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POISSON STRUCTURES ON AFFINE SPACES AND FLAG VARIETIES. II

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Dedicated to the memory of our colleague Xu-Dong Liu (1962-2005)

Abstract. The standard Poisson structures on the flag varieties G/P of a complex reductive algebraic group G are investigated. It is shown that the orbits of symplectic leaves in G/P under a fixed maximal torus of G are smooth irreducible locally closed subvarieties of G/P, isomorphic to intersections of dual Schubert cells in the full flag variety G/B of G, and their Zariski closures are explicitly computed. Two different proofs of the former result are presented. The first is in the framework of Poisson homogeneous spaces, and the second one uses an idea of weak splittings of surjective Poisson submersions, based on the notion of Poisson–Dirac submanifolds. For a parabolic subgroup P with abelian unipotent radical (in which case G/P is a Hermitian symmetric space of compact type), it is shown that all orbits of the standard Levi factor L of P on G/P are complete Poisson subvarieties which are quotients of L, equipped with the standard Poisson structure. Moreover, it is proved that the Poisson structure on G/P vanishes at all special base points for the L-orbits on G/P constructed by Richardson, Röhrle, and Steinberg.

Introduction

In this paper, we investigate the geometry of the standard Poisson structures on flag varieties of complex reductive algebraic groups G. We prove that, in such flag varieties, there are only finitely many orbits of symplectic leaves with respect to a fixed maximal torus H of G. These H-orbits are isomorphic to intersections of dual Schubert cells in the full flag variety of G, and their closures are explicitly determined. Further, these H-orbits are parametrized by explicit subsets of the double Weyl group of G, and inclusions of closures are determined by an explicit relation in terms of the Bruhat order. Additional structure is obtained in the case of Hermitian symmetric spaces of compact type, i.e., for flag varieties G/P where P is a parabolic subgroup of G with abelian unipotent radical.

A more precise statement of our main results is given in Theorem 0.4 below. First, however, we need to fix some notation.

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0.1. Let G be a connected reductive algebraic group over $\mathbb{C}$. Fix a pair of dual Borel subgroups $B^{\pm}$ in G, and set $H = B^+ \cap B^-$ for the corresponding maximal torus. Denote the corresponding Lie algebras by $\mathfrak{g}$, $\mathfrak{b}^{\pm}$, and $\mathfrak{h}$. Denote by Δ and Δ^+ the set of all roots, respectively all positive roots, of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let Γ be the set of all positive simple roots of $\mathfrak{g}$. For a subset J of Γ, let $P_J^{\pm}$ be the standard parabolic subgroups of G, containing respectively the Borel subgroups $B^{\pm}$. Let $L_J = P_J^+ \cap P_J^-$ be the common Levi factor of $P_J^{\pm}$. Denote by $U^{\pm}$ and $U_J^{\pm}$ the unipotent radicals of $B^{\pm}$ and $P_J^{\pm}$, respectively. Set $\mathfrak{p}_J^{\pm} = \operatorname{Lie} P_J^{\pm}$, $\mathfrak{l}_J = \operatorname{Lie} L_J$, $\mathfrak{n}^{\pm} = \operatorname{Lie} U^{\pm}$, and $\mathfrak{n}_J^{\pm} = \operatorname{Lie} U_J^{\pm}$.

0.2. We fix a nondegenerate bilinear invariant form $\langle .,. \rangle$ on $\mathfrak{g}$ for which the square of the length of a long root is equal to 2. Recall [3, eq. (1.1)] that the standard r-matrix of $\mathfrak{g}$ is given by

$$r_{\mathfrak{g}} = \sum_{\alpha \in \Delta_+} \frac{\langle \alpha, \alpha \rangle}{2} e_\alpha \wedge f_\alpha, \tag{0.1}$$

where $\{e_\alpha\}_{\alpha \in \Delta^+}$ and $\{f_\alpha\}_{\alpha \in \Delta^+}$ are any sets of root vectors of $\mathfrak{g}$, normalized by $[e_\alpha, f_\alpha] = \alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$. (In the last equation, $\mathfrak{h}$ and $\mathfrak{h}^*$ are identified via the restriction of the form $\langle .,. \rangle$; for $\alpha \in \Delta^+$, e_α and f_α are root vectors for the roots α and $-\alpha$, respectively.) The standard Poisson structure on G is given by

$$\pi_G = L(r_{\mathfrak{g}}) - R(r_{\mathfrak{g}}) = \chi^R(r_{\mathfrak{g}}) - \chi^L(r_{\mathfrak{g}}). \tag{0.2}$$

Here $L(r_{\mathfrak{g}})$ and $R(r_{\mathfrak{g}})$ refer to the left and right invariant bivector fields on G associated to $r_{\mathfrak{g}} \in \bigwedge^2 \mathfrak{g} \cong \bigwedge^2 T_e G$.

For any subset $J \subseteq \Gamma$, the standard parabolic subgroup P_J^+ is a Poisson algebraic subgroup of $(G, \pi_{\mathfrak{g}})$. The natural projection

$$\eta_J : G \to G/P_J^+ \tag{0.3}$$

induces the following Poisson structure on the flag variety G/P_J^+:

$$\pi_J := \eta_{J*}(\pi) = -\chi(r_{\mathfrak{g}}); \tag{0.4}$$

see [3, Theorem 1.8] and Proposition 1.3 below. Throughout the paper, for an action of G on a variety M, we denote the extension to $\bigwedge \mathfrak{g}$ of the infinitesimal action of $\mathfrak{g}$ on M by

$$\chi : \bigwedge \mathfrak{g} \to \Gamma(M, \bigwedge TM).$$

For brevity, we set $\eta = \eta_\emptyset$ and $\pi = \pi_\emptyset$.

Since the Poisson structure π_G vanishes on H, the left action of H on G/P_J^+ preserves π_J.

0.3. In this paper we investigate the geometry of the Poisson structures π_J on the flag varieties G/P_J^+. This continues our work with Brown in [3], which we shall refer to as Part I.

Before we state the main results of the paper, we introduce some notation on Weyl groups. The Weyl group of the pair (G, H) will be denoted by W. For $w \in W$, we will denote by $\dot{w}$ a representative of w in the normalizer of H. If a formula does not depend on the choice of this representative the dot will be omitted. The Weyl group of (L_J, H), naturally thought of as a subgroup of W, will be denoted by W_J. Recall that each coset in W/W_J has unique minimal and maximal length representatives. The sets of those will be denoted by $W^J_{\min}$ and $W^J_{\max}$ respectively. Denote the longest elements of W and W_J by $w_\circ$ and $w_\circ^J$.

For $w \in W$ and $J \subseteq \Gamma$, set

$$x^J_w = wP^+_J \in G/P^+_J, \qquad x_w = x^{\emptyset}_w = wB^+ \in G/B^+. \tag{0.5}$$

0.4. The following Theorem summarizes some of our results (Theorems 1.5, 1.8, 4.6, and Proposition 4.2). Recall from Part I, [3, eq. (2.11)], the notation

$$\mathcal{U}_{\dot{w}_1,w_2} = U^-_{w_1}\dot{w}_1 \cap B^+ w_2 B^+, \quad w_1, w_2 \in W, \tag{0.6}$$

where for $w \in W$,

$$U^-_w = U^- \cap \mathrm{Ad}_w(U^-). \tag{0.7}$$

Theorem. (i) *There are only finitely many H-orbits of symplectic leaves on $(G/P^+_J, \pi_J)$, parametrized by pairs $(w_1, w_2) \in W^J_{\max} \times W$ such that $w_1 \le w_2$ in the Bruhat order. The torus orbit corresponding to the pair (w_1, w_2) is given by*

$$\mathcal{S}^J_{w_1,w_2} = \mathcal{U}_{\dot{w}_1,w_2}.P^+_J$$

and is biregularly isomorphic to the intersection $\mathcal{B}_{w_1,w_2} = B^- x_{w_1} \cap B^+ x_{w_2}$ of dual Schubert cells in the generalized full flag variety G/B^+. Thus, the H-orbits of symplectic leaves on $(G/B^+, \pi)$ are exactly the intersections of dual Schubert cells $\mathcal{B}_{w_1,w_2}$.

(ii) *For each pair (w_1, w_2) as above, the Zariski closure of $\mathcal{S}^J_{w_1,w_2}$ in $(G/P^+_J, \pi_J)$ is equal to the union of $\mathcal{S}^J_{v_1,v_2}$ over those pairs $(v_1, v_2) \in W^J_{\max} \times W$ with $v_1 \le v_2$ for which there exists $z \in W_J$ such that $w_1 \le v_1 z$ and $w_2 \ge v_2 z$.*

(iii) *If P^+_J is a parabolic subgroup of G with abelian unipotent radical (in which case G/P^+_J is a Hermitian symmetric space of compact type), then all L_J-orbits on G/P^+_J are complete Poisson subvarieties of $(G/P^+_J, \pi_J)$, and all of them are quotients of the standard Poisson group structure on L_J.*

In addition, we prove that in the case of Hermitian symmetric spaces of compact type the Poisson structure π_J vanishes at the base points for the L_J-orbits constructed by Richardson, Röhrle, and Steinberg [19]. In this case, we also characterize explicitly the H-orbits of symplectic leaves which fall within a given L_J-orbit. This relates parts (i) and (iii) of the above Theorem. It is done in Theorem 4.13 and will not be formulated here due to the amount of notation it requires.

The partition of the partial flag variety G/P^+_J into H-orbits of leaves coincides with Lusztig's partition [14] of G/P^+_J into locally closed subvarieties which are isomorphic to intersections of dual Schubert cells in the full flag variety G/B^+. More precisely, $\mathcal{S}^J_{w_1,w_2} = \mathcal{P}^J_{w_1,w'_2,w''_2}$ (in the notation of [20, Section 5]), where w'_2 and w''_2 are the unique elements in $W^J_{\min}$ and W_J such that $w_2 = w'_2 w''_2$. This follows, e.g., from the discussion in Rietsch [20, Section 5] and §1.5 below.

We also derive explicit formulas for the restrictions of the Poisson structures π_J to the open B^--orbit in Hermitian symmetric spaces of compact type for classical groups and show that in all cases those are the quasiclassical limits of classes of quadratic algebras that attracted a lot of attention in the theory of quantum groups (see Section 5). While those classes were previously studied case by case, our work conceptually unifies them. Finally, in the exceptional case E_6 we find a new interesting quadratic Poisson structure on a 16 dimensional affine space, related to a half-spin representation of $\mathfrak{so}_{10}$.

0.5. We offer two different proofs of the first part of Theorem 0.4. The first one, which appears in Section 1, uses the theory of Poisson homogeneous spaces. The second one (see Section 3) is more geometric and is based upon the notion of Poisson–Dirac submanifolds [26], [5]. The latter are submanifolds of a Poisson manifold (M, Π) which in general might not be Poisson submanifolds but have the property that their symplectic leaves are exactly the connected components of their intersections with the symplectic leaves of (M, Π). The second approach uses the idea of *weak splittings of surjective Poisson submersions*, developed in Section 3. Briefly, if $p : (M, \Pi) \to (N, \pi)$ is such a submersion, a weak splitting of it is a partition of $N = \bigsqcup_{\alpha \in A} N_\alpha$ into complete Poisson submanifolds and liftings $i : N_\alpha \to M$ of the restrictions of the submersions $p|_{p^{-1}(N_\alpha)}$ such that $(i_\alpha(N_\alpha), i_{\alpha *}(\pi|_{N_\alpha}))$ are Poisson–Dirac submanifolds of (M, Π). If such a weak splitting exists, then the symplectic leaves of the base (N, π) are just the connected components of the inverse images under the maps i_α of the symplectic leaves of (M, Π). It is interesting to note that in the present situation the needed partition of $(G/P_J^+, \pi_J)$ is exactly the partition into Schubert cells. The second proof of Theorem 0.4 grew out of our attempt to understand geometrically the observation [3, Remark 3.10].

0.6. As was noted earlier, many interesting quadratic algebras are quantizations of the algebras of functions on particular Schubert cells in particular flag varieties. The idea of weak splittings of surjective Poisson submersions also suggests that the primitive ideals of those algebras can be obtained as push forwards from the primitive ideals of localizations of quotients of the quantized algebras of functions on simple groups under nonalgebra (!) maps. We plan to return to this in a forthcoming publication.

In the case when the parabolic subgroup P_J^+ has abelian unipotent radical, there exists a real form G_0 of G for which $G_0 \cap L_J$ is a maximal compact subgroup of both G_0 and L_J. In [19], an explicit order-reversing bijection is constructed between the L_J-orbits on G/P_J^+ and the G_0-orbits on G/P_J^+ which were studied in great detail in the framework of symmetric spaces by Wolf [24], Takeuchi [23], and others. It is interesting to understand whether this bijection can be further refined to a bijection between torus orbits of symplectic leaves of the Poisson structure π_J and orbits of leaves of a real Poisson structure on G/P_J^+, e.g. the one of Foth and Lu [8].

Finally, let us note that parts (i) and (ii) of Theorem 0.4 show that each intersection of dual Schubert varieties in G/P_J^+ (i.e., each Richardson variety [2] in G/P_J^+) is the Zariski closure of a single H-orbit of symplectic leaves. This suggests the possibility to construct explicit degenerations of Richardson varieties by deforming algebraically the Poisson structure π_J and looking at how torus orbits of symplectic leaves deform. This could provide a Poisson geometric approach to Schubert calculus.

0.7. We conclude the Introduction with some notation to be used later in the paper.

If $\{\alpha_1, \dots, \alpha_N\}$ denotes the set of positive simple roots of the reductive Lie algebra $\mathfrak{g}$ and $\gamma = \sum_i n_i \alpha_i$ is an arbitrary root of $\mathfrak{g}$, then the α_j *height* of γ is defined by $n_{\alpha_j}(\gamma) = n_j$. The *support* of γ is defined by

$$\operatorname{supp} \gamma = \{\alpha_j \mid 1 \le j \le N, \;\; n_{\alpha_j}(\gamma) \ne 0\}.$$

If V is a vector space, then for a given $\pi \in \bigwedge^2 V$ we will use the standard notation for the linear map

$$\pi^\sharp : V^* \to V, \quad \pi^\sharp(\xi) = \xi \rfloor \pi. \tag{0.8}$$

For a subspace $V_1 \subset V$, we set

$$(V_1)^0 = \{\xi \in V^* \mid \xi(v) = 0 \text{ for all } v \in V_1\}. \tag{0.9}$$

In particular, this notation will be used for a Poisson bivector $\pi \in (M, \bigwedge^2 TM)$, in which case $\pi^\sharp : T^*M \to TM$ is a bundle map. A submanifold X of (M, π) will be called a *complete Poisson submanifold* if it is stable under all Hamiltonian flows, that is, if it is a union of symplectic leaves.

Given an algebraic group G and an element $g \in G$, we will denote by Ad_g both the conjugation action of g on G, $\mathrm{Ad}_g(h) = ghg^{-1}$, and the adjoint action of g on $\mathrm{Lie}\, G$. As usual, ad_x will be used for the adjoint action of $\mathrm{Lie}\, G$ on itself.

As in Part I, we will use the following convention to distinguish between double cosets and orbits in homogeneous spaces. For two subgroups C and D of a group G and an element $g \in G$: (1) The notation CgD will denote the double coset of g in G, and (2) the notation $C.gD$ will denote the C-orbit of $gD \in G/D$.

Finally, for a locally closed subvariety Y of an algebraic variety X and a subset Z of Y, $\mathrm{Cl}_Y(Z)$ will denote the Zariski closure of Z in Y.

1. Torus orbits of symplectic leaves in flag varieties via Poisson homogeneous spaces

1.1. In this section we study the geometry of the Poisson structure π_J on the flag variety G/P_J^+ by the techniques of Poisson homogeneous spaces; cf. [3, Section 1].

First we introduce some more notation to be used in the rest of the paper and recall basic facts on minimal and maximal length representatives for cosets in Weyl groups. Denote the set of all roots of $(\mathfrak{l}_J, \mathfrak{h})$ by Δ_J and the corresponding subset of positive roots by $\Delta_J^+ = \Delta_J \cap \Delta^+$. Recall that the (unique) minimal length and maximal length representatives w of a coset in W/W_J are characterized respectively by

$$w(\Delta_J^+) \subset \Delta^+ \tag{1.1}$$

and

$$w(\Delta_J^+) \subset -\Delta^+. \tag{1.2}$$

Recall that there are two natural bijections between $W^J_{\min}$ and $W^J_{\max}$ given by

$$v \in W^J_{\min} \mapsto v w_\circ^J \in W^J_{\max} \qquad \text{and} \qquad v \in W^J_{\min} \mapsto w_\circ v \in W^J_{\max}, \tag{1.3}$$

where $w_\circ$ and $w_\circ^J$ are the longest elements of W and W_J.

1.2. Recall that, in the terminology of [3, §1.3-1.4], the standard Poisson algebraic group (G, π_G) is a part of the algebraic Manin triple $(G \times G, G_{\mathrm{diag}}, F)$, where G_{diag} denotes the diagonal subgroup of $G \times G$ and F is the dual Poisson algebraic group given by

$$F = \{(hu^+, h^{-1}u^-) \mid h \in H,\ u^\pm \in U^\pm\}. \tag{1.4}$$

On the level of Lie algebras, we have the standard Manin triple $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}_{\mathrm{diag}}, \mathfrak{g}^*)$, where $\mathfrak{g}^* = \mathrm{Lie}\, F$ and the bilinear form on $\mathfrak{g} \oplus \mathfrak{g}$ is

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle \tag{1.5}$$

in terms of the nondegenerate bilinear form $\langle ., . \rangle$ on $\mathfrak{g}$ fixed in §0.2.

1.3. Proposition. (a) *The orthogonal complement of* $\mathfrak{p}_J^+$ *in the dual Lie bialgebra* $\mathfrak{g}^*$ *for the standard Lie bialgebra structure on* $\mathfrak{g}$ *is* $(\mathfrak{p}_J^+)^\perp = \mathfrak{n}_J^+ \oplus \{0\}$.

(b) *The standard parabolic subgroup* P_J^+ *of* G *is a Poisson algebraic subgroup for the standard Poisson structure* π_G.

(c) *The pair* $\big(G/P_J^+, \eta_{J*}(\pi_G)\big)$ *is a Poisson homogeneous space for the standard Poisson algebraic group* (G, π_G). *The Poisson structure* $\eta_{J*}(\pi_G)$ *is equal to* $-\chi(r_\mathfrak{g})$ *and will be denoted by* π_J *for brevity.*

(d) *The Drinfeld Lagrangian subalgebra* (*cf.* [7]) *of the base point* $x_1^J = eP_J^+$ *of the Poisson homogeneous space in* (c) *is*

$$\bar{\mathfrak{l}}_J = \{(l + n_1^+, l + n_2^+) \mid l \in \mathfrak{l}_J,\ n_i^+ \in \mathfrak{n}_J^+\} \subset \mathfrak{g} \oplus \mathfrak{g} \cong D(\mathfrak{g}). \tag{1.6}$$

It is the tangent Lie algebra of the connected algebraic subgroup

$$\overline{L}_J = \{(lu_1^+, lu_2^+) \mid l \in L_J,\ u_i^+ \in U_J^+\} \tag{1.7}$$

of $G \times G$. *In particular,* $\big(G/P_J^+, \pi_J\big)$ *is an algebraic Poisson homogeneous space for the standard Poisson algebraic group* (G, π_G) *in the terminology of* [3, Definition 1.7].

The proof of Proposition 1.3 is analogous to that of [3, Proposition 3.2] and will be omitted.

Below we will need the following well known Lemma.

1.4. Lemma. *All* B^-*-orbits on the flag variety* G/P_J^+ *are parametrized by* $W^J_{\max}$ *by* $w \mapsto B^- x_w^J$. *Moreover* $B^- x_w^J$ *is biregularly isomorphic to* U_w^- (*recall* (0.7)) *by*

$$u \in U_w^- \mapsto u x_w^J.$$

For completeness, we will sketch the proof of the second part of Lemma 1.4. For any $w \in W$, one has $B^- = U_w^-(B^- \cap \mathrm{Ad}_w(B^+))$. If $w \in W^J_{\max}$, then (1.2) implies that $\mathfrak{n}^- \cap \mathrm{Ad}_w(\mathfrak{n}^-) \subset \mathrm{Ad}_w(\mathfrak{n}_J^-)$. Since U_w^- and $\mathrm{Ad}_w(U_J^-)$ are connected subgroups of G, it follows that $U_w^- \subset \mathrm{Ad}_w(U_J^-)$, which easily implies the second statement in Lemma 1.4.

Set

$$\Omega^J = \{(w_1, w_2) \in W^J_{\max} \times W \mid w_1 \le w_2\}. \tag{1.8}$$

1.5. Theorem. *There are finitely many* H*-orbits of symplectic leaves on* $(G/P_J^+, \pi_J)$, *bijectively parametrized by* Ω^J, *and all of them are smooth irreducible locally closed subvarieties of* G/P_J^+. *The* H*-orbit of leaves corresponding to* $(w_1, w_2) \in \Omega^J$ *is explicitly given by*

$$\mathcal{S}^J_{w_1,w_2} = \mathcal{U}_{\dot{w}_1,w_2}.P_J^+ \tag{1.9}$$

(*recall* (0.6)) *and is biregularly isomorphic to the intersection* $\mathcal{B}_{w_1,w_2} = B^- x_{w_1} \cap B^+ x_{w_2}$ *of dual Schubert cells in the full flag variety* G/B^+.

In particular, the H*-orbits of symplectic leaves on* $(G/B^+, \pi)$ *are exactly the intersections of dual Schubert cells* $\mathcal{B}_{w_1,w_2}$, *indexed by pairs* $(w_1, w_2) \in W \times W$ *such that* $w_1 \le w_2$.

It is easy to see that, while $\mathcal{U}_{\dot{w}_1,w_2}$ depends on the representative of w_1 in the normalizer of H, the set $\mathcal{S}^J_{w_1,w_2}$ does not.

Proof. It follows from [3, Theorem 1.10] that the H-orbits of symplectic leaves in $(G/P_J^+, \pi_J)$ are smooth locally closed subsets of G/P_J^+. Moreover the same

Theorem implies that they are exactly the irreducible components of the inverse images under the map

$$\Delta : G/P_J^+ \hookrightarrow (G\times G)/\overline{L}_J, \quad \Delta(gP_J^+) = (g,g)\overline{L}_J \tag{1.10}$$

of the $(B^+\times B^-)$-orbits on $(G\times G)/\overline{L}_J$. The map Δ is an embedding because

$$G_{\rm diag} \cap \overline{L}_J = (P_J^+)_{\rm diag}.$$

(Recall that $G_{\rm diag}$ denotes the diagonal subgroup of $G\times G$.)

Applying [3, Theorem 8.1], we see that each such orbit passes through the coset of $(\dot{w}_2, \dot{w}_1)$ for some $(w_1, w_2) \in W^J_{\min} \times W$ and that all such orbits are distinct. Because $(\dot{w}^J_\circ, \dot{w}^J_\circ) \in \overline{L}_J$, using (1.3) we obtain that

$$\{(\dot{w}_2, \dot{w}_1)\overline{L}_J \mid w_1 \in W^J_{\max},\ w_2 \in W\}$$

is a complete, irredundant set of representatives for all $(B^+\times B^-)$-orbits on $(G\times G)/\overline{L}_J$. Note that

$$\Delta^{-1}\left((B^+\times B^-)(\dot{w}_2,\dot{w}_1)\overline{L}_J\right) \subseteq B^- x^J_{w_1}.$$

Thus, $\Delta^{-1}\left((B^+\times B^-)(\dot{w}_2,\dot{w}_1)\overline{L}_J\right)$ consists of all points $u^- x^J_{w_1}$, with $u^- \in U^-_w$, for which there exist $b^\pm \in B^\pm$, $u_1^+, u_2^+ \in U_J^+$, and $l \in L_J$ such that

$$u^-\dot{w}_1 = b^-\dot{w}_1 l u_2^+ = b^+\dot{w}_2 l u_1^+. \tag{1.11}$$

From the first equality we get that $lu_2^+ = \mathrm{Ad}^{-1}_{\dot{w}_1}((b^-)^{-1}u^-) \in \mathrm{Ad}^{-1}_{w_1}(B^-)$. Since $w_1 \in W^J_{\max}$ we have $\mathrm{Ad}_{\dot{w}_1}(u_2^+) \in B^-$; thus $l \in L_J \cap \mathrm{Ad}_{w_1}(B^-)$. For arbitrary $l \in L_J\cap \mathrm{Ad}_{w_1}(B^-)$ and $u^- \in U^-_w$ there exist $b^- \in B^-$ and $u_2^+ \in U_J^+$ that satisfy the first equality in (1.11). Then the second equality in (1.11) implies

$$\Delta^{-1}\left((B^+\times B^-)(\dot{w}_2,\dot{w}_1)\overline{L}_J\right) = \left(U^-_{w_1}\dot{w}_1 \cap B^+ w_2 (L_J\cap \mathrm{Ad}^{-1}_{w_1}(B^-))U_J^+\right).P_J^+.$$

From (1.2) one has $\mathrm{Ad}^{-1}_{w_1}(B^-) = (L_J\cap B^+)(U_J^-\cap \mathrm{Ad}^{-1}_{w_1}(B^-))$. Thus $L_J\cap\mathrm{Ad}^{-1}_{w_1}(B^-) = L_J \cap B^+$ and

$$\Delta^{-1}\left((B^+\times B^-)(\dot{w}_2,\dot{w}_1)\overline{L}_J\right) = \left(U^-_{w_1}\dot{w}_1 \cap B^+ w_2 B^+\right).P_J^+ = \mathcal{U}_{\dot{w}_1,w_2}.P_J^+.$$

It is known that $\mathcal{U}_{\dot{w}_1,w_2}$ is nonempty if and only if $w_2 \geq w_1$ (this follows from results in [12] and is proved explicitly in [6, Corollary 1.2]). On the other hand if $w_2 \geq w_1$, then $\mathcal{S}^J_{w_1,w_2}$ is irreducible and biregularly isomorphic to $\mathcal{U}_{\dot{w}_1,w_2}$ and $\mathcal{B}_{w_1,w_2}$ because of Lemma 1.4 and [3, Theorem 2.4]. □

1.6. Next we describe the relation between the Poisson structures on torus orbits of leaves in different flag varieties for a fixed reductive group G. For a subset $J\subseteq\Gamma$, denote by

$$\mu_J : G/B^+ \to G/P_J^+ \tag{1.12}$$

the natural projection. The composition $G \xrightarrow{\eta} G/B^+ \xrightarrow{\mu_J} G/P_J^+$ coincides with η_J (recall (0.3)). Since the surjective maps η and η_J are Poisson, the projection μ_J is Poisson as well.

For all $J\subseteq\Gamma$ and $w\in W$, the sets $B^\pm x^J_w$ are complete Poisson subvarieties of $(G/P_J^+, \pi_J)$. This follows from the facts that $(G/P_J^+,\pi_J)$ is a quotient of (G,π_G) and $B^\pm$ are Poisson algebraic subgroups of (G,π_G). (The statement is also a simple corollary of Theorem 1.5.)

Proposition. *For every subset $J \subseteq \Gamma$ and every $w \in W^J_{\max}$, the projection μ_J restricts to an isomorphism of Poisson varieties*

$$\mu_J|_{B^- x_w} : (B^- x_w, \pi) \xrightarrow{\cong} (B^- x^J_w, \pi_J).$$

In particular, for $(w_1, w_2) \in \Omega^J$ this restricts to the Poisson isomorphism

$$\mu_J|_{\mathcal{S}_{w_1,w_2}} : (\mathcal{S}_{w_1,w_2}, \pi) \xrightarrow{\cong} (\mathcal{S}^J_{w_1,w_2}, \pi_J),$$

where for simplicity we set $\mathcal{S}_{w_1,w_2} := \mathcal{S}^{\emptyset}_{w_1,w_2}$.

Proof. Lemma 1.4 guarantees that μ_J is an isomorphism between the affine spaces $B^- x_w \subset G/B^+$ and $B^- x^J_w \subset G/P^+_J$. It was shown above that $B^- x_w$ and $B^- x^J_w$ are complete Poisson subvarieties of $(G/B^+, \pi)$ and $(G/P^+_J, \pi_J)$, respectively. The first statement now follows from the fact that μ_J is a Poisson mapping, and the second one is a corollary of the first. □

1.7. Remark. One can first establish Proposition 1.6 and then deduce the general case of Theorem 1.5 from the case $J = \emptyset$. The proof of Theorem 1.5 in this special case is easier than the general case. This gives a somewhat simpler proof of Theorem 1.5.

Finally we describe the Zariski closures of the torus orbits of symplectic leaves $\mathcal{S}^J_{w_1,w_2}$ of $(G/P^+_J, \pi_J)$.

1.8. Theorem. *For all $(w_1, w_2) \in \Omega^J$, the Zariski closure of the H-orbit of symplectic leaves $\mathcal{S}^J_{w_1,w_2}$ in $(G/P^+_J, \pi_J)$ is equal to the union of $\mathcal{S}^J_{v_1,v_2}$ over those $(v_1, v_2) \in \Omega^J$ for which there exists $z \in W_J$ such that $w_1 \le v_1 z$ and $w_2 \ge v_2 z$.*

Note that in the special case of the generalized full flag variety G/B^+, Theorem 1.8 reduces to the well known fact that for the Richardson varieties,

$$\overline{B^- x_{w_1} \cap B^+ x_{w_2}} = \overline{B^- x_{w_1}} \cap \overline{B^+ x_{w_2}}.$$

The inclusion relations among the closures of the H-orbits $\mathcal{S}^J_{w_1,w_2}$ were also determined (independently) by Rietsch [20, Proposition 7.2], with different combinatorics. Moreover, Rietsch determined these relations for the nonnegative parts of the $\mathcal{S}^J_{w_1,w_2}$ in the sense of Lusztig [20, Theorem 6.1]. Our derivation leads to a simpler final result and finds a relation with orbit closures in wonderful group compactifications [21], [13].

Proof. By modifying Springer's Lemma 2.2 [21] or directly using [13, Lemmas 4.17 and 4.22], one gets that for all $(w_1, w_2) \in W^J_{\min} \times W$ inside $(G \times G)/\overline{L}_J$:

(1.13)
$$\overline{(B^+ \times B^+).(w_2, w_1)\overline{L}_J} = \bigsqcup \{ (B^+ \times B^+).(v_2, v_1)\overline{L}_J \mid (v_1, v_2) \in W^J_{\min} \times W, \; \exists z \in W_J \text{ such that } w_1 \ge v_1 z, \; w_2 \ge v_2 z \}.$$

Acting on (1.13) by $(e, \dot{w}_\circ)$ and using the fact that $\mathrm{Ad}_{w_\circ}(B^+) = B^-$ and that for $w, v \in W$, we have $w \le v$ if and only if $w_\circ w \ge w_\circ v$, we see that for $(w_1, w_2) \in W^J_{\min} \times W$,

$$\overline{(B^+ \times B^-).(w_2, w_\circ w_1)\overline{L}_J} = \bigsqcup \{ (B^+ \times B^-).(v_2, w_\circ v_1)\overline{L}_J \mid (v_1, v_2) \in W^J_{\min} \times W, \; \exists z \in W_J \text{ such that } w_\circ w_1 \le w_\circ v_1 z, \; w_2 \ge v_2 z \}.$$

We deduce from the second correspondence between $W^J_{\min}$ and $W^J_{\max}$ in (1.3) that for all $(w_1, w_2) \in W^J_{\max} \times W$,

(1.14)
$$\overline{(B^+ \times B^-).(w_2, w_1)\overline{L}_J} = \bigsqcup \{(B^+ \times B^-).(v_2, v_1)\overline{L}_J \mid (v_1, v_2) \in W^J_{\max} \times W, \\ \exists z \in W_J \text{ such that } w_1 \le v_1 z,\ w_2 \ge v_2 z\}.$$

Now we apply [3, Lemma 2.6] for Y equal to the image of the embedding Δ from (1.1) and the stratification of $(G \times G)/\overline{L}_J$ by $(B^+ \times B^-)$-orbits. Note that $\Delta(G/P^+_J)$ intersects each $(B^+ \times B^-)$-orbit transversally since the diagonal of $\mathfrak{g} \oplus \mathfrak{g}$ and $\mathfrak{b}_+ \oplus \mathfrak{b}_-$ span $\mathfrak{g} \oplus \mathfrak{g}$. It follows that

$$(1.15) \quad \mathrm{Cl}_{\Delta(G/P^+_J)} \left(\Delta(G/P^+_J) \cap ((B^+ \times B^-).(w_2, w_1)\overline{L}_J)\right) \\ = \Delta(G/P^+_J) \cap \overline{(B^+ \times B^-).(w_2, w_1)\overline{L}_J}.$$

Recall from the proof of Theorem 1.5 that for $(w_1, w_2) \in W^J_{\max} \times W$,

$$\Delta(G/P^+_J)((B^+ \times B^-).(w_2, w_1)\overline{L}_J) \neq \emptyset \quad \text{if and only if} \quad w_1 \le w_2,$$

in which case

$$\mathcal{S}^J_{w_1, w_2} = \Delta^{-1}\left((B^+ \times B^-).(w_2, w_1)\overline{L}_J\right).$$

The Lemma now follows from (1.14), (1.15), and the fact that Δ is an embedding. □

The Zariski closures of the H-orbits of symplectic leaves inside each Schubert cell in $(G/P^+_J, \pi_J)$ have a particularly simple form.

1.9. Proposition. *For each $w \in W^J_{\max}$ the following hold:*

(i) *The H-orbits of symplectic leaves in the Schubert cell $(B^- x^J_w \subset G/P^+_J, \pi_J|_{B^- x^J_w})$ are parametrized by $W^{\ge w} = \{w_2 \in W \mid w_2 \ge w\}$ by*

$$w_2 \in W^{\ge w} \mapsto \mathcal{S}^J_{w, w_2} = \mathcal{U}_{w, w_2}.P^+_J.$$

(ii) *For each $w_2 \in W^{\ge w}$, the Zariski closure of $\mathcal{S}^J_{w,w_2}$ inside the Schubert cell $B^- x^J_w$ consists of all $\mathcal{S}^J_{w,v_2}$ for $v_2 \in W$ such that $w \le v_2 \le w_2$.*

This proposition generalizes Theorems 3.9 and 3.13 from [3].

Proof. The first part follows from

$$\mathcal{S}^J_{w,w_2} = \mathcal{U}_{w,w_2}.P^+_J \subset (U^-_w w).P^+_J = B^- x^J_w$$

and the fact that the Schubert cells $B^- x^J_w$ for $w \in W^J_{\max}$ partition G/P^+_J.

It is easy to deduce the second part from Theorem 1.8, but we offer a direct proof which better explains the result.

In terms of the isomorphism $\mu_J|_{B^- x_w} : B^- x_w \to B^- x^J_w$ (cf. §1.6), $\mathcal{S}^J_{w,w_2}$ is given by

$$\mathcal{S}^J_{w,w_2} = (\mu_J|_{B^- x_w})^{-1} (B^- x_w \cap B^+ x_{w_2}).$$

Applying [3, Lemma 2.6] for the partition $X = G/B^+ = \bigsqcup_{w_1 \in W} B^+ x_{w_1}$ and $Y = B^- x_w$, and using the standard formulas for closures of Schubert cells, leads to

$$\mathrm{Cl}_{B^- x_w}(B^- x_w \cap B^+ x_{w_2}) = \bigsqcup_{v_2 \in W, w \le v_2 \le w_2} (B^- x_w \cap B^+ x_{v_2}).$$

The second part of the Proposition follows from this, applying once again the isomorphism $\mu_J|_{B^- x_w}$. □

2. Weak splittings of surjective Poisson submersions

2.1. First we recall the definition of a Poisson–Dirac submanifold of a Poisson manifold, given by Crainic and Fernandes in [5, Section 9].

Definition. Assume that (M, Π) is a smooth (real or complex) Poisson manifold. A submanifold X of M is called a Poisson–Dirac submanifold if the following two conditions are satisfied:

(i) For each symplectic leaf S of (M, Π), the intersection $S \cap X$ is clean (i.e., it is smooth and $T_x(S \cap X) = T_x S \cap T_x X$ for all $x \in S \cap X$) and $S \cap X$ is a symplectic submanifold of $(S, (\Pi|_S)^{-1})$.

(ii) The family of symplectic structures $(\Pi|_S)^{-1}|_{S \cap X}$ is induced by a smooth Poisson structure π on X.

Here and below, for a nondegenerate Poisson structure π_0 we denote by $(\pi_0)^{-1}$ the corresponding symplectic form.

In the setting of the above Definition, the symplectic leaves of (X, π) are exactly the connected components of the intersections of the symplectic leaves of (M, Π) with X.

2.2. The following simple criterion was proved in [5].

Proposition. *Assume that (M, Π) is a Poisson manifold and that X is a submanifold for which there exists a subbundle E of $T_X M$ such that*

(i) $T_X M = TX \oplus E$ *and*

(ii) *the restriction of the Poisson tensor Π to X splits as*

$$\Pi|_X = \pi + \pi_E$$

for some smooth bivector fields $\pi \in \Gamma(X, \bigwedge^2 TX)$ and $\pi_E \in \Gamma(X, \bigwedge^2 E)$.

Then X is a Poisson–Dirac submanifold of (M, Π), and the induced Poisson structure on it coincides with π.

Crainic and Fernandes call submanifolds satisfying the conditions of Proposition 2.2 *Poisson–Dirac submanifolds admitting a Dirac projection.* Earlier, Xu [26] investigated such submanifolds with an extra property, namely that E^0 is a Lie subalgebroid of T^*M, equipped with the standard cotangent bundle algebroid structure (recall (0.9)).

2.3. Definition. Assume that (M, Π) and (N, π) are Poisson manifolds and that $p : (M, \Pi) \to (N, \pi)$ is a surjective Poisson submersion. A weak splitting of p is a partition

$$N = \bigsqcup_{\alpha \in A} N_\alpha \tag{2.1}$$

of (N, π) into complete Poisson submanifolds such that for each $\alpha \in A$ there exists a smooth lifting $i_\alpha : N_\alpha \to M$ (of $p|_{p^{-1}(N_\alpha)} : p^{-1}(N_\alpha) \to N_\alpha$) with the properties:

(i) $i_\alpha(N_\alpha)$ is a Poisson–Dirac submanifold of (M, Π) and

(ii) the induced Poisson structure on $i_\alpha(N_\alpha)$ is $i_{\alpha *}(\pi|_{N_\alpha})$.

Note that i_α *is not required to be a Poisson map.* An important special case is illustrated in algebraic terms in Proposition 2.6.

2.4. Remark. If a surjective Poisson submersion $p : (M, \Pi) \to (N, \pi)$ admits a weak splitting as in Definition 2.3, then the symplectic foliation of (N, π) is easily described in terms of the symplectic foliation of (M, Π). Namely, each symplectic leaf of (N, π) lies entirely in one of the submanifolds N_α and is of the type $i_\alpha^{-1}(S \cap i_\alpha(N_\alpha))^\circ$, where S is a symplectic leaf of M and $(S \cap i_\alpha(N_\alpha))^\circ$ is a connected component of $(S \cap i_\alpha(N_\alpha))$.

2.5. The following Proposition provides a sufficient condition for the condition (ii) in Definition 2.3 which is easier to check.

Proposition. *Assume that* $p : (M, \Pi) \to (N, \pi)$ *is a surjective Poisson submersion. Let*

$$N = \bigsqcup_{\alpha \in A} N_\alpha$$

be a partition of (N, π) *into complete Poisson submanifolds such that for each* $\alpha \in A$*, there exists a smooth lifting* $i_\alpha : N_\alpha \to M$ *(of* $p|_{p^{-1}(N_\alpha)} : p^{-1}(N_\alpha) \to N_\alpha$*) whose image is a Poisson–Dirac submanifold admitting a Dirac projection with respect to a subbundle* E_α *of* $T_{i_\alpha(N_\alpha)}M$*, cf. Proposition* 2.2*. If the* E_α *are tangent to the fibers of* p *(i.e., the* E_α *contain the tangent spaces to the fibers of* p*), then condition* (ii) *in Definition* 2.3 *is satisfied and the families* $\{N_\alpha\}_{\alpha \in A}$, $\{i_\alpha\}_{\alpha \in A}$ *provide a weak splitting of* p.

Proof. For $m \in i_\alpha(N_\alpha)$ denote the fiber of p through m by $F_m = p^{-1}(p(m))$. Since $p : (M, \Pi) \to (N, \pi)$ is Poisson and i_α is a lifting of $p|_{p^{-1}(N_\alpha)} : p^{-1}(N_\alpha) \to N_\alpha$, we have

$$T_m M = T_m(i_\alpha(N_\alpha)) \oplus T_m F_m \tag{2.2}$$

and

$$\Pi_m - i_{\alpha *}(\pi|_{N_\alpha})_m \in T_m F_m \wedge T_m M. \tag{2.3}$$

On the other hand, the fact that $i_\alpha(N_\alpha) \subset M$ satisfies the conditions of Proposition 2.2 implies

(2.4)

$$\Pi|_{i_\alpha(N_\alpha)} = \pi_\alpha + \pi_{E_\alpha},$$

$$\text{for some } \pi_\alpha \in \Gamma(i_\alpha(N_\alpha), \textstyle\bigwedge^2 T(i_\alpha(N_\alpha))),\ \pi_{E_\alpha} \in \Gamma(i_\alpha(N_\alpha), \textstyle\bigwedge^2 E_\alpha).$$

Putting together (2.2)–(2.4) and $T_m F_m \subseteq E_\alpha(m)$ gives $\pi_\alpha = i_{\alpha *}(\pi|_{N_\alpha})$, which is exactly condition (ii) in Definition 2.4 (taking into account Proposition 2.2). □

2.6. If $p : (M, \Pi) \to (N, \pi)$ is a Poisson map, then the pull back $p^* : (C^\infty(N), \{.,.\}_\pi) \to (C^\infty(M), \{.,.\}_\Pi)$ is a homomorphism of Poisson algebras and turns $C^\infty(M)$ into a module for the Poisson algebra $(C^\infty(N), \{.,.\}_\pi)$. The following Proposition provides an algebraic characterization of an important special case of weak splittings of surjective Poisson submersions. Its proof is simple and will be left to the reader.

Proposition. *Assume that* $p : (M, \Pi) \to (N, \pi)$ *is a surjective Poisson submersion and* $i : N \to M$ *a smooth lifting of* p*. Denote by* Tp *the subbundle of* TM *whose fibers are the tangent spaces to the fibers of* p*. Then the trivial partition of* N *with one stratum and the map* $i : N \to M$ *provide a weak splitting of* p *such that* $\Pi|_{i(N)} \in \bigwedge^2 Ti(N) \oplus \bigwedge^2 Tp|_{i(N)}$ *if and only if* $i^* : (C^\infty(M), \{.,.\}_\Pi) \to (C^\infty(N), \{.,.\}_\pi)$ *is a morphism of* $(C^\infty(N), \{.,.\}_\pi)$ *modules.*

2.7. All weak splittings of surjective Poisson submersions considered in this paper will be in the category of (complex) quasiprojective Poisson varieties. This means that in the setting of Definition 2.1, we require X to be a (smooth) locally closed subvariety of the smooth quasiprojective Poisson variety M. In Proposition 2.2, we require E to be an algebraic subbundle of $T_X M$. Finally, in the algebraic setting, in Definition 2.3 we require (2.1) to be an algebrogeometric stratification of M (in the sense of [3, §0.8]) and the maps i_α to be algebraic.

3. Weak splittings of surjective Poisson submersions for flag varieties

3.1. Since the Poisson structure π_G vanishes on the maximal torus H of G, the left and right regular actions of H on G preserve it, and thus π_G descends to a Poisson structure on G/H. One can see this in another way: H is a Poisson algebraic subgroup of (G, π_G); thus π_G descends to a Poisson structure on the homogeneous space G/H because of [3, Theorem 1.8]. Denote the standard projection by

$$\tau : G \to G/H, \quad \text{and set} \quad \pi_{G/H} = \tau_*(\pi_G). \tag{3.1}$$

It is clear that the projections

$$\nu_J : (G/H, \pi_{G/H}) \to (G/P_J^+, \pi_J), \quad \nu_J(gH) = gP_J^+ \tag{3.2}$$

are surjective Poisson submersions. For brevity, set $\nu = \nu_\emptyset$. Finally, for $w \in W$, set

$$y_w = wH \in G/H. \tag{3.3}$$

The following Theorem contains the main result in this section. Based on it, a second proof of Theorem 1.5 is given in §3.10.

3.2. Theorem. *Assume that G is an arbitrary complex reductive algebraic group and J is a subset of the set of positive simple roots Γ. The partition into Schubert cells*

$$G/P_J^+ = \bigsqcup_{w \in W^J_{\max}} B^- x_w$$

and the morphisms

$$i_w^J : B^- x_w^J \to G/H, \quad \textit{given by} \quad i_w^J(u^- x_w^J) \to u^- y_w, \ \textit{for}\ u^- \in U_w^-, \tag{3.4}$$

provide a weak splitting, in the sense of Definition 2.3, of the surjective Poisson submersion $\nu_J : (G/H, \pi_{G/H}) \to (G/P_J^+, \pi_J)$ (recall (0.5), (0.7), and (3.1)–(3.3)).

Theorem 3.2 has the following important Corollary.

3.3. Corollary. *Each symplectic leaf of $(G/P_J^+, \pi_J)$ is a symplectic submanifold of a symplectic leaf of $(G/H, \pi_{G/H})$. More precisely, the symplectic submanifold is precisely the image of the symplectic leaf under the corresponding morphism i_w^J.*

The proof of Theorem 3.2 will be split into several lemmas.

3.4. We will use the identification of vector spaces

$$T_e G \oplus T_e^* G = \mathfrak{g} \oplus \mathfrak{g}^* \cong D(\mathfrak{g}) \cong \mathfrak{g} \oplus \mathfrak{g} = T_e G \oplus T_e G \tag{3.5}$$

coming from the embeddings of $\mathfrak{g}$ and $\mathfrak{g}^*$ in the double $D(\mathfrak{g}) \cong \mathfrak{g} \oplus \mathfrak{g}$ of the Lie bialgebra $\mathfrak{g}$; cf. §1.2. This induces the identifications

$$T_g G \oplus T_g^* G \cong R_g(\mathfrak{g} \oplus \mathfrak{g}^*) \cong R_g(\mathfrak{g} \oplus \mathfrak{g}) = T_g G \oplus T_g G, \quad \text{for } g \in G. \tag{3.6}$$

Here and below, L_g and R_g refer to the left and right translations $a \mapsto ga$ and $a \mapsto ag$, for $a \in G$. For simplicity of the notation, we denote in the same way the induced tangent maps $T_a \to T_{ga}$ and $T_a \to T_{ag}$. In the identifications (3.6), the tangent and cotangent spaces at g correspond respectively to

$$\begin{aligned} T_g G &\cong R_g(\mathfrak{g}_{\mathrm{diag}}), \\ T_g^* G &\cong R_g(\mathrm{Lie}\, F) = R_g\{(h+n^+, -h+n^-) \mid h \in \mathfrak{h},\ n^\pm \in \mathfrak{n}^\pm\}; \end{aligned} \tag{3.7}$$

recall the notation in §1.2. The pairing between them is given by (1.5).

The main reason for using the identifications (3.6) is that the graph of $\pi^\sharp_{G,g} : T_g^* G \to T_g G$ (cf. (0.8)) under those identifications corresponds to the Drinfeld Lagrangian subalgebra [7] $\mathfrak{l}_g$ of $D(\mathfrak{g})$ for the base point g of (G, π_G), considered as a homogeneous space over itself. Since π_G vanishes at e, one has $\mathfrak{l}_e = \mathrm{Lie}\, F$. Moreover, because the map $g \mapsto \mathfrak{l}_g \subset D(\mathfrak{g})$ is G-equivariant with respect to the adjoint action of G on $D(\mathfrak{g}) \cong \mathfrak{g} \oplus \mathfrak{g}$ (see [7]), one has

$$\mathfrak{l}_g = \mathrm{Ad}_g(\mathrm{Lie}\, F) = \mathrm{Ad}_g\left(\{(h+n^+, -h+n^-) \mid h \in \mathfrak{h},\ n^\pm \in \mathfrak{n}^\pm\}\right).$$

Taking into account $R_g \circ \mathrm{Ad}_g = L_g$, this leads us to the following result.

3.5. Lemma. *In the identifications* (3.6)*, the graph of* $\pi^\sharp_{G,g} : T_g^* G \to T_g G$ *(cf.* (0.8)*), corresponds to the subspace*

$$R_g(\mathrm{Ad}_g F) = L_g(\mathrm{Lie}\, F) = L_g\left(\{(h+n^+, -h+n^-) \mid h \in \mathfrak{h},\ n^\pm \in \mathfrak{n}^\pm\}\right) \subset T_g G \oplus T_g G.$$

We will further need the following well known result.

3.6. Lemma. *If* V_1 *and* V_2 *are subspaces of a finite dimensional vector space* V *and* $\pi \in \bigwedge^2 V$*, then*

$$\pi^\sharp(V_1^0) \subseteq V_2 \quad \Longleftrightarrow \quad \pi \in \bigwedge^2 V_1 + \bigwedge^2 V_2.$$

Sketch of the proof. The standard nondegenerate pairing between $V \oplus V$ and $V^* \oplus V^*$ restricts to a nondegenerate pairing between $\bigwedge^2 V$ and $\bigwedge^2 V^*$. Then $\pi^\sharp(V_1^0) \subseteq V_2$ if and only if $\pi \in (V_1^0 \wedge V_2)^0$. One easily checks that for all subspaces V_1 and V_2 of V, one has $(V_1^0 \wedge V_2)^0 = \bigwedge^2 V_1 + \bigwedge^2 V_2$. □

3.7. For all $w \in W$, define the following algebraic subbundles $\widetilde{E}_w$ and E_w of $T_{HU_w^- w} G$ and $T_{U_w^- y_w} G/H$, respectively:

$$\begin{aligned} \widetilde{E}_w(bw) &= L_{bw}(\mathfrak{b}^+) + R_{bw}(\mathfrak{n}^+ \cap \mathrm{Ad}_w(\mathfrak{n}^-)) \subset T_{bw}(G), \quad \text{for } b \in HU_w^-, \\ E_w &= \tau_*\left(\widetilde{E}_w\right) \subset T_{U_w^- y_w} G/H; \end{aligned} \tag{3.8}$$

recall (3.4). It is easy to see that the push-forward in (3.8) does not depend on the choice of preimage. For a subvariety $N \subset G$ such that $N = \tau^{-1}\tau(N)$, denote by $T(\tau|_N)$ the bundle over N whose fibers are the tangent spaces to the fibers of $\tau|_N$:

$$T(\tau|_N)(g) = L_g(\mathfrak{h}) \subset T_g G, \quad g \in N. \tag{3.9}$$

The fact that the E_w are algebraic bundles follows from $\widetilde{E}_w \supset T(\tau|_{HU_w^- w})$: for all $w \in W$ and $b \in HU_w^-$, one has $\widetilde{E}_w(bw) \supset L_{bw}(\mathfrak{b}^+) \supset L_{bw}(\mathfrak{h}) = T(\tau|_{HU_w^- w})(bw)$.

Proposition. *For every $w \in W$, the following hold:*

(1) $T(HU_w^- w) + \widetilde{E}_w = T_{HU_w^- w} G$,
$T(HU_w^- w) \cap \widetilde{E}_w = T(\tau|_{HU_w^- w})$,
$\pi_G|_{HU_w^- w} \in \bigwedge^2 T(HU_w^- w) + \bigwedge^2 \widetilde{E}_w$, *and*

(2) $T(U_w^- y_w) \oplus E_w = T_{U_w^- y_w} G$,
$\pi_{G/H}|_{U_w^- y_w} \in \bigwedge^2 T(U_w^- y_w) \oplus \bigwedge^2 E_w$;

recall (0.4), (3.3), (3.4), (3.8), *and* (3.9).

Proof. Part (2) follows from part (1). The following inclusions imply the first statement in part (1):

$$\begin{aligned} T_{bw}(HU_w^- w) + \widetilde{E}_w(uw) &\supset R_{bw}\big(\mathfrak{b}^- \cap \mathrm{Ad}_w(\mathfrak{b}^-) + \mathfrak{n}^+ \cap \mathrm{Ad}_w(\mathfrak{n}^-)\big) + L_{bw}(\mathfrak{b}^+) \\ &= R_{bw}(\mathrm{Ad}_w(\mathfrak{b}^-)) + L_{uw}(\mathfrak{b}^+) = R_{bw}(\mathrm{Ad}_{bw}(\mathfrak{b}^-)) + L_{bw}(\mathfrak{b}^+) \\ &= L_{bw}(\mathfrak{b}^- + \mathfrak{b}^+) = T_{bw}(G), \end{aligned}$$

where $w \in W$ and $b \in U_w^-$. We used the fact that $b \in HU_w^- \subset \mathrm{Ad}_w(B^-)$ and thus $\mathrm{Ad}_{bw}(\mathfrak{b}^-) = \mathrm{Ad}_w(\mathfrak{b}^-)$.

It is clear that

$$T(HU_w^- w) \cap \widetilde{E}_w \supset T(\tau|_{HU_w^- w}) \tag{3.10}$$

because $HU_w^- w = U_w^- wH$. Next we show the opposite inclusion. Fix $b \in HU_w^-$. If $x \in \mathfrak{g}$ and $R_{bw}(x) \in T_{bw}(HU_w^- w) \cap \widetilde{E}_w$, then $x = x_1 = \mathrm{Ad}_{bw}(y) + x_2$ for some $y \in \mathfrak{b}^+$ and $x_1 \in \mathfrak{b}^- \cap \mathrm{Ad}_w(\mathfrak{b}^-)$, and $x_2 \in \mathfrak{n}^+ \cap \mathrm{Ad}_w(\mathfrak{n}^-)$. So

$$\mathrm{Ad}_{bw}(y) = x_1 - x_2 \in \mathfrak{b}^- \cap \mathrm{Ad}_w(\mathfrak{b}^-) + \mathfrak{n}^+ \cap \mathrm{Ad}_w(\mathfrak{n}^-) = \mathrm{Ad}_w(\mathfrak{b}^-) = \mathrm{Ad}_{bw}(\mathfrak{b}^-).$$

Taking into account $\mathrm{Ad}_{bw}(y) \in \mathrm{Ad}_{bw}(\mathfrak{b}^+)$, one obtains $y \in h$ and $x_2 = 0$, and consequently $x \in \mathfrak{h}$. This proves the opposite inclusion to (3.10) and completes the proof of the second statement in (1).

In the remainder of this proof we show the third property in part (1). Under the identifications (3.6), the tangent space $T_{bw}(HU_w^- w)$ corresponds to

$$R_{bw}\left((\mathfrak{b}^- \cap \mathrm{Ad}_w(\mathfrak{b}^-))_{\mathrm{diag}}\right) \quad \text{for all } b \in HU_w^-.$$

Recall that the images of the tangent and the cotangent spaces to G in (3.6) are given by (3.7) and the pairing between them is given by (1.5); see §3.4. By a direct computation, one checks that under the identifications (3.6), $(T_{bw}(HU_w^- w))^0$ corresponds to $R_{bw}((\mathfrak{n}^+ \cap \mathrm{Ad}_w(\mathfrak{n}^-)) \oplus \mathfrak{n}^-)$; see (0.8)–(0.9). (Here and below, for two subalgebras $\mathfrak{f}_1, \mathfrak{f}_2 \subseteq \mathfrak{g}$, by $\mathfrak{f}_1 \oplus \mathfrak{f}_2$ we denote the canonical direct sum subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$ and not the possible direct sum inside $\mathfrak{g}$.) Recall that the graph of $\pi_G^\sharp : T_{bw}^* G \to T_{bw} G$ is given by Lemma 3.5. If $R_{bw}(x) \in \pi_G^\sharp\left((T_{bw}(HU_w^- w))^0\right)$, then there exist $n^+ \in \mathfrak{n}^+ \cap \mathrm{Ad}_w(\mathfrak{n}^-)$ and $n^- \in \mathfrak{n}^-$ such that

$$(x + n^+, x + n^-) \in \mathrm{Ad}_{bw}(\mathrm{Lie}\, F) \subset \mathrm{Ad}_{bw}(\mathfrak{b}^+ \oplus \mathfrak{b}^-).$$

Comparing the first components gives $x \in \mathrm{Ad}_{bw}(\mathfrak{b}^+) + \mathfrak{n}^+ \cap \mathrm{Ad}_w(\mathfrak{n}^-)$, and consequently

$$\begin{aligned} \pi_G^\sharp\left((T_{bw}(HU_w^- w))^0\right) &\subset R_{uw}\left(\mathrm{Ad}_{bw}(\mathfrak{b}^+) + \mathfrak{n}^+ \cap \mathrm{Ad}_w(\mathfrak{n}^-)\right) \\ &= L_{bw}(\mathfrak{b}^+) + R_{bw}(\mathfrak{n}^+ \cap \mathrm{Ad}_w(\mathfrak{n}^-)) = \widetilde{E}_w(bw). \end{aligned}$$

Now the third statement in part (1) follows from Lemma 3.6. □

3.8. Proof of Theorem 3.2. First we prove the Theorem in the case $J = \emptyset$. For all $w \in W$, the tangent spaces to the fibers of $\nu_\emptyset$ inside $U_w^- y_w$ are $\eta_*(L_{bw}(\mathfrak{b}^+))$ for $b \in HU_w^-$, and they are contained in E_w. Proposition 2.5 and part (2) of Proposition 3.7 imply that

$$\pi_{G/H}|_{U_w^- y_w} - i^\emptyset_{w*}(\pi|_{B^- x_w}) \in \bigwedge^2 E_w, \quad \text{for all } w \in W, \tag{3.11}$$

which establishes the Theorem in the case $J = \emptyset$.

For the general case, in addition to Proposition 3.7 (2), we need to prove that

$$\pi_{G/H}|_{U_w^- y_w} - i^J_{w*}(\pi_J|_{B^- x_w^J}) \in \bigwedge^2 E_w, \quad \text{for all } J \subset \Gamma,\ w \in W^J. \tag{3.12}$$

Because $i^J_w = i^\emptyset_w \circ (\mu_J|_{B^- x_w})^{-1}$ and

$$\mu_J|_{B^- x_w} : (B^- x_w, \pi|_{B^- x_w}) \to (B^- x^J_w, \pi_J|_{B^- x^J_w})$$

is a Poisson isomorphism for all $w \in W^J$ (recall (1.12) and Proposition 1.6), we get that

$$i^J_{w*}(\pi_J|_{B^- x^J_w}) = i^\emptyset_{w*}(\pi|_{B^- x_w}), \quad \text{for all } w \in W^J. \tag{3.13}$$

Equations (3.11) and (3.13) imply (3.12), and this completes the proof of the Theorem. □

3.9. Lemma. *The H-orbits of symplectic leaves of $(G/H, \pi_{G/H})$ are exactly the projections $\tau(B^- w_1 B^- \cap B^+ w_2 B^+)$ of the double Bruhat cells of G onto G/H, for $w_1, w_2 \in W$.*

Sketch of the proof. The proof of the Lemma is analogous to the well known fact that the H-orbits of symplectic leaves of (G, π_G) are the double Bruhat cells of G. The Drinfeld Lagrangian subalgebra of the base point eH of G/H is $\mathrm{Lie}(H_{\mathrm{diag}}(U^+ \times U^-))$; see [3, Theorem 1.8]. The H-orbits of symplectic leaves of $(G/H, \pi_{G/H})$ are the inverse images of the $(B^+ \times B^-)$-orbits on $(G \times G)/H_{\mathrm{diag}}(U^+ \times U^-)$ under the map

$$G/H \to (G \times G)/H_{\mathrm{diag}}(U^+ \times U^-), \quad gH \mapsto (g, g)H_{\mathrm{diag}}(U^+ \times U^-). \tag{3.14}$$

By the Bruhat Lemma, the $(B^+ \times B^-)$-orbits on $(G \times G)/H_{\mathrm{diag}}(U^+ \times U^-)$ are parametrized by $W \times W$ via

$$W \times W \ni (w_1, w_2) \longmapsto (B^+ \times B^-)(w_2, w_1)H_{\mathrm{diag}}(U^+ \times U^-).$$

Finally, the inverse images of the above orbits under (3.14) are exactly the projections $\tau(B^- w_1 B^- \cap B^+ w_2 B^+)$. □

3.10. A second proof of Theorem 1.8. For $w \in W^J$ and $w_1, w_2 \in W$, the intersection of $\tau(B^- w_1 B^- \cap B^+ w_2 B^+)$ with $i^J_w(B^- x^J_w) = U^-_w y_w$ is nonempty only if $w_1 = w$ and $w_2 \geq w$, because it lies inside $\tau(B^- w \cap B^- w_1 B^-)$ (thus $w_1 = w$) and consequently inside $B^- w B^+ \cap B^+ w_2 B^+$ (thus $w_2 \geq w$ by [6, Corollary 1.2]). If $w_1 = w$ and $w_2 \geq w$, then the intersection of $\tau(B^- w_1 B^- \cap B^+ w_2 B^+)$ with the image of i^J_w is $\tau(U_{\dot{w}, w_2})$ (cf. (0.6)) and is a nonempty irreducible subvariety of G/H by [3, Theorem 2.3]. Theorem 3.2, Lemma 3.9, and the argument of the proof of [3, Theorem 1.8] imply that for each $w_2 \in W$ with $w_2 \geq w$, the set $\tau(\mathcal{U}_{\dot{w}, w_2})$ is a (single) H-orbit of symplectic leaves of $(i^J_w(B^- x^J_w), i^J_{w*}(\pi_J|_{B^- x^J_w}))$. Thus, the

H-orbits of symplectic leaves of $(B^- x^J_w, \pi_J|_{B^- x^J_w})$ are

$$(i^J_w)^{-1} \circ \tau(\mathcal{U}_{\dot{w},w_2}) = \mathcal{U}_{\dot{w},w_2}.P^+_J = \mathcal{S}^J_{w,w_2}, \quad w_2 \in W,\ w_2 \geq w.$$

This completes the second proof of the Theorem. □

Let us note that although in the second proof of Theorem 1.8 we still used the theory of Poisson homogeneous spaces in Lemma 3.9, we avoided the combinatorics arguments from the first proof. The latter were replaced by the geometric construction of weak splittings of Poisson submersions. In addition, in a subsequent publication we will demonstrate that those geometric arguments can be extended to the quantum situation, while we are not aware of any quantum version of the dressing orbit method that provides a classification of H-invariant prime ideals of some class of associative algebras.

4. Hermitian symmetric spaces of compact type

4.1. In this section we investigate the Poisson structure π_J on G/P^+_J for the case when the unipotent radical U^+_J of P^+_J is abelian and G is a simple algebraic group. The flag varieties G/P^+_J of this type exhaust all irreducible Hermitian symmetric spaces of compact type, [24]. We show that in this case all L_J-orbits on G/P^+_J are complete Poisson subvarieties with respect to the Poisson structure π_J. We then use the results of Richardson, Röhrle, and Steinberg [19] on special representatives for the L_J-orbits on G/P^+_J. We prove that the Poisson structure π_J vanishes at all such special base points, and as a result of this $(G/P^+_J, \pi_J)$ stratifies into complete Poisson subvarieties, each of which is a quotient of L_J, equipped with the standard Poisson structure.

First, set

$$r_J = \sum_{\alpha \in \Delta^+_J} \frac{\langle \alpha, \alpha \rangle}{2} e_\alpha \wedge f_\alpha \quad \text{and} \quad \pi_{L_J} = L(r_J) - R(r_J); \tag{4.1}$$

cf. (0.1) and (0.2). These are respectively the standard r-matrix and the standard Poisson structure on the reductive group L_J. It is well known that (L_J, π_J) is a Poisson algebraic subgroup of (G, π_G). Also set

$$\check{r}_J = \sum_{\alpha \in \Delta^+ \setminus \Delta^+_J} \frac{\langle \alpha, \alpha \rangle}{2} e_\alpha \wedge f_\alpha. \tag{4.2}$$

Observe that $r_{\mathfrak{g}} = r_J + \check{r}_J$.

4.2. Proposition. *Assume that $J \subset \Gamma$ is such that U^+_J is abelian. Then the following properties hold:*

(i) *The Poisson structure π_J on G/P^+_J is given by*

$$\pi_J = -\chi(r_J). \tag{4.3}$$

In particular, all L_J-orbits on the flag variety G/P^+_J are complete Poisson subvarieties of $(G/P^+_J, \pi_J)$.

(ii) *Under the identification $\Psi_J : \mathfrak{n}^-_J \xrightarrow{\cong} B^-.P^+_J \subset G/P^+_J$, $\Psi_J(x) = \exp(x)P^+_J$, of L_J-spaces (where L_J acts on the first term by the adjoint action), the restriction of the Poisson structure π_J to $B^-.P^+_J$ corresponds to $-\chi(r_J) \in \Gamma(\mathfrak{n}^-_J, \bigwedge^2 T\mathfrak{n}^-_J)$. Here $\chi : \bigwedge^2 \mathfrak{l}_J \to \Gamma(\mathfrak{n}^-_J, \bigwedge^2 T\mathfrak{n}^-_J)$ is derived from the adjoint action of L_J on $\mathfrak{n}^-_J$.*

Proof. (i) For all $u = \exp(x) \in U_J^-$, $x \in \mathfrak{n}_J^-$,

$$-R_u(\check{r}_J) = -L_u(\mathrm{Ad}_u^{-1}(\check{r}_J)) \equiv -L_u(\mathrm{ad}_x^2(\check{r}_J))/2 \mod L_u(\mathfrak{p}_J^+ \wedge \mathfrak{g})$$

because $[\mathfrak{n}_J^+, \mathfrak{n}_J^-] \subseteq \mathfrak{l}_J$. We have $\mathrm{ad}_x^2(\check{r}_J) = 0$ since for all $y, z \in \mathfrak{n}_J^+$,

$$\langle \mathrm{ad}_x^2(\check{r}_J), y \otimes z \rangle = \langle \mathrm{ad}_x^2(y), z \rangle - \langle y, \mathrm{ad}_x^2(z) \rangle = 0.$$

Therefore (4.3) holds on the open B^- orbit on G/P_J^+, and thus on the full G/P_J^+ since both sides of (4.3) are algebraic.

The second statement in (i) directly follows from (4.3). It is also a consequence of Theorem 1.1(b) in [19], stating that each L_J-orbit on G/P_J^+ is the intersection of a P_J^+-orbit and a P_J^--orbit. (The latter are complete Poisson subvarieties of $(G/P_J^+, \pi_J)$ as was shown in §1.6.)

Part (ii) follows from part (i) by noting that $B^-.P_J^+ = U_J^-.P_J^+$ is L_J stable and Ψ_J intertwines the actions of L_J on $\mathfrak{n}_J^-$ and $B^-.P_J^+$. □

4.3. The L_J-orbits on G/P_J^+ were classified by Richardson, Röhrle, and Steinberg [19], and previously the L_J orbits on the unipotent radical $U_J^- \cong B^-.P_J^+ \subset G/P_J^+$ were treated by Wolf [25] and Muller, Rubenthaller, and Schiffmann [15]. We recall the parametrization of the L_J-orbits on G/P_J^+ from [19, Theorem 1.2]. Fix a maximal sequence $(\beta_1, \ldots, \beta_k)$ of long roots in $\Delta^+ \backslash \Delta_J^+$ which are mutually orthogonal. Denote by $u_{-\beta_i}$ a nontrivial element in the one-parameter unipotent subgroup of G corresponding to the root $-\beta_i$. Denote by $w_{\beta_i} \in W$ the reflection corresponding to β_i.

Note that $\{w_{\beta_i}\}_{i=1}^k$ mutually commute because $\{\beta_i\}_{i=1}^k$ are mutually orthogonal. (The elements $\{u_{-\beta_i}\}_{i=1}^k$ also mutually commute because U_J^- is abelian.) Let $\dot{w}_{\beta_i}$ be a representative of w_{β_i} in the normalizer of the maximal torus H of G. Finally, set

$$x_{st} = \prod_{i=1}^{t} \dot{w}_{\beta_i} \prod_{j=t+1}^{s} u_{-\beta_j} \quad \text{for} \quad 0 \le t \le s \le k.$$

Theorem (Richardson–Röhrle–Steinberg [19])**.** *If the parabolic subgroup P_J^+ of G has abelian unipotent radical, then $\{x_{st}P_J^+ \mid 0 \le t \le s \le k\}$ is a system of representatives for the L_J-orbits on G/P_J^+.*

4.4. Let us recall that P_J^+ has an abelian unipotent radical only if it is a maximal parabolic subgroup of G. In addition, if $J = \Gamma \backslash \{\alpha'\}$, then P_J^+ has abelian unipotent radical if and only the α' height of the longest root θ of $\mathfrak{g}$ is equal to 1. (We fix this root α' for the remainder of this section.) If this condition is satisfied, then the α'-height of any root $\gamma \in \Delta$ is equal to $0, \pm 1$. Moreover $n_{\alpha'}(\gamma) = 1$, 0, or -1 if γ is a root of $\mathfrak{n}_J^+$, $\mathfrak{l}_J$, or $\mathfrak{n}_J^-$, respectively (i.e., $\gamma \in \Delta^+ \backslash \Delta_J^+$, Δ_J, or $-(\Delta^+ \backslash \Delta_J^+)$).

4.5. We will work with a special maximal set $(\beta_1, \ldots, \beta_k)$ of mutually orthogonal long roots in $\Delta^+ \backslash \Delta_J^+$. We proceed analogously to the proof of [19, Proposition 2.8], defining inductively β_i and subsets Γ_i of the set Γ of positive simple roots of $\mathfrak{g}$. Let $\beta_1 = \theta$ be the highest root of $\mathfrak{g}$ and $\Gamma_1 = \Gamma$. Assume that for some $i \le k$, we have already defined β_i and Γ_i. Let $\widetilde{\Gamma}_{i+1}$ be the set of all roots in Γ_i that are orthogonal to β_i. (Since β_i is dominant in the root system defined by Γ_i, all roots in it that are orthogonal to β_i are combinations of simple roots in $\widetilde{\Gamma}_{i+1}$.) If $k > 1$, then $\widetilde{\Gamma}_{i+1}$ contains α'. Denote by Γ_{i+1} the connected component of $\widetilde{\Gamma}_{i+1}$ containing α'. (Here we identify Γ with the Dynkin graph of $\mathfrak{g}$ and view $\widetilde{\Gamma}_{i+1}$ as a subgraph of it.) Finally, set β_{i+1} to be the highest root of the root system defined by Γ_{i+1}.

This sequence has the properties that

$$\operatorname{supp}\beta_j \subseteq \operatorname{supp}\beta_i = \Gamma_i \quad \text{for} \quad j \geq i \tag{4.4}$$

(cf. §0.7) and that β_i is the longest root in the root system defined by Γ_i.

For simplicity of the exposition, we set

$$u_{-\beta_i} = \exp(f_{\beta_i}), \tag{4.5}$$

but we note that all proofs work for general $u_{-\beta_i}$.

We will also use special representatives for the reflections $w_{\beta_i} \in W$ in the normalizer of the maximal torus H of G (and will thus omit the dot on top of them). Set

$$w_{\beta_i} = \exp(e_{\beta_i})\exp(-f_{\beta_i})\exp(e_{\beta_i}). \tag{4.6}$$

This normalization is not necessary for the proof of the main result in Theorem 4.6, but it simplifies the exposition. Since β_i and β_j are orthogonal for $i \neq j$ and $\beta_i + \beta_j$ is not a root of $\mathfrak{g}$, we have that $w_{\beta_i}(\beta_j) = 0$ and $[f_{\beta_i}, f_{\beta_j}] = 0$, and consequently

$$[e_{\beta_i}, f_{\beta_j}] = 0, \qquad \mathrm{Ad}_{w_{\beta_i}}(f_{\beta_j}) = f_{\beta_j},$$
$$\mathrm{Ad}_{w_{\beta_i}}(u_{-\beta_j}) = u_{-\beta_j}, \qquad \mathrm{Ad}_{w_{\beta_i}}(e_{\beta_j}) = e_{\beta_j}$$

for the special representatives (4.6) of w_{β_i}.

The first main result in this section is contained in the following Theorem.

4.6. Theorem. *If P_J^+ is a parabolic subgroup with abelian unipotent radical in the complex simple algebraic group G, then the Poisson structure π_J vanishes at the base points $x_{st}P_J^+$ of the L_J-orbits on G/P_J^+ $(0 \leq t \leq s \leq k)$. Therefore, all L_J-orbits on G/P_J^+ are complete Poisson subvarieties of $(G/P_J^+, \pi_J)$ and are quotients of (L_J, π_{L_J}); cf.* [3, Theorem 1.8].

Even in the case of Grassmannians, Theorem 4.6 contains new information, compared to Part I [3], where we dealt with the open B^--orbits on Grassmannians.

Let us note that Theorem 4.6 is not valid for an arbitrary maximal set $(\beta_1, \dots, \beta_k)$ of mutually orthogonal long roots of $\Delta^+ \backslash \Delta_J^+$. For instance, in the case of A_4 and $\alpha' = \epsilon_3 - \epsilon_2$, take $\beta_1 = \epsilon_3 - \epsilon_1$ and $\beta_2 = \epsilon_4 - \epsilon_2$. Then π_J does not vanish at $u_{-\beta_1}u_{-\beta_2}P_J^+$. The easiest way to see this is to note that the Poisson structure $\pi_{2,2}$ on the matrix affine Poisson space $M_{2,2}$ (cf. [3, eq. (1.7)]) does not vanish at $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and then to use the embedding from [3, Proposition 3.4].

For the proof of Theorem 4.6, we will need several lemmas.

4.7. Lemma. *Assume that β is a long root in $\Delta^+ \backslash \Delta_J^+$ and set*

$$w_\beta = \exp(e_\beta)\exp(-f_\beta)\exp(e_\beta).$$

Let α be any root of $\mathfrak{g}$ and $y_\alpha \in \mathfrak{g}$ a nonzero vector in the corresponding root space. If $[f_\beta, y_\alpha] \neq 0$, then $w_\beta(\alpha) = \alpha - \beta$ and

$$[f_\beta, y_\alpha] = -\mathrm{Ad}_{w_\beta}(y_\alpha).$$

Proof. First, there is no root γ of $\mathfrak{g}$ such that $\gamma - \beta$ and $\gamma - 2\beta$ are roots as well. For if this happens, then the α' height of γ needs to be equal to 1, i.e., $\gamma \in \Delta^+ \backslash \Delta_J^+$, because the α' heights of all roots of $\mathfrak{g}$ are 0 or ± 1. Consequently,

$\gamma+\beta$ would not be a root and $\langle\gamma,\beta^\vee\rangle\geq 2$. This would be a contradiction, since $|\langle\gamma,\beta^\vee\rangle|\leq 1$ because β is a long root. From this we get that the only roots of $\mathfrak{g}$ of the form $\alpha+i\beta$ for $i\in\mathbb{Z}$ are α and $\alpha-\beta$; thus $w_\beta(\alpha)=\alpha-\beta$, $\mathrm{ad}_{e_\beta}(x_\alpha)=0$, and $\mathrm{ad}^2_{f_\beta}(x_\alpha)=0$. Consequently, $\mathrm{Span}\{y_\alpha,[f_\beta,y_\alpha]\}$, under the adjoint action, is isomorphic to the vector representation of the $\mathfrak{sl}_2$ triple $\{e_\beta,\beta^\vee=[e_\beta,f_\beta],f_\beta\}$. By a standard computation, in this basis w_β acts by $\left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right]$, so $\mathrm{Ad}_{w_\beta}(y_\alpha)=-[f_\beta,y_\alpha]$. □

4.8. Lemma. *If $\alpha\in\Delta_J^+$ is such that*

$$[f_{\beta_i},e_\alpha]\neq 0\quad\textit{and}\quad[f_{\beta_j},f_\alpha]\neq 0$$

for some $i\neq j\leq k$, then $\beta_i-\alpha-\beta_j\in\Delta_J^+$ and

$$[f_{\beta_i},e_\alpha]\wedge[f_{\beta_j},f_\alpha]+[f_{\beta_i},e_{\beta_i-\alpha-\beta_j}]\wedge[f_{\beta_j},f_{\beta_i-\alpha-\beta_j}]=0. \tag{4.7}$$

Proof. It follows from Lemma 4.7 that

$$w_{\beta_i}(\alpha)=\alpha-\beta_i\quad\text{and}\quad w_{\beta_j}(-\alpha)=-\alpha-\beta_j.$$

Since β_i and β_j are orthogonal, $w_{\beta_j}(\beta_i)=0$ and

$$w_{\beta_j}w_{\beta_i}(\alpha)=w_{\beta_j}(\alpha-\beta_i)=\alpha+\beta_j-\beta_i.$$

Therefore $\beta_i-\alpha-\beta_j$ is a root of $\mathfrak{g}$. Because $\alpha-\beta_i$ is a root of α' height equal to -1, it belongs to $-(\Delta^+\backslash\Delta_J^+)$. As a consequence of this, $\operatorname{supp}\alpha\subset\operatorname{supp}\beta_i=\Gamma_i$; recall §4.5. If $i<j$, then $\operatorname{supp}\alpha\subset\Gamma_i\subset\Gamma_j$, and the fact that $\alpha+\beta_j$ is a root would contradict the property that β_j is the highest root of the root system of Γ_j; cf. §4.5. Thus, $i>j$ and $\alpha+\beta_j$ is a positive root in the root system of Γ_i. Since β_i is the highest root in this root system, $\beta_i-\alpha-\beta_j$ needs to be a positive root.

Because $\mathrm{Ad}_{w_{\beta_j}w_{\beta_i}}$ preserves the bilinear form $\langle .,.\rangle$ on $\mathfrak{g}$,

$$\mathrm{Ad}_{w_{\beta_j}w_{\beta_i}}(e_\alpha)=af_{\beta_i-\alpha-\beta_j}\quad\text{and}\quad\mathrm{Ad}_{w_{\beta_j}w_{\beta_i}}(f_\alpha)=a^{-1}e_{\beta_i-\alpha-\beta_j} \tag{4.8}$$

for some $a\in\mathbb{C}^\times$. Equation (4.6) implies that $\mathrm{Ad}_{w_{\beta_i}}(f_{\beta_i})=-e_{\beta_i}$. Combining the above facts and using the fact that $\mathrm{Ad}_{w_{\beta_j}}(e_{\beta_i})=e_{\beta_i}$ (cf. §4.5) leads to

$$\mathrm{Ad}_{w_{\beta_j}w_{\beta_i}}([f_{\beta_i},e_\alpha])=-a[e_{\beta_i},f_{\beta_i-\alpha-\beta_j}].$$

Thus, $[e_{\beta_i},f_{\beta_i-\alpha-\beta_j}]\neq 0$, and consequently $[f_{\beta_i},e_{\beta_i-\alpha-\beta_j}]\neq 0$. Applying Lemma 4.7 and (4.8) twice, we get that

$$\begin{aligned}[f_{\beta_i},e_{\beta_i-\alpha-\beta_j}]&=-\mathrm{Ad}_{w_{\beta_i}}(e_{\beta_i-\alpha-\beta_j})=-a\,\mathrm{Ad}_{w^2_{\beta_i}w_{\beta_j}}(f_\alpha)\\&=a\,\mathrm{Ad}_{w_{\beta_j}}(f_\alpha)=-a[f_{\beta_j},f_\alpha].\end{aligned}$$

In the third equality, we used the fact that $\mathrm{Ad}_{w_{\beta_j}}(f_\alpha)$ is the lowest weight vector for the vector representation of the $\mathfrak{sl}_2$ triple $\{e_{\beta_i},\beta_i^\vee=[e_{\beta_i},f_{\beta_i}],f_{\beta_i}\}$ (under the adjoint action), and in this representation $w^2_{\beta_i}$ acts by $-\mathrm{id}$; recall (4.6). Analogously,

$$\begin{aligned}[f_{\beta_j},f_{\beta_i-\alpha-\beta_j}]&=-\mathrm{Ad}_{w_{\beta_j}}(f_{\beta_i-\alpha-\beta_j})=-a^{-1}\,\mathrm{Ad}_{w_{\beta_i}w^2_{\beta_j}}(e_\alpha)\\&=a^{-1}\,\mathrm{Ad}_{w_{\beta_i}}(e_\alpha)=-a^{-1}[f_{\beta_i},e_\alpha].\end{aligned}$$

Hence,

$$\begin{aligned}[f_{\beta_i},e_{\beta_i-\alpha-\beta_j}]\wedge[f_{\beta_j},f_{\beta_i-\alpha-\beta_j}]&=(-a[f_{\beta_j},f_\alpha])\wedge(-a^{-1}[f_{\beta_i},e_\alpha])\\&=-[f_{\beta_i},e_\alpha]\wedge[f_{\beta_j},f_\alpha].\end{aligned}$$

□

4.9. Lemma. *For all* $0 \le t \le s \le k$,

$$\mathrm{Ad}^{-1}_{u_{-\beta_{t+1}}\dots u_{-\beta_s}}(r_J) = r_J - \sum_{j=t+1}^{s} \sum_{\alpha\in\Delta_J^+} \frac{\langle\alpha,\alpha\rangle}{2}\left([f_{\beta_j}, e_\alpha]\wedge f_\alpha + e_\alpha \wedge [f_{\beta_j}, f_\alpha]\right).$$

Proof. For arbitrary $0 \le i \le k$ and $0 \le j \le k$ and $\alpha \in \Delta_J$, the sum $\alpha + \beta_i + \beta_j$ is not a root of $\mathfrak{g}$ since its α' height is -2. Taking into account that $u_{-\beta_j} = \exp(f_{\beta_j})$, we get

$$\begin{aligned}&\mathrm{Ad}^{-1}_{u_{-\beta_{t+1}}\dots u_{-\beta_s}}(r_J)\\ &\quad= \sum_{\alpha\in\Delta_J^+}\left(e_\alpha - [f_{\beta_{t+1}}, e_\alpha] - \ldots - [f_{\beta_s}, e_\alpha]\right)\wedge\left(f_\alpha - [f_{\beta_{t+1}}, f_\alpha] - \ldots - [f_{\beta_s}, f_\alpha]\right).\end{aligned}$$

The Lemma will follow if we show that for all $s \le i, j \le k$,

$$\sum_{\alpha\in\Delta_J^+} [f_{\beta_i}, e_\alpha]\wedge[f_{\beta_j}, f_\alpha] = 0.$$

This is a corollary of Lemma 4.8, which implies that all $\alpha \in \Delta_J^+$ for which $[f_{\beta_i}, e_\alpha] \neq 0$ and $[f_{\beta_j}, f_\alpha] \neq 0$ can be grouped in pairs such the sum of the corresponding expressions $[f_{\beta_i}, e_\alpha]\wedge[f_{\beta_j}, f_\alpha]$ will be equal to 0. □

In the setting of §4.3, for each $0 \le t \le k$ set

$$w_t = \prod_{i=1}^{t} w_i. \tag{4.9}$$

The same notation will be used for the representative of w_t in the normalizer of H in G which is the product of the representatives (4.6).

4.10. Proof of Theorem 4.6. We will prove that

$$\mathrm{Ad}^{-1}_{w_t}\left([f_{\beta_j}, e_\alpha])\wedge f_\alpha\right) \in \mathfrak{p}_J^+\wedge\mathfrak{g}, \quad \text{for } \alpha\in\Delta_J^+,\ 0\le t<j\le k. \tag{4.10}$$

Analogously, one shows that

$$\mathrm{Ad}^{-1}_{w_t}\left(e_\alpha\wedge[f_{\beta_j}, f_\alpha]\right) \in \mathfrak{p}_J^+\wedge\mathfrak{g}, \quad \text{for } \alpha\in\Delta_J^+,\ 0\le t<j\le k. \tag{4.11}$$

It is clear that

$$\mathrm{Ad}^{-1}_{w_t}(e_\alpha\wedge f_\alpha) \in \mathfrak{n}^-\wedge\mathfrak{n}^+, \quad \text{for } \alpha\in\Delta_J^+,\ 0\le t\le k. \tag{4.12}$$

Lemma 4.9, (4.10)–(4.12), and the commutativity of w_t and $u_{-\beta_{t+1}}\dots u_{-\beta_s}$ (cf. §4.5) imply that

$$\mathrm{Ad}_{x_{st}}(r_J) \in \mathfrak{p}_J^+\wedge\mathfrak{g}, \quad \text{for } 0\le t\le s\le k,$$

which is equivalent to the vanishing of π_J at $x_{st}P_J^+$.

Thus, we are left with showing (4.10). We will make use of the following fact [19, Lemma 2.10 (b)]:

(*) *For* $\gamma \in -(\Delta^+\backslash\Delta_J^+)$, *the set of all* β_i, $0 \le i \le t$, *not orthogonal to* γ *has cardinality* 0, 1, *or* 2, *and accordingly* $w_t(\gamma) \in -(\Delta^+\backslash\Delta_J^+)$, Δ_J, *or* $\Delta^+\backslash\Delta_J^+$.

If $w_t^{-1}(-\alpha) \notin -(\Delta^+\backslash\Delta_J^+)$, then $\mathrm{Ad}^{-1}_{w_t}(f_\alpha) \in \mathfrak{p}_J^+$, and we are done. If $w_t(\alpha)^{-1} \in -(\Delta^+\backslash\Delta_J^+)$, then applying (*) for $\gamma = w_t^{-1}(-\alpha)$ one gets that there exists $0 \le i \le t$ such that $w_t^{-1}(-\alpha) = w_{\beta_i}(-\alpha) = -\alpha - \beta_i$. This means that $\langle\beta_i^\vee, \alpha\rangle = -1$, and consequently $\langle \beta_i^\vee, \alpha - \beta_j \rangle = -1$ ($\langle \beta_i, \beta_j \rangle = 0$ since $j > t \geq i$). If $[f_{\beta_j}, e_\alpha] = 0$, then we are done. If $[f_{\beta_j}, e_\alpha] \neq 0$, then using (*) again, this time for $\gamma = \alpha - \beta_j$, we get $w_t^{-1}(\alpha - \beta_j) \notin -(\Delta^+ \backslash \Delta_J^+)$; therefore $\mathrm{Ad}_{w_t}^{-1}[f_{\beta_j}, e_\alpha]) \in \mathfrak{p}_J^+$. This establishes (4.10) and thus the Theorem. □

4.11. In the last part of this section (§§4.11–4.13), for a parabolic subgroup P_J^+ with abelian unipotent radical, we characterize the symplectic leaves of (G, π_J^+) within each L_J-orbit. First we recall some facts on minimal length representatives in Weyl groups. For two subsets I and J of the set of positive simple roots Γ of G denote by ${}^I W_{\mathrm{min}}$ and ${}^I W_{\mathrm{min}}^J$ the set of (unique) minimal length representatives of the cosets in $W_I \backslash W$ and $W_I \backslash W / W_J$. Recall the following standard facts; see e.g. [4, Lemma 4.3].

Lemma. *For arbitrary $I, J \subset \Gamma$ the following hold:*

(i) *Every element of W_{min}^J can be uniquely represented as a product vw for some $w \in {}^I W_{\mathrm{min}}^J$ and $v \in W_I \cap W_{\mathrm{min}}^{I \cap w(J)}$.*

(ii) *Every element of W can be uniquely represented as a product $v_1 w v_2$ for some $w \in {}^I W_{\mathrm{min}}^J$, $v_2 \in W_J$ and $v_1 \in W_I \cap W_{\mathrm{min}}^{I \cap w(J)}$.*

4.12. We will need the following results from [19]; recall (4.9).

Theorem (Richardson–Röhrle–Steinberg)**.** *If $P_J^+ \subset G$ has abelian unipotent radical, then the following hold:*

(i) *For any given P_J^+-orbit and any given P_J^--orbit on G/P_J^+, the two orbits are either disjoint or else intersect in a single L_J-orbit.*

(ii) *The set $\{w_s\}_{s=0}^k$ coincides with ${}^J W_{\mathrm{min}}^J$ and thus $G/P_J^+ = \bigsqcup_{0 \leq s \leq k} P_J^+ . w_s P_J^+ = \bigsqcup_{0 \leq s \leq k} P_J^- . w_s P_J^+$.*

(iii) *For all $0 \leq t \leq s \leq k$, we have*

$$L_J . x_{st} P_J^+ = P_J^- . w_t P_J^+ \cap P_J^+ . w_s P_J^+;$$

cf. (4.9).

Part (i) is [19, Theorem 1.1 (b)], part (ii) is [19, Proposition 2.11], and part (iii) is [19, Lemma 3.5 (d)]. We will illustrate the inclusion

$$L_J . x_{st} P_J^+ \subset P_J^- . w_t P_J^+ \cap P_J^+ . w_s P_J^+ .$$

Since $\dot{w}_{\beta_i}$ and $u_{-\beta_j}$ commute for $i \neq j$ (cf. §4.5),

$$L_J . x_{st} P_J^+ = L_J \prod_{j=t+1}^{s} u_{-\beta_j} . \prod_{i=1}^{t} \dot{w}_i P_J^+ \subset P_J^- w_t P_J^+ .$$

Because $[e_{\beta_i}, f_{\beta_j}] = 0$ for $i \neq j$ and $w_{\beta_i}^2 = 1$,

$$\begin{aligned} L_J . x_{st} P_J^+ &= L_J \prod_{i=1}^{t} \exp(e_{\beta_i}) \left(\prod_{i=1}^{t} \exp(-e_{\beta_i}) \exp(f_{\beta_i}) \exp(-e_{\beta_i}) \right) \prod_{j=t+1}^{s} w_{\beta_j} P_J^+ \\ &\subset P_J^+ . w_s P_J^+ . \end{aligned}$$

The second main result of this section is contained in the following Theorem.

4.13. Theorem. *Assume that P_J^+ is a parabolic subgroup of G with abelian unipotent radical. For all $0 \leq s \leq t \leq k$, the H-orbits of symplectic leaves of $(L_J.x_{st}P_J^+, \pi_J|_{L_J.x_{st}P_J^+})$ are $\mathcal{S}^J_{(v_1 w_t w_\circ^J, v_2 w_s v_3)}$ (cf. (1.9)) for unique*

$$v_3 \in W_J, \qquad v_1 \in W_J \cap W_{\min}^{J\cap w_t(J)}, \qquad v_2 \in W_J \cap W_{\min}^{J\cap w_s(J)} \tag{4.13}$$

such that $v_1 w_t w_\circ^J \leq v_2 w_s v_3$.

As an example, let us note that the L_J-orbits inside the open B^--orbit on G/P_J^+ are the orbits $L_J.x_{0s}P_J^+$, for $0 \leq s \leq k$. Theorem 4.13 implies that the symplectic leaves of $(L_J.x_{0s}P_J^+, \pi_J|_{L_J.x_{0s}P_J^+})$ are $\mathcal{S}^J_{(w_\circ^J, v_2 w_t v_3)}$ for some $v_3 \in W_J$ and $v_2 \in W_J \cap W_{\min}^{J\cap w_s(J)}$ such that $w_\circ^J \leq v_2 w_s v_3$.

Proof. Because $\mathcal{U}_{\dot{w}_1,w_2} \subset B^- w_1 \cap B^+ w_2 B^+$ (recall (0.6)), we have

$$\mathcal{S}^J_{w_1,w_2} = \mathcal{U}_{\dot{w}_1,w_2}.P_J^+ \subset B^- x^J_{w_1} \cap B^+ x^J_{w_2} \subset P_J^- x^J_{w_1} \cap P_J^+ x^J_{w_2}.$$

It follows from Lemma 4.11, Theorem 4.12 (ii), and (1.3) that every element of Ω^J can be uniquely represented in the form $(v_1 w_t w_\circ^J, v_2 w_s v_3)$ for some $0 \leq s, t \leq k$ and v_3, v_1, v_2 as in (4.13) such that $v_1 w_t w_\circ^J \leq v_2 w_s v_3$. For such a pair, we have

$$\mathcal{S}^J_{(v_1 w_t w_\circ^J, v_2 w_s v_3)} \subset P_J^-.v_1 w_t w_\circ^J P_J^+ \cap P_J^+.v_2 w_s v_3 P_J^+ = P_J^+.w_t P_J^+ \cap P_J^-.w_s P_J^+.$$

Theorem 4.12 implies that the last intersection is nontrivial only if $t \leq s$, in which case it is equal to $L_J.x_{ts}P_J^+$. This completes the proof of the Theorem. □

5. The open B^--orbits in compact Hermitian symmetric spaces

In this section, we treat in detail the restriction of the Poisson structure π_J to the open B^--orbit of G/P_J^+ in the case when G is a complex simple group and P_J^+ is a parabolic subgroup with abelian unipotent radical. First, in §5.1, we obtain general formulas for $\pi_J|_{B^-.P_J^+}$. Then, in §§5.3–5.6, we use it to derive explicit formulas for all classical groups and show that all such Poisson structures are quasiclassical limits of interesting classes of quadratic algebras, of the type known as *quantized coordinate rings* of classical varieties. (See, e.g., [10] for a general survey of quantized coordinate rings.) In §5.7, we show that the E_6 case gives rise to a new quadratic Poisson structure on a 16 dimensional affine space, related to a half-spin representation of $\mathfrak{so}_{10}$.

5.1. For a general reductive Lie algebra $\mathfrak{g}$ with fixed dual Borel subalgebras $\mathfrak{b}^\pm$ as in the Introduction, consider an irreducible representation $V_\mathfrak{g}^\lambda$ of $\mathfrak{g}$ with highest weight λ. Define the bivector field

$$\pi_\mathfrak{g}^\lambda = \chi(r_\mathfrak{g}) \in \Gamma((V_\mathfrak{g}^\lambda)^*, \bigwedge^2 T(V_\mathfrak{g}^\lambda)^*) \tag{5.1}$$

(cf. (0.1) and §0.2), where χ is derived from the action of $\mathfrak{g}$ on $(V_\mathfrak{g}^\lambda)^*$. If $\pi_\mathfrak{g}^\lambda$ is Poisson, then the corresponding Poisson bracket on the algebra of regular functions on $(V_\mathfrak{g}^\lambda)^*$, identified with the symmetric algebra $S(V_\mathfrak{g}^\lambda)$, is induced by

$$\{v_1, v_2\} = m(r_\mathfrak{g}(v_1 \otimes v_2)), \quad v_1, v_2 \in V_\mathfrak{g}^\lambda, \tag{5.2}$$

where $m : V_\mathfrak{g}^\lambda \otimes V_\mathfrak{g}^\lambda \to S(V_\mathfrak{g}^\lambda)$ is the multiplication map, $m(v_1, v_2) = v_1 v_2$.

In the setting of §0.2, for $\alpha \in \Delta_J^+$ and $\beta \in \Delta^+ \backslash \Delta_J^+$ set

$$(5.3) \qquad [e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}, \quad [f_\alpha, e_\beta] = N_{-\alpha,\beta} e_{-\alpha+\beta}, \quad N_{\alpha,\beta}, N_{-\alpha,\beta} \in \mathbb{C}.$$

For a Lie algebra $\mathfrak{g}$, we will denote by $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ its derived subalgebra.

Finally, for arbitrary $J \subset \Gamma$, denote by $\theta|_J$ the restriction of the highest root of $\mathfrak{g}$ to (a dominant weight of) $\mathfrak{l}_J$.

Proposition. *Assume that G is a complex simple algebraic group and P_J^+ is a parabolic subgroup of G with abelian unipotent radical. Then the following hold:*

(i) *Under the adjoint action of $\mathfrak{l}_J$, the space $\mathfrak{n}_J^+$ is an irreducible representation of $\mathfrak{l}'_J$ with highest weight $\theta|_J$. The restriction of the Poisson structure π_J to $B^-.P_J^+ \subset G/P_J^+$, identified with $\mathfrak{n}_J^-$ by $x \mapsto \exp(x)P_J^+$, coincides with $-\pi_{\mathfrak{l}'_J}^{\theta|_J}$; cf.* (5.1)–(5.2).

(ii) *The Poisson structure on $\mathfrak{n}_J^- = \{\sum_{\beta \in \Delta^+ \backslash \Delta_J^+} y_\beta f_\beta \mid y_\beta \in \mathbb{C}\} \cong \mathbb{A}^{|\Delta^+| - |\Delta_J^+|}$ from the first part is also is given by*

$$\{y_\beta, y_\gamma\} = \sum_{\alpha \in \Delta_J^+} \frac{\langle \alpha, \alpha \rangle}{2} \big(-N_{\alpha,\beta} N_{-\alpha,\gamma} y_{\alpha+\beta} y_{-\alpha+\gamma} + N_{-\alpha,\beta} N_{\alpha,\gamma} y_{-\alpha+\beta} y_{\alpha+\gamma} \big),$$
$$\beta, \gamma \in \Delta^+ \backslash \Delta_J^+.$$

Proof. The first statement of part (i) is well known. The simplest way to show it here is to observe that L_J has finitely many orbits on $\mathfrak{n}_J^-$ (acting by the adjoint action) because it has finitely many orbits on G/P_J^+. This implies that $\mathfrak{n}_J^-$ is an irreducible representation of $\mathfrak{l}'_J$ (under the adjoint action) and it must have highest weight $\theta|_J$.

The second statement in part (i) now follows from the first one, Proposition 4.2(ii), and the definition (5.1).

Part (ii) is an immediate corollary of part (i); see (5.2). □

5.2. Assume that the positive simple roots $\alpha_1, \ldots, \alpha_N$ of the simple algebraic group G are enumerated as in [1]. Recall that only maximal parabolic subgroups P_J^+ of G can have an abelian unipotent radical. Moreover, those are exactly the parabolic subgroups $P^+_{\Gamma \backslash \{\alpha_m\}}$ for which the simple root α_m appears with multiplicity 1 in the expansion of the highest root θ of $\mathfrak{g}$ in terms of the positive simple roots, i.e., the α_m height of θ is equal to 1. In other words, α_m should be a cominuscule root. This leads to the following choices for α_m according to the type of G:

A_N	$\alpha_1, \ldots, \alpha_N$,
B_N	α_1,
C_N	α_N,
D_N	$\alpha_1, \alpha_{N-1}, \alpha_N$,
E_6	α_1, α_6,
E_7	α_7.

For the other types (E_8, F_4, G_2), there are no parabolic subgroups with abelian unipotent radicals. Below we will denote by ω_m the m-th fundamental weight of G.

5.3. A_{N-1} **case.** In this case, $\mathfrak{g} = \mathfrak{sl}_N$,

$$\theta = \alpha_1 + \ldots + \alpha_N = \omega_1 + \omega_N,$$

and all simple roots α_m $(m = 1, \ldots, N)$ are cominuscule. The derived subalgebra of the Levi factor of the parabolic subalgebra $\mathfrak{p}^+_{\Gamma\setminus\{\alpha_m\}}$ is $\mathfrak{l}'_{\Gamma\setminus\{\alpha_m\}} \cong \mathfrak{sl}_m \oplus \mathfrak{sl}_{N-m+1}$. Under the adjoint action of $\mathfrak{l}'_{\Gamma\setminus\{\alpha_m\}}$, the nilradical $\mathfrak{n}^-_{\Gamma\setminus\{\alpha_m\}}$ is identified with $V^{\omega_1}_{\mathfrak{sl}_m} \otimes \left(V^{\omega_1}_{\mathfrak{sl}_{N-m+1}}\right)^*$, where $V^{\omega_1}_{\mathfrak{sl}_l}$ denotes the vector representation of $\mathfrak{sl}_l$. In Part I, we showed that, after applying the twist [3, eq. (3.14)], the induced Poisson structure on $\mathfrak{n}^-_{\Gamma\setminus\{\alpha_m\}}$ corresponds to the standard quadratic Poisson structure on the space of rectangular matrices $M_{N-m+1,m}$ (cf. [3, eq. (1.7)]). It is the quasiclassical limit of the quadratic algebra of *quantum matrices*.

5.4. B_{N+1} **case.** In this case, $\mathfrak{g} \cong \mathfrak{so}_{2N+1}$,

$$\theta = \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_N = \omega_2,$$

and the only root that is cominuscule is α_1. Furthermore, $\mathfrak{l}'_{\Gamma\setminus\{\alpha_1\}} \cong \mathfrak{so}_{2N-1}$, and under the adjoint action the nilradical $\mathfrak{n}^-_{\Gamma\setminus\{\alpha_1\}}$ corresponds to the vector representation $V^{\omega_1}_{\mathfrak{so}_{2N-1}}$ of $\mathfrak{so}_{2N-1}$. If we identify

$$\mathfrak{n}^-_{\Gamma\setminus\{\alpha_1\}} \cong \{\sum_{i=2}^{N} x_i(E_{N+i,1} - E_{N+1,i}) + \sum_{j=2}^{N} y_j(E_{j,1} - E_{N+1,N+j}) \\ + z(E_{2N+1,1} - E_{N+1,2N+1}) \bigm| x_i, y_j, z \in \mathbb{C}\bigr\} = \mathbb{A}^{2N-1},$$

then the induced Poisson structure on $\mathfrak{n}^-_{\Gamma\setminus\{\alpha_1\}}$ is given by

$$\begin{aligned} 2\{x_i, x_j\} &= x_i x_j, && i < j, \\ 2\{y_i, y_j\} &= -y_i y_j, && i < j, \\ 2\{z, x_i\} &= -z x_i, && \text{for all } z, \\ 2\{z, y_j\} &= z y_j, && \text{for all } j, \\ 2\{x_i, y_j\} &= x_i y_j, && i \neq j, \\ 2\{x_i, y_i\} &= -2\sum_{l>i} x_l y_l - z^2, && \text{for all } i. \end{aligned}$$

This is the quasiclassical limit of the *odd dimensional quantum Euclidean space*, $O^{2N-1}_{q^{1/2}}(\mathbb{C})$, introduced by Reshetikhin, Takhtadzhyan, and Faddeev, [18, Definition 12]. (See [16, §§2.1, 2.2] for a simplified set of relations.)

5.5. C_{N+1} **case.** In this case, $\mathfrak{g} \cong \mathfrak{sp}_{2N}$ and

$$\theta = 2\alpha_1 + \ldots + 2\alpha_{N-1} + \alpha_N = 2\omega_1.$$

The only cominuscule root is α_N, and $\mathfrak{l}'_{\Gamma\setminus\{\alpha_N\}} \cong \mathfrak{sl}_N$. Then under the adjoint action the nilradical $\mathfrak{n}^+_{\Gamma\setminus\{\alpha_N\}}$ corresponds to the second symmetric power $S^2(V^{\omega_1}_{\mathfrak{sl}_N}) \cong V^{2\omega_1}_{\mathfrak{sl}_N}$ of the vector representation of $\mathfrak{sl}_N$. If we identify $\mathfrak{n}^-_{\Gamma\setminus\{\alpha_N\}}$ with the space of symmetric matrices of size N by

$$\mathfrak{n}^-_{\Gamma\setminus\{\alpha_N\}} = \left\{ \left[\begin{smallmatrix} 0 & 0 \\ Y & 0 \end{smallmatrix}\right] \,\middle|\, Y = (y_{ij})^N_{i,j=1},\ y_{ij} = y_{ji} \right\},$$

then the Poisson structure on $\mathfrak{n}^-_{\Gamma\setminus\{\alpha_N\}}$ is given by

$$\begin{aligned}\{y_{ij}, y_{lm}\} = &\big(\mathrm{sign}(m-i) + \mathrm{sign}(l-j)\big) y_{il} y_{jm}\\ &+ \big(\mathrm{sign}(l-i) + \mathrm{sign}(m-j)\big) y_{im} y_{jl}.\end{aligned}$$

Interestingly, this Poisson structure is not, as one might expect after seeing §5.4, the quasiclassical limit of a quantum symplectic space (see [18, Definition 14] or [16, §1.1]). Instead, it is the quasiclassical limit of the algebra of *quantum symmetric matrices* introduced by Noumi in [17, Theorem 4.3, Proposition 4.4, and comments following the proof] (with the parameters a_k all set equal to 1), and of the one given by Kamita [11, Theorem 0.2] (with q and q^{-1} interchanged).

5.6. D_N **case.** In this case, $\mathfrak{g} = \mathfrak{so}_{2N}$,

$$\theta = \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{N-2} + \alpha_{N-1} + \alpha_N = \omega_2,$$

and the cominuscule roots are α_1, α_{N-1}, and α_N. (We will assume $N \geq 4$.) Below we consider those three cases.

(a) Root α_1**.** Here $\mathfrak{l}'_{\Gamma\setminus\{\alpha_1\}} \cong \mathfrak{so}_{2(N-1)}$ and $\mathfrak{n}^-_{\Gamma\setminus\{\alpha_1\}}$, considered an $\mathfrak{l}'_{\Gamma\setminus\{\alpha_1\}}$-module under the adjoint action, is isomorphic to the vector representation $V^{\omega_1}_{\mathfrak{so}_{2(N-1)}}$ of $\mathfrak{so}_{2(N-1)}$. We identify

$$\begin{aligned}\mathfrak{n}^-_{\Gamma\setminus\{\alpha_1\}} &= \{\sum_{i=2}^{N} x_i (E_{N+i,1} - E_{N+1,i}) + \sum_{j=2}^{N} y_j (E_{j,1} - E_{N+1,N+j}) \bigm| x_i, y_j \in \mathbb{C}\}\\ &= \mathbb{A}^{2(N-1)}.\end{aligned}$$

Then the induced Poisson structure on $\mathfrak{n}^-_{\Gamma\setminus\{\alpha_1\}}$ is given by

$$\begin{aligned}2\{x_i, x_j\} &= x_i x_j, && i < j,\\ 2\{y_i, y_j\} &= -y_i y_j, && i < j,\\ 2\{x_i, y_j\} &= x_i y_j, && i \neq j,\\ \{x_i, y_i\} &= -\sum_{l>i} x_l y_l, && \text{for all } i.\end{aligned}$$

It is the quasiclassical limit of the *even dimensional quantum Euclidean space* $O^{2(N-1)}_{q^{1/2}}(\mathbb{C})$; see [18], [16].

(b) Root α_N**.** In this case $\mathfrak{l}'_{\Gamma\setminus\{\alpha_N\}} \cong \mathfrak{sl}_N$, and under the adjoint action $\mathfrak{n}^+_{\Gamma\setminus\{\alpha_N\}}$ is isomorphic to $V^{\omega_2}_{\mathfrak{sl}_N} \cong \bigwedge^2 V^{\omega_1}_{\mathfrak{sl}_N}$, where $V^{\omega_1}_{\mathfrak{sl}_N}$ is the vector representation of $\mathfrak{sl}_N$. Identify $\mathfrak{n}^-_{\Gamma\setminus\{\alpha_N\}}$ with the space of skew-symmetric matrices of size N by

$$\mathfrak{n}^-_{\Gamma\setminus\{\alpha_N\}} = \left\{ \left[\begin{smallmatrix} 0 & 0 \\ Y & 0 \end{smallmatrix}\right] \,\middle|\, Y = (y_{ij})_{i,j=1}^N,\ y_{ij} = -y_{ji} \right\}.$$

Then the Poisson structure on $\mathfrak{n}^-_{\Gamma\setminus\{\alpha_N\}}$ is given by

$$\begin{aligned}2\{y_{ij}, y_{lm}\} = &\big(\mathrm{sign}(l-j) + \mathrm{sign}(m-i)\big) y_{il} y_{jm}\\ &- \big(\mathrm{sign}(l-i) + \mathrm{sign}(m-j)\big) y_{im} y_{jl}.\end{aligned}$$

This is the quasiclassical limit of the algebra of *quantum antisymmetric matrices* introduced by Strickland in [22, Section 1] (with q replaced by $q^{1/2}$).

(c) Root α_{N-1}. One can lift the involutive automorphism of the Dynkin graph D_N (preserving $\alpha_1, \dots, \alpha_{N-2}$, and interchanging α_{N-1} and α_N) to an automorphism of $\mathfrak{so}_{2N}$ that restricts to an isomorphism between $\mathfrak{l}_{\Gamma \setminus \{\alpha_{N-1}\}}$ and $\mathfrak{l}_{\Gamma \setminus \{\alpha_N\}}$ (interchanging their r-matrices) and their modules $\mathfrak{n}^-_{\Gamma \setminus \{\alpha_{N-1}\}}$ and $\mathfrak{n}^-_{\Gamma \setminus \{\alpha_N\}}$. As a result, that automorphism of $\mathfrak{so}_{2N}$ restricts to an isomorphism between the Poisson structures on $\mathfrak{n}^-_{\Gamma \setminus \{\alpha_{N-1}\}}$ and $\mathfrak{n}^-_{\Gamma \setminus \{\alpha_N\}}$.

5.7. *E_6* **case.** With this example, we show that compact Hermitian symmetric spaces for exceptional groups give rise to new interesting quadratic Poisson structures on affine spaces.

The highest root of the simple Lie algebra $\mathfrak{e}_6$ of type E_6 is

$$\theta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 = \omega_2,$$

and $\mathfrak{e}_6$ has two cominuscule roots: α_1 and α_6. Similarly to §5.6(c), one lifts the involutive automorphism of the Dynkin graph E_6 that interchanges α_1 and α_6 and fixes the other nodes to an automorphism of $\mathfrak{e}_6$ that restricts to an isomorphism between $\mathfrak{l}'_{\Gamma \setminus \{\alpha_1\}}$ and $\mathfrak{l}'_{\Gamma \setminus \{\alpha_6\}}$, and between their modules $\mathfrak{n}^-_{\Gamma \setminus \{\alpha_1\}}$ and $\mathfrak{n}^-_{\Gamma \setminus \{\alpha_6\}}$. This linear map provides an isomorphism between the induced Poisson structures on $\mathfrak{n}^-_{\Gamma \setminus \{\alpha_1\}}$ and $\mathfrak{n}^-_{\Gamma \setminus \{\alpha_6\}}$.

In the case of the root α_1, we have $\mathfrak{l}'_{\Gamma \setminus \{\alpha_1\}} \cong \mathfrak{so}_{10}$, and as an $\mathfrak{l}'_{\Gamma \setminus \{\alpha_1\}}$-module $\mathfrak{n}^+_{\Gamma \setminus \{\alpha_1\}}$ is isomorphic to one of the half-spin representations $V^{\omega_5}_{\mathfrak{so}_{10}}$ of $\mathfrak{so}_{10}$. As a vector space, $V^{\omega_5}_{\mathfrak{so}_{10}}$ is identified with

$$V \oplus \bigwedge^3 V \oplus \bigwedge^5 V$$

for a 5 dimensional vector space V; cf. [9, Section 20]. A basis $\{v_1, \dots, v_5\}$ of V gives rise to the basis $\{v_{i_1} \wedge \dots \wedge v_{i_n} \mid n \text{ odd}, \ i_1 < \dots < i_n\}$ of $V^{\omega_5}_{\mathfrak{so}_{10}}$. We view it as a set of coordinate functions

$$\{y_I \mid I \subset \{1, \dots, 5\}, \ |I| \text{ odd}\}$$

on $(V^{\omega_5}_{\mathfrak{so}_{10}})^*$. In terms of those coordinates, the induced quadratic Poisson structure on this 16 dimensional affine space is given by

$$\begin{aligned}\{y_I, y_J\} = &\sum_{\substack{i \in I \setminus J \\ j \in J \setminus I \\ i \neq j}} \operatorname{sign}(j-i) a^{I,J}_{i,j} y_{(I \setminus \{i\}) \cup \{j\}} y_{(J \setminus \{j\}) \cup \{i\}} \\ &- \frac{1}{2} \sum_{\substack{\{i,j\} \subset I \setminus J \\ i<j}} a^{I,J}_{i,j} y_{I \setminus \{i,j\}} y_{J \cup \{i,j\}} + \frac{1}{2} \sum_{\substack{\{i,j\} \subset J \setminus I \\ i<j}} a^{I,J}_{i,j} y_{I \cup \{i,j\}} y_{J \setminus \{i,j\}}.\end{aligned}$$

Here, for two subsets I and J, and two elements i and j of $\{1, \dots, 5\}$, we set

$$I(i,j) = \{l \in I \mid i < l < j \ \text{ or } \ j < l < i\}$$

and

$$a^{I,J}_{i,j} = (-1)^{|I(i,j)| + |J(i,j)|}.$$

For example, for $i < j$,

$$\{y_{\{i\}}, y_{\{j\}}\} = y_{\{i\}} y_{\{j\}}.$$

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