \subset M^-_{cut}} (-1)^F \cdot \overline{N}_F\left(\beta - \frac{1}{2}\mu_F\right) = 0.$$
Note that the summands in the two sums above are different. In the first, we regard F as a subset of M^+_{cut}, and in the second, as a subset of M^-_{cut}.

Choose a connected component $F \subset Z/S^1$. Note that F is oriented by the reduced orientation. Since F can be regarded as a subset of both M^+_{cut} and M^-_{cut}, we will add a superscript $F^{\pm}$ to emphasize that F is being thought of as a subspace of the corresponding cut space.

It suffices to show that

$$(*)\qquad (-1)^{F^+} \cdot \overline{N}_{F^+}\left(\beta - \frac{1}{2}\mu_{F^+}\right) + (-1)^{F^-} \cdot \overline{N}_{F^-}\left(\beta - \frac{1}{2}\mu_{F^-}\right) = 0.$$

Recall that $Z \subset M$ is of (real) codimension 1, and so $Z/S^1 \subset M^{\pm}_{cut}$ is of (real) codimension 2. Therefore, the normal bundle $NF^{\pm}$ to Z/S^1 in the cut spaces has rank 2. We can turn the bundles $NF^{\pm}$ to complex line bundles using Corollary 5.1, and then the weight of the action $S^1 \circlearrowleft NF^{\pm}$ will be -1 for NF^+ and $+1$ for NF^-.

This is, however, not good, since in order to write Kostant's formula, we need our weights to be polarized. Therefore, we will use for NF^- the complex structure coming from the isomorphism

$$NF^- \xrightarrow{\simeq} Z \times_{S^1} \mathbb{C}\ ,$$

and for NF^+ we will use the complex structure which is *opposite* to the one induced by the isomorphism

$$NF^+ \xrightarrow{\simeq} Z \times_{S^1} \mathbb{C}\ .$$

With this convention, the bundles $NF^{\pm}$ become isomorphic as equivariant complex line bundles, and the weight of the S^1-action on those bundles is $+1$.

Also, Lemma 5.2 implies that the determinant line bundles $\mathbb{L}^{\pm}_{cut}$, when restricted to F, are isomorphic as equivariant complex line bundles, and the weight of the S^1-action on the fibers of $\mathbb{L}^{\pm}_{cut}|_F$ is $+1$.

Recall now (see §5.3) that the explicit expression for $\overline{N}_{F^{\pm}}\left(\beta - \frac{1}{2}\mu_{F^{\pm}}\right)$ involves only the following ingredients:

- $\mu_{F^{\pm}}$, which are equal to each other ($\mu_{F^{\pm}} = +1$), since $\mathbb{L}^+_{cut}|_F \simeq \mathbb{L}^-_{cut}|_F$.
- $c_1(NF^{\pm})$, which are equal since $NF^{\pm}$ are isomorphic as complex line bundles, by our previous remark.
- $\hat{A}(TF)$, which are equal, since $F^+ = F^-$ as manifolds.

This means that the terms $\overline{N}_{F^{\pm}}$ in equation $(*)$ above are the same.

So all that is left is to explain why

$$(-1)^{F^+} + (-1)^{F^-} = 0\ .$$

But this follows easily from Claim 5.1. This claim implies that the orientation on F^-, followed by the one of NF^-, gives the orientation of M^-_{cut}. Hence, $(-1)^{F^-} = 1$. Since we switched the original orientation for NF^+, composing the orientation of F^+ with the one of NF^+ will give the opposite orientation on M^+_{cut}, and hence $(-1)^{F^+} = -1$. The additivity result follows. □

6. An example: The two-sphere

In this section we give an example which illustrates the additivity of spin^c-quantization under cutting.

In this example, the manifold is the standard two-sphere $M = S^2 \subset \mathbb{R}^3$, with the outward orientation and the standard Riemannian structure. The circle group $S^1 \subset \mathbb{C}$ acts effectively on the two sphere by rotations about the z-axis.

We will need the following lemma.

Lemma 6.1. *Let M be an oriented Riemannian manifold, on which a Lie group G acts transitively by orientation preserving isometries. Choose a point $x \in M$ and denote by G_x the stabilizer at x and by $\sigma\colon G_x \to SO(T_xM)$ the isotropy representation. Then:*

(1) *G_x acts on $SO(T_xM)$ by $g \cdot A = \sigma(g) \circ A$.*
(2) *The map*
$$G \to M, \qquad g \mapsto g \cdot x$$
is a principal G_x-bundle (where G_x acts on G by right multiplication).
(3) *The principal $SO(T_xM)$-bundle $G \times_{G_x} SO(T_xM)$ is isomorphic to $SOF(M)$, the bundle of oriented orthonormal frames on M.*

Proof. (1) is easy. (2) follows from Proposition B.18 in [2] (with $H = G_x$), together with the fact that G/G_x is diffeomorphic to M. To show (3), consider the map taking an element $[g, A] \in G \times_{G_x} SO(T_xM)$ to the frame $g_* \circ A\colon T_xM \xrightarrow{\cong} T_{g\cdot x}M$. This map can be easily checked to be an isomorphism of principal $SO(T_xM)$-bundles. $\square$

6.1. **The trivial S^1-equivariant spin^c-structure on S^2.**

To define an S^1-equivariant spin^c-structure on S^2, one needs to describe the space P and the maps in a commutative diagram of the following form (see Remark 8.3):

$$\begin{array}{ccccc}
S^1 \times P & \longrightarrow & P & \longleftarrow & P \times Spin^c(2) \\
\downarrow & & \Lambda\downarrow & & \downarrow \\
S^1 \times SOF(S^2) & \longrightarrow & SOF(S^2) & \longleftarrow & SOF(S^2) \times SO(2) \\
\downarrow & & \pi\downarrow & & \\
S^1 \times S^2 & \longrightarrow & S^2 & &
\end{array}$$

Set $P = Spin^c(3)$. By the above lemma, the choice of a point $x = (0,0,1) \in S^2$ and a basis for T_xS^2 give an isomorphism between the frame bundle of S^2 and $SO(3) \times_{SO(2)} SO(2) = SO(3)$. Thus $SOF(S^2) \cong SO(3)$, and our diagram becomes

$$\begin{array}{ccccc}
S^1 \times Spin^c(3) & \longrightarrow & Spin^c(3) & \longleftarrow & Spin^c(3) \times Spin^c(2) \\
\downarrow & & \Lambda\downarrow & & \downarrow \\
S^1 \times SO(3) & \longrightarrow & SO(3) & \longleftarrow & SO(3) \times SO(2) \\
\downarrow & & \pi\downarrow & & \\
S^1 \times S^2 & \longrightarrow & S^2 & &
\end{array}$$

Now we describe the maps in this diagram. Denote

$$C_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The map $S^1 \times S^2 \to S^2$ is a rotation about the vertical axis, i.e., $(e^{i\theta}, v) \mapsto C_\theta \cdot v$.

The second horizontal row gives the actions of S^1 and $SO(2)$ on the frame bundle $SO(3)$. Those are given by left and right multiplication by C_θ, respectively. The covering map $\pi\colon SO(3)\to S^2$ is given by $A\mapsto A\cdot x$, and Λ is the natural map from the spin^c-group to the special orthogonal group.

All that is left is to describe the actions of S^1 and $Spin^c(2)$ on $Spin^c(3)$ (the top row in the diagram). Since $Spin^c(2)\subset Spin^c(3)$, this group will act by right-multiplication. The S^1-action on $Spin^c(3)$ is given by

$$(e^{i\theta}\ ,\ [A,z])\mapsto [x_{\theta/2}\cdot A\ ,\ e^{i\theta/2}\cdot z], \tag{1}$$

where $x_\theta=\cos\theta+\sin\theta\cdot e_1e_2\in Spin(3)$. Note that $x_{\theta/2}$ and $e^{i\theta/2}$ are defined only up to sign, but the equivalence class $[x_{\theta/2}\ ,\ e^{i\theta/2}]$ is a well defined element in $Spin^c(3)$.

We will call this S^1-equivariant spin^c-structure *the trivial spin^c-structure on the S^1-manifold S^2*, and denote it by P_0. The reason for using the word 'trivial' is justified by the following lemma.

Lemma 6.2. *The determinant line bundle of the trivial spin^c-structure P_0 is isomorphic to the trivial complex line bundle $\mathbb{L}\cong S^2\times\mathbb{C}$, with the nontrivial S^1-action*

$$S^1\times\mathbb{L}\to\mathbb{L},\qquad (e^{i\theta},(v,z))\mapsto(C_\theta\cdot v,e^{i\theta}\cdot z).$$

Proof. It is easy to check that the map

$$\mathbb{L}=Spin^c(3)\times_{Spin^c(2)}\mathbb{C}\to S^2\times\mathbb{C},\qquad [[A,z],w]\mapsto(\lambda(A)\cdot x,z^2w)\ ,$$

where $\lambda\colon Spin(3)\to SO(3)$ is the double cover and $x=(0,0,1)$ is the north pole, is an isomorphism of complex line bundles. The fact that S^1 acts on $\mathbb{L}$ via (1) and that $\lambda(x_{\theta/2})=C_\theta$ implies that the S^1 action on $S^2\times\mathbb{C}$, induced by the above isomorphism, is the one stated in the lemma. □

Another reason for calling P_0 a trivial spin^c-structure is that the quantization $Q(S^2)$ (with respect to P_0) is the zero space. We do not prove this fact now, since it will follow from a more general statement (see Claim 6.3).

6.2. Classifying all spin^c-structures on S^2.

Quantizing the trivial spin^c-structure on S^2 is not interesting, since the quantization is the zero space. However, once we have an equivariant spin^c-structure on a manifold, we can generate all the other equivariant spin^c-structures by twisting it with complex equivariant Hermitian line bundles (or, equivalently, with equivariant principal $U(1)$-bundles). For details on this process, see Appendix D, §2.7 in [2]. We will use this technique to construct all spin^c-structures on our S^1-manifold S^2.

It is known that all (nonequivariant) complex Hermitian line bundles over S^2 are classified by $H^2(S^2;\mathbb{Z})\cong\mathbb{Z}$, i.e., by the integers. The S^1-equivariant line bundles over S^2 are classified by a pair of integers (for instance, the weights of the S^1-action on the fibers at the poles). This is well known, but because we couldn't find a direct reference, we will give a direct proof of this fact.

Here is an explicit construction of an equivariant line bundle over S^2, determined by a pair of integers.

Definition 6.1. Given a pair of integers (k,n), define an S^1-equivariant complex Hermitian line bundle $L_{k,n}$ as follows:

(1) As a complex line bundle,

$$L_{k,n}=Spin(3)\times_{Spin(2)}\mathbb{C}\cong S^3\times_{S^1}\mathbb{C}\ ,$$

where $Spin(2) \cong S^1$ acts on $\mathbb{C}$ with weight n and on $Spin(3)$ by right multiplication.

(2) The circle group S^1 acts on $L_{k,n}$ by

$$S^1 \times L_{k,n} \to L_{k,n}, \qquad \big(e^{i\theta}, [A, z]\big) \mapsto [x_{\theta/2} \cdot A, e^{\frac{i\theta}{2}(n+2k)} \cdot z],$$

where $x_\theta = \cos\theta + \sin\theta \cdot e_1 e_2 \in Spin(2) \subset Spin(3)$.

Now we prove:

Claim 6.1. Every S^1 equivariant line bundle over S^2 is isomorphic to $L_{k,n}$, for some $k, n \in \mathbb{Z}$.

Proof. Let L be an S^1-equivariant line bundle over S^2. Since L is, in particular, an ordinary line bundle, we can assume it is of the form $L = S^3 \times_{S^1} \mathbb{C}$, where S^1 acts on $\mathbb{C}$ with weight n. Also, since L is an equivariant line bundle, we have a map

$$\rho\colon S^1 \times L \to L, \qquad (e^{i\theta}, x) \mapsto e^{i\theta} \cdot x \ .$$

Define a map

$$\eta\colon S^1 \times L \to L, \qquad (e^{i\theta}, [A, z]) \mapsto [x_{-\theta/2} \cdot A, e^{-i\theta n/2} z] \ .$$

This map is well defined.

By composing ρ and η we get a third map,

$$\delta\colon S^1 \times L \to L,$$

which lifts the trivial action on S^2. Since S^2 is connected, this composed action will act on all the fibers of L with one fixed weight k. Therefore, we get

$$e^{i\theta} \cdot [x_{-\theta/2} \cdot A, e^{-i\theta n/2} z] = [A, e^{ik\theta} z],$$

and after a change of variables, the given action $S^1 \circlearrowright L$ is

$$e^{i\theta} \cdot [B, w] = [x_{\theta/2} \cdot B, e^{i\theta n/2 + ik\theta} w] \ .$$

This means that L is isomorphic to $L_{k,n}$. □

We now 'twist' the trivial spin^c-structure by $U(L_{k,n})$, the unit circle bundle of $L_{k,n}$, to get nontrivial spin^c-structures on S^2. Observe that the group $U(1)$ acts on $Spin^c(3)$ from the right by multiplication by elements of the form $[1, c] \in Spin^c(3)$.

Definition 6.2.

$$P_{k,n} = P_0 \times_{U(1)} U(L_{k,n}),$$

where we quotient by the anti-diagonal action of $U(1)$.

This is an S^1-equivariant spin^c-structure on S^2. The principal action of $Spin^c(2)$ comes from acting from the right on the $P_0 \cong Spin^c(3)$ component, and the left S^1-action is induced from the diagonal action on $P_0 \times L_{k,n}$.

Claim 6.2. Fix $(k, n) \in \mathbb{Z}^2$, and denote by $\mathbb{L} = \mathbb{L}_{k,n}$ the determinant line bundle associated to the spin^c-structure $P_{k,n}$ on S^2. Let $N = (0,0,1)$, $S = (0,0,-1) \in S^2$ be the north and the south poles.

Then S^1 acts on $\mathbb{L}|_N$ with weight $2k + 2n + 1$ and on $\mathbb{L}|_S$ with weight $2k + 1$.

Proof. The determinant line bundle is

$$\mathbb{L} = P_{k,n} \times_{Spin^c(2)} \mathbb{C} = \big[Spin^c(3) \times_{U(1)} \big(S^3 \times_{S^1} S^1\big)\big] \times_{Spin^c(2)} \mathbb{C} \ .$$

An element of $\mathbb{L}$ can be written in the form $[[[A, 1], [A, 1]], u]$, where $A \in Spin(3) \cong S^3$ and $u \in \mathbb{C}$.

(1) For the north pole $N=(0,0,1)$ we can choose $A=1\in Spin(3)$; hence an element of $\mathbb{L}|_N$ will have the form $[[[1,1],[1,1]],u]$. Let $e^{i\theta}\in S^1$ act on $\mathbb{L}|_N$ to get

$$\left[\left[[x_{\theta/2},e^{i\theta/2}],[x_{\theta/2},e^{i\theta(n+2k)/2}]\right],u\right]=\left[\left[[1,1],[x_{\theta/2},e^{i\theta(n+2k)/2}]\right],e^{i\theta}u\right]$$
$$=\left[\left[[1,1],[x_{\theta/2},e^{i\theta\cdot n/2}]\right],e^{i\theta(1+2k)}u\right]=\left[\left[[1,1],[1,e^{i\theta\cdot n/2}\cdot e^{i\theta\cdot n/2}]\right],e^{i\theta(1+2k)}u\right]$$
$$=\left[[[1,1],[1,1]],e^{i\theta(1+2k+2n)}u\right],$$

and therefore the weight on $\mathbb{L}|_N$ is $1+2k+2n$.

(2) For the south pole $S=(0,0,-1)$ choose $A=e_2e_3$. We again compute the action of an element $e^{i\theta}$ on $[[[A,1],[A,1]],u]$ and use the identity $A\cdot x_\theta = x_{-\theta}\cdot A$ for any $x_\theta\in Spin(2)\subset Spin(3)$:

$$\left[\left[[x_{\theta/2}A,e^{i\theta/2}],[x_{\theta/2}A,e^{i\theta(n+2k)/2}]\right],u\right]$$
$$=\left[\left[[A,1],[Ax_{-\theta/2},e^{i\theta(n+2k)/2}]\right],e^{i\theta}u\right]$$
$$=\left[\left[[A,1],[Ax_{-\theta/2},e^{i\theta\cdot n/2}]\right],e^{i\theta(1+2k)}u\right]$$
$$=\left[\left[[A,1],[A,e^{-i\theta\cdot n/2}\cdot e^{i\theta\cdot n/2}]\right],e^{i\theta(1+2k)}u\right]$$
$$=\left[[[A,1],[A,1]],e^{i\theta(1+2k)}u\right],$$

and therefore the weight on $\mathbb{L}|_S$ is $2k+1$. □

Remark 6.1. Note that the $2k+2n+1$ and $2k+1$ are both odd numbers. This is not surprising in view of Lemma 3.1. The isotropy weight at N (or at S) is ± 1, and its sum with the weight on $\mathbb{L}_N$ (or on $\mathbb{L}_S$) must be even. This implies that the weights of $S^1 \circlearrowright \mathbb{L}_{\{N,S\}}$ must be odd.

Remark 6.2. The above claim implies that the determinant line bundle of the spin^c-structure $P_{k,n}$ is isomorphic to $L_{2k+1,2n}$, i.e., $\mathbb{L}_{k,n}\cong L_{2k+1,2n}$.

Claim 6.3. Fix $(k,n)\in\mathbb{Z}^2$ and denote by $Q_{k,n}(S^2)$ the quantization of the spin^c-structure $P_{k,n}$ on S^2. Then the multiplicity of a weight $\beta\in Lie(S^1)^*\cong\mathbb{Z}$ in $Q_{k,n}(S^2)$ is given by

$$\#(\beta,Q_{k,n}(S^2))=\begin{cases}1 & 0<\beta-k\le n,\\ -1 & n<\beta-k\le 0,\\ 0 & \text{otherwise.}\end{cases}$$

In particular, if $n=0$, then $Q_{k,0}(S^2)$ is the zero representation.

Proof. By the Kostant formula for spin^c-quantization (Theorem 3.1) the multiplicity is given by

$$\#(\beta,Q_{k,n}(S^2))=\overline{N}_{(0,0,1)}\left(\beta-\frac{1+2k+2n}{2}\right)-\overline{N}_{(0,0,-1)}\left(\beta-\frac{1+2k}{2}\right).$$

The definition of $\overline{N}_p$ implies that

$$\overline{N}_{(0,0,1)}\left(\beta-\frac{1+2k+2n}{2}\right)=\begin{cases}1 & \beta-k\le n,\\ 0 & \beta-k>n,\end{cases}$$

and similarly,

$$\overline{N}_{(0,0,-1)}\left(\beta - \frac{1+2k}{2}\right) = \begin{cases} 1 & \beta - k \leq 0, \\ 0 & \beta - k > 0. \end{cases}$$

Using that, one can compute $\#(\beta, Q_{k,n}(S^2))$ and get the required result. □

6.3. **Cutting a spin^c-structure on S^2.** Now we get to the cutting of the spin^c-structure $P_{k,n}$ on S^2. Let $\mathbb{L}$ be the determinant line bundle of $P_{k,n}$. We take the equator $Z = \{(\cos\alpha, \sin\alpha, 0)\} \subset S^2$ to be our reducible splitting hypersurface (see §2.4). The cut spaces $M^{\pm}_{cut}$ are both diffeomorphic to S^2, and we would like to know what are $(P_{k,n})^{\pm}_{cut}$. Because the cut spaces are spheres again, we must have

$$(P_{k,n})^{\pm}_{cut} = P_{k^{\pm},n^{\pm}} \qquad \text{for some integers} \qquad k^{\pm}, n^{\pm} \ .$$

Corollary 5.2 implies that S^1 acts on $\mathbb{L}^-_{cut}|_N$ and on $\mathbb{L}^+_{cut}|_S$ with weight $+1$. Lemma 5.3 implies that the weight of the S^1 action on $\mathbb{L}|_N$ and $\mathbb{L}|_S$ will be equal to the weight of the action on $\mathbb{L}^+_{cut}|_N$ and $\mathbb{L}^-_{cut}|_S$, respectively. From this we get the equations

$$2k^+ + 1 = 1, \qquad 2k^+ + 2n^+ + 1 = 2k + 2n + 1,$$
$$2k^- + 2n^- + 1 = 1, \qquad 2k^- + 1 = 2k + 1,$$

which yield $k^+ = 0,\ n^+ = k + n,\ k^- = k,\ n^- = -k$. Therefore we obtain

$$(P_{k,n})^+_{cut} = P_{0,k+n}, \qquad\qquad (P_{k,n})^-_{cut} = P_{k,-k} \ .$$

Remark 6.3. We see that there is no symmetry between the spin^c-structures on the '+' and '−' cut spaces as one might expect. This is because the definition of the covering map $SO(3) \to S^2$ involved a choice of a point (in our case, the north pole), which 'broke' the symmetry of the two-sphere.

The quantization of the cut spaces is thus obtained from Claim 6.3. For the '+' cut space we get, for any weight $\beta \in \mathbb{Z}$,

$$\#(\beta, Q^+_{k,n}(S^2)) = \#(\beta, Q_{0,k+n}(S^2)) = \begin{cases} 1 & -k < \beta - k \leq n, \\ -1 & n < \beta - k \leq -k, \\ 0 & \text{otherwise} \end{cases}$$

and for the '−' cut space,

$$\#(\beta, Q^-_{k,n}(S^2)) = \#(\beta, Q_{k,-k}(S^2)) = \begin{cases} 1 & 0 < \beta - k \leq -k, \\ -1 & -k < \beta - k \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is an easy exercise to check that

$$\#(\beta, Q_{k,n}(S^2)) = \#(\beta, Q^+_{k,n}(S^2)) + \#(\beta, Q^-_{k,n}(S^2)),$$

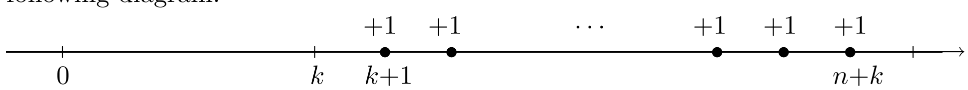
and this implies that as virtual S^1-representations, we have

$$Q_{k,n}(S^2) = Q^-_{k,n}(S^2) \oplus Q^+_{k,n}(S^2).$$

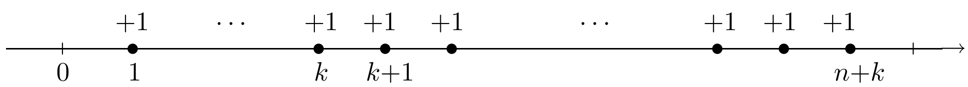
As expected, we have additivity of spin^c-quantization under cutting in this example.

6.4. **Multiplicity diagrams.** The S^1-equivariant spin^c-quantization of a manifold M can be described using multiplicity diagrams as follows. Above each integer on the real line, we write the multiplicity of the weight represented by this integer, if it is nonzero.

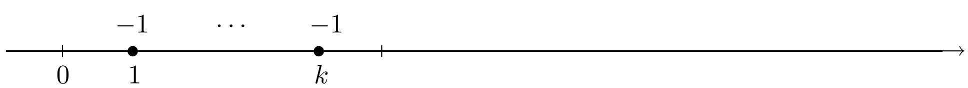
For example, if $n, k > 0$, then the quantization $Q_{k,n}$ of S^2 is given by the following diagram:



The quantization of the '+' cut space, $Q^+_{k,n}$, which is equal to $Q_{0,k+n}$, will have the following diagram:

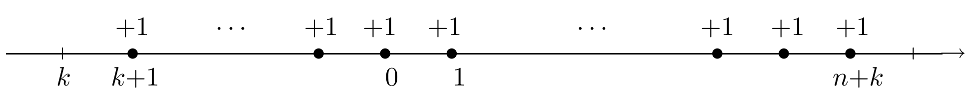


Finally, $Q^-_{k,n} = Q_{k,-k}$ is given by

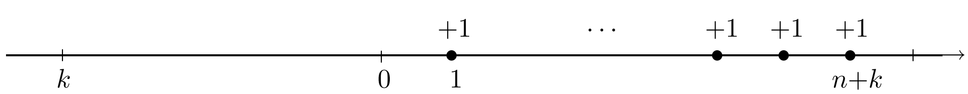


Clearly, one can see that the diagram of $Q_{k,n}$ is the 'sum' of the diagrams of $Q^\pm_{k,n}$.

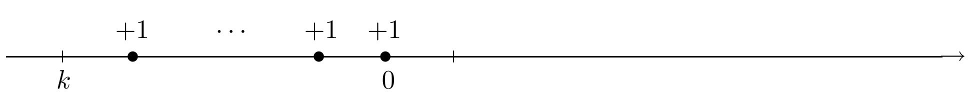
Let us present another case, where only positive multiplicities occur in the quantization of all three spaces (the original manifold S^2 and the cut spaces). This happens if $k < 0 < n + k$. In this case, the diagram for $Q_{k,n}$ is as follows:



The diagram for $Q^+_{k,n} = Q_{0,n+k}$ is



and for $Q^-_{k,n} = Q_{k,-k}$ we have



and again the additivity is clear.

The additivity is clearer in the last set of diagrams, as we can actually see the diagram of $Q_{k,n}$ being cut into two parts. It seems like the diagram was cut at some point between 0 and 1. The point at which the cutting is done depends on the spin^c-structure on $\mathbb{C}$ that was chosen during the cutting process (see §2.4).

7. Relation to symplectic cutting

The cutting construction was originally defined for symplectic manifolds (see [4]). In this paper we followed [6] and defined cutting for manifolds which are not necessarily symplectic. However, our work can be related to symplectic cutting as described in [11]. We outline the main ideas of this procedure.

Assume that a symplectic manifold (M, ω) is endowed with a spin^c-structure $P \to SOF(M) \to M$ (with respect to an orientation and a Riemannian structure). When a connection 1-form $\theta \in \Omega^1(P; \mathfrak{u}(1))$ on $P \to SOF(M)$ is given, then the following compatibility condition between the symplectic structure, the spin^c-structure, and the connection may be imposed:

$$d\theta = \pi^*(-i \cdot \omega) \ ,$$

where $\pi \colon P \to M$ is the projection. When this condition (and one more technical assumption) are satisfied, then (P, θ) is called *a spin^c-prequantization* for (M, ω) (alternatively, given an oriented Riemannian manifold M, we 'prequantize' the manifold (M, ω), where ω is a closed two-form, determined by the above equality). If all those structures respect a G-action on the bundles $P \to SOF(M) \to M$, then G acts on M in a Hamiltonian fashion, with a 'natural' moment map $\Phi \colon M \to \mathfrak{g}^*$ given by

$$\pi^*\left(\Phi^{\xi}\right) = -i\left(\iota_{\xi_P}\theta\right), \qquad \xi \in \mathfrak{g} \ .$$

In the case where $G = S^1$, we can cut the manifold M along a level set of the moment map Φ. The cutting construction can be extended to the spin^c-prequantization (P, θ), and so we end up with two pairs $(P^{\pm}_{cut}, \theta^{\pm}_{cut})$ for the cut spaces $(M^{\pm}_{cut}, \omega^{\pm}_{cut})$.

The cutting construction for spin^c-prequantization involves a choice of an odd integer $\ell \in \mathbb{Z}$. It turns out that $(P^{\pm}_{cut}, \theta^{\pm}_{cut})$ are spin^c-prequantizations for $(M^{\pm}_{cut}, \omega^{\pm}_{cut})$ if and only if the cutting was done along the submanifold $Z = \Phi^{-1}(\ell/2)$.

On the level of multiplicity diagrams, the diagram of P will be cut at $\ell/2$ to give the diagrams for $P^{\pm}_{cut}$. The fact that $\ell/2$ is not an integer is the reason for having additivity.

Details about spin^c-prequantization for symplectic manifolds and the corresponding cutting construction will be available in [11].

8. Appendix: Spin^c-structures, Clifford algebras and spin^c-quantization

This section contains a review of standard material, which is used throughout the paper. We review the definitions of spin^c-structures (including the equivariant case), Clifford algebras, Clifford multiplication, and spinor bundles. We then proceed and define the spin^c-Dirac operator associated to a spin^c-structure as well as the concept of spin^c-quantization. We also fix some notation and terminology.

8.1. **Spin^c-structures.**

Definition 8.1. Let V be a finite-dimensional vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, equipped with a symmetric bilinear form $B : V \times V \to \mathbb{K}$. The *Clifford algebra*

$Cl(V,B)$ is the quotient $T(V)/I(V,B)$, where $T(V)$ is the tensor algebra of V and $I(V,B)$ is the ideal generated by $\{v\otimes v - B(v,v)\cdot 1 \ : \ v\in V\}$.

Definition 8.2. If $V=\mathbb{R}^k$ and B is minus the standard inner product on V, then define the following objects:

(1) $C_k := Cl(V,B)$, and $C_k^c := Cl(V,B)\otimes\mathbb{C}$.

(2) The *spin group*

$$Spin(k)=\{v_1v_2\dots v_l \ : \ v_i\in\mathbb{R}^k,\ ||v_i||=1 \text{ and } 0\le l \text{ is even}\}\subset C_k.$$

(3) The *spin^c-group*

$$Spin^c(k)=(Spin(k)\times U(1))\diagup K,$$

where $U(1)\subset\mathbb{C}$ is the unit circle and $K=\{(1,1),(-1,-1)\}$.

Remark 8.1. Equivalently, one can define the group $Spin^c(k)$ as

$$\left\{c\cdot v_1\cdots v_l \ : \ v_i\in\mathbb{R}^k,\ ||v_i||=1,\ 0\le l \text{ is even, and } c\in U(1)\right\}\subset C_k^c.$$

Proposition 8.1. (1) *There is a linear map $C_k\to C_k$, $x\mapsto x^t$, characterized by $(v_1\dots v_l)^t=v_l\dots v_1$ for all $v_1,\dots,v_l\in\mathbb{R}^k$.*

(2) *For each $x\in Spin(k)$ and $y\in\mathbb{R}^k$, we have $xyx^t\in\mathbb{R}^k$.*

(3) *For each $x\in Spin(k)$, the map $\lambda(x):\mathbb{R}^k\to\mathbb{R}^k$, $y\mapsto xyx^t$, is in $SO(k)$, and $\lambda: Spin(k)\to SO(k)$ is a double covering for $k\ge 1$. It is a universal covering map for $k\ge 3$.*

For the proof, see page 16 in [1].

Definition 8.3. Let M be a manifold and Q a principal $SO(k)$-bundle on M. A *spin^c-structure* on Q is a principal $Spin^c(k)$-bundle $P\to M$, together with a map $\Lambda: P\to Q$, such that the following diagram commutes:

$$\begin{array}{ccc} P\times Spin^c(k) & \longrightarrow & P \\ \Big\downarrow{\scriptstyle \Lambda\times\lambda^c} & & \Big\downarrow{\scriptstyle \Lambda} \\ Q\times SO(k) & \longrightarrow & Q \end{array}$$

Here, the maps corresponding to the horizontal arrows are the principal actions, and $\lambda^c: Spin^c(k)\to SO(k)$ is given by $[x,z]\mapsto\lambda(x)$, where $\lambda: Spin(k)\to SO(k)$ is the double covering. Also:

- *A spin^c-structure on an oriented Riemannian vector bundle E* is a spin^c-structure on the associated bundle of oriented orthonormal frames, $SOF(E)$.
- *A spin^c-structure on an oriented Riemannian manifold* is a spin^c-structure on its tangent bundle.
- Given a spin^c-structure on $Q\to M$, *its determinant line bundle* is $\mathbb{L}=P\times_{Spin^c(k)}\mathbb{C}$, where the left action of $Spin^c(k)$ on $\mathbb{C}$ is given by $[x,z]\cdot w=z^2w$. This is a hermitian line bundle over M.

8.2. **Equivariant spin^c-structures.**

Definition 8.4. Let G,H be Lie groups. A *G-equivariant principal H-bundle* is a principal H-bundle $\pi: Q\to M$ together with left G-actions on Q and M, such that π intertwines the G-action on Q and M, and the G-action on Q commutes with the principal H-action.

Remark 8.2. A G-equivariant principal H-bundle gives rise to the following commuting diagram (the horizontal arrows correspond to the G and H actions):

$$\begin{array}{ccccc} G \times Q & \longrightarrow & Q & \longleftarrow & Q \times H \\ {\scriptstyle Id\times\pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} & & \\ G \times M & \longrightarrow & M & & \end{array}$$

Definition 8.5. Let $\pi : E \to M$ be a fiberwise oriented Riemannian vector bundle, and let G be a Lie group. If a G-action on $E \to M$ is given that preserves the orientations and the inner products of the fibers, we will call E a *G-equivariant oriented Riemannian vector bundle.*

Definition 8.6. Let $\pi : Q \to M$ be a G-equivariant principal $SO(k)$-bundle. *A G-equivariant spin^c-structure* on Q is a spin^c-structure $\Lambda : P \to Q$ on Q, together with a left action of G on P, such that Λ intertwines the G-actions on P and Q, and the G-action on P commutes with the principal Spin^c-action.

Remark 8.3.

(1) We have the following commuting diagram (where the horizontal arrows correspond to the principal and the G-actions):

$$\begin{array}{ccccc} G \times P & \longrightarrow & P & \longleftarrow & P \times Spin^c(k) \\ {\scriptstyle Id\times\Lambda}\big\downarrow & & {\scriptstyle \Lambda}\big\downarrow & & {\scriptstyle \Lambda\times\lambda^c}\big\downarrow \\ G \times Q & \longrightarrow & Q & \longleftarrow & Q \times SO(k) \\ {\scriptstyle Id\times\pi}\big\downarrow & & {\scriptstyle \pi}\big\downarrow & & \\ G \times M & \longrightarrow & M & & \end{array}$$

(2) The bundle $P \to M$ is a G-equivariant principal $Spin^c(k)$-bundle.
(3) The determinant line bundle $\mathbb{L} = P\times_{Spin^c(k)}\mathbb{C}$ is a G-equivariant Hermitian line bundle.

8.3. **Clifford multiplication and spinor bundles.**

Proposition 8.2. *The number of inequivalent irreducible (complex) representations of the algebra $C_k^c = C_k \otimes \mathbb{C}$ is 1 if k is even and 2 if k is odd.*

For a proof, see Theorem I.5.7 in [3].
Note that, for all k, $\mathbb{R}^k \subset C_k \subset C_k^c$.

Definition 8.7. Let k be a positive integer. Define a *Clifford multiplication map*

$$\mu : \mathbb{R}^k \otimes \Delta_k \to \Delta_k \qquad \text{by} \quad \mu(x \otimes v) = \rho_k(x)v,$$

where $\rho_k : C_k^c \to End(\Delta_k)$ is an irreducible representation of C_k^c.

Definition 8.8. Let k be a positive integer and ρ_k an irreducible representation of C_k^c. The restriction of ρ_k to the group $Spin(k) \subset C_k \subset C_k^c$ is called *the complex spin representation* of $Spin(k)$. It will also be denoted by ρ_k.

Remark 8.4. For an odd integer k, the complex spin representation is independent of the choice of an irreducible representation of C_k^c (see Proposition I.5.15 in [3]).

The following proposition summarizes a few facts about the complex spin representation. Proofs can be found in [1] and in [3].

Proposition 8.3. *Let $\rho_k : Spin(k) \to End(\Delta_k)$ be the complex spin representation. Then*

(1) $dim_{\mathbb{C}}\Delta_k = 2^l$*, where* $l = k/2$ *if* k *is even and* $l = (k-1)/2$ *if* k *is odd.*
(2) ρ_k *is a faithful representation of* $Spin(k)$*.*
(3) *If* k *is odd, then* ρ_k *is irreducible.*
(4) *If* k *is even, then* ρ_k *is reducible and splits as a sum of two inequivalent irreducible representations of the same dimension,*

$$\rho_k^+ : Spin(k) \to End(\Delta_k^+) \qquad \text{and} \qquad \rho_k^- : Spin(k) \to End(\Delta_k^-) \ .$$

Remark 8.5. The representation ρ_k extends to a representation of the group $Spin^c(k)$ and will also be denoted by ρ_k. Explicitly, $\rho_k([x,z])v = z \cdot \rho_k(x)v$.

Definition 8.9. Let P be a spin^c-structure on an oriented Riemannian manifold M. Then the *spinor bundle* of the spin^c-structure is the complex vector bundle $S = P \times_{Spin^c(m)} \Delta_m$, where $m = dim(M)$.

If P is a G-equivariant spin^c-structure, then S will be a G-equivariant complex vector bundle.

Remark 8.6. It is possible to choose a Hermitian inner product on Δ_k which is preserved by the action of the group $Spin^c(k)$. This induces a Hermitian inner product on the spinor bundle. In the G-equivariant case, G will act on the fibers of S by Hermitian transformations.

From Proposition 8.3 we get

Proposition 8.4. *Let P be a (G-equivariant) spin^c-structure on an oriented Riemannian manifold M of even dimension, and let S be the corresponding spinor bundle. Then S splits as a sum $S = S^+ \oplus S^-$ of two (G-equivariant) complex vector bundles.*

Remark 8.7. If M is an oriented Riemannian manifold, equipped with a spin^c-structure and a corresponding spinor bundle S, then a Clifford multiplication map $\mu : \mathbb{R}^k \otimes \Delta_k \to \Delta_k$ induces a map on the associated bundles $TM \otimes S \to S$. This map is also called Clifford multiplication and will be denoted by μ as well.

8.4. **The spin^c-Dirac operator.** The following is a reformulation of Proposition D.11 from [3]:

Proposition 8.5. *Let M be an oriented Riemannian manifold of dimension $m \geq 1$, $P \to SOF(M)$ a spin^c-structure on M, and $P_1 = P/Spin(m)$. Then*

(1) *P_1 is a principal $U(1)$-bundle over M, and $P \to SOF(M) \times P_1$ is a double cover.*
(2) *The determinant line bundle of the spin^c-structure is naturally isomorphic to $\mathbb{L} = P_1 \times_{U(1)} \mathbb{C}$.*
(3) *If $A : TP_1 \to i\mathbb{R}$ is an invariant connection and $Z : T(SOF(M)) \to \mathfrak{so}(m)$ the Levi-Civita connection on M, then the $SO(m) \times U(1)$-invariant connection $Z \times A$ on $SOF(M) \times P_1$ lifts to a unique $Spin^c(m)$-invariant connection on its double cover P.*

Definition 8.10. Assume the following data is given:

(1) An oriented Riemannian manifold M of dimension m.
(2) A spin^c-structure $P \to SOF(M)$ on M, with the spinor bundle S.

(3) A connection on $P_1 = P/Spin(m)$ which gives rise to a covariant derivative $\nabla : \Gamma(S) \to \Gamma(T^*M \otimes S)$.

The *Dirac spin^c operator* (or simply, the *Dirac operator*) associated to this data is the composition

$$D : \Gamma(S) \xrightarrow{\nabla} \Gamma(T^*M \otimes S) \xrightarrow{\simeq} \Gamma(TM \otimes S) \xrightarrow{\mu} \Gamma(S) \ ,$$

where the isomorphism is induced by the Riemannian metric (which identifies $T^*M \simeq TM$) and μ is the Clifford multiplication.

Remark 8.8.

(1) When k is odd, a choice for μ is to be made, in order to get a well-defined Dirac operator.
(2) If G acts on M by orientation preserving isometries, the spin^c-structure on M is G-equivariant, and the connection on P_1 is G-invariant, then the Dirac operator D will commute with the G-action on $\Gamma(S)$.
(3) If $dim(M)$ is even, then the Dirac operator decomposes into a sum of two operators $D^{\pm} : \Gamma(S^{\pm}) \to \Gamma(S^{\mp})$, which are also called Dirac operators.
(4) If the manifold M is complete, then the Dirac operator is essentially self-adjoint on $L^2(S)$ (see Theorem II.5.7 in [3] or chapter 4 in [1]).

8.5. **spin^c-quantization.** We now restrict our attention to the case of an even-dimensional oriented Riemannian manifold M which is also compact. Since the concept of spin^c-quantization will be defined as the index of the operator D^+, it makes sense to define it only for even-dimensional manifolds. The compactness is used to ensure that $dim(ker(D^+))$ and $dim(coker(D^+))$ are finite.

Definition 8.11. Assume that the following data is given:

(1) An oriented compact Riemannian manifold M of dimension $2m$.
(2) G a Lie group that acts on M by orientation preserving isometries.
(3) $P \to SOF(M)$ a G-equivariant spin^c-structure.
(4) A $U(1)$-invariant connection on $P_1 = P/Spin(2m)$.

Then the *spin^c-quantization of M*, with respect to the above date, is the virtual complex G-representation $Q(M) = ker(D^+) - coker(D^+)$.

The *index* of D^+ is the integer $index(D^+) = dim(ker(D^+)) - dim(coker(D^+))$.

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