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LOCAL WELL-POSEDNESS FOR THE MODIFIED KDV EQUATION IN ALMOST CRITICAL $\widehat{H^r_s}$-SPACES

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ABSTRACT. We study the Cauchy problem for the modified KdV equation

$$u_t + u_{xxx} + (u^3)_x = 0, \qquad u(0) = u_0$$

for data u_0 in the space $\widehat{H^r_s}$ defined by the norm

$$\|u_0\|_{\widehat{H^r_s}} := \|\langle \xi \rangle^s \widehat{u_0}\|_{L^{r'}_\xi}.$$

Local well-posedness of this problem is established in the parameter range $2 \geq r > 1$, $s \geq \frac{1}{2} - \frac{1}{2r}$, so the case $(s, r) = (0, 1)$, which is critical in view of scaling considerations, is almost reached. To show this result, we use an appropriate variant of the Fourier restriction norm method as well as bi- and trilinear estimates for solutions of the Airy equation.

1. INTRODUCTION AND MAIN RESULT

In this paper we study the local well-posedness (LWP) of the Cauchy problem for the modified KdV equation

$$(1) \qquad u_t + u_{xxx} + (u^3)_x = 0, \qquad u(0) = u_0, \qquad x \in \mathbb{R}.$$

As long as data u_0 in the classical Sobolev spaces H^s_x are considered, this problem is known to be well-posed for $s \geq \frac{1}{4}$ and ill-posed (in the C^0-uniform sense) for $s < \frac{1}{4}$. Both the positive and the negative results were shown by Kenig, Ponce, and the second author; see [KPV93, Theorem 2.4] and [KPV01, Theorem 1.3], respectively. The situation remains the same when the defocusing modified KdV equation, i.e. (1) with a negative sign in front of the nonlinearity, is considered. In this case the proof of the well-posedness result remains identically valid, while the ill-posedness result here is due to Christ, Colliander and Tao; cf. [CCT03, Theorem 4]. In both cases the standard scaling argument suggests LWP for $s > -\frac{1}{2}$, so - on the H^s_x-scale - there is a considerable gap of $\frac{3}{4}$ derivatives between the scaling prediction and the optimal LWP result.

This gap could be closed partially by the first author in [G04], where data in the spaces $\widehat{H^r_s}$ are considered, which are defined by the norms

$$\|u_0\|_{\widehat{H^r_s}} := \|\langle \xi \rangle^s \widehat{u_0}\|_{L^{r'}_\xi},$$

where $\widehat{u_0}$ denotes the Fourier transform of u_0, $\langle \xi \rangle = (1 + \xi^2)^{\frac{1}{2}}$ and $\frac{1}{r} + \frac{1}{r'} = 1$. The choice of these norms was motivated by earlier work of Cazenave, Vilela and

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the second author on nonlinear Schrödinger equations (see [CVV01]), yet another alternative class of data spaces has been considered in [VV01].

The main result in [G04] was LWP for (1) in the parameter range $2 \geq r > \frac{4}{3}$, $s \geq s(r) := \frac{1}{2} - \frac{1}{2r}$, which coincides for $r = 2$ with the optimal result on the H^s_x-scale. The proof used an appropriate variant of Bourgain's Fourier restriction norm method; cf. [B93]. Especially the function spaces $X^r_{s,b}$, defined by

$$\|f\|_{X^r_{s,b}} := \left(\int d\xi d\tau \langle \xi \rangle^{sr'} \langle \tau - \xi^3 \rangle^{br'} |\hat{f}(\xi, \tau)|^{r'} \right)^{\frac{1}{r'}}, \quad \frac{1}{r} + \frac{1}{r'} = 1,$$

were utilised, as well as the time restriction norm spaces

$$X^r_{s,b}(\delta) := \{ f = \tilde{f}|_{[-\delta,\delta]\times\mathbb{R}} : \tilde{f} \in X^r_{s,b} \}$$

with norm

$$\|f\|_{X^r_{s,b}(\delta)} := \inf\{ \|\tilde{f}\|_{X^r_{s,b}} : \tilde{f}|_{[-\delta,\delta]\times\mathbb{R}} = f \}.$$

A key estimate in [G04] was the following Airy-version of the Fefferman-Stein-estimate (cf. [F70] and [G04, Corollary 3.6]):

$$\|e^{-t\partial^3 u_0}\|_{L^{3r}_{xt}} \leq c \|I^{-\frac{1}{3r}} u_0\|_{\widehat{L^r_x}}, \qquad r > \frac{4}{3}. \tag{2}$$

Here and below I (J) denotes the Riesz (Bessel) potential operator of order -1 and $\widehat{L^r_x} = \widehat{H^r_0}$. This estimate fails to be true for $r \leq \frac{4}{3}$, which explains the restriction $r > \frac{4}{3}$ in [G04].

It is the aim of the present paper to show how this difficulty can be overcome by using bi- and trilinear estimates for solutions of the Airy equation (instead of linear and bilinear ones). This allows us to extend the LWP result for (1) to the parameter range $2 \geq r > 1$, $s \geq s(r)$. More precisely, the following theorem is the main result of this paper.

Theorem 1. *Let $2 \geq r > 1$, $s \geq s(r) = \frac{1}{2} - \frac{1}{2r}$ and $u_0 \in \widehat{H^r_s}$. Then there exist $b > \frac{1}{r}$, $\delta = \delta(\|u_0\|_{\widehat{H^r_s}}) > 0$ and a unique solution $u \in X^r_{s,b}(\delta)$ of (1). This solution is persistent, and the flow map $S : u_0 \mapsto u$, $\widehat{H^r_s} \to X^r_{s,b}(\delta_0)$ is locally Lipschitz continuous for any $\delta_0 \in (0, \delta)$.*

Theorem 1 is sharp in the sense that, for given $r \in (1, 2]$, we have ill-posedness in the C^0-uniform sense for $\frac{1}{r} - 1 < s < s(r)$. This can be seen by using the counterexample from [KPV01], as it was discussed in [G04, section 5]. Combined with scaling considerations - observe that $\widehat{H^r_s}$ scales like H^σ_x if $s - \frac{1}{r} = \sigma - \frac{1}{2}$ - this shows that the case $(s, r) = (0, 1)$ becomes critical in our setting and that our result covers the whole subcritical range. Unfortunately, our argument breaks down - even for small data - in the critical case, and we must leave this as an open problem. Notice, however, that for specific data

$$u_0 = a\ \delta + \mu\ p.v.\frac{1}{x} \qquad (a, \mu \ \text{ small})$$

of critical regularity the existence of global solutions of (1) was shown in [PV05, Theorem 1.2]. By the general LWP Theorem [G04, Theorem 2.3] the proof of the following estimate is sufficient to establish Theorem 1.

Theorem 2. *Let $2 \geq r > 1$ and $s \geq s(r) = \frac{1}{2} - \frac{1}{2r}$. Then for all $b' < 0$ and $b > \frac{1}{r}$ the estimate*

$$\left\| \partial_x \left(\prod_{i=1}^{3} u_i \right) \right\|_{X^r_{s,b'}} \leq c \prod_{i=1}^{3} \|u_i\|_{X^r_{s,b}} \tag{3}$$

holds true.

Remarks.

i) (On the lifespan of local solutions) Using [G05, Lemma 5.2], we have for u_1, u_2, u_3 supported in $[-\delta, \delta] \times \mathbb{R}$ $(0 < \delta \leq 1)$ the estimate

$$\left\| \partial_x \left(\prod_{i=1}^{3} u_i \right) \right\|_{X^r_{s,b-1}} \leq c\delta^{1-\frac{1}{r}-\varepsilon} \prod_{i=1}^{3} \|u_i\|_{X^r_{s,b}},$$

provided $2 \geq r > 1, s \geq s(r), b > \frac{1}{r}, \varepsilon > 0$. Inserting this estimate, especially the specific power of δ, into the proof of the local result, we obtain a lifespan of size $\delta \sim \|u_0\|_{\widehat{H}^r_s}^{-\frac{2r}{r-1}-\varepsilon'}$. For $r = 2$ this coincides - up to ε' - with the result in [KPV93] (see also [FLP99, Theorem 1.1]).

ii) Concerning related results for the one-dimensional cubic NLS and DNLS equations we refer to [G05].

2. Bi- and trilinear Airy estimates

Throughout this section we consider solutions $u(t) = e^{-t\partial_x^3}u_0$, $v(t) = e^{-t\partial_x^3}v_0$ and $w(t) = e^{-t\partial_x^3}w_0$ of the Airy equation with data u_0, v_0 and w_0, respectively. Certain bi- and trilinear expressions involving these solutions will be estimated in the spaces $\widehat{L^p_x}(\widehat{L^q_t})$ and $\widehat{L^r_{xt}} := \widehat{L^r_x}(\widehat{L^r_t})$, where

$$\|f\|_{\widehat{L^q_x}(\widehat{L^p_t})} := \left(\int \left(\int |\widehat{f}(\xi,\tau)|^{p'} d\tau \right)^{\frac{q'}{p'}} d\xi \right)^{\frac{1}{q'}}, \quad \frac{1}{q} + \frac{1}{q'} = \frac{1}{p} + \frac{1}{p'} = 1.$$

(Below we will always write p', q', etc. to indicate conjugate Hölder exponents, $\widehat{f}$ or $\mathcal{F}f$ will denote the Fourier transform of f, while for the partial Fourier transform in the space variable the symbol $\mathcal{F}_x$ will be used.) We begin with the following bilinear estimate, which we state and prove in a slightly more general version than actually needed.

Lemma 1. *Let I^s denote the Riesz potential of order $-s$ and let $I^s_-(f,g)$ be defined by its Fourier transform (in the space variable):*

$$\mathcal{F}_x I^s_-(f,g)(\xi) := \int_* d\xi_1 |\xi_1 - \xi_2|^s \mathcal{F}_x f(\xi_1) \mathcal{F}_x g(\xi_2),$$

where $\int_*$ is shorthand for $\int_{\xi_1+\xi_2=\xi}$. Then we have

$$\|I^{\frac{1}{p}} I^{\frac{1}{p}}_-(u,v)\|_{\widehat{L^q_x}(\widehat{L^p_t})} \leq c \|u_0\|_{\widehat{L^{r_1}_x}} \|v_0\|_{\widehat{L^{r_2}_x}},$$

provided $1 \leq q \leq r_{1,2} \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r_1} + \frac{1}{r_2}$.

Proof. Taking the Fourier transform first in space and then in time we obtain

$$\mathcal{F}_x I^{\frac{1}{p}} I_-^{\frac{1}{p}}(u,v)(\xi,t) = c|\xi|^{\frac{1}{p}} \int_* d\xi_1 |\xi_1 - \xi_2|^{\frac{1}{p}} e^{it(\xi_1^3+\xi_2^3)} \mathcal{F}_x u_0(\xi_1) \mathcal{F}_x v_0(\xi_2)$$

and

$$\mathcal{F} I^{\frac{1}{p}} I_-^{\frac{1}{p}}(u,v)(\xi,\tau) = c|\xi|^{\frac{1}{p}} \int_* d\xi_1 |\xi_1 - \xi_2|^{\frac{1}{p}} \delta(\tau - \xi_1^3 - \xi_2^3) \mathcal{F}_x u_0(\xi_1) \mathcal{F}_x v_0(\xi_2),$$

respectively. We use $\delta(g(x)) = \sum_n \frac{1}{|g'(x_n)|}\delta(x - x_n)$, where the sum is taken over all simple zeros of g, which in our case is

$$g(\xi_1) = \tau - \xi_1^3 - \xi_2^3 = \tau - \xi^3 + 3\xi\xi_1(\xi - \xi_1)$$

with the zeros

$$\xi_1^{\pm} = \frac{\xi \pm y}{2}, \qquad y := 2\sqrt{\frac{\tau}{3\xi} - \frac{\xi^2}{12}}$$

and the derivative

$$g'(\xi_1^{\pm}) = 3\xi(\xi - 2\xi_1^{\pm}) = \mp 3\xi y.$$

Hence

$$\begin{aligned} (4) \qquad & \mathcal{F} I^{\frac{1}{p}} I_-^{\frac{1}{p}}(u,v)(\xi,\tau) \\ = \quad & c|\xi|^{-\frac{1}{p'}} y^{-\frac{1}{p'}} \left(\mathcal{F}_x u_0(\frac{\xi+y}{2}) \mathcal{F}_x v_0(\frac{\xi-y}{2}) + \mathcal{F}_x u_0(\frac{\xi-y}{2}) \mathcal{F}_x v_0(\frac{\xi+y}{2}) \right). \end{aligned}$$

Using $d\tau = 3|\xi| y dy$, we see that the $L_\tau^{p'}$-norm of the first contribution equals

$$\left(\int dy |\mathcal{F}_x u_0(\frac{\xi+y}{2}) \mathcal{F}_x v_0(\frac{\xi-y}{2})|^{p'} \right)^{\frac{1}{p'}} = c \left(|\mathcal{F}_x u_0|^{p'} * |\mathcal{F}_x v_0|^{p'}(\xi) \right)^{\frac{1}{p'}}.$$

Now Young's inequality is applied to see that

$$\left(\int d\xi (|\mathcal{F}_x u_0|^{p'} * |\mathcal{F}_x v_0|^{p'}(\xi))^{\frac{q'}{p'}} \right)^{\frac{1}{q'}} \leq c \|u_0\|_{\widehat{L_x^{r_1}}} \|v_0\|_{\widehat{L_x^{r_2}}}$$

(cf. the proof of [G05, Lemma 1]), which is the desired bound. Finally we observe that the second contribution in (4) can be treated in precisely the same manner with r_1 and r_2 interchanged. □

Arguing similarly as in the proof of Lemma 2.1 in [G04] we obtain:

Corollary 1. *For p, q, $r_{1,2}$ as in the previous lemma and $b_i > \frac{1}{r_i}$, the estimate*

$$\|I^{\frac{1}{p}} I_-^{\frac{1}{p}}(u_1,u_2)\|_{\widehat{L_x^q}(\widehat{L_t^p})} \leq c\|u_1\|_{X_{0,b_1}^{r_1}} \|u_2\|_{X_{0,b_2}^{r_2}}$$

is valid.

The next step is to dualize the preceding corollary. For that purpose we recall the bilinear operator I_+^s, defined by

$$\mathcal{F}_x I_+^s(f,g)(\xi) := \int_{\xi_1+\xi_2=\xi} d\xi_1 |\xi + \xi_2|^s \mathcal{F}_x f(\xi_1) \mathcal{F}_x g(\xi_2),$$

and the linear operators

$$M_u^s v := I_-^s(u,v) \qquad \text{and} \qquad N_u^s w := I_+^s(w, \overline{u}),$$

which are formally adjoint w.r.t. the inner product on L^2_{xt} (cf. [G04, p. 3299]). With this notation, Corollary 1 expresses the boundedness of

$$I^{\frac{1}{p}} M^{\frac{1}{p}}_{u_1} : X^{r_2}_{0,b_2} \longrightarrow \widehat{L^q_x}(\widehat{L^p_t})$$

with operator norm $\le c\|u_1\|_{X^{r_1}_{0,b_1}}$. By duality, under the additional hypothesis $1 < p, q, r_{1,2} < \infty$, it follows that

$$N^{\frac{1}{p}}_{u_1} I^{\frac{1}{p}} : \widehat{L^{q'}_x}(\widehat{L^{p'}_t}) \longrightarrow X^{r'_2}_{0,-b_2}$$

is bounded with the same norm. Thus we obtain the following estimate:

Corollary 2. *Let $1 < q \le r_{1,2} \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r_1} + \frac{1}{r_2}$ and $b_i > \frac{1}{r_i}$. Then*

$$\|I^{\frac{1}{p}}_{+}(I^{\frac{1}{p}} u_2, \overline{u}_1)\|_{X^{r'_2}_{0,-b_2}} \le c\|u_1\|_{X^{r_1}_{0,b_1}} \|u_2\|_{\widehat{L^{q'}_x}(\widehat{L^{p'}_t})}. \tag{5}$$

Remark. Since the phase function $\phi(\xi) = \xi^3$ is odd, we have $\|u_1\|_{X^r_{s,b}} = \|\overline{u}_1\|_{X^r_{s,b}}$, and we may replace $\overline{u}_1$ by u_1 in the left hand side of (5).

The special case in (5), where $p = q = r_{1,2}$, will be sufficient for our purposes. In this case, (5) can be written as

$$\|I^{\frac{1}{r'}}_{+}(I^{\frac{1}{r'}} u_2, u_1)\|_{X^r_{0,b'}} \le c\|u_1\|_{X^{r'}_{0,-b'}} \|u_2\|_{\widehat{L^r_{xt}}}, \tag{6}$$

provided $1 < r < \infty$, $b' < -\frac{1}{r'}$. Combining this with the trivial endpoint of the Hausdorff-Young inequality, i.e.

$$\|u_2 u_1\|_{\widehat{L^r_{xt}}} \le c\|u_1\|_{\widehat{L^\infty_{xt}}} \|u_2\|_{\widehat{L^r_{xt}}},$$

we obtain by elementary Hölder estimates

$$\|I^{\frac{1}{\rho'}}_{+}(I^{\frac{1}{\rho'}} u_2, u_1)\|_{X^r_{0,\beta}} \le c\|u_1\|_{X^{\rho'}_{0,-\beta}} \|u_2\|_{\widehat{L^r_{xt}}}, \tag{7}$$

where $0 \le \frac{1}{\rho'} \le \frac{1}{r'}$ and $\beta < -\frac{1}{\rho'}$. In this form we shall actually make use of Corollary 2.

Now we turn to the trilinear estimates. Again we take the Fourier transform first in x and then in t to obtain

$$\mathcal{F}_x(uvw)(\xi, t) = c \int_* d\xi_1 d\xi_2 e^{it(\xi_1^3 + \xi_2^3 + \xi_3^3)} \mathcal{F}_x u_0(\xi_1) \mathcal{F}_x v_0(\xi_2) \mathcal{F}_x w_0(\xi_3)$$

(where now $\int_* = \int_{\xi_1+\xi_2+\xi_3=\xi}$) and

$$\mathcal{F}(uvw)(\xi, \tau) = c \int_* d\xi_1 d\xi_2 \delta(\xi_1^3 + \xi_2^3 + \xi_3^3 - \tau) \mathcal{F}_x u_0(\xi_1) \mathcal{F}_x v_0(\xi_2) \mathcal{F}_x w_0(\xi_3).$$

Now the argument of δ, that is,

$$g(\xi_2) = 3(\xi - \xi_1)\xi_2^2 - 3(\xi - \xi_1)^2 \xi_2 - 3\xi\xi_1(\xi - \xi_1) + \xi^3 - \tau,$$

has exactly two zeros

$$\xi_2^{\pm} = \frac{\xi - \xi_1}{2} \pm \sqrt{\frac{(\xi + \xi_1)^2}{4} + \frac{\tau - \xi^3}{3(\xi - \xi_1)}} =: \frac{\xi - \xi_1}{2} \pm y, \tag{8}$$

with

$$|g'(\xi_2^{\pm})| = 6|\xi - \xi_1|\sqrt{\frac{(\xi+\xi_1)^2}{4} + \frac{\tau - \xi^3}{3(\xi - \xi_1)}} = 6|\xi - \xi_1|y.$$

Using $\delta(g(\xi_2)) = \sum_{g(x_n)=0} \frac{\delta(\xi_2 - x_n)}{|g'(x_n)|}$, where the sum is taken over all simple zeros of g, we see that

$$\mathcal{F}(uvw)(\xi,\tau) = c(K_+(\xi,\tau) + K_-(\xi,\tau)),$$

where

$$K_\pm(\xi,\tau) = \int d\xi_1 \frac{1}{|\xi-\xi_1|y}\mathcal{F}_x u_0(\xi_1)\mathcal{F}_x v_0(\frac{\xi-\xi_1}{2} \pm y)\mathcal{F}_x w_0(\frac{\xi-\xi_1}{2} \mp y)$$

with y as defined in (8).

In order to estimate $\|uvw\|_{\widehat{L^r_{xt}}} = \|\mathcal{F}(uvw)\|_{L^{r'}_{\xi\tau}}$ we distinguish between three cases depending on the relative size of the frequencies ξ_1, ξ_2 and ξ_3:

i) $|\xi_1| \sim |\xi_2| \gg \langle \xi_3 \rangle$,
ii) $|\xi_2 - \xi_3| \geq |\xi_2 + \xi_3|$,
iii) $1 \leq |\xi_2 - \xi_3| \leq |\xi_2 + \xi_3|$.

To treat the first case we define the trilinear operator T by

$$\mathcal{F}_x T(f,g,h) := \int_* d\xi_1 d\xi_2 \mathcal{F}_x f(\xi_1)\mathcal{F}_x g(\xi_2)\mathcal{F}_x h(\xi_3)\chi_{\{|\xi_1|\sim|\xi_2|\gg\langle\xi_3\rangle\}},$$

where again $\int_* = \int_{\xi_1+\xi_2+\xi_3=\xi}$. In this case we have:

Lemma 2. *Let* $1 \leq r \leq 2$ *and* $s_1 > \frac{1}{4r'} - \frac{1}{2}$, $s_2 \geq \frac{1}{2r'}$. *Then*

$$\|T(u,v,w)\|_{\widehat{L^r_{xt}}} \leq c\|u_0\|_{\widehat{H^r_{s_1}}}\|v_0\|_{\widehat{H^r_{s_1}}}\|w_0\|_{\widehat{H^r_{s_2}}}.$$

Proof. By the above computation we have

$$\mathcal{F}T(u,v,w)(\xi,\tau) = c(K^+(\xi,\tau) + K^-(\xi,\tau))$$

with

$$K^\pm(\xi,\tau) = \int_{A_\pm} d\xi_1 \frac{1}{|\xi-\xi_1|y}\mathcal{F}_x u_0(\xi_1)\mathcal{F}_x v_0(\frac{\xi-\xi_1}{2} \pm y)\mathcal{F}_x w_0(\frac{\xi-\xi_1}{2} \mp y),$$

where $A_\pm = \{|\xi_1| \sim |\frac{\xi-\xi_1}{2} \pm y| \gg \langle\frac{\xi-\xi_1}{2} \mp y\rangle\}$ and y is defined by (8). Since in $A_\pm$ the inequality $|\xi_1||\frac{\xi-\xi_1}{2} \pm y| \leq c|\xi - \xi_1|y$ holds true, we get the upper bound

$$K^\pm(\xi,\tau) \leq c\int d\xi_1 \mathcal{F}_x J^{-1}u_0(\xi_1)\mathcal{F}_x J^{-1}v_0(\frac{\xi-\xi_1}{2} \pm y)\mathcal{F}_x w_0(\frac{\xi-\xi_1}{2} \mp y),$$

leading to

$$\|T(u,v,w)\|_{\widehat{L^1_{xt}}} \leq c\|J^{-1}u_0\|_{\widehat{L^\infty_x}}\|J^{-1}v_0\|_{\widehat{L^1_x}}\|w_0\|_{\widehat{L^1_x}}.$$

By symmetry between the first two factors and multilinear interpolation we obtain

$$\|T(u,v,w)\|_{\widehat{L^1_{xt}}} \leq c\|J^{-1}u_0\|_{L^2_x}\|J^{-1}v_0\|_{L^2_x}\|w_0\|_{\widehat{L^1_x}}. \tag{9}$$

On the other hand we have

$$\|uvw\|_{L^2_{xt}} \leq c\|u\|_{L^8_x(L^4_t)}\|v\|_{L^8_x(L^4_t)}\|w\|_{L^4_x(L^\infty_t)}$$

with

$$\|w\|_{L^4_x(L^\infty_t)} \leq c\|I^{\frac{1}{4}}u_0\|_{L^2_x}, \tag{10}$$

which is the maximal function estimate from [S87, Thm. 3]. Concerning the first two factors we interpolate between the sharp version of Kato's smoothing effect, i.e. $\|Iu\|_{L^\infty_x(L^2_t)} = c\|u_0\|_{L^2_x}$ (see [KPV91, Thm. 4.1]) and (10) to obtain

$$\|I^{\frac{3}{8}}u\|_{L^8_x(L^4_t)} \le c\|u_0\|_{L^2_x},$$

such that

$$\|T(u,v,w)\|_{L^2_{xt}} \le c\|J^{-\frac{3}{8}}u_0\|_{L^2_x}\|J^{-\frac{3}{8}}v_0\|_{L^2_x}\|J^{\frac{1}{4}}w_0\|_{L^2_x}. \tag{11}$$

Using multilinear interpolation again, now between (9) and (11) we finally see that, for $1 \le r \le 2$,

$$\begin{aligned}\|T(u,v,w)\|_{\widehat{L^r_{xt}}} &\le c\|J^{\frac{5}{4r'}-1}u_0\|_{L^2_x}\|J^{\frac{5}{4r'}-1}v_0\|_{L^2_x}\|J^{\frac{1}{2r'}}w_0\|_{\widehat{L^r_x}}\\ &\le c\|u_0\|_{\widehat{H^r_{s_1}}}\|v_0\|_{\widehat{H^r_{s_1}}}\|w_0\|_{\widehat{H^r_{s_2}}},\end{aligned}$$

where in the last step we have used the Sobolev type embedding $\widehat{H^r_s} \subset \widehat{H^\rho_\sigma}$, which holds true for $s - \frac{1}{r} > \sigma - \frac{1}{\rho}$, $r \le \rho$. □

Corollary 3. *For r, $s_{1,2}$ as in the previous lemma and $b > \frac{1}{r}$ the estimate*

$$\|T(u_1,u_2,u_3)\|_{\widehat{L^r_{xt}}} \le c\|u_1\|_{X^r_{s_1,b}}\|u_2\|_{X^r_{s_1,b}}\|u_3\|_{X^r_{s_2,b}}$$

holds true.

Next we introduce $T_\ge$ ($T_\le$) by

$$\mathcal{F}_x T_\ge(f,g,h) := \int_* d\xi_1 d\xi_2 \mathcal{F}_x f(\xi_1)\mathcal{F}_x g(\xi_2)\mathcal{F}_x h(\xi_3)\chi_{\{|\xi_2-\xi_3|\ge|\xi_2+\xi_3|\}}$$

and

$$\mathcal{F}_x T_\le(f,g,h) := \int_* d\xi_1 d\xi_2 \mathcal{F}_x f(\xi_1)\mathcal{F}_x g(\xi_2)\mathcal{F}_x h(\xi_3)\chi_{\{1\le|\xi_2-\xi_3|\le|\xi_2+\xi_3|\}}.$$

Lemma 3. *Let $1 < p_1 < p < p_0 < \infty$, $p < p_0'$, $\frac{3}{p} = \frac{1}{p_0} + \frac{2}{p_1}$ and $\frac{2}{p_1} < 1 + \frac{1}{p}$. Then the estimate*

$$\|T_\ge(u,v,w)\|_{\widehat{L^p_{xt}}} \le c\|u_0\|_{\widehat{L^{p_0}_x}}\|I^{-\frac{1}{2p}}v_0\|_{\widehat{L^{p_1}_x}}\|I^{-\frac{1}{2p}}w_0\|_{\widehat{L^{p_1}_x}}$$

is valid.

Proof. For the Fourier transform of $T_\ge(u,v,w)$ in both variables we obtain

$$\mathcal{F}T_\ge(u,v,w)(\xi,\tau) = c(K^+_\ge(\xi,\tau) + K^-_\ge(\xi,\tau)),$$

where

$$K^\pm_\ge(\xi,\tau) = \int_{\{2y\ge|\xi-\xi_1|\}} d\xi_1 \frac{1}{|\xi-\xi_1|y}\mathcal{F}_x u_0(\xi_1)\mathcal{F}_x v_0(\frac{\xi-\xi_1}{2} \pm y)\mathcal{F}_x w_0(\frac{\xi-\xi_1}{2} \mp y),$$

with y as in (8) again. By symmetry we may restrict ourselves to the estimation of $K^+_\ge$. Using $|\frac{\xi-\xi_1}{2} \pm y| \le 2y$ and Hölder's inequality, we see that

$$\begin{aligned}&K^+_\ge(\xi,\tau) \le c\left(\int d\xi_1 \frac{|\mathcal{F}_x u_0(\xi_1)|^p}{|\xi-\xi_1|^{(1-\theta)p}}\right)^{\frac{1}{p}}\\ &\times\left(\int \frac{d\xi_1}{|\xi-\xi_1|^{\theta p'}y}|\mathcal{F}_x I^{-\frac{1}{2p}}v_0(\frac{\xi-\xi_1}{2}+y)\mathcal{F}_x I^{-\frac{1}{2p}}w_0(\frac{\xi-\xi_1}{2}-y)|^{p'}\right)^{\frac{1}{p'}},\end{aligned}$$

where $\theta = \frac{3}{p'} - \frac{2}{p_1'}$ ($\in (0,1)$ by our assumptions). Taking the $L^{p'}_\tau$-norm of both sides and using $d\tau = 6|\xi - \xi_1| y dy$ we arrive at

$$\|\mathcal{F}T_{\geq}(u,v,w)(\xi,\cdot)\|_{L^{p'}_\tau} \leq c(|\mathcal{F}_x u_0|^p * |\xi|^{(\theta-1)p})^{\frac{1}{p}}$$
$$\times \left(\int \frac{d\xi_1 dy}{|\xi-\xi_1|^{\theta p'-1}} |\mathcal{F}_x I^{-\frac{1}{2p}} v_0(\frac{\xi-\xi_1}{2}+y)\mathcal{F}_x I^{-\frac{1}{2p}} w_0(\frac{\xi-\xi_1}{2}-y)|^{p'}\right)^{\frac{1}{p'}}.$$

Changing variables ($z_\pm := \frac{\xi-\xi_1}{2} \pm y$) we see that the second factor equals

$$\left(\int \frac{dz_+ dz_-}{|z_+ + z_-|^{\theta p'-1}} |\mathcal{F}_x I^{-\frac{1}{2p}} v_0(z_+)\mathcal{F}_x I^{-\frac{1}{2p}} w_0(z_-)|^{p'}\right)^{\frac{1}{p'}} \leq c\|I^{-\frac{1}{2p}} v_0\|_{\widehat{L^{p_1}_x}} \|I^{-\frac{1}{2p}} w_0\|_{\widehat{L^{p_1}_x}},$$

by the Hardy-Littlewood-Sobolev-inequality, requiring θ to be chosen as above and $1 < \theta p' < 2$, which follows from our assumptions. It remains to estimate the $L^{p'}_\xi$-norm of the first factor, that is,

$$\begin{aligned} & \||\mathcal{F}_x u_0|^p * |\xi|^{(\theta-1)p}\|^{\frac{1}{p}}_{L_\xi^{\frac{p'}{p}}} \\ \leq \quad & c(\||\mathcal{F}_x u_0|^p\|_{L_\xi^{\frac{p_0'}{p}}} \||\xi|^{(\theta-1)p}\|_{L_\xi^{\frac{1}{(1-\theta)p},\infty}})^{\frac{1}{p}} \\ \leq \quad & c\|u_0\|_{\widehat{L^{p_0}_x}}, \end{aligned}$$

where the HLS inequality was used again. For its application we need

$$0 < (1-\theta)p < 1; \quad 1 < \frac{p_0'}{p} < \frac{1}{1-(1-\theta)p} \quad \text{and} \quad \theta = \frac{1}{p_0'},$$

which follows from the assumptions, too. □

Corollary 4. *For $1 < r < 2$ there exist $s_{0,1} \geq 0$ with $s_0 + 2s_1 = \frac{1}{r}$, such that*

$$\|T_{\geq}(u,v,w)\|_{\widehat{L^r_{xt}}} \leq c\|I^{-s_0}u_0\|_{\widehat{L^r_x}} \|I^{-s_1}v_0\|_{\widehat{L^r_x}} \|I^{-s_1}w_0\|_{\widehat{L^r_x}}. \tag{12}$$

In addition, for $b > \frac{1}{r}$ we have

$$\|T_{\geq}(u_1,u_2,u_3)\|_{\widehat{L^r_{xt}}} \leq c\|I^{-s_0}u_1\|_{X^r_{0,b}} \|I^{-s_1}u_2\|_{X^r_{0,b}} \|I^{-s_1}u_3\|_{X^r_{0,b}}.$$

Proof of (12). Using Hölder's inequality and the Airy-version of the Fefferman-Stein-estimate, that is,

$$\|u\|_{L^{3q}_{xt}} \leq c\|I^{-\frac{1}{3q}}u_0\|_{\widehat{L^q_x}}, \qquad q > \frac{4}{3} \tag{13}$$

(see [G04, Corollary 3.6]), we get for

$$\frac{4}{3} < q_0 < 2 < q_1 \qquad \text{with} \qquad \frac{3}{2} = \frac{1}{q_0} + \frac{2}{q_1} \tag{14}$$

that

$$\|T_{\geq}(u,v,w)\|_{L^2_{xt}} \leq \|uvw\|_{L^2_{xt}} \leq c\|I^{-\frac{1}{3q_0}}u_0\|_{\widehat{L^{q_0}_x}} \|I^{-\frac{1}{3q_1}}v_0\|_{\widehat{L^{q_1}_x}} \|I^{-\frac{1}{3q_1}}w_0\|_{\widehat{L^{q_1}_x}}. \tag{15}$$

Multilinear interpolation of (15) with Lemma 3 yields (12), provided $p, p_0, p_1; q_0, q_1$, defined by the interpolation conditions

$$\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{2} = \frac{1-\theta}{p_0} + \frac{\theta}{q_0} = \frac{1-\theta}{p_1} + \frac{\theta}{q_1},$$

fulfill the assumptions of Lemma 3 and (14), respectively, which can be guaranteed by choosing θ sufficiently small. Now $s_{0,1}$ are obtained from

$$s_0 = \frac{\theta}{3q_0} \qquad \text{and} \qquad s_1 = \frac{1-\theta}{2p} + \frac{\theta}{3q_1},$$

which gives

$$s_0 + 2s_1 = \frac{1-\theta}{p} + \frac{\theta}{3}\Big(\frac{1}{q_0} + \frac{2}{q_1}\Big) = \frac{1}{r},$$

as desired. □

Remark. By (13), Corollary 4 still holds true for $r \geq 2$ (with $s_0 = s_1 = \frac{1}{3r}$).

Lemma 4. *Let $1 \leq r < \rho \leq \infty$. Then*

$$\|T_{\leq}(u,v,w)\|_{\widehat{L^r_{xt}}} \leq c\|u_0\|_{\widehat{L^\rho_x}} \|I^{-\frac{1}{2r}} v_0\|_{\widehat{L^r_x}} \|I^{-\frac{1}{2r}} w_0\|_{\widehat{L^r_x}}.$$

Proof. We have

$$\mathcal{F}T_{\leq}(u,v,w)(\xi,\tau) = c(K^+_{\leq}(\xi,\tau) + K^-_{\leq}(\xi,\tau)),$$

where

$$K^{\pm}_{\leq}(\xi,\tau) = \int_{\{1\leq 2y \leq |\xi-\xi_1|\}} \frac{d\xi_1}{|\xi-\xi_1|y} \mathcal{F}_x u_0(\xi_1)\mathcal{F}_x v_0\Big(\frac{\xi-\xi_1}{2} \pm y\Big)\mathcal{F}_x w_0\Big(\frac{\xi-\xi_1}{2} \mp y\Big)$$

with y as defined in (8). By symmetry between v and w it suffices to treat $K^+_{\leq}$, which we decompose dyadically with respect to y to obtain the upper bound:

$$\begin{aligned} & c\sum_{j=0}^{\infty} \int_{\{1\leq 2y\leq|\xi-\xi_1|\,,\ y\sim 2^j\}} \frac{d\xi_1}{|\xi-\xi_1|y} \mathcal{F}_x u_0(\xi_1)\mathcal{F}_x v_0\Big(\frac{\xi-\xi_1}{2} + y\Big)\mathcal{F}_x w_0\Big(\frac{\xi-\xi_1}{2} - y\Big) \\ \leq\ & c\sum_{j=0}^{\infty} 2^{-j} \int_{\{y\sim 2^j\}} d\xi_1 \mathcal{F}_x u_0(\xi_1) \mathcal{F}_x I^{-\frac{1}{2}} v_0\Big(\frac{\xi-\xi_1}{2} + y\Big)\mathcal{F}_x I^{-\frac{1}{2}} w_0\Big(\frac{\xi-\xi_1}{2} - y\Big) \\ \leq\ & c\sum_{j=0}^{\infty} 2^{-j} \|u_0\|_{\widehat{L^p_x}} \lambda(\{y\sim 2^j\})^{\frac{1}{p'}} \|I^{-\frac{1}{2}} v_0\|_{\widehat{L^1_x}} \|I^{-\frac{1}{2}} w_0\|_{\widehat{L^1_x}}, \end{aligned}$$

where $\lambda(\{y \sim 2^j\})$ denotes the Lebesgue measure of $\{\xi_1 : y(\xi_1) \sim 2^j\}$, which is bounded by $c2^j$.[1] Hence, for any $p > 1$,

$$\begin{aligned} \|K^+_{\leq}\|_{L^\infty_{\xi\tau}} &\leq c\sum_{j=0}^{\infty} 2^{-\frac{j}{p'}} \|u_0\|_{\widehat{L^p_x}} \|I^{-\frac{1}{2}} v_0\|_{\widehat{L^1_x}} \|I^{-\frac{1}{2}} w_0\|_{\widehat{L^1_x}} \\ &\leq c\|u_0\|_{\widehat{L^p_x}} \|I^{-\frac{1}{2}} v_0\|_{\widehat{L^1_x}} \|I^{-\frac{1}{2}} w_0\|_{\widehat{L^1_x}}. \end{aligned} \tag{16}$$

[1] To see this, we write $\{\xi_1 : y(\xi_1) \sim 2^j\} = S_1 \cup S_2$, where in S_1 we assume that $|\xi - \xi_1| \lesssim 2^j$, $|\xi+\xi_1| \lesssim 2^j$ or $|\xi - 3\xi_1| \lesssim 2^j$. Then S_1 consists of a finite number of intervals of total length bounded by $c2^j$. For S_2 we have $|\xi-\xi_1| \gg 2^j$, $|\xi+\xi_1| \gg 2^j$ and $|\xi - 3\xi_1| \gg 2^j$, implying that

$$\left|\frac{dy}{d\xi_1}\right| = \frac{1}{2y|\xi-\xi_1|}\left|\frac{(\xi+\xi_1)(\xi-3\xi_1)}{4} + y^2\right| \gtrsim \frac{|\xi+\xi_1||\xi-3\xi_1|}{y|\xi-\xi_1|} \gtrsim 1\ ,$$

which gives

$$\lambda(S_2) = \int_{S_2} d\xi_1 \leq \int \frac{d\xi_1}{dy}\chi_{\{y\sim 2^j\}} dy \leq c2^j.$$

On the other hand, by integration with respect first to $d\tau = 6y(\xi - \xi_1)dy$, then to $d\xi$, and finally to $d\xi_1$, we see that

$$\|K^+_{\leq}\|_{L^1_{\xi\tau}} \leq c\|u_0\|_{\widehat{L^\infty_x}}\|v_0\|_{\widehat{L^\infty_x}}\|w_0\|_{\widehat{L^\infty_x}}. \tag{17}$$

Now multilinear interpolation between (16) and (17) leads to

$$\|K^+_{\leq}\|_{L^{r'}_{\xi\tau}} \leq c\|u_0\|_{\widehat{L^\rho_x}}\|I^{-\frac{1}{2r}}v_0\|_{\widehat{L^r_x}}\|I^{-\frac{1}{2r}}w_0\|_{\widehat{L^r_x}},$$

which gives the desired result. □

Corollary 5. *Let $1 \leq r < \rho \leq \infty$, $\beta > \frac{1}{\rho}$, $b > \frac{1}{r}$ and $\varepsilon > 0$. Then*

$$\|T_{\leq}(u_1,u_2,u_3)\|_{\widehat{L^r_{xt}}} \leq c\|u_1\|_{X^\rho_{0,\beta}}\|I^{-\frac{1}{2r}}u_2\|_{X^r_{0,b}}\|I^{-\frac{1}{2r}}u_3\|_{X^r_{0,b}}$$

and

$$\|T_{\leq}(u_1,u_2,u_3)\|_{\widehat{L^r_{xt}}} \leq c\|u_1\|_{X^r_{\varepsilon,b}}\|I^{-\frac{1}{2r}}u_2\|_{X^r_{0,b}}\|I^{-\frac{1}{2r}}u_3\|_{X^r_{0,b}}$$

are valid.

3. Proof of Theorem 2

Without loss of generality we may assume that $s = s(r)$. Then we rewrite the left hand side of (3) as

$$\left\|\langle\tau-\xi^3\rangle^{b'}\langle\xi\rangle^s|\xi|\int d\nu\prod_{i=1}^3\widehat{u_i}(\xi_i,\tau_i)\right\|_{L^{r'}_{\xi,\tau}},$$

where $d\nu = d\xi_1 d\xi_2 d\tau_1 d\tau_2$ and $\sum_{i=1}^3(\xi_i,\tau_i) = (\xi,\tau)$.

In the sequel, we shall use the following notation:

- ξ_{max}, ξ_{med}, ξ_{min} are defined by $|\xi_{max}| \geq |\xi_{med}| \geq |\xi_{min}|$,
- p denotes the projection on low frequencies, i.e. $\widehat{pf}(\xi) = \chi_{\{|\xi|\leq 1\}}\widehat{f}(\xi)$,
- $f \preceq g$ is shorthand for $|\widehat{f}| \leq c|\widehat{g}|$,
- for the mixed weights coming from the $X^r_{s,b}$-norms we shall write $\sigma_0 := \tau - \xi^3$ and $\sigma_i := \tau_i - \xi_i^3$, $1 \leq i \leq 3$, respectively,
- the Fourier multiplier associated with these weights is denoted by $\Lambda^b := \mathcal{F}^{-1}\langle\tau-\xi^3\rangle^b\mathcal{F}$,
- for a real number x we write $x\pm$ to denote $x \pm \varepsilon$ for arbitrarily small $\varepsilon > 0$; $\infty-$ stands for an arbitrarily large real number.

Apart from the trivial region where $|\xi_{max}| \leq 1$, whose contribution can be estimated by

$$\left\|\prod_{i=1}^3 pu_i\right\|_{\widehat{L^r_{xt}}} \leq c\prod_{i=1}^3\|pu_i\|_{\widehat{L^{3r}_{xt}}} \leq c\prod_{i=1}^3\|pu_i\|_{X^r_{0,b}} \leq c\prod_{i=1}^3\|u_i\|_{X^r_{s,b}},$$

we consider three cases:

1. the nonresonant case, where $|\xi_{max}| \gg |\xi_{med}|$,
2. the semiresonant case with $|\xi_{max}| \sim |\xi_{med}| \gg |\xi_{min}|$ and, finally,
3. the resonant case, where $|\xi_{max}| \sim |\xi_{min}|$.

1. In the nonresonant case we assume without loss of generality that $|\xi_1| \geq |\xi_2| \geq |\xi_3|$. Then we have for this region

$$\begin{aligned} J^s\partial_x(u_1u_2u_3) &\preceq \partial_x(J^su_1J^su_2J^{-s}u_3) \\ &\preceq I^{\frac{1}{r}}I_-^{\frac{1}{r}}(J^su_1, J^su_2)J^{1-s-\frac{2}{r}}u_3 \\ &\preceq I_+^{0+}(I^{\frac{1}{r}+}I_-^{\frac{1}{r}}(J^su_1, J^su_2), J^{1-s-\frac{2}{r}-}u_3). \end{aligned}$$

Now the dual version (7) of the bilinear estimate is applied to obtain

$$\begin{aligned} &\|I_+^{0+}(I^{\frac{1}{r}+}I_-^{\frac{1}{r}}(J^su_1, J^su_2), J^{1-s-\frac{2}{r}-}u_3)\|_{X^r_{0,b'}} \\ \leq\ & c\|I^{\frac{1}{r}}I_-^{\frac{1}{r}}(J^su_1, J^su_2)\|_{\widehat{L^r_{xt}}}\|J^{1-s-\frac{2}{r}-}u_3\|_{X^{\infty-}_{0,0+}} \leq c\prod_{i=1}^{3}\|u_i\|_{X^r_{s,b}}, \end{aligned}$$

where in the last step we have used the bilinear estimate itself (Corollary 1) for the first and Sobolev-type embeddings for the second factor.

2. In the semiresonant case we assume again $|\xi_1| \geq |\xi_2| \geq |\xi_3|$ and consider two subcases: If, in addition, $|\xi_1+\xi_2| \leq 1$ (so that $\langle\xi\rangle \leq c\langle\xi_3\rangle$), we can argue as in case 1, with u_1 and u_3 interchanged:

$$\begin{aligned} J^s\partial_x(u_1u_2u_3) &\preceq \partial_x(J^{-s}u_1J^su_2J^su_3) \\ &\preceq I_+^{0+}(I^{\frac{1}{r}+}I_-^{\frac{1}{r}}(J^su_3, J^su_2), J^{1-s-\frac{2}{r}-}u_1), \end{aligned}$$

which can be treated as above by applying (7), Sobolev-type embeddings and Corollary 1. On the other hand, if $|\xi_1+\xi_2| \geq 1$, we have

$$|\sigma_0 - \sigma_1 - \sigma_2 - \sigma_3| = 3|\xi_1+\xi_2||\xi_2+\xi_3||\xi_3+\xi_1| \gtrsim \langle\xi_1\rangle\langle\xi_2\rangle,$$

and hence, for any $\varepsilon > 0$,

$$\langle\xi_1\rangle^\varepsilon\langle\xi_2\rangle^\varepsilon \leq c\prod_{i=0}^{3}\langle\sigma_i\rangle^\varepsilon.$$

So, in this subcase, we have the upper bound

$$\|T(J^{\frac{s+1}{2}-}\Lambda^{0+}u_1, J^{\frac{s+1}{2}-}\Lambda^{0+}u_2, \Lambda^{0+}u_3)\|_{\widehat{L^r_{xt}}} \leq c\prod_{i=1}^{3}\|u_i\|_{X^r_{s,b}}$$

by Corollary 3.

3. In the resonant case we distinguish several subcases:

3.1. At least for one pair (i,j) we have $|\xi_i - \xi_j| \geq |\xi_i + \xi_j|$.

Here we may assume by symmetry that $|\xi_2 - \xi_3| \geq |\xi_2 + \xi_3|$. Then we have for nonnegative $s_{0,1}$ with $s_0 + 2s_1 = \frac{1}{r}$

$$\partial_x J^s(u_1u_2u_3) \preceq T_{\geq}(J^{s+s_0}u_1, J^{s+s_1}u_2, J^{s+s_1}u_3),$$

so that Corollary 4 leads to the desired bound.

3.2. $|\xi_1-\xi_2|\le|\xi_1+\xi_2|$, $|\xi_2-\xi_3|\le|\xi_2+\xi_3|$ and $|\xi_3-\xi_1|\le|\xi_3+\xi_1|$, so that all the ξ_i have the same sign, which implies

$$|\xi_1|^3\sim|\xi_2|^3\sim|\xi_3|^3\le\prod_{i=0}^3\langle\sigma_i\rangle.$$

3.2.1. At least one of the $|\xi_i-\xi_j|\ge 1$.

By symmetry we may assume that $|\xi_2-\xi_3|\ge 1$. Gaining a $\langle\xi\rangle^\varepsilon$ from the σ's we obtain as an upper bound for this subcase

$$\|T_{\le}(J^{s-}\Lambda^{0+}u_1,J^{\frac12}\Lambda^{0+}u_2,J^{\frac12}\Lambda^{0+}u_3)\|_{\widehat{L^r_{xt}}}\le c\prod_{i=1}^3\|u_i\|_{X^r_{s,b}},$$

where we have used the second part of Corollary 5.

3.2.2. $|\xi_i-\xi_j|\le 1$ for all $1\le i\ne j\le 3$.

Again, we can gain a $\langle\xi\rangle^\varepsilon$ from the σ's. Now, writing

$$f_i(\xi,\tau)=\langle\xi\rangle^s\langle\tau-\xi^3\rangle^b\mathcal{F}u_i(\xi,\tau),\quad 1\le i\le 3,\quad\text{such that}\quad\|f_i\|_{L^{r'}_{\xi\tau}}=\|u_i\|_{X^r_{s,b}},$$

it suffices to show

$$\left\|\langle\xi\rangle^{s-}|\xi|\int_A d\nu\prod_{i=1}^3\langle\xi_i\rangle^{-s}\langle\tau_i-\xi_i^3\rangle^{-\frac1r-}f_i(\xi_i,\tau_i)\right\|_{L^{r'}_{\xi\tau}}\le c\prod_{i=1}^3\|f_i\|_{L^{r'}_{\xi\tau}},\tag{18}$$

where in A all the differences $|\xi_k-\xi_j|$, $1\le k\ne j\le 3$, are bounded by 1 and $|\xi|\sim|\xi_i|\sim\langle\xi_i\rangle$ for all $1\le i\le 3$. By Hölder's inequality and Fubini's Theorem the proof of (18) is reduced to show that

$$\sup_{\xi,\tau}\langle\xi\rangle^{1-2s-}\left(\int_A d\nu\prod_{i=1}^3\langle\tau_i-\xi_i^3\rangle^{-1-}\right)^{\frac1r}<\infty.\tag{19}$$

Using [GTV97, Lemma 4.2] twice, we see that

$$\int_A d\nu\prod_{i=1}^3\langle\tau_i-\xi_i^3\rangle^{-1-}\le c\int_{A'}d\xi_1d\xi_2\langle\tau-\xi^3+3(\xi_1+\xi_2)(\xi-\xi_1)(\xi-\xi_2)\rangle^{-1-},$$

where A' is simply the projection of A onto $\mathbb{R}^2$. We decompose

$$A'=A_0\cup A_1\cup\bigcup_{0\le k,j\le c\ln(|\xi|)}A_{kj},$$

where in A_0 (A_1) we have that $|\xi_1+\xi_2-\frac{2\xi}{3}|\le\frac{100}{|\xi|}$ ($|\xi_1+\xi_3-\frac{2\xi}{3}|\le\frac{100}{|\xi|}$), so that the contributions of these subregions are bounded by $\frac{c}{|\xi|}$, while in A_{kj} it should hold that $|\xi_1+\xi_2-\frac{2\xi}{3}|\sim 2^{-k}$ and $|\xi_1+\xi_3-\frac{2\xi}{3}|\sim 2^{-j}$. By symmetry we may assume $k\le j$. To estimate the integral over A_{kj}, we introduce new variables $x_1:=\xi_1+\xi_2-\frac{2\xi}{3}$ and $x_2:=\xi_1-\xi_2$, such that

$$|x_1|\sim 2^{-k}\qquad\text{and}\qquad |x_2|=|\xi_1+\xi_2-\frac{2\xi}{3}+2(\xi_1+\xi_3-\frac{2\xi}{3})|\lesssim 2^{-k}.$$

Then

$$\int_{A_{kj}} d\xi_1 d\xi_2 \langle \tau - \xi^3 + 3(\xi_1+\xi_2)(\xi-\xi_1)(\xi-\xi_2)\rangle^{-1-}$$
$$\leq \int_{|x_2|\lesssim 2^{-k}} dx_2 \int_{|x_1|\sim 2^{-k}} dx_1 \langle \tau - \xi^3 + 3(x_1 + \tfrac{2\xi}{3})(\tfrac{x_1+x_2}{2} - \tfrac{2\xi}{3})(\tfrac{x_1-x_2}{2} - \tfrac{2\xi}{3})\rangle^{-1-}.$$

Substituting $z := (x_1 + \frac{2\xi}{3})(\frac{x_1+x_2}{2} - \frac{2\xi}{3})(\frac{x_1-x_2}{2} - \frac{2\xi}{3})$, so that

$$\left|\frac{dz}{dx_1}\right| = \left|\frac{3x_1^2 - x_2^2}{4} - x_1\xi\right| \sim |x_1\xi| \sim |\xi| 2^{-k},$$

we see that the latter is bounded by

$$\int_{|x_2|\lesssim 2^{-k}} dx_2 \int 2^k \frac{dz}{|\xi|} \langle \tau - \xi^3 + 3z\rangle^{-1-} \leq \frac{c}{|\xi|}.$$

Finally, summing up over j and k, we have

$$\int_{A'} d\xi_1 d\xi_2 \langle \tau - \xi^3 + 3(\xi_1+\xi_2)(\xi-\xi_1)(\xi-\xi_2)\rangle^{-1-} \leq c\frac{(\ln|\xi|)^2}{|\xi|} \leq c|\xi|^{-1+},$$

which gives (19). □

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References

[B93] Bourgain, J.: Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, GAFA 3 (1993), 107–156 and 209–262. MR1215780 (95d:35160b)

[CVV01] Cazenave, T., Vega, L., Vilela, M. C.: A note on the nonlinear Schrödinger equation in weak L^p spaces, Communications in contemporary Mathematics, Vol. 3, No. 1 (2001), 153-162. MR1820017 (2002g:35192)

[CCT03] Christ, M., Colliander, J., Tao, T.: Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations, Amer. J. Math. 125 (2003), no. 6, 1235–1293. MR2018661 (2005d:35223)

[F70] Fefferman, C.: Inequalities for strongly singular convolution operators, Acta Math. 124 (1970), 9–36. MR0257819 (41:2468)

[FLP99] Fonseca, G., Linares, F., Ponce, G.: Global well-posedness for the modified Korteweg-de Vries equation, Comm. PDE, 24 (1999), 683–705. MR1683054 (2000a:35210)

[GTV97] Ginibre, J., Tsutsumi, Y., Velo, G.: On the Cauchy Problem for the Zakharov System, J. of Functional Analysis 151 (1997), 384–436. MR1491547 (2000c:35220)

[G04] Grünrock, A.: An improved local well-posedness result for the modified KdV equation, IMRN 2004, No. 61, 3287–3308. MR2096258 (2006f:35242)

[G05] Grünrock, A.: Bi- and trilinear Schrödinger estimates in one space dimension with applications to cubic NLS and DNLS, IMRN 2005, No. 41, 2525–2558. MR2181058 (2007b:35298)

[KPV91] Kenig, C., Ponce, G., Vega, L.: Oscillatory Integrals and Regularity of dispersive equations, Indiana University Math. J., 40, 1991, 33–69. MR1101221 (92d:35081)

[KPV93] Kenig, C., Ponce, G., Vega, L.: Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle, CPAM 46 (1993), 527–620. MR1211741 (94h:35229)

[KPV01] Kenig, C., Ponce, G., Vega, L.: On the illposedness of some canonical dispersive equations, Duke Math. J. 106 (2001), 617–633. MR1813239 (2002c:35265)

[PV05] Perelman, G., Vega, L.: Self-similar planar curves related to modified Korteweg-de Vries equation, preprint. MR2309566 (2008a:35245)

[S87] Sjölin, P.: Regularity of solutions to the Schrödinger equation, Duke Math. J. 55 (1987), no. 3, 699–715. MR904948 (88j:35026)

[VV01] Vargas, A., Vega, L.: Global wellposedness for $1D$ nonlinear Schrödinger equation for data with an infinite L^2 norm, J. Math. Pures Appl. 80, 10(2001), 1029–1044. MR1876762 (2002j:35290)

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